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Non-Parametric Tests for Testing of Scale Parameters

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One of the fundamental problems in testing of equality of populations is of testing the equality of scale parameters. The subsequent usages for scale are dispersion, spread and variability. In this paper, we proposed non-parametric tests based on U-Statistics for the testing of equality of scale parameters. The null distribution of proposed tests is developed and its Pitman efficiency is worked out to compare proposed tests with respect to some existing tests. Simulation study is carried out to compute the asymptotic power of proposed tests. An illustrative example is also provided.

Keywords: Two-sample scale, common quantile, null distribution, Pitman efficiency, illustrative example, asymptotic power

Introduction

Let $X_1,\ldots,X_{n_1}$ and $Y_1,\ldots,Y_{n_2}$ be independent random samples from two populations *X* and *Y* having absolutely continuous cumulative distribution functions (cdfs) F(x) and G(y), respectively. We assumed that both of these distributions are alike except differing in their scale parameters with common known quantile ξ_q of order q, $0 \le q \le 1$. Thus, if we take $G(y) = F(y / \theta)$, then we wish to test the null hypothesis

$$\mathrm{H}_0 : \theta = 1, \text{ with } \mathrm{F}\left(\xi_q\right) = \mathrm{G}\left(\xi_q\right) = q$$

against the alternative hypothesis

$$\mathrm{H}_\mathrm{A} : \theta > 1, \text{ with } \mathrm{F}\left(\xi_q\right) = \mathrm{G}\left(\xi_q\right) = q .$$

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Under the null hypothesis, *X*s and *Y*s are alike, but under alternative, *Y*s will have more variation than *X*s. Without loss of generality, we assume that ξ_q is zero for pre-specified *q*.

If both the distributions have common quantile of order 0.5, then a number of non-parametric tests are available in literature including Mood (1954), Sukhatme (1957), Ansari and Bradley (1960), Klotz (1962), Tamura (1966), Yanagawa (1970), Kochar and Gupta (1986), Kössler (1994, 1999), Öztürk (2001), Kössler and Kumar (2010), and Kumar and Goyal (2018).

Deshpande and Kusum (1984) proposed a test under the assumption that the two distributions have a common known quantile of order *q*, $0 \leq q \leq 1$, which was further modified by Mahajan et al. (2011), Kössler and Kumar (2016), and Goyal and Kumar (2018). This type of problem has a number of practical applications in agriculture, engineering, business, industries, biology, atmospheric and chemical sciences.

The Proposed Tests

Consider *m* as a fixed non-negative integer and *k* as a fixed positive integer such that $1 \leq 2m + 1 \leq n_1$ and $1 \leq k \leq n_2$. Define the following two symmetrical kernels, $h^{(1)}$ and $h^{(2)}$, as

$$h^{(1)}\left(X_1,\ldots,X_{2m+1};Y_1,\ldots,Y_k\right)=\begin{cases}1 & \text{if } 0\leq \text{Med}_X \leq \text{Min}_Y \text{ and } \text{Min}_X \geq 0 \\ & \text{or } \text{Max}_Y \leq \text{Med}_X \leq 0 \text{ and } \text{Max}_X \leq 0 \\ 0 & \text{otherwise}\end{cases}$$

$$h^{(2)}\left(X_1,\ldots,X_{2m+1};Y_1,\ldots,Y_k\right)=\begin{cases}1 & \text{if } 0\leq \text{Med}_X \leq \text{Min}_Y \text{ and } \text{Min}_X \geq 0 \\ & \text{or } \text{Max}_Y \leq \text{Med}_X \leq 0 \text{ and } \text{Max}_X \leq 0 \\ -1 & \text{if } 0\leq \text{Min}_Y \leq \text{Med}_X \text{ and } \text{Min}_X \geq 0 \\ & \text{or } \text{Med}_X \leq \text{Max}_Y \leq 0 \text{ and } \text{Max}_X \leq 0 \\ 0 & \text{otherwise}\end{cases} \tag{1}$$

where Med_X = Median of $X_1,\ldots, X_{2m+1}$, Min_X (or Max_X) = Minimum (or Maximum) of $X_1,\ldots, X_{2m+1}$, and Min_Y (or Max_Y) = Minimum (or Maximum) of $Y_1,\ldots, Y_k$.

The *U*-Statistics associated with kernel $h^{(c)}$, $c = 1, 2$, and of degree $(2m + 1, k)$, is defined as

$$U_{m,k}^{(c)}\left(X_1,\ldots,X_{n_1};Y_1,\ldots,Y_{n_2}\right)= \left[\binom{n_1}{2m+1}\binom{n_2}{k}\right]^{-1}\sum_s h^{(c)}\left(X_{i_1},\ldots,X_{i_{2m+1}};Y_{j_1},\ldots,Y_{j_k}\right) \tag{2}$$

where s is summation extended over all possible combinations $(i_1,\ldots, i_{2m+1})$ of $2m + 1$ integers chosen from $(1,\ldots, n_1)$ and all possible combinations $(j_1,\ldots, j_k)$ of k integers chosen from $(1,\ldots, n_2)$.

The test rejects H_0 in favor of H_A for large values of $U_{m,k}^{(c)}$; $c = 1, 2$. In particular,

1) For $m = 0$ and $k = 1$, the test statistic $U_{m,k}^{(1)}$ corresponds to test statistics proposed by Sukhatme (1957).
2) For $m = 0$ and $k = 1$, the test statistic $U_{m,k}^{(2)}$ corresponds to test statistics proposed by Deshpande and Kusum (1984).
3) For $k = 1$, the test statistic $U_{m,k}^{(c)}$; $c = 1, 2$. corresponds to test statistics proposed by Goyal and Kumar (2018).

Distribution of the Test Statistics

The expectation of test statistic $U_{m,k}^{(c)}$ is

$$\begin{aligned}\mathrm{E}\left(U_{m,k}^{(c)}\right)&=\left[\binom{n_1}{2m+1}\binom{n_2}{k}\right]^{-1}\sum_s h^{(c)}\left(X_{i_1},\ldots,X_{i_{2m+1}};Y_{j_1},\ldots,Y_{j_k}\right)\\ &=\mathrm{E}\left[h^{(c)}\left(X_1,\ldots,X_{2m+1};Y_1,\ldots,Y_k\right)\right]\end{aligned} \tag{3}$$

For $c = 1$,

$$\mathrm{E}\left(U_{m,k}^{(1)}\right)=\int_0^{\infty}\mathrm{P}\left(\mathrm{Med}_X\le t\right)\mathrm{P}\left(\mathrm{Min}_Y=t\right)dt+\int_{-\infty}^{0}\mathrm{P}\left(\mathrm{Med}_X\ge t\right)\mathrm{P}\left(\mathrm{Max}_Y=t\right)dt\,.$$

Under H_0,

$$\mathrm{E}\left(U_{m,k}^{(1)}\right)=\frac{k\left[\left(1-q\right)^{2m+k+1}+q^{2m+k+1}\right]}{\left(2m+k+1\right)}\left\{\sum_{i=m+1}^{2m+1}\frac{\binom{2m+1}{i}}{\binom{2m+k}{i}}\right\}. \tag{4}$$

For $c = 2$,

$$\mathrm{E}\left(U_{m,k}^{(2)}\right)=\int_0^\infty \mathrm{P}\left(\mathrm{Med}_X \le t\right)\mathrm{P}\left(\mathrm{Min}_Y = t\right)dt+\int_{-\infty}^0 \mathrm{P}\left(\mathrm{Med}_X \ge t\right)\mathrm{P}\left(\mathrm{Max}_Y = t\right)dt$$
$$-\int_0^\infty \mathrm{P}\left(\mathrm{Med}_X \ge t\right)\mathrm{P}\left(\mathrm{Min}_Y = t\right)dt-\int_{-\infty}^0 \mathrm{P}\left(\mathrm{Med}_X \le t\right)\mathrm{P}\left(\mathrm{Max}_Y = t\right)dt$$

Under H_0,

$$\mathrm{E}\left(U_{m,k}^{(2)}\right)=\frac{k\left[\left(1-q\right)^{2m+k+1}+q^{2m+k+1}\right]}{\left(2m+k+1\right)}\sum_{i=m+1}^{2m+1}\binom{2m+1}{i}\left[\frac{1}{\binom{2m+k}{i}}-\frac{1}{\binom{2m+k}{i+k-1}}\right]. \tag{5}$$

The following theorem provides us the asymptotic normality of $U_{m,k}^{(c)}$, which follows from the well-known theory of *U*-Statistics given by Lehmann (1963).

Theorem 1. Let $N = n_1 + n_2$. The asymptotic distribution of $N^{1/2}\left[U_{m,k}^{(c)}-\mathrm{E}\left(U_{m,k}^{(c)}\right)\right]$ as $N \to \infty$ in such a way that that $(n_1 / N) \to \lambda$, $0 < \lambda < 1$, is normal with mean zero and variance, $\sigma^2\left(U_{m,k}^{(c)}\right)$, given by

$$\sigma^2\left(U_{m,k}^{(c)}\right)=\frac{\left(2m+1\right)^2\zeta_{10}^{(c)}}{\lambda}+\frac{\zeta_{01}^{(c)}}{1-\lambda}. \tag{6}$$

Here,

$$\zeta_{10}^{(c)}=\mathrm{E}\left[\left(h^{(c)}\left(x,X_2,\ldots,X_{2m+1};Y_1,Y_2,\ldots,Y_k\right)\right)^2\right]-\left[\mathrm{E}\left(U_{m,k}^{(c)}\right)\right]^2$$

and

$$\zeta_{01}^{(c)} = \mathrm{E}\left[\left(h^{(c)}\left(X_1, X_2, \ldots, X_{2m+1}; y, Y_2, \ldots, Y_k\right)\right)^2\right] - \left[\mathrm{E}\left(U_{m,k}^{(c)}\right)\right]^2,$$

where

$$h^{(c)}\left(x, X_2, \ldots, X_{2m+1}; Y_1, Y_2, \ldots, Y_k\right) = \mathrm{E}\left[h^{(c)}\left(X_1, X_2, \ldots, X_{2m+1}; Y_1, Y_2, \ldots, Y_k\right) \mid X_1 = x\right]$$

and

$$h^{(c)}\left(X_1, X_2, \ldots, X_{2m+1}; y, Y_2, \ldots, Y_k\right) = \mathrm{E}\left[h^{(c)}\left(X_1, X_2, \ldots, X_{2m+1}; Y_1, Y_2, \ldots, Y_k\right) \mid Y_1 = y\right]$$

.

Under H_0, after some complex but standard computations, we elaborate the asymptotic null variance, $\sigma_0^2\left(U_{m,k}^{(c)}\right)$, as

$$\sigma_0^2\left(U_{m,k}^{(c)}\right) = \frac{(2m+1)^2\left\{k^2\rho_{m,k}^{(c)}\left[(1-q)^{4m+2k+1} + q^{4m+2k+1}\right] - \left[\mathrm{E}\left(U_{m,k}^{(c)}\right)\right]^2\right\}}{\lambda(1-\lambda)} \tag{7}$$

where, for $c = 1$,

$$\begin{aligned}\rho_{m,k}^{(1)} = &\binom{2m}{m}^2 \sum_{i=0}^{m}\sum_{j=0}^{m}\binom{m}{i}\binom{m}{j}\frac{(-1)^{i+j}(4m+2k-i-j+1)^{-1}}{(2m+k-i)(2m+k-j)} \\ &+ \frac{2\binom{2m}{m}}{(2m+k)}\left[\sum_{i=m+1}^{2m}\frac{\binom{2m}{i}}{\binom{2m+k-1}{i}}\right]\left[\sum_{j=0}^{m}\binom{m}{j}\frac{(-1)^{m+j}(2m+k-j+1)^{-1}}{(2m+k-j)}\right] \\ &+ \frac{1}{(2m+k)^2}\left[\sum_{i=m+1}^{2m}\frac{\binom{2m}{i}}{\binom{2m+k-1}{i}}\right]^2\end{aligned}$$

and for $c = 2$,

$$\rho_{m,k}^{(2)} = 4\binom{2m}{m}^2 \sum_{i=0}^{m}\sum_{j=0}^{m}\binom{m}{i}\binom{m}{j}\frac{(-1)^{i+j}\left(4m+2k-i-j+1\right)^{-1}}{\left(2m+k-i\right)\left(2m+k-j\right)}$$

$$+\frac{4\binom{2m}{m}}{\left(2m+k\right)}\left[\sum_{i=m+1}^{2m}\frac{\binom{2m}{i}}{\binom{2m+k-1}{i}}-\sum_{i=m}^{2m}\frac{\binom{2m}{i}}{\binom{2m+k-1}{i+k-1}}\right]\left[\sum_{j=0}^{m}\binom{m}{j}\frac{(-1)^{m+j}\left(2m+k-j+1\right)^{-1}}{\left(2m+k-j\right)}\right]$$

$$+\frac{1}{\left(2m+k\right)^2}\left[\sum_{i=m+1}^{2m}\frac{\binom{2m}{i}}{\binom{2m+k-1}{i}}-\sum_{i=m}^{2m}\frac{\binom{2m}{i}}{\binom{2m+k-1}{i+k-1}}\right]^2$$

Asymptotic Relative Efficiency

In order to see the performance of the proposed tests with respect to existing tests, we compute the Pitman asymptotic relative efficiencies (AREs) of $U_{m,k}^{(c)}$ tests relative to some existing tests, namely Sukhatme (1957) test (S), Mahajan et al. (2011) test (MGA), and some members of Kössler and Kumar (2016) test (T_l). Moreover, we also compare the proposed tests $U_{m,k}^{(1)}$ and $U_{m,k}^{(2)}$ with each other as well.

Remark 1. Pitman AREs of the proposed tests with respect to Deshpande and Kusum (1984) test is same as that of AREs with respect to Kössler and Kumar (2016) test (T_l) for $l = 1$.

The efficacy of the test statistics $U_{m,k}^{(c)}$ under the sequence of local alternatives, $\theta_N = N^{(-1/2)}\theta$, is

$$e^2\left(U_{m,k}^{(c)}\right)=\lim_{N\to\infty}\frac{\left[\frac{d}{d\theta}E\left(U_{m,k}^{(c)}\right)\Big|_{\theta=1}\right]^2}{N\sigma_0^2\left(U_{m,k}^{(c)}\right)}$$

$$=\frac{c^2\left[(2m+1)\binom{2m}{m}k\right]^2}{\sigma_0^2\left(U_{m,k}^{(c)}\right)}\left[\int_0^\infty\left(F(y)-q\right)^m\left(1-F(y)\right)^{m+k-1}f^2(y)\,dy\right.$$
$$\left.-\int_{-\infty}^0\left(F(y)\right)^{m+k-1}\left(q-F(y)\right)^m f^2(y)\,dy\right]^2 \quad (8)$$

The efficacies of the *S*, *MGA*, and T_l tests are

$$e^2(S)=\frac{\lambda(1-\lambda)}{\frac{1}{12}-q^2(1-q)^2}\left\{\int_{-\infty}^{\infty}|y|\,f^2(y)\,dy\right\}^2$$

$$e^2(MGA)=\frac{831600\,\lambda(1-\lambda)}{131\left(q^{11}+(1-q)^{11}\right)}\left\{\int_{-\infty}^{0}y\,F^2(y)\left[q-F(y)\right]^2 f^2(y)\,dy\right.$$
$$\left.-\int_0^\infty y\left[F(y)-q\right]^2\left[1-F(y)\right]^2 f^2(y)\,dy\right\}^2 \quad (9)$$

$$e^2(T_l)=\frac{4l^2(4l-1)\lambda(1-\lambda)}{\left(q^{4l-1}+(1-q)^{4l-1}\right)}\left\{\int_{-\infty}^{\infty}|y|\left[F(y)-q\right]^{2l-2}f^2(y)\,dy\right\}^2$$

AREs of $U_{m,k}^{(1)}$ and $U_{m,k}^{(2)}$ tests with respect to (w.r.t.) competing tests for some underlying distributions are given in Tables 1-10.

Table 1. AREs of $U^{(1)}_{m,k}$ w.r.t. competing tests for Uniform distribution

		q or (1 – *q*)								
		0.1			**0.3**			**0.5**		
Test	***m***	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**
S	0	1.0000	0.4500	0.2908	1.0000	0.3757	0.2260	1.0000	0.4167	0.2593
	1	0.2913	0.3118	0.2702	0.1256	0.1525	0.1465	0.6863	0.4242	0.3056
	2	0.1313	0.2119	0.2256	0.0421	0.0804	0.0987	0.5878	0.4222	0.3287
MGA	0	1.4079	0.6336	0.4095	1.7154	0.6445	0.3878	1.7013	0.7089	0.4411
	1	0.4101	0.4390	0.3804	0.2154	0.2616	0.2513	1.1676	0.7218	0.5198
	2	0.1849	0.2984	0.3176	0.0722	0.1380	0.1693	1.0000	0.7184	0.5593
T_1	0	0.8086	0.3639	0.2352	0.7859	0.2952	0.1776	1.0000	0.4167	0.2593
	1	0.2355	0.2521	0.2185	0.0987	0.1198	0.1151	0.6863	0.4242	0.3056
	2	0.1062	0.1714	0.1824	0.0331	0.0632	0.0776	0.5878	0.4222	0.3287
T_2	0	0.3546	0.1595	0.1031	0.4105	0.1542	0.0928	0.4286	0.1786	0.1111
	1	0.1033	0.1106	0.0958	0.0515	0.0626	0.0601	0.2941	0.1818	0.1310
	2	0.0466	0.0751	0.0800	0.0173	0.0330	0.0405	0.2519	0.1810	0.1409
T_3	0	0.2257	0.1016	0.0656	0.2750	0.1033	0.0622	0.2727	0.1136	0.0707
	1	0.0657	0.0704	0.0610	0.0345	0.0419	0.0403	0.1872	0.1157	0.0833
	2	0.0296	0.0478	0.0509	0.0116	0.0221	0.0271	0.1603	0.1152	0.0897
$U^{(2)}_{m,k}$	0	0.8086	0.9185	1.0000	0.7859	0.8518	1.0000	1.0000	1.0000	1.0000
	1	0.3511	0.7823	1.3354	0.1753	0.6426	1.6456	1.0000	1.0000	1.0000
	2	0.1849	0.7442	1.9680	0.0722	0.6471	2.5552	1.0000	1.0000	1.0000

Table 2. AREs of $U^{(1)}_{m,k}$ w.r.t. competing tests for Normal distribution

		q or (1 – *q*)								
		0.1			**0.3**			**0.5**		
Test	***m***	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**
S	0	1.0000	0.7179	0.5370	1.0000	0.6122	0.4356	1.0000	0.6699	0.4946
	1	0.3977	0.5278	0.5059	0.1752	0.2650	0.2836	0.9150	0.7088	0.5739
	2	0.1999	0.3696	0.4251	0.0657	0.1437	0.1911	0.8672	0.7196	0.6123
MGA	0	0.9251	0.6642	0.4968	1.0998	0.6733	0.4791	1.1531	0.7724	0.5703
	1	0.3679	0.4883	0.4680	0.1927	0.2915	0.3119	1.0551	0.8173	0.6617
	2	0.1849	0.3420	0.3933	0.0722	0.1581	0.2101	1.0000	0.8298	0.7060
T_1	0	0.8086	0.5805	0.4343	0.7859	0.4811	0.3423	1.0000	0.6699	0.4946
	1	0.3216	0.4268	0.4091	0.1377	0.2083	0.2229	0.9150	0.7088	0.5739
	2	0.1616	0.2989	0.3438	0.0516	0.1130	0.1502	0.8672	0.7196	0.6123
T_2	0	0.5866	0.4212	0.3151	0.6525	0.3995	0.2842	0.7232	0.4845	0.3577
	1	0.2333	0.3096	0.2968	0.1143	0.1729	0.1851	0.6618	0.5126	0.4150
	2	0.1173	0.2168	0.2494	0.0429	0.0938	0.1247	0.6272	0.5204	0.4428
T_3	0	0.5485	0.3938	0.2946	0.6273	0.3840	0.2732	0.6690	0.4481	0.3309
	1	0.2181	0.2895	0.2775	0.1099	0.1662	0.1779	0.6122	0.4742	0.3839
	2	0.1096	0.2027	0.2332	0.0412	0.0902	0.1199	0.5802	0.4814	0.4096
$U^{(2)}_{m,k}$	0	0.8086	0.9185	1.0000	0.7859	0.8518	1.0000	1.0000	1.0000	1.0000
	1	0.3511	0.7823	1.3354	0.1753	0.6426	1.6456	1.0000	1.0000	1.0000
	2	0.1849	0.7442	1.9680	0.0722	0.6471	2.5552	1.0000	1.0000	1.0000

Table 3. AREs of $U_{m,k}^{(1)}$ w.r.t. competing tests for Logistic distribution

		***q* or (1 – *q*)**								
		0.1			**0.3**			**0.5**		
Test	***m***	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**
S	0	1.0000	0.7685	0.5900	1.0000	0.6552	0.4809	1.0000	0.7124	0.5419
	1	0.4129	0.5705	0.5581	0.1816	0.2859	0.3129	0.9385	0.7549	0.6251
	2	0.2112	0.4018	0.4698	0.0692	0.1557	0.2105	0.9002	0.7673	0.6647
MGA	0	0.8756	0.6728	0.5166	1.0441	0.6841	0.5021	1.1109	0.7914	0.6020
	1	0.3615	0.4995	0.4886	0.1897	0.2986	0.3267	1.0426	0.8386	0.6944
	2	0.1849	0.3518	0.4114	0.0722	0.1625	0.2198	1.0000	0.8524	0.7384
T_1	0	0.8086	0.6214	0.4771	0.7859	0.5149	0.3779	1.0000	0.7124	0.5419
	1	0.3339	0.4613	0.4512	0.1427	0.2247	0.2459	0.9385	0.7549	0.6251
	2	0.1708	0.3249	0.3799	0.0544	0.1223	0.1655	0.9002	0.7673	0.6647
T_2	0	0.6438	0.4948	0.3799	0.7073	0.4635	0.3401	0.7858	0.5598	0.4258
	1	0.2658	0.3673	0.3593	0.1285	0.2023	0.2213	0.7375	0.5932	0.4912
	2	0.1360	0.2587	0.3025	0.0489	0.1101	0.1489	0.7074	0.6030	0.5224
T_3	0	0.6475	0.4976	0.3821	0.7229	0.4737	0.3476	0.7687	0.5476	0.4166
	1	0.2674	0.3694	0.3614	0.1313	0.2067	0.2262	0.7215	0.5803	0.4805
	2	0.1367	0.2602	0.3042	0.0500	0.1125	0.1522	0.6920	0.5898	0.5110
$U_{m,k}^{(2)}$	0	0.8086	0.9185	1.0000	0.7859	0.8518	1.0000	1.0000	1.0000	1.0000
	1	0.3511	0.7823	1.3354	0.1753	0.6426	1.6456	1.0000	1.0000	1.0000
	2	0.1849	0.7442	1.9680	0.0722	0.6471	2.5552	1.0000	1.0000	1.0000

Table 4. AREs of $U_{m,k}^{(1)}$ w.r.t. competing tests for Laplace distribution

		***q* or (1 – *q*)**								
		0.1			**0.3**			**0.5**		
Test	***m***	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**
S	0	1.0000	0.8791	0.7105	1.0000	0.7425	0.5982	1.0000	0.7407	0.5833
	1	0.4436	0.6786	0.6972	0.1839	0.3212	0.3817	0.9341	0.7732	0.6573
	2	0.2372	0.4908	0.6011	0.0705	0.1730	0.2518	0.8941	0.7795	0.6897
MGA	0	0.7794	0.6852	0.5538	1.0247	0.7609	0.6130	1.1185	0.8285	0.6524
	1	0.3458	0.5289	0.5434	0.1884	0.3291	0.3912	1.0447	0.8648	0.7351
	2	0.1849	0.3825	0.4685	0.0722	0.1773	0.2581	1.0000	0.8719	0.7713
T_1	0	0.8086	0.7108	0.5745	0.7859	0.5835	0.4701	1.0000	0.7407	0.5833
	1	0.3587	0.5487	0.5637	0.1445	0.2524	0.3000	0.9341	0.7732	0.6573
	2	0.1918	0.3968	0.4860	0.0554	0.1360	0.1979	0.8941	0.7795	0.6897
T_2	0	0.7937	0.6978	0.5640	0.8433	0.6262	0.5045	0.8216	0.6086	0.4793
	1	0.3521	0.5386	0.5534	0.1550	0.2709	0.3219	0.7675	0.6353	0.5401
	2	0.1883	0.3895	0.4771	0.0594	0.1459	0.2124	0.7346	0.6405	0.5666
T_3	0	0.9290	0.8167	0.6601	0.9259	0.6875	0.5539	0.8107	0.6005	0.4729
	1	0.4121	0.6304	0.6477	0.1702	0.2974	0.3534	0.7573	0.6268	0.5329
	2	0.2204	0.4559	0.5584	0.0653	0.1602	0.2332	0.7249	0.6320	0.5591
$U_{m,k}^{(2)}$	0	0.8086	0.9185	1.0000	0.7859	0.8518	1.0000	1.0000	1.0000	1.0000
	1	0.3511	0.7823	1.3354	0.1753	0.6426	1.6456	1.0000	1.0000	1.0000
	2	0.1849	0.7442	1.9680	0.0722	0.6471	2.5552	1.0000	1.0000	1.0000

Table 5. AREs of $U_{m,k}^{(1)}$ w.r.t. competing tests for Cauchy distribution

		***q* or (1 – *q*)**								
		0.1			**0.3**			**0.5**		
Test	***m***	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**
S	0	1.0000	1.0286	0.8835	1.0000	0.8958	0.7676	1.0000	0.9375	0.8253
	1	0.4798	0.8026	0.8643	0.2086	0.4010	0.4940	1.0145	0.9799	0.9117
	2	0.2649	0.5818	0.7395	0.0845	0.2203	0.3286	1.0121	0.9897	0.9441
MGA	0	0.6981	0.7180	0.6168	0.8547	0.7656	0.6561	0.9880	0.9262	0.8154
	1	0.3349	0.5603	0.6033	0.1783	0.3428	0.4222	1.0023	0.9682	0.9008
	2	0.1849	0.4061	0.5162	0.0722	0.1883	0.2808	1.0000	0.9778	0.9328
T_1	0	0.8086	0.8317	0.7144	0.7859	0.7040	0.6032	1.0000	0.9375	0.8253
	1	0.3879	0.6490	0.6989	0.1640	0.3152	0.3882	1.0145	0.9799	0.9117
	2	0.2142	0.4705	0.5979	0.0664	0.1731	0.2582	1.0121	0.9897	0.9441
T_2	0	1.0503	1.0803	0.9280	1.0879	0.9745	0.8351	1.2117	1.1360	1.0000
	1	0.5039	0.8430	0.9078	0.2270	0.4363	0.5374	1.2293	1.1874	1.1048
	2	0.2782	0.6111	0.7767	0.0919	0.2397	0.3574	1.2264	1.1992	1.1440
T_3	0	1.6322	1.6788	1.4421	1.5549	1.3929	1.1936	1.5807	1.4819	1.3045
	1	0.7831	1.3101	1.4107	0.3244	0.6236	0.7682	1.6037	1.5490	1.4412
	2	0.4323	0.9496	1.2069	0.1314	0.3425	0.5109	1.5999	1.5644	1.4924
$U_{m,k}^{(2)}$	0	0.8086	0.9185	1.0000	0.7859	0.8518	1.0000	1.0000	1.0000	1.0000
	1	0.3511	0.7823	1.3354	0.1753	0.6426	1.6456	1.0000	1.0000	1.0000
	2	0.1849	0.7442	1.9680	0.0722	0.6471	2.5552	1.0000	1.0000	1.0000

Table 6. AREs of $U_{m,k}^{(2)}$ w.r.t. competing tests for Uniform distribution

		***q* or (1 – *q*)**								
		0.1			**0.3**			**0.5**		
Test	***m***	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**
S	0	1.2367	0.4899	0.2908	1.2724	0.4411	0.2260	1.0000	0.4167	0.2593
	1	0.8295	0.3986	0.2023	0.7165	0.2373	0.0890	0.6863	0.4242	0.3056
	2	0.7103	0.2848	0.1146	0.5829	0.1243	0.0386	0.5878	0.4222	0.3287
MGA	0	1.7412	0.6898	0.4095	2.1828	0.7566	0.3878	1.7013	0.7089	0.4411
	1	1.1679	0.5611	0.2848	1.2292	0.4070	0.1527	1.1676	0.7218	0.5198
	2	1.0000	0.4009	0.1614	1.0000	0.2132	0.0663	1.0000	0.7184	0.5593
T_1	0	1.0000	0.3962	0.2352	1.0000	0.3466	0.1776	1.0000	0.4167	0.2593
	1	0.6708	0.3223	0.1636	0.5631	0.1865	0.0700	0.6863	0.4242	0.3056
	2	0.5743	0.2303	0.0927	0.4581	0.0977	0.0304	0.5878	0.4222	0.3287
T_2	0	0.4385	0.1737	0.1031	0.5223	0.1810	0.0928	0.4286	0.1786	0.1111
	1	0.2941	0.1413	0.0717	0.2941	0.0974	0.0365	0.2941	0.1818	0.1310
	2	0.2518	0.1010	0.0406	0.2393	0.0510	0.0159	0.2519	0.1810	0.1409
T_3	0	0.2791	0.1106	0.0656	0.3499	0.1213	0.0622	0.2727	0.1136	0.0707
	1	0.1872	0.0900	0.0457	0.1970	0.0653	0.0245	0.1872	0.1157	0.0833
	2	0.1603	0.0643	0.0259	0.1603	0.0342	0.0106	0.1603	0.1152	0.0897
$U_{m,k}^{(1)}$	0	1.2367	1.0887	1.0000	1.2724	1.1740	1.0000	1.0000	1.0000	1.0000
	1	2.8479	1.2782	0.7489	5.7056	1.5562	0.6077	1.0000	1.0000	1.0000
	2	5.4083	1.3437	0.5081	13.8442	1.5453	0.3914	1.0000	1.0000	1.0000

Table 7. AREs of $U_{m,k}^{(2)}$ w.r.t. competing tests for Normal distribution

		***q* or (1 – *q*)**								
		0.1			**0.3**			**0.5**		
Test	***m***	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**
S	0	1.2367	0.7817	0.5370	1.2724	0.7187	0.4356	1.0000	0.6699	0.4946
	1	1.1325	0.6746	0.3789	0.9995	0.4124	0.1723	0.9150	0.7088	0.5739
	2	1.0810	0.4967	0.2160	0.9092	0.2221	0.0748	0.8672	0.7196	0.6123
MGA	0	1.1441	0.7231	0.4968	1.3995	0.7905	0.4791	1.1531	0.7724	0.5703
	1	1.0477	0.6241	0.3505	1.0993	0.4536	0.1896	1.0551	0.8173	0.6617
	2	1.0000	0.4595	0.1998	1.0000	0.2443	0.0822	1.0000	0.8298	0.7060
T_1	0	1.0000	0.6320	0.4343	1.0000	0.5648	0.3423	1.0000	0.6699	0.4946
	1	0.9157	0.5455	0.3064	0.7855	0.3241	0.1354	0.9150	0.7088	0.5739
	2	0.8741	0.4016	0.1747	0.7146	0.1746	0.0588	0.8672	0.7196	0.6123
T_2	0	0.7255	0.4586	0.3151	0.8303	0.4690	0.2842	0.7232	0.4845	0.3577
	1	0.6644	0.3958	0.2223	0.6522	0.2691	0.1125	0.6618	0.5126	0.4150
	2	0.6341	0.2914	0.1267	0.5933	0.1450	0.0488	0.6272	0.5204	0.4428
T_3	0	0.6783	0.4287	0.2946	0.7982	0.4508	0.2732	0.6690	0.4481	0.3309
	1	0.6212	0.3700	0.2078	0.6270	0.2587	0.1081	0.6122	0.4742	0.3839
	2	0.5929	0.2724	0.1185	0.5703	0.1393	0.0469	0.5802	0.4814	0.4096
$U_{m,k}^{(1)}$	0	1.2367	1.0887	1.0000	1.2724	1.1740	1.0000	1.0000	1.0000	1.0000
	1	2.8479	1.2782	0.7489	5.7056	1.5562	0.6077	1.0000	1.0000	1.0000
	2	5.4083	1.3437	0.5081	13.8442	1.5453	0.3914	1.0000	1.0000	1.0000

Table 8. AREs of $U_{m,k}^{(2)}$ w.r.t. competing tests for Logistic distribution

		***q* or (1 – *q*)**								
		0.1			**0.3**			**0.5**		
Test	***m***	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**
S	0	1.2367	0.8367	0.5900	1.2724	0.7692	0.4809	1.0000	0.7124	0.5419
	1	1.1759	0.7292	0.4179	1.0363	0.4450	0.1901	0.9385	0.7549	0.6251
	2	1.1421	0.5398	0.2387	0.9577	0.2405	0.0824	0.9002	0.7673	0.6647
MGA	0	1.0828	0.7326	0.5166	1.3286	0.8032	0.5021	1.1109	0.7914	0.6020
	1	1.0296	0.6385	0.3659	1.0821	0.4646	0.1985	1.0426	0.8386	0.6944
	2	1.0000	0.4727	0.2090	1.0000	0.2511	0.0860	1.0000	0.8524	0.7384
T_1	0	1.0000	0.6765	0.4771	1.0000	0.6045	0.3779	1.0000	0.7124	0.5419
	1	0.9508	0.5896	0.3379	0.8145	0.3497	0.1494	0.9385	0.7549	0.6251
	2	0.9235	0.4365	0.1930	0.7527	0.1890	0.0648	0.9002	0.7673	0.6647
T_2	0	0.7962	0.5387	0.3799	0.9000	0.5441	0.3401	0.7858	0.5598	0.4258
	1	0.7571	0.4695	0.2691	0.7330	0.3147	0.1345	0.7375	0.5932	0.4912
	2	0.7353	0.3476	0.1537	0.6774	0.1701	0.0583	0.7074	0.6030	0.5224
T_3	0	0.8008	0.5418	0.3821	0.9198	0.5561	0.3476	0.7687	0.5476	0.4166
	1	0.7614	0.4722	0.2706	0.7492	0.3217	0.1374	0.7215	0.5803	0.4805
	2	0.7396	0.3496	0.1546	0.6923	0.1739	0.0596	0.6920	0.5898	0.5110
$U_{m,k}^{(1)}$	0	1.2367	1.0887	1.0000	1.2724	1.1740	1.0000	1.0000	1.0000	1.0000
	1	2.8479	1.2782	0.7489	5.7056	1.5562	0.6077	1.0000	1.0000	1.0000
	2	5.4083	1.3437	0.5081	13.8442	1.5453	0.3914	1.0000	1.0000	1.0000

Table 9. AREs of $U_{m,k}^{(2)}$ w.r.t. competing tests for Laplace distribution

		***q* or (1 – *q*)**								
		0.1			**0.3**			**0.5**		
Test	***m***	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**
S	0	1.2367	0.9571	0.7105	1.2724	0.8718	0.5982	1.0000	0.7407	0.5833
	1	1.2634	0.8674	0.5221	1.0490	0.4998	0.2320	0.9341	0.7732	0.6573
	2	1.2830	0.6594	0.3054	0.9759	0.2674	0.0986	0.8941	0.7795	0.6897
MGA	0	0.9639	0.7460	0.5538	1.3039	0.8933	0.6130	1.1185	0.8285	0.6524
	1	0.9847	0.6760	0.4069	1.0749	0.5122	0.2377	1.0447	0.8648	0.7351
	2	1.0000	0.5139	0.2380	1.0000	0.2740	0.1010	1.0000	0.8719	0.7713
T_1	0	1.0000	0.7739	0.5745	1.0000	0.6851	0.4701	1.0000	0.7407	0.5833
	1	1.0216	0.7014	0.4222	0.8244	0.3928	0.1823	0.9341	0.7732	0.6573
	2	1.0375	0.5332	0.2470	0.7670	0.2101	0.0775	0.8941	0.7795	0.6897
T_2	0	0.9816	0.7597	0.5640	1.0731	0.7352	0.5045	0.8216	0.6086	0.4793
	1	1.0028	0.6884	0.4144	0.8846	0.4215	0.1956	0.7675	0.6353	0.5401
	2	1.0184	0.5234	0.2424	0.8230	0.2255	0.0831	0.7346	0.6405	0.5666
T_3	0	1.1489	0.8891	0.6601	1.1781	0.8071	0.5539	0.8107	0.6005	0.4729
	1	1.1736	0.8058	0.4850	0.9712	0.4628	0.2148	0.7573	0.6268	0.5329
	2	1.1919	0.6126	0.2837	0.9036	0.2476	0.0913	0.7249	0.6320	0.5591
$U_{m,k}^{(1)}$	0	1.2367	1.0887	1.0000	1.2724	1.1740	1.0000	1.0000	1.0000	1.0000
	1	2.8479	1.2782	0.7489	5.7056	1.5562	0.6077	1.0000	1.0000	1.0000
	2	5.4083	1.3437	0.5081	13.8442	1.5453	0.3914	1.0000	1.0000	1.0000

Table 10. AREs of $U_{m,k}^{(2)}$ w.r.t. competing tests for Cauchy distribution

		***q* or (1 – *q*)**								
		0.1			**0.3**			**0.5**		
Test	***m***	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**	***k* = 1**	***k* = 2**	***k* = 3**
S	0	1.2367	1.1198	0.8835	1.2725	1.0517	0.7677	1.0000	0.9375	0.8253
	1	1.3663	1.0259	0.6473	1.1905	0.6241	0.3002	1.0145	0.9799	0.9117
	2	1.4326	0.7818	0.3757	1.1701	0.3404	0.1286	1.0121	0.9897	0.9441
MGA	0	0.8633	0.7817	0.6168	1.0876	0.8989	0.6561	0.9880	0.9262	0.8154
	1	0.9537	0.7161	0.4518	1.0175	0.5335	0.2566	1.0023	0.9682	0.9008
	2	1.0000	0.5457	0.2623	1.0000	0.2910	0.1099	1.0000	0.9778	0.9328
T_1	0	1.0000	0.9055	0.7144	1.0000	0.8266	0.6033	1.0000	0.9375	0.8253
	1	1.1048	0.8296	0.5234	0.9356	0.4905	0.2359	1.0145	0.9799	0.9117
	2	1.1584	0.6321	0.3038	0.9195	0.2676	0.1011	1.0121	0.9897	0.9441
T_2	0	1.2990	1.1762	0.9280	1.3843	1.1442	0.8351	1.2117	1.1360	1.0000
	1	1.4351	1.0776	0.6798	1.2951	0.6790	0.3266	1.2293	1.1874	1.1048
	2	1.5047	0.8211	0.3947	1.2729	0.3704	0.1399	1.2264	1.1992	1.1440
T_3	0	2.0186	1.8278	1.4421	1.9787	1.6354	1.1937	1.5807	1.4819	1.3045
	1	2.2301	1.6745	1.0564	1.8511	0.9705	0.4668	1.6037	1.5490	1.4412
	2	2.3382	1.2760	0.6133	1.8194	0.5294	0.2000	1.5999	1.5644	1.4924
$U_{m,k}^{(1)}$	0	1.2367	1.0887	1.0000	1.2724	1.1740	1.0000	1.0000	1.0000	1.0000
	1	2.8479	1.2782	0.7489	5.7056	1.5562	0.6077	1.0000	1.0000	1.0000
	2	5.4083	1.3437	0.5081	13.8442	1.5453	0.3914	1.0000	1.0000	1.0000

From the tables of AREs, one can observe that the performance of the $U_{m,k}^{(1)}$ and $U_{m,k}^{(2)}$ tests depend upon the tail behavior of the underlying distribution. Also, these tests perform better than competing tests for some specified choices of *m* and *k*. We observe that these tests attain maximum efficiency as follows:

1) For Uniform, Normal, Logistic, and Laplace distributions, $U_{m,k}^{(1)}$ and $U_{m,k}^{(2)}$ tests attain maximum efficiency at $m = 0$, $k = 1$ for any q. Moreover, for Laplace distribution, $U_{m,k}^{(2)}$ attained maximum efficiency at *m* as large as possible and $k = 1$ for $q \notin (0.3, 0.7)$.
2) For the Cauchy distribution, $U_{m,k}^{(1)}$ and $U_{m,k}^{(2)}$ tests attain maximum efficiency at $m = 1$, $k = 1$ for $q = 0.5$. Now, for $q \neq 0.5$ and $q \in (0.3, 0.7)$, $U_{m,k}^{(1)}$ and $U_{m,k}^{(2)}$ attained maximum efficiency at $m = 0$, $k = 1$. Moreover, for $q \notin (0.3, 0.7)$, $U_{m,k}^{(1)}$ attained maximum efficiency at $m = 0$, $k = 2$ and $U_{m,k}^{(2)}$ attained maximum efficiency at m as large as possible and $k = 1$.
3) For the comparison of $U_{m,k}^{(1)}$ and $U_{m,k}^{(2)}$ tests with respect to each other, AREs doesn't depend upon the underlying distribution. In order to attain more efficiency, one should consider $U_{m,k}^{(2)}$ test in comparison to $U_{m,k}^{(1)}$ test.

An Illustrative Example

To see the execution of the tests $U_{m,k}^{(c)}$, we consider the data of Jung and Parekh (1970), in which the authors worked out a simple method for the determination of the total iron-binding capacity using Hyland Control Sera. They provide the total iron-binding capacity (TIBC) values using the proposed test (abbreviated as Jung-Parekh method) and the Ramsay method for a sample of 20 observations.

Now, it is of interest to check if there is more variation in determination of TIBC by using Jung-Parekh method in comparison to Ramsay method. Both data sets have common quantile of order 0.1, i.e., $q = 0.1$. The computed values $U_{m,k}^{(1)}$ and $U_{m,k}^{(2)}$, test statistics along with their *p*-values, are given in Table 11.

Table 11. Computed values of test statistics and corresponding *p*-values

		$U^{(1)}_{m,k}$		$U^{(2)}_{m,k}$	
k	***m***	**test statistic**	***p*-value**	**test statistic**	***p*-value**
1	0	0.4000	0.4541	-0.0200	0.4490
	1	0.3058	0.4311	-0.0330	0.4152
	2	0.2323	0.4147	-0.0330	0.4029
2	0	0.2261	0.4109	-0.2730	0.4197
	1	0.1521	0.3896	-0.2720	0.4099
	2	0.1069	0.3760	-0.2310	0.4358
3	0	0.1475	0.3988	-0.3490	0.4351
	1	0.0844	0.3677	-0.3440	0.4344
	2	0.0529	0.3486	-0.2890	0.4699

From the Table 11, it can be seen that both $U^{(1)}_{m,k}$ and $U^{(2)}_{m,k}$ tests statistics don't reject the null hypothesis. Thus, we conclude that in determination of TIBC by using Jung-Parekh method has not more variation in comparison to Ramsay method at 5% level of significance.

Simulation Study

We have proposed two test statistics, $U^{(1)}_{m,k}$ and $U^{(2)}_{m,k}$, for testing the hypothesis that the scale parameter is equal to one or greater than one, when distributions have common known quantile ξ_q of order q, $0 \le q \le 1$. In order to assess the performance of the proposed test statistics, we compute the asymptotic power of $U^{(1)}_{m,k}$ and $U^{(2)}_{m,k}$ tests by using the Monte Carlo simulation technique. Data is generated from Normal distribution with sample sizes n_1 and n_2 as n_1, n_2 = 10, 20, and 30. The scale parameters are considered as θ = 2, 2.5, and 3. The number of repetitions carried out is 10,000 and the level of significance is fixed at 5%. The asymptotic power of $U^{(1)}_{m,k}$ and $U^{(2)}_{m,k}$ tests for different values of *m* and *k* is given in Tables 12-17.

Table 12. Asymptotic power of $U^{(1)}_{m,k}$ for q = 0.1

		***k* = 1**			***k* = 2**			***k* = 3**		
***n*₁, *n*₂**	***θ***	***m* = 0**	***m* = 1**	***m* = 2**	***m* = 0**	***m* = 1**	***m* = 2**	***m* = 0**	***m* = 1**	***m* = 2**
10, 10	2.0	0.5541	0.2896	0.2508	0.5494	0.4080	0.3261	0.5013	0.3907	0.3853
	2.5	0.6804	0.3203	0.2903	0.6616	0.5124	0.4071	0.6198	0.5070	0.4608
	3.0	0.7915	0.3953	0.3267	0.7602	0.5714	0.4461	0.6956	0.5529	0.5225
20, 10	2.0	0.6093	0.3615	0.3390	0.6077	0.4940	0.3944	0.5486	0.4695	0.4398
	2.5	0.7320	0.4052	0.3950	0.7161	0.6091	0.4635	0.6634	0.5734	0.5229
	3.0	0.8914	0.4910	0.4056	0.8122	0.6796	0.4773	0.7421	0.6312	0.5537
20, 20	2.0	0.6775	0.4562	0.4175	0.6441	0.5602	0.4522	0.5791	0.5385	0.4970
	2.5	0.8604	0.5145	0.4837	0.7812	0.6728	0.5278	0.7197	0.6358	0.5528
	3.0	0.9503	0.6235	0.5649	0.8680	0.7654	0.5871	0.7948	0.7101	0.6095
30, 20	2.0	0.7012	0.5309	0.4967	0.6945	0.6199	0.5372	0.6144	0.5972	0.5766
	2.5	0.8937	0.6209	0.5685	0.8510	0.7380	0.5860	0.8043	0.6994	0.6009
	3.0	0.9627	0.6874	0.6103	0.9005	0.8489	0.6512	0.8679	0.7972	0.6770
30, 30	2.0	0.7429	0.6283	0.5635	0.7248	0.6784	0.6016	0.7080	0.6556	0.6192
	2.5	0.9313	0.7010	0.6239	0.9112	0.7820	0.6427	0.8509	0.7421	0.6714
	3.0	0.9735	0.8017	0.7258	0.9523	0.8965	0.7490	0.9304	0.8506	0.7613

Table 13. Asymptotic power of $U^{(1)}_{m,k}$ for q = 0.3

		***k* = 1**			***k* = 2**			***k* = 3**		
***n*₁, *n*₂**	***θ***	***m* = 0**	***m* = 1**	***m* = 2**	***m* = 0**	***m* = 1**	***m* = 2**	***m* = 0**	***m* = 1**	***m* = 2**
10, 10	2.0	0.3990	0.1527	0.1075	0.3621	0.2297	0.1606	0.3310	0.2576	0.2217
	2.5	0.5603	0.1813	0.1327	0.4435	0.2734	0.1942	0.4012	0.3013	0.2574
	3.0	0.7407	0.2085	0.1624	0.5081	0.3013	0.2028	0.4624	0.3456	0.2885
20, 10	2.0	0.4785	0.1906	0.1218	0.4110	0.2552	0.1914	0.3862	0.2909	0.2603
	2.5	0.6285	0.2117	0.1730	0.5138	0.3021	0.2196	0.4690	0.3449	0.2892
	3.0	0.8454	0.2329	0.2011	0.5730	0.3415	0.2375	0.5212	0.3902	0.3510
20, 20	2.0	0.6194	0.2215	0.2119	0.5130	0.2886	0.2318	0.4596	0.3184	0.2984
	2.5	0.8126	0.2410	0.2250	0.6385	0.3447	0.2592	0.5791	0.3910	0.3379
	3.0	0.9539	0.2752	0.2568	0.7437	0.3976	0.2864	0.6789	0.4393	0.4090
30, 20	2.0	0.6872	0.3127	0.2910	0.5581	0.3543	0.3205	0.4992	0.3921	0.3646
	2.5	0.8574	0.3135	0.2890	0.6891	0.3892	0.3610	0.6224	0.4291	0.4113
	3.0	0.9677	0.3508	0.3137	0.7812	0.4290	0.4035	0.7328	0.4880	0.4398
30, 30	2.0	0.7790	0.3842	0.3217	0.6288	0.4107	0.3516	0.5703	0.4522	0.3997
	2.5	0.9390	0.4355	0.3668	0.7813	0.4745	0.3958	0.7036	0.5290	0.4620
	3.0	0.9831	0.4904	0.4105	0.8742	0.5484	0.4697	0.8145	0.5762	0.5004

Table 14. Asymptotic power of $U_{m,k}^{(1)}$ for q = 0.5

		***k* = 1**			***k* = 2**			***k* = 3**		
n_1, n_2	θ	m = 0	m = 1	m = 2	m = 0	m = 1	m = 2	m = 0	m = 1	m = 2
10, 10	2.0	0.4971	0.3110	0.2837	0.2406	0.2538	0.2437	0.1953	0.2308	0.2362
	2.5	0.6518	0.3891	0.3039	0.2581	0.2764	0.2760	0.2261	0.2533	0.2603
	3.0	0.7481	0.4709	0.3269	0.2807	0.2987	0.3013	0.2497	0.2691	0.2789
20, 10	2.0	0.5798	0.3866	0.3413	0.3086	0.3225	0.3217	0.2650	0.2699	0.2894
	2.5	0.7360	0.4898	0.3904	0.3487	0.3573	0.3584	0.3088	0.3192	0.3241
	3.0	0.8346	0.5733	0.4513	0.4113	0.4276	0.4295	0.3634	0.3708	0.4014
20, 20	2.0	0.7033	0.4662	0.3925	0.3527	0.3719	0.3753	0.2942	0.3116	0.3302
	2.5	0.8824	0.6124	0.4674	0.4301	0.4495	0.4431	0.3722	0.4096	0.4256
	3.0	0.9511	0.7083	0.6205	0.5717	0.5861	0.5874	0.4751	0.5289	0.5508
30, 20	2.0	0.7791	0.5966	0.4790	0.4399	0.4613	0.4615	0.3762	0.4046	0.4197
	2.5	0.9232	0.7603	0.5712	0.5244	0.5393	0.5407	0.4610	0.5014	0.5201
	3.0	0.9714	0.8562	0.7036	0.6730	0.6916	0.6887	0.5617	0.6209	0.6416
30, 30	2.0	0.8610	0.6376	0.5611	0.5224	0.5396	0.5408	0.4599	0.4928	0.5031
	2.5	0.9655	0.8028	0.6920	0.6419	0.6622	0.6643	0.5708	0.6227	0.6395
	3.0	0.9814	0.8931	0.8203	0.7690	0.7784	0.7823	0.6543	0.6796	0.6937

Table 15. Asymptotic power of $U_{m,k}^{(2)}$ for q = 0.1

		***k* = 1**			***k* = 2**			***k* = 3**		
n_1, n_2	θ	m = 0	m = 1	m = 2	m = 0	m = 1	m = 2	m = 0	m = 1	m = 2
10, 10	2.0	0.6428	0.5127	0.5092	0.6060	0.4021	0.3195	0.3688	0.2714	0.2018
	2.5	0.6957	0.6780	0.6612	0.6669	0.5695	0.4914	0.5370	0.4733	0.3923
	3.0	0.8084	0.7843	0.7719	0.7637	0.6843	0.5623	0.6164	0.5310	0.4809
20, 10	2.0	0.6870	0.6235	0.6114	0.6739	0.5235	0.4476	0.4839	0.3612	0.3247
	2.5	0.8123	0.7810	0.7634	0.7667	0.6784	0.5850	0.6320	0.5320	0.4538
	3.0	0.8860	0.8813	0.8635	0.8138	0.7950	0.6324	0.6791	0.6007	0.5617
20, 20	2.0	0.7514	0.7429	0.7297	0.7383	0.6357	0.5680	0.5969	0.4819	0.4202
	2.5	0.9109	0.8996	0.8689	0.8750	0.7890	0.6973	0.7312	0.6325	0.5460
	3.0	0.9706	0.9615	0.9562	0.9162	0.8894	0.7912	0.8435	0.6884	0.6457
30, 20	2.0	0.8129	0.8075	0.7782	0.7955	0.6996	0.6142	0.6477	0.5660	0.4814
	2.5	0.9336	0.9209	0.9028	0.8964	0.8740	0.8180	0.8358	0.7536	0.6691
	3.0	0.9798	0.9790	0.9693	0.9385	0.9235	0.8562	0.8890	0.7995	0.7170
30, 30	2.0	0.8864	0.8758	0.8640	0.8543	0.7675	0.6795	0.7166	0.6198	0.5709
	2.5	0.9872	0.9744	0.9609	0.9420	0.9363	0.8879	0.9102	0.8244	0.7482
	3.0	0.9955	0.9937	0.9882	0.9568	0.9441	0.9118	0.9325	0.8648	0.8195

Table 16. Asymptotic power of $U_{m,k}^{(2)}$ for q = 0.3

		k = 1			k = 2			k = 3		
n_1, n_2	θ	m = 0	m = 1	m = 2	m = 0	m = 1	m = 2	m = 0	m = 1	m = 2
10, 10	2.0	0.5167	0.4910	0.3206	0.3185	0.2943	0.2722	0.3017	0.2640	0.1638
	2.5	0.6829	0.6419	0.4383	0.4075	0.3798	0.3509	0.3830	0.3321	0.2544
	3.0	0.7920	0.7686	0.5202	0.5109	0.5007	0.4812	0.5066	0.4667	0.3792
20, 10	2.0	0.5918	0.5823	0.3916	0.3528	0.3367	0.3106	0.3492	0.2998	0.1875
	2.5	0.7652	0.7335	0.4955	0.4467	0.4058	0.3845	0.4079	0.3610	0.2867
	3.0	0.8555	0.8412	0.5867	0.5440	0.5225	0.5026	0.5331	0.4924	0.3833
20, 20	2.0	0.7431	0.7089	0.4368	0.3990	0.3710	0.3373	0.3885	0.3236	0.2493
	2.5	0.9037	0.8564	0.5712	0.5018	0.4692	0.4384	0.4817	0.4175	0.3328
	3.0	0.9656	0.9423	0.6598	0.6029	0.5535	0.5244	0.5730	0.5103	0.4390
30, 20	2.0	0.8143	0.7719	0.4918	0.4504	0.4066	0.3680	0.4283	0.3592	0.2609
	2.5	0.9401	0.8837	0.6509	0.5792	0.5271	0.4966	0.5590	0.4799	0.3814
	3.0	0.9804	0.9512	0.7183	0.6640	0.6146	0.5797	0.6376	0.5584	0.4725
30, 30	2.0	0.8846	0.8562	0.5367	0.4912	0.4485	0.4002	0.4618	0.3925	0.3017
	2.5	0.9830	0.9420	0.6795	0.6278	0.5890	0.5330	0.5979	0.5228	0.4439
	3.0	0.9966	0.9837	0.7795	0.7137	0.6650	0.6243	0.6825	0.6210	0.5523

Table 17. Asymptotic power of $U_{m,k}^{(2)}$ for q = 0.5

		k = 1			k = 2			k = 3		
n_1, n_2	θ	m = 0	m = 1	m = 2	m = 0	m = 1	m = 2	m = 0	m = 1	m = 2
10, 10	2.0	0.5028	0.3068	0.2875	0.2515	0.2640	0.2631	0.2023	0.2317	0.2402
	2.5	0.6579	0.3880	0.3010	0.2691	0.2809	0.2820	0.2297	0.2562	0.2640
	3.0	0.7715	0.4793	0.3234	0.2898	0.3043	0.3085	0.2512	0.2705	0.2811
20, 10	2.0	0.6169	0.3854	0.3484	0.3101	0.3257	0.3266	0.2697	0.2775	0.2933
	2.5	0.7684	0.4967	0.3898	0.3523	0.3629	0.3615	0.3113	0.3201	0.3298
	3.0	0.8890	0.6053	0.4566	0.4184	0.4305	0.4310	0.3692	0.3719	0.4050
20, 20	2.0	0.7332	0.4720	0.3901	0.3569	0.3767	0.3795	0.2978	0.3190	0.3348
	2.5	0.8935	0.6109	0.4713	0.4387	0.4528	0.4509	0.3824	0.4184	0.4276
	3.0	0.9676	0.7166	0.6189	0.5758	0.5865	0.5889	0.4770	0.5308	0.5523
30, 20	2.0	0.8125	0.5592	0.4850	0.4413	0.4621	0.4637	0.3780	0.4083	0.4202
	2.5	0.9413	0.7024	0.5662	0.5275	0.5406	0.5418	0.4639	0.5032	0.5210
	3.0	0.9819	0.8093	0.7234	0.6743	0.6975	0.6990	0.5698	0.6290	0.6497
30, 30	2.0	0.8740	0.6405	0.5638	0.5277	0.5405	0.5440	0.4674	0.4966	0.5095
	2.5	0.9725	0.8012	0.6933	0.6480	0.6678	0.6699	0.5725	0.6297	0.6404
	3.0	0.9911	0.8964	0.8198	0.7703	0.7815	0.7886	0.6584	0.6812	0.6970

Based on the computation of the asymptotic power for Normal distribution, we conclude that:

1) The change in scale of order of 3 is detected for random samples of size ≥ 20 at $m = 0$ and $k = 1$ for both $U_{m,k}^{(1)}$ and $U_{m,k}^{(2)}$ tests for any q.
2) For all other $U_{m,k}^{(1)}$ and $U_{m,k}^{(2)}$ tests (except at 1 above), a larger sample size is needed to detect the change of scale of the same order.
3) The $U_{m,k}^{(2)}$ test detects the change of scale with more power in comparison to the $U_{m,k}^{(1)}$ test. This validates the computations of AREs as well.

Conclusion

In this paper, we proposed two classes of distribution-free tests for testing the equality of the scale parameter with common known quantile ξ_q of order q, $0 \le q \le 1$. The proposed classes of tests $U_{m,k}^{(1)}$ and $U_{m,k}^{(2)}$, are generalization of some existing tests. The null distribution of the proposed tests is derived. We compared the proposed class of tests with some existing tests in terms of Pitman asymptotic relative efficiency. It is observed that for some underlying distributions, the proposed tests perform better than competing tests for some choices of m and k. We applied the proposed tests on real life data set of Jung and Parekh (1970). The power of the proposed tests is assessed using Monte Carlo simulation study and some conclusions are made for practical implementation.

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