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Spectra and leading directions for differential-algebraic equations

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Spectra and leading directions for differential-algebraic equations

Vu Hoang Linh * Volker Mehrmann †

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Abstract

The state of the art in the spectral theory of linear time-varying differential-algebraic equations (DAEs) is surveyed. To characterize the asymptotic behavior and the growth rate of solutions, basic spectral notions such as Lyapunov- and Bohl exponents, and Sacker-Sell spectra are discussed. For DAEs in strangeness-free form, the results extend those for ordinary differential equations, but only under additional conditions. This has consequences concerning the boundedness of solutions of inhomogeneous equations. Also, linear subspaces of leading directions are characterized, which are associated with spectral intervals and which generalize eigenvectors and invariant subspaces as they are used in the linear time-invariant setting.

Keywords: differential-algebraic equation, strangeness index, Lyapunov exponent, Bohl exponent, Sacker-Sell spectrum, exponential dichotomy, leading direction.

AMS(MOS) subject classification: 65L07, 65L80, 34D08, 34D09

1 Introduction

Differential-algebraic equations (DAEs) are a very convenient modeling concept in many different application areas see [5, 25, 28, 30, 43, 44] and the

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references therein. But many numerical difficulties arise due to the fact that the solution depends on derivatives of the data and that the dynamics is constrained to a manifold, which often is only given implicitly.

In this chapter we survey the spectral theory for linear DAEs with variable coefficients of the form

$$E(t)\dot{x} = A(t)x + f(t), \tag{1}$$

on the half-line $\mathbb{I} = [0, \infty)$, together with an initial condition

$$x(0) = x_0. \tag{2}$$

We assume that $E, A \in C(\mathbb{I}, \mathbb{R}^{n\times n})$, and $f \in C(\mathbb{I}, \mathbb{R}^n)$ are sufficiently smooth, using the notation $C(\mathbb{I}, \mathbb{R}^{n\times n})$ to denote the space of continuous functions from $\mathbb{I}$ to $\mathbb{R}^{n\times n}$. In the following we leave off the explicit dependence of the coefficients on the time t.

Linear systems of the form (1) arise directly in many applications and, via linearization [8], they describe the local behavior in the neighborhood of a solution for general implicit nonlinear system of DAEs

$$F(t, x, \dot{x}) = 0, \quad t \in \mathbb{I}. \tag{3}$$

For linear systems with constant coefficients, the asymptotic behavior and the directions of growth or decay as well as oscillatory behavior can be characterized via the eigenvalues and eigenvectors of the matrix pencil $\lambda E - A$ and many good numerical methods are available, see [26, 46]. For systems with variable coefficients, however, different concepts are necessary such as the Lyapunov [41], Bohl [1, 4, 13] and Sacker-Sell spectra [45], which were designed for ordinary differential equations (ODEs) to analyze the qualitative behavior of solutions of differential equations as time tends to infinity. For a long time the numerical computation of these spectra in the variable coefficient ODE case was considered unfeasible, but recently large progress has been made, see [15, 19, 21] and the references therein.

The spectral theory and numerical methods for the computation of spectra in the case of DAEs, however, is still in its infancy. In [11, 12] results on Lyapunov exponents and Lyapunov regularity was studied, in [35] the concept of exponential dichotomy was used in the numerical solution of DAE boundary value problems, and in [24, 23] robustness results in the context of exponential stability and Bohl exponents were studied. All these papers use the tractability index approach of [27, 42] and consider linear systems of DAEs of tractability index at most one, only. Recently, in [31, 36, 37, 39, 40], the classical spectral theory and the numerical methods (such as QR and SVD methods) for the computation of Lyapunov, Bohl and Sacker-Sell spectra were developed for general DAEs in so-called strangeness-free formulation. We systematically survey these results and emphasize the difficulties that arise for DAEs with some simple illustrative examples.

After a brief review of the DAE theory based on the strangeness index approach [30], we present the basic properties of Lyapunov characteristic exponents in Section 3. Then, in Section 4, we introduce the notions for characterizing the uniform growth rate: Bohl exponents, exponential dichotomy, and Sacker-Sell spectrum. The connection between the spectra and the existence of bounded solutions for both homogeneous and inhomogeneous equations is also described. The analysis of leading directions and leading subspaces is given in Section 5.

2 A review of DAE theory

In this section we briefly recall some concepts from the theory of differential-algebraic equations, see e.g. [5, 30, 43]. We follow [30] in notation and style.

A function $x : \mathbb{I} \to \mathbb{R}^n$ is called a *solution* of (1) if $x \in C^1(\mathbb{I}, \mathbb{R}^n)$ and x satisfies (1) pointwise. It is called a *solution of the initial value problem* (1)–(2) if x is a solution of (1) and satisfies (2). An initial condition (2) is called *consistent* if the corresponding initial value problem has at least one solution.

In this paper, we restrict ourselves to *regular* DAEs, i. e., we require that (1) (or (3) locally) has a unique solution for sufficiently smooth E, A, f (F) and appropriately chosen (consistent) initial conditions.

2.1 Reduction to strangeness-free DAEs

The concept of strangeness-index is based on *derivative arrays* associated with (1) as first introduced in [7]. Consider the inflated system

$$M_\ell \dot{z}_\ell = N_\ell z_\ell + g_\ell, \tag{4}$$

where

$$\begin{aligned}
(M_\ell)_{i,j} &= \binom{i}{j} E^{(i-j)} - \binom{i}{j+1} A^{(i-j-1)}, \; i,j = 0,\dots,\ell, \\
(N_\ell)_{i,j} &= \begin{cases} A^{(i)} & \text{for } i = 0,\dots,\ell, \; j = 0, \\ 0 & \text{otherwise,} \end{cases} \\
(z_\ell)_j &= x^{(j)}, \; j = 0,\dots,\ell, \\
(g_\ell)_i &= f^{(i)}, \; i = 0,\dots,\ell,
\end{aligned}$$

using the convention that $\binom{i}{j} = 0$ for $i < 0$, $j < 0$ or $j > i$. To guarantee existence and uniqueness of solutions, the following hypothesis is used, see [29, 30].

Hypothesis 1 *There exist integers μ, a, and d such that the inflated pair (M_μ, N_μ) in (4) associated with the given pair of matrix functions (E, A) has the following properties:*

1. *For all $t \in \mathbb{I}$ we have $\operatorname{rank} M_\mu(t) = (\mu+1)n - a$ such that there exists a smooth matrix function Z_2 of size $(\mu+1)n \times a$ and pointwise maximal rank satisfying $Z_2^T M_\mu = 0$.*

2. *For all $t \in \mathbb{I}$ we have $\operatorname{rank} \hat{A}_2(t) = a$, where $\hat{A}_2 = Z_2^T N_\mu [I_n \; 0 \; \cdots \; 0]^T$ such that there exists a smooth matrix function T_2 of size $n \times d$, $d - a$, and pointwise maximal rank satisfying $\hat{A}_2 T_2 = 0$.*

3. *For all $t \in \mathbb{I}$ we have $\operatorname{rank} E(t) T_2(t) = d$ such that there exists a smooth matrix function Z_1 of size $n \times d$ and pointwise maximal rank satisfying $\operatorname{rank} \hat{E}_1 T_2 = d$ with $\hat{E}_1 = Z_1^T E$.*

The smallest possible μ for which Hypothesis 1 holds is called the *strangeness index* of (1). Systems with vanishing strangeness index are called *strangeness-free*. It has been shown in [29, 30], that the strangeness index is closely related to the *differentiation index*, see [5], but allowing over- and underdetermined systems, and that under some constant rank conditions, every uniquely solvable (regular) linear DAE of the form (1) with sufficiently smooth E, A satisfies Hypothesis 1 and that there exists a (pointwise) numerically computable *reduced system*

$$\begin{array}{lrcl} (a) & E_1 \dot{x} & = & A_1 x + f_1, \\ (b) & 0 & = & A_2 x + f_2, \end{array} \tag{5}$$

with $E_1 = Z_1^T E$, $A_1 = Z_1^T A \in C(\mathbb{I}, \mathbb{R}^{d \times n})$, $A_2 = Z_2^T N_\mu [\, I_n \;\; 0 \;\; \cdots \;\; 0 \,]^T \in C(\mathbb{I}, \mathbb{R}^{a \times n})$, $f_1 = Z_1^T f \in C(\mathbb{I}, \mathbb{R}^d)$, and $f_2 = Z_2^T g_\mu \in C(\mathbb{I}, \mathbb{R}^a)$. This implies that the matrix function

$$\bar{E} := \begin{bmatrix} E_1 \\ A_2 \end{bmatrix} \tag{6}$$

is invertible for all $t \in \mathbb{I}$ and therefore also that E_1 and A_2 are of full row-rank.

System (5) is a reformulation of system (1) (of differentiation index less than or equal to one) that displays all the algebraic constraints explicitly in (5 (b)). It follows that an initial vector $x_0 \in \mathbb{R}^n$ is *consistent* if and only if

$A_2(0)x_0 + f_2(0) = 0$. It has also been shown in [29, 30], that for this system implicit Runge-Kutta and BDF methods behave as for ODEs and that this approach can also be extended to over- and underdetermined systems as well as locally to general nonlinear systems.

2.2 Essentially underlying implicit ODEs

In the following, we assume that the homogeneous linear DAE in consideration

$$E\dot{x} = Ax, \quad t \in \mathbb{I}, \tag{7}$$

is already strangeness-free and the coefficients are as in (5). But it is often convenient to transform (7) into another form which is easier to handle but its solutions have the same asymptotic behavior as those (7). Suppose that $P \in C(\mathbb{I}, \mathbb{R}^{n\times n})$ and $Q \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ are nonsingular matrix functions such that Q and Q^{-1} are bounded. Then the transformed DAE system

$$\tilde{E}(t)\dot{\tilde{x}} = \tilde{A}(t)\tilde{x}, \tag{8}$$

with $\tilde{E} = PEQ$, $\tilde{A} = PAQ - PE\dot{Q}$ and $x = Q\tilde{x}$ is called *globally kinematically equivalent* to (7) and the transformation is called a *global kinematic equivalence transformation.* If $P \in C^1(\mathbb{I}, \mathbb{R}^{n\times n})$ and, furthermore, also P and P^{-1} are bounded then we call this a *strong global kinematic equivalence transformation.*

The following key lemma is a modification of [36, Lemma 7].

Lemma 2 *Consider a strangeness-free DAE system of the form (7) with continuous coefficients E, A. Let $U \in C^1(\mathbb{I}, \mathbb{R}^{n\times d})$ be an arbitrary orthonormal basis of the solution space of (7). Then there exists a matrix function $V \in C(\mathbb{I}, \mathbb{R}^{n\times d})$ with pointwise orthonormal columns such that by the change of variables $x = Uz$ and multiplication of both sides of (7) from the left by V^T, one obtains the system*

$$\mathcal{E}\dot{z} = \mathcal{A}z, \tag{9}$$

where $\mathcal{E} := V^TEU$, $\mathcal{A} := V^TAU - V^TE\dot{U}$ and $\mathcal{E}$ is upper triangular.

Proof. The proof is given in [37] and [39] and is similar to that of [40]. □

System (9) is called *essentially underlying implicit ODE system* (EUODE) of (7). It can be made explicit by multiplying with $\mathcal{E}^{-1}$ from the left, see also [2] for constructing EUODEs of so-called *properly-stated DAEs.*

Note that for a fixed U, the matrix function V that leads to the EUODE is not unique. In fact, any V for which V^TEU is invertible yields an implicit

EUODE. However, obviously $\mathcal{E}^{-1}\mathcal{A}$ is unique, i. e., with a given basis, the explicit EUODE provided by Lemma 2 is unique.

We also often use the special case of of *s*emi-implicit strangeness-free DAEs with coefficients of the form

$$E := \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \quad A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{10}$$

with E_{11} pointwise nonsingular. This DAE is strangeness-free if and only if A_{22} is pointwise invertible and by inserting $x_2 = -A_{22}^{-1}A_{21}x_1$ into the first equation, we obtain an implicit ODE

$$E_{11}\dot{x}_1 = \tilde{A}_{11}x_1, \tag{11}$$

where $\tilde{A}_{11} = A_{11} - A_{12}A_{22}^{-1}A_{21}$. It is easy to show that if $A_{22}^{-1}A_{21}$ is bounded, then (11) and (9) are globally kinematically equivalent. Furthermore, if E_1 is sufficiently smooth, then (7) can always be transformed into the form (10) by an appropriate kinematical equivalence transformation [36].

3 Lyapunov spectral theory for DAEs

In this section we review results on the qualitative behavior of solutions of DAEs from [36, 37, 40]. For a non-vanishing function $f : [0,\infty) \longrightarrow \mathbb{R}^n$, the quantities $\chi^u(f) = \limsup_{t\to\infty} \frac{1}{t}\ln\|f(t)\|$, $\chi^\ell(f) = \liminf_{t\to\infty} \frac{1}{t}\ln\|f(t)\|$, are called *upper and lower Lyapunov exponents of* f, respectively. It is well-known, see e.g. [1] how the Lyapunov exponents characterize the growth of a function. Let $f : [0,\infty) \longrightarrow \mathbb{R}$ be a non-vanishing function. Then $\chi^u(f) = \alpha \neq \pm\infty$ if and only if for any $\varepsilon > 0$ the following two conditions hold simultaneously:

$$\lim_{t\to\infty} \frac{|f(t)|}{\exp(\alpha+\varepsilon)t} = 0, \; \limsup_{t\to\infty} \frac{|f(t)|}{\exp(\alpha-\varepsilon)t} = \infty.$$

Analogously, $\chi^\ell(f) = \beta \neq \pm\infty$ if and only if for any $\varepsilon > 0$ the following two conditions hold simultaneously:

$$\liminf_{t\to\infty} \frac{|f(t)|}{\exp(\beta+\varepsilon)t} = 0, \; \lim_{t\to\infty} \frac{|f(t)|}{\exp(\beta-\varepsilon)t} = \infty.$$

For a constant $c \neq 0$ and non-vanishing functions $f, f_1, \ldots, f_n$, the Lyapunov exponents satisfy

$$\chi^u(cf) = \chi^u(f), \quad \chi^\ell(cf) = \chi^\ell(f),$$

and if $f_1, \dots f_n$ are non-vanishing scalar functions, then

$$\chi^u\left(\sum_{i=1}^n f_i\right) \le \max_{i=1,\dots n} \chi^u(f_i), \quad \chi^\ell\left(\sum_{i=1}^n f_i\right) \le \max_{i=1,\dots n} \chi^\ell(f_i),$$

where equality holds if the maximal upper/lower Lyapunov exponent is attained by only one function.

3.1 Lyapunov exponents and spectral intervals

To study the qualitative behavior of DAE solutions we determine the Lyapunov exponents of the columns of fundamental solution matrices.

A matrix function $X \in C^1(\mathbb{I}, \mathbb{R}^{n\times k})$, with $d \le k \le n$, is called *fundamental solution matrix of the strangeness-free DAE (7)* if each of its columns is a solution to (7) and $\operatorname{rank} X(t) = d$, for all $t \ge 0$. A fundamental solution matrix is said to be *minimal* if $k = d$.

One may construct a minimal fundamental matrix solution by solving initial value problems for (7) with d linearly independent, consistent initial vectors. Given an EUODE (9), any minimal fundamental solution matrix X of (7) has the form $X = UZ$, where Z is the corresponding fundamental solution matrix of (9).

Definition 3 ([36]) *For a given minimal fundamental solution matrix X of a strangeness-free DAE system of the form (7), and for $1 \le i \le d$, we introduce*

$$\lambda_i^u = \limsup_{t\to\infty} \frac{1}{t} \ln \|X(t)e_i\|, \ \lambda_i^\ell = \liminf_{t\to\infty} \frac{1}{t} \ln \|X(t)e_i\|,$$

where e_i denotes the i-th unit vector and $\|\cdot\|$ denotes the Euclidean norm. The columns of a minimal fundamental solution matrix form a normal basis *if $\Sigma_{i=1}^d \lambda_i^u$ is minimal. The λ_i^u, $i = 1, 2, ..., d$ belonging to a normal basis are called* (upper) Lyapunov exponents. *If $\Sigma_{i=1}^d \lambda_i^\ell$ is minimal, too, then the intervals $[\lambda_i^\ell, \lambda_i^u]$, $i = 1, 2, ..., d$, are called* Lyapunov spectral intervals. *The union of the Lyapunov spectral intervals is called the* Lyapunov spectrum *of (7) and denoted by Σ_L.*

Example 4 If (7) is time-invariant, i. e., E, A are constant matrices, then the Lyapunov spectrum of (7) is the set of the real parts of generalized eigenvalues, i. e., $\Sigma_L = \{\operatorname{Re}\lambda, \det(\lambda E - A) = 0\}$.

From the properties of the Lyapunov exponents, it is easy to establish the following observations.

- If the largest upper Lyapunov exponent is negative, then (7) is asymptotically stable.
- If the largest upper Lyapunov exponent is positive, then (7) is unstable.
- If at least one upper Lyapunov exponent is negative, then there exist a bounded solution of (7), which tends to 0 exponentially.
- If the smallest lower Lyapunov exponent is positive, then all solutions of (7) are unbounded and exponentially increasing.
- If the largest upper Lyapunov exponent is zero, then nothing can be concluded about the stability, since the system (7) may have unbounded solutions.

Similar as in the case of ODEs, a normal basis for (7) exists and it can be constructed from any (minimal) fundamental matrix solution.

Proposition 5 *[37, 39] For any given minimal fundamental matrix X of (7), for which the Lyapunov exponents of the columns are ordered decreasingly, there exists a constant, nonsingular, and upper triangular matrix $C \in \mathbb{R}^{d\times d}$ such that the columns of XC form a normal basis for (7).*

Example 6 For the DAE

$$\begin{array}{rcl} \dot{x}_1 &=& x_1, \\ \dot{x}_2 &=& -x_2, \\ 0 &=& x_1 + x_2 - x_3, \end{array} \quad .$$

we have fundamental solution matrices

$$X_1(t) = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \\ e^t & e^{-t} \end{bmatrix}, \; X_2(t) = \begin{bmatrix} e^t & e^t \\ 0 & e^{-t} \\ e^t & e^{-t} + e^t \end{bmatrix}.$$

Here X_1 is normal and has $\Sigma_L = \{\pm 1\}$ and X_2 is not normal, but $X_1 = X_2 C$ with $C = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

Global kinematic equivalence transformations preserve the Lyapunov exponents as well as the normality of a basis formed by the columns of a fundamental solution matrix.

Theorem 7 *[37, 40] Let X be a normal basis for (7). Then the Lyapunov spectrum of the DAE (7) and that of the ODE (9) are the same. If $\mathcal{E}, \mathcal{A}$ are as in (9) and if $\mathcal{E}^{-1}\mathcal{A}$ is bounded, then all the Lyapunov exponents of (7) are finite. Furthermore, the spectrum of (9) does not depend on the choice of the basis U and the matrix function V.*

As a consequence of this theorem and the Lyapunov inequality [41, Theorem 2.5.1], we have the following corollary.

Corollary 8 *Let $\{\lambda_i^u\}_{i=1}^d$ be the upper Lyapunov exponents of (7). Then*

$$\sum_{i=1}^{d} \lambda_i^u \geq \limsup_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} \mathcal{E}^{-1}\mathcal{A}(s)ds. \tag{12}$$

Here $\operatorname{tr} \mathcal{E}^{-1}\mathcal{A}(s)$ denotes the trace of the matrix function.

As in [12], we say that the DAE system (7) is *Lyapunov-regular* if its EUODE (9) is Lyapunov-regular, i. e., if

$$\sum_{i}^{d} \lambda_i^u = \liminf_{t\to\infty} \frac{1}{t} \int_0^t \operatorname{tr} \mathcal{E}^{-1}\mathcal{A}(s)ds.$$

Remark 9 The Lyapunov-regularity of a strangeness-free DAE system (7) is well-defined, since it does not depend on the construction of (9). If (7) is Lyapunov-regular, then for any nontrivial solution x, the limit $\lim_{t\to\infty} \frac{1}{t} \ln \|x(t)\|$ exists. Hence, we have $\lambda_i^l = \lambda_i^u$, i. e., the Lyapunov spectrum of (7) is a point spectrum. Note that unlike in [12], where certain inherent ODEs of the same size as the original DAE are used, here the spectral analysis is based on the essentially underlying ODEs, which have reduced size and can be constructed numerically.

In the following we consider the adjoint equation of (7), given by

$$-E^T \dot{y} = (A + \dot{E})^T y, \tag{13}$$

see e.g., [9, 32, 33], and also a slightly different formulation in [2].

The following result gives the relation between the EUODEs of (7) and (13).

Proposition 10 *Let the orthonormal columns of the matrix U form a basis of the solution subspace of (7). Then there exists $V \in C^1(\mathbb{I}, \mathbb{R}^{n\times d})$ such that the columns of V form an orthonormal basis for the solution subspace of*

(13). Furthermore, via the change of variables $y = Vw$ and multiplication of both sides of (13) by U^T, the EUODE for the adjoint system (13) is given by

$$-\mathcal{E}^T \dot{w} = (\mathcal{A} + \dot{\mathcal{E}})^T w, \tag{14}$$

which is exactly the adjoint of (9). If U is such that the matrix $\mathcal{E}$ is upper triangular with positive diagonal elements, then the corresponding V is unique.

Proof. To prove uniqueness, suppose that there exist matrix functions V and $\hat{V}$ with orthonormal columns, such that $\mathcal{E}\dot{z} = \mathcal{A}z$ and $\hat{\mathcal{E}}\dot{z} = \hat{\mathcal{A}}z$, respectively, where both $\mathcal{E}$ and $\hat{\mathcal{E}}$ are upper triangular with positive diagonal elements. Since the columns of V and $\hat{V}$ are bases of the same subspace, there exists $S \in C(\mathbb{I}, \mathbb{R}^{n\times d})$, such that $V = \hat{V}S$. We have $S^T S = V^T(\hat{V}\hat{V}^T V) = V^T V = I_d$ and thus S is orthogonal. On the other hand, by the construction of the EUODE, we have $S^T\hat{\mathcal{E}} = \mathcal{E}$ and $\hat{\mathcal{E}}$ is invertible, which implies that S^T is upper triangular. Hence, S is a diagonal matrix with diagonal elements $+1$ or -1. But since $\mathcal{E}$ and $\hat{\mathcal{E}}$ have positive diagonals, then $S = I$.

For the existence we give a constructive proof. From the proof of Lemma 2, V is determined via $EU = V\mathcal{E}$. Due to the special form of E, we can first determine an auxiliary pair $\tilde{V}, \tilde{\mathcal{E}}$ such that $EU = \begin{bmatrix} E_1 U \\ 0 \end{bmatrix} = \tilde{V}\tilde{\mathcal{E}}$, which implies that $\tilde{V} = \begin{bmatrix} \tilde{V}_1 \\ 0 \end{bmatrix}$. Here $\tilde{V}_1$ and $\tilde{\mathcal{E}}$ are determined by, e.g., a smooth QR decomposition of $E_1 U$. Unfortunately, in general such a $\tilde{V}$ is not a basis of the solution subspace of (13) yet. But we observe that we can replace the zero block by any $\tilde{V}_2$ and the relation $\tilde{V}^T EU = \mathcal{E}$ still holds. Hence, we look for an appropriate $\tilde{V}_2$ block so that $\tilde{V} = \begin{bmatrix} \tilde{V}_1 \\ \tilde{V}_2 \end{bmatrix}$ satisfies the algebraic constraint of (13). It remains to orthonormalize the columns of $\tilde{V}$. The adjoint DAE (13) has the form

$$\begin{bmatrix} E_{11}^T & 0 \\ E_{12}^T & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -(A_{11} + \dot{E}_{11})^T & -A_{21}^T \\ -(A_{12} + \dot{E}_{12})^T & -A_{22}^T \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

and, since the adjoint of (7) is again strangeness-free, see [9, 32, 33], we can reorder the equations so that the left upper $d \times d$-block of the coefficient matrix on the left-hand side is nonsingular and then eliminate the left lower block giving

$$\begin{bmatrix} \tilde{E}_{11}^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} -\tilde{A}_{11}^T & -\tilde{A}_{21}^T \\ -\tilde{A}_{12}^T & -\tilde{A}_{22}^T \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

In this system, the algebraic constraint is given explicitly by $0 = \tilde{A}_{12}^T y_1 + \tilde{A}_{22}^T y_2$, where $\tilde{A}_{22}^T$ is nonsingular. Hence, let $\tilde{V}_2 = -\tilde{A}_{22}^{-T} \tilde{A}_{12}^T \tilde{V}_1$. Finally, applying a Gram-Schmidt orthogonalization to $\tilde{V}$, we obtain a basis of (13), denoted by V, which also fulfils $V^T EU = \mathcal{E}$, where $\mathcal{E}$ is upper triangular. One easily verifies that the so obtained EUODE of the adjoint equation (13) is exactly the adjoint of EUODE (9) of the original DAE (7).

With these preparations we obtain a generalization of [36, Theorem 19].

Theorem 11 *Suppose that the matrix function $\mathcal{E} = V^T EU$ and its inverse are bounded on $\mathbb{I}$, where the columns of U, V form bases of the solution spaces in Proposition 10. System (7) is Lyapunov regular if and only if (13) is Lyapunov regular, and in this case we have the* Perron identity

$$\lambda_i + \mu_i = 0, \quad i = 1, 2, ..., d,$$

where λ_i are the Lyapunov exponents of (7) in decreasing order and μ_i are the Lyapunov exponents of the adjoint system (13) in increasing order.

Proof. Due to Proposition 10, it suffices to consider two implicit EUODEs which are adjoint of each other. The assertion then follows from the *Lagrange identity* $W^T(t)\mathcal{E}(t)Z(t) = W^T(0)\mathcal{E}(0)Z(0)$, where Z and W are fundamental solutions of EUODE (9) and its adjoint (14), respectively.

Note that if either $\mathcal{E}$ or its inverse is not bounded, then the Perron identity may not hold.

Example 12 Consider the ODE

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= (\sin(\ln t) + \cos(\ln t))x_2 \end{aligned}$$

together with its adjoint system and the fundamental solution matrices

$$X_1 = \begin{bmatrix} e^t & 0 \\ 0 & e^{t \sin \ln t} \end{bmatrix}, \; X_2 = \begin{bmatrix} e^t & e^t \\ 0 & e^{t \sin \ln t} \end{bmatrix}$$

then X_1 is normal and $\Sigma_L = \{1, [-1, 1]\}$. X_1^{-T} is a normal basis of the adjoint system whose Lyapunov spectrum is $\{-1, [-1, 1]\}$. The columns of X_2 form a normal basis as well, but the columns of

$$X_2^{-T} = \begin{bmatrix} e^{-t} & 0 \\ e^{-t \sin \ln t} & e^{-t \sin \ln t} \end{bmatrix}$$

do not form a normal basis of the adjoint system.

3.2 Stability of Lyapunov exponents

It is, in general, very difficult to compute Lyapunov exponents via numerical methods, since they may be very sensitive under small changes in the coefficient matrices. In order to study this sensitivity for DAEs, we consider a DAE in the form (7) and the specially perturbed system

$$[E+G]\dot{x} = [A+H]x, \quad t \in \mathbb{I}, \tag{15}$$

where

$$G = \begin{bmatrix} G_1 \\ 0 \end{bmatrix}, \; H(t) = \begin{bmatrix} H_1(t) \\ H_2(t) \end{bmatrix},$$

and where G_1 and H_1, H_2 are assumed to have the same order of smoothness as E_1 and A_1, A_2, respectively.

Perturbations of this special structure are called *admissible perturbations*, generalizing the concept for constant coefficient DAEs studied in [6]. The DAE (7) is said to be *robustly strangeness-free* if it stays strangeness-free under all sufficiently small admissible perturbations. It is easy to see that the DAE (7) is robustly strangeness-free under admissible perturbations if and only if the matrix function $\bar{E}$ in (6) is boundedly invertible.

The upper Lyapunov exponents $\lambda_1^u \geq ... \geq \lambda_d^u$ of (7) are said to be *stable* if for any $\epsilon > 0$, there exists $\delta > 0$ such that the conditions $\sup_t \|F(t)\| < \delta$, $\sup_t \|H(t)\| < \delta$, and $\sup_t \left\|\dot{H}_2(t)\right\| < \delta$ on the perturbations imply that the perturbed DAE system (15) is strangeness-free and

$$|\lambda_i^u - \gamma_i^u| < \epsilon, \quad \text{for all } i = 1, 2, ..., d,$$

where the γ_i^u are the ordered upper Lyapunov exponents of an admissibly perturbed system (15).

It is clear that the stability of upper Lyapunov exponents is invariant under strong global kinematic equivalence transformations. Compared with the ODE case, however, the boundedness of $\dot{H}_2$ is needed.

Example 13 [37, 39] Consider the system $\dot{x}_1 = x_1$, $0 = x_2$, which is easily seen to be robustly strangeness-free and Lyapunov regular with Lyapunov exponent $\lambda = 1$. For the perturbed DAE

$$(1+\varepsilon^2 \sin(2nt))\,\dot{x}_1 - \varepsilon\cos(nt)\,\dot{x}_2 = x_1, \; 0 = -2\varepsilon\sin(nt)\,x_1 + x_2, \tag{16}$$

where ε is a small parameter and n is a given integer, from the second equation of (16), we obtain $x_2 = 2\varepsilon \sin nt\, x_1$. Differentiating this expression for x_2 and inserting the result into the first equation, after some elementary calculations, we obtain $\dot{x}_1 = (1+n\varepsilon^2+n\varepsilon^2\cos(2nt))\,x_1$. Explicit integration yields $x_1 = e^{(1+n\varepsilon^2)t+\varepsilon^2\sin(2nt)/2}$, from which the only Lyapunov exponent $\hat{\lambda} = 1+n\varepsilon^2$ is calculated. Clearly, though ε is small (hence the perturbations in the coefficient matrices are small), the difference between two Lyapunov exponents may be made arbitrarily large by choosing large n.

A minimal fundamental solution matrix X for (7) is called *integrally separated* if for $i = 1, 2, ..., d-1$ there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$\frac{\|X(t)e_i\|}{\|X(s)e_i\|} \cdot \frac{\|X(s)e_{i+1}\|}{\|X(t)e_{i+1}\|} \geq c_2 e^{c_1(t-s)},$$

for all t, s with $t \geq s \geq 0$.

The integral separation property is invariant under strong global kinematic equivalence transformations. Furthermore, if a fundamental solution X of (7) is integrally separated, then so is the corresponding fundamental solution Z of (9) and vice versa.

By using a global kinematic equivalence transformation, (7) can be transformed to a special structured form, where the block A_{21} becomes zero, see [36, Remark 13]. The advantage of this form is that the associated EUODE then reads $E_{11}\dot{x}_1 = A_{11}x_1$. Therefore, for the perturbation analysis, we may assume that (7) is already given with $A_{21} = 0$.

Theorem 14 *[36] Consider (7) with $A_{21} = 0$. Suppose that the matrix $\bar{E}$ in (6) is boundedly invertible and that $E_{11}^{-1}A_{11}$, $A_{12}A_{22}^{-1}$ and the derivative of A_{22} are bounded on $[0,\infty)$. Then, the upper Lyapunov exponents of (7) are distinct and stable if and only if the system has the integral separation property.*

Remark 15 Example 13 and Theorem 14 demonstrate that, unlike in the perturbation analysis of time-invariant DAEs [6], that of time-varying DAEs requires more restrictive conditions. However, for some classes of structured problems, see [10] and [36, Section 3.2], part of these conditions can be relaxed.

4 Bohl exponents and Sacker-Sell spectrum

Since in general Lyapunov spectra are unstable, other spectral concepts such as Bohl exponents [4, 13] and Sacker-Sell spectra [45] were introduced. The extension of these concepts to DAEs has been first presented in [36].

Definition 16 *Let x be a nontrivial solution of (7). The* (upper) Bohl exponent $\kappa_B^u(x)$ *of this solution is the greatest lower bound of all those values ρ for which there exist constants $N_\rho > 0$ such that*

$$\|x(t)\| \leq N_\rho e^{\rho(t-s)} \|x(s)\|$$

for any $t \geq s \geq 0$. If such numbers ρ do not exist, then one sets $\kappa_B^u(x) = +\infty$. Similarly, the lower Bohl exponent $\kappa_B^\ell(x)$ *is the least upper bound of all those values ρ' for which there exist constants $N'_\rho > 0$ such that*

$$\|x(t)\| \geq N'_\rho e^{\rho'(t-s)} \|x(s)\|, \quad 0 \leq s \leq t.$$

Lyapunov exponents and Bohl exponents are related via

$$\kappa_B^\ell(x) \leq \lambda^\ell(x) \leq \lambda^u(x) \leq \kappa_B^u(x),$$

i. e., the Bohl spectral intervals include the Lyapunov intervals. If the largest upper Bohl exponent of all the solutions of (7) is negative, then the system is (uniformly) exponentially stable. Formulas characterizing Bohl exponents for ODEs, see e.g. [13], immediately extend to DAEs, i. e.

$$\kappa_B^u(x) = \limsup_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}, \quad \kappa_B^\ell(x) = \liminf_{s,t-s\to\infty} \frac{\ln\|x(t)\| - \ln\|x(s)\|}{t-s}.$$

Moreover, unlike the Lyapunov exponents, the Bohl exponents are stable for admissible perturbations without the integral separation assumption, see [10, 36]. The third spectral concept of Sacker-Sell spectra is defined via exponential dichotomy.

Definition 17 *The DAE (7) is said to have* exponential dichotomy *if for any minimal fundamental solution X there exist a projection $\Pi \in \mathbb{R}^{d\times d}$ and positive constants K and α such that*

$$\begin{array}{rcl} \|X(t)\Pi X^+(s)\| & \leq & Ke^{-\alpha(t-s)}, \quad t \geq s, \\ \|X(t)(I_d - \Pi)X^+(s)\| & \leq & Ke^{\alpha(t-s)}, \quad s > t, \end{array} \tag{17}$$

where X^+ denotes the generalized Moore-Penrose inverse of X.

Since we have that for a fundamental solution matrix X of 7) and an orthonormal basis of the solution subspace represented by a matrix U that $X = UZ$, where Z is the fundamental solution matrix of (9) and hence invertible, it follows that $X^+ = Z^{-1}U^T$, and hence $\|X(t)\Pi X^+(s)\| = \left\|Z(t)\Pi Z^{-1}(s)\right\|$, $\|X(t)(I_d - \Pi)X^+(s)\| = \left\|Z(t)(I_d - \Pi)Z^{-1}(s)\right\|$. From this it follows that the

DAE (7) has exponential dichotomy if and only if its corresponding EUODE (9) has exponential dichotomy.

The projector Π can be chosen to be orthogonal [15, 19]. It projects to a subspace of the complete solution subspace in which all the solutions are uniformly exponentially decreasing, while the solutions belonging to the complementary subspace are uniformly exponentially increasing.

In order to extend the concept of exponential dichotomy spectrum to DAEs, we need *shifted DAE systems*

$$E(t)\dot{x} = [A(t) - \lambda E(t)]x, \quad t \in \mathbb{I}, \tag{18}$$

where $\lambda \in \mathbb{R}$, which immediately leads to a shifted EUODE $\mathcal{E}\dot{z} = (\mathcal{A} - \lambda\mathcal{E})z$.

Definition 18 *The* Sacker-Sell (or exponential dichotomy) spectrum *of the DAE system (7) is defined by*

$$\Sigma_S := \{\lambda \in \mathbb{R},\ \text{the shifted DAE (18) does not have an exponential dichotomy}\}\,.$$

The complement of Σ_S is called the resolvent set *for the DAE system (7), denoted by $\rho(E, A)$.*

Theorem 19 *[36] The Sacker-Sell spectrum of (7) is exactly the Sacker-Sell spectrum of its EUODE (9). Furthermore, the Sacker-Sell spectrum of (7) consists of at most d closed intervals.*

Note that the most right endpoint of the Sacker-Sell spectral interval is the largest Bohl exponent of the system.

Example 20 Consider the following DAE

$$\begin{array}{rcl} \dot{x}_1 & = & \lambda x_1, \\ \dot{x}_2 & = & (\sin(\ln t) + \cos(\ln t))x_2, \qquad \lambda \in \mathbb{R}, t \geq t_0 > 0. \\ 0 & = & x_1 + x_2 - x_3, \end{array}$$

A simple computation gives $\Sigma_L = \{\lambda\} \cup [-1, 1]$, and $\Sigma_S = \{\lambda\} \cup [-\sqrt{2}, \sqrt{2}]$, i. e., $\Sigma_L \subset \Sigma_S$. If $\lambda \in [-\sqrt{2}, \sqrt{2}]$, then the Sacker-Sell spectrum reduces to one single interval. For $\lambda > \sqrt{2}$ or $\lambda < -\sqrt{2}$, the endpoints of the Sacker-Sell spectral intervals are the lower/upper Bohl exponents of the columns of the fundamental solution matrix

$$X(t) = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{t \sin \ln t} \\ e^{\lambda t} & e^{t \sin \ln t} \end{bmatrix}.$$

It is shown in [36, Section 3.4], that under some boundedness conditions, the Sacker-Sell spectrum of the DAE (7) is stable with respect to admissible perturbations. Theorem 50 in [36] also states that if X is an integrally separated fundamental matrix of (7), then the Sacker-Sell spectrum of the system is exactly given by the d (not necessarily disjoint) Bohl intervals associated with the columns of X.

In the remainder of the paper, we assume that Σ_S consists of $p \leq d$ pairwise disjoint spectral intervals, i. e., $\Sigma_S = \cup_{i=1}^{p}[a_i, b_i]$, and $b_i < a_{i+1}$ for all $1 \leq i \leq p$.

4.1 Boundedness of solutions of inhomogeneous equations

The Sacker-Sell spectrum can be used to study the boundedness of solutions of the inhomogeneous equation (1). For simplicity of presentation, we assume that the coefficient matrices E and A are given in the semi-implicit form (10). Decomposing f into two parts f_1 and f_2, solving for x_2, and inserting gives the underlying inhomogeneous ODE

$$E_{11}\dot{x}_1 = \tilde{A}_{11}x_1 + f_1 - A_{12}A_{22}^{-1}f_2, \tag{19}$$

where $\tilde{A}_{11}$ is defined as in (11).

Theorem 21 *Suppose that E_{11} and A_{22} are boundedly invertible, and that A_{11}, A_{12}, A_{21} are bounded. Then, for all continuous bounded functions f, the following statements hold:*

i) If $\Sigma_S \cap [0, \infty) = \emptyset$, i. e., the largest Bohl exponent is negative, then all the solutions of (1) are bounded.

ii) If $0 \notin \Sigma_S$, i. e., the system (7) has an exponential dichotomy, then there exists at least one bounded solution for (1). If in addition $\Sigma_S \cap (-\infty, 0) \neq \emptyset$, i. e. the projection Π in (17) is not trivially zero, then there exist infinitely many bounded solutions of (1) by adding an arbitrary bounded nontrivial solution of the corresponding homogenous equation.

Proof. Since $A_{22}^{-1}A_{21}$ is bounded, the Sacker-Sell spectrum of (10) and that of (11) are the same. Furthermore, the boundedness of x_1 implies that of x_2. Thus, it suffices to consider the underlying inhomogeneous ODE (19). Under the given assumptions, (19) reduces to

$$\dot{x}_1 = E_{11}^{-1}\tilde{A}_{11}x_1 + E_{11}^{-1}\left(f_1 - A_{12}A_{22}^{-1}f_2\right),$$

where $E_{11}^{-1}\tilde{A}_{11}$ and $E_{11}^{-1}\left(f_1 - A_{12}A_{22}^{-1}f_2\right)$ are bounded. By invoking [13, Theorem 5.2, p. 129], it follows that all the solutions x_1 of (19) are bounded if the largest Bohl exponent is negative. Since the algebraic component x_2 is bounded as well, this implies the boundedness of all the solutions of (1).

Similarly, if (11) has an exponential dichotomy, then due to [13, Theorem 3.2, p. 168], (19) has at least a bounded solution, which can be explicitly defined by the Green function of (11). As before, the boundedness of x_1 implies that of x_2 and the whole solution x.

If $\Sigma_S \cap (-\infty, 0) \neq \emptyset$, then there exists at least one negative upper Lyapunov exponent, which implies the existence of bounded nontrivial solutions of the corresponding homogenous equation. Furthermore, these bounded solutions create a linear subspace [15]. Finally, due to the solution theory of linear ODEs, adding an arbitrary (bounded) nontrivial solution of the corresponding homogenous equation to a (bounded) solution of the inhomogeneous equation gives another (bounded) solution. □

5 Leading directions associated with spectral intervals

As we have noted before, initial vectors of (7) must be chosen in a consistent way and they form a d-dimensional subspace in $\mathbb{R}^n$. Furthermore, the solutions of (7) also form a d-dimensional subspace of functions in $C^1(\mathbb{I}, \mathbb{R}^n)$. We denote these spaces by $\mathbb{S}_0$ and $\mathbb{S}(t)$, respectively, and for $x_0 \in \mathbb{S}_0$ we denote by $x(t; x_0)$ the (unique) solution of (7) that satisfies $x(0; x_0) = x_0$.

Assume that the upper Lyapunov exponents $\{\lambda_i^u\})_{i=1}^d$ are given in decreasing order. For $j = 1, \ldots d$, define the set W_j of all consistent initial conditions w such that the upper Lyapunov exponent of the solution $x(t; w)$ of (7) satisfies $\chi^u(x(\cdot; w)) \leq \lambda_j^u$, i. e.,

$$W_j = \left\{ w \in \mathbb{S}_0 : \chi^u\left(x(\cdot; w)\right) \leq \lambda_j^u \right\}, \quad j = 1, \ldots, d.$$

Let the columns of $U(\cdot)$ form a smoothly varying basis of the solution subspace $\mathbb{S}(\cdot)$ of (7) and consider an associated EUODE (9). Then instead of W_j, the corresponding set of all initial conditions for (9) that lead to Lyapunov exponents not greater than λ_j^u can be considered. In this way it is obvious that all results for ODEs in [15] apply to EUODEs of the form (9) and, as a consequence of Theorem 7, we obtain analogous statements for (7).

Proposition 22 *Let d_j be the largest number of linearly independent solutions x of (7) such that $\limsup_{t\to\infty} \frac{1}{t}\ln\|x(t)\| = \lambda_j^u$. Then W_j is a d_j-dimensional linear subspace of $\mathbb{S}_0$. Furthermore, the spaces W_j, $j = 1, 2, \ldots$, form a* filtration *of $\mathbb{S}_0$, i. e., if p is the number of distinct upper Lyapunov exponents of the system, then we have*

$$\mathbb{S}_0 = W_1 \supset W_2 \supset \ldots \supset W_p \supset W_{p+1} = \{0\}\,.$$

It follows that $\limsup_{t\to\infty} \frac{1}{t}\ln\|x(t;w)\| = \lambda_j^u$ if and only if $w \in W_j \backslash W_{j+1}$. Moreover, if we have d distinct upper Lyapunov exponents, then the dimension of W_j is $d - j + 1$.

If Y_j is defined as the orthogonal complement of W_{j+1} in W_j, i. e.,

$$W_j = W_{j+1} \oplus Y_j, \quad Y_j \perp W_{j+1},$$

then $\mathbb{S}_0 = Y_1 \oplus Y_2 \oplus \ldots \oplus Y_p$, and

$$\limsup_{t\to\infty} \frac{1}{t}\ln\|x(t;w)\| = \lambda_j^u \text{ if and only if } w \in Y_j.$$

Thus, if we have $p = d$ distinct Lyapunov exponents, then $\dim(Y_j) = 1$ for all $j = 1, \ldots, d$. Generalizing results of [15, 16] to DAEs, in [37] a numerical method for computing the spaces Y_j via smooth singular value decompositions of fundamental solutions was suggested.

Considering the resolvent set $\rho(E, A)$, for $\mu \in \rho(E, A)$, we can define the *stable* set associated with (7) as

$$\mathcal{S}_\mu = \left\{ w \in \mathbb{S}_0 : \lim_{t\to\infty} e^{-\mu t}\|x(t;w)\| = 0 \right\}.$$

Then, for $\mu_1, \mu_2 \in \rho(E, A),\ \mu_1 < \mu_2$, we have $\mathcal{S}_{\mu_1} \subseteq \mathcal{S}_{\mu_2}$.

In the following we study the EUODE (9) associated with (7), and for simplicity, we assume that Z is the *principal matrix solution*, i. e., $Z(0) = I_d$. This can always achieved by an appropriate kinematic equivalence transformation.

Following the construction for ODEs in [15, 45], we can characterize the stable and unstable subspaces

$$\mathcal{S}_\mu^d = \left\{ v \in \mathbb{R}^d : \lim_{t\to\infty} e^{-\mu t}\|Z(t)v\| = 0 \right\}, \mathcal{U}_\mu^d = \left\{ v \in \mathbb{R}^d : \lim_{t\to\infty} e^{\mu t}\left\|Z(t)^{-T}v\right\| = 0 \right\}$$

associated with (9).

Recalling that p is the number of disjoint spectral intervals, we choose a set of values $\mu_0 < \mu_1 < \ldots < \mu_p$, such that $\mu_j \in \rho(E, A)$ and $\Sigma_S \cap (\mu_{j-1}, \mu_j) = [a_j, b_j]$ for $j = 1, \ldots, p$, i. e.,

$$\mu_0 < a_1 \le b_1 < \mu_1 < \cdots < \mu_{j-1} < a_j \le b_j < \mu_j < \cdots < \mu_{p-1} < a_p \le b_p < \mu_p.$$

Let U be an orthonormal basis of the solution subspace for (7) and introduce the sets

$$\mathcal{N}_j = U(0)\mathcal{N}_j^d = \left\{ w \in \mathbb{S}_0 : w = U(0)v,\ v \in \mathcal{N}_j^d \right\},\ j = 1, \ldots, p. \tag{20}$$

Proposition 23 *([37]) Consider the EUODE (9) associated with (7), and the sets $\mathcal{N}_j$ defined in (20), $j = 1, \ldots, p$. If $w \in \mathcal{N}_j \setminus \{0\}$ and*

$$\limsup_{t\to\infty} \frac{1}{t} \ln \|x(t; w)\| = \chi^u, \quad \liminf_{t\to\infty} \frac{1}{t} \ln \|x(t; w)\| = \chi^\ell,$$

then $\chi^\ell, \chi^u \in [a_j, b_j]$.

This result means that $\mathcal{N}_j$ is the subspace of initial conditions associated with solutions of (7) whose upper and lower Lyapunov exponents are inside $[a_j, b_j]$.

Theorem 24 *Consider the EUODE (9) associated with (7), and the sets $\mathcal{N}_j$ defined in (20). Then $w \in \mathcal{N}_j \setminus \{0\}$ if and only if*

$$\frac{1}{K_{j-1}} e^{a_j(t-s)} \le \frac{\|x(t; w)\|}{\|x(s; w)\|} \le K_j e^{b_j(t-s)}, \quad \text{for all } t \ge s \ge 0, \tag{21}$$

and some positive constants K_{j-1}, K_j.

Proof. Due to the construction of the EUODE (9), see Lemma 2, we have $x(t; w) = U(t)Z(t)v$, where $v = U(0)^T w$, and thus $\|x(t; w)\| = \|Z(t)v\|$. Theorem 3.9 and Remark 3.10 of [15] state that $v \in \mathcal{N}_j^d$ if and only if

$$\frac{1}{K_{j-1}} e^{a_j(t-s)} \le \frac{\|Z(t)v\|}{\|Z(s)v\|} \le K_j e^{b_j(t-s)}, \quad \text{for all } t \ge s \ge 0,$$

and some positive constants K_{j-1}, K_j. Hence, (21) follows immediately. For more details see [37]. $\square$

The sets N_j and the Bohl exponents are closely related.

Corollary 25 *Consider the EUODE (9) associated with (7) and the sets $\mathcal{N}_j$ defined in (20). Then for all $j = 1, \dots, p$, one has $w \in \mathcal{N}_j \setminus \{0\}$ if and only if $a_j \le \kappa^\ell(x(\cdot; w)) \le \kappa^u(x(\cdot; w)) \le b_j$, where κ^ℓ, κ^u are the Bohl exponents.*

Proof. The proof follows from Theorem 24 and Definition 16. ◻

The stable and unstable sets are then characterized as follows.

Proposition 26 *Consider the EUODE (9) associated with (7). For all $j = 1, \dots, p$, we have*

(i) $\mathcal{S}_{\mu_j} = U(0)\mathcal{S}^d_{\mu_j}$.

(ii) Let the unstable sets for (7) be defined by $\mathcal{U}_{\mu_j} = U(0)\mathcal{U}^d_{\mu_j}$. *Then* $\mathcal{S}_{\mu_j} \oplus \mathcal{U}_{\mu_j} = \mathbb{S}_0$ *and* $\mathcal{N}_j = \mathcal{S}_{\mu_j} \cap \mathcal{U}_{\mu_{j-1}}$.

(iii) $\mathbb{S}_0 = \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \cdots \oplus \mathcal{N}_p$.

Proof. (i) First we prove
$$U(0)\mathcal{S}^d_{\mu_j} \subseteq \mathcal{S}_{\mu_j}.$$
To this end, take an arbitrary $w \in U(0)\mathcal{S}^d_{\mu_j}$. Then the corresponding initial value for (9) defined by $v = U(0)^T w$ clearly belongs to $\mathcal{S}^d_{\mu_j}$ and $w = U(0)v$ holds. By considering the one-to-one relation between the solutions of (7) and those of its associated EUODE (9) and invoking the equality $\|x(t; w)\| = \|Z(t)v\|$, $v \in \mathcal{S}^d_{\mu_j}$ implies $w \in \mathcal{S}_{\mu_j}$. Conversely, take an arbitrary $w \in \mathcal{S}_{\mu_j}$. Then there exists a unique $v \in \mathbb{R}^d$ which satisfies $w = U(0)v$. Using again that $\|x(t; w)\| = \|Z(t)v\|$, from the definition of $\mathcal{S}_{\mu_j}$ and that of $\mathcal{S}^d_{\mu_j}$, the claim $v \in \mathcal{S}^d_{\mu_j}$ follows.

(ii) As a consequence of Theorem 3.4 in [15], we have $\mathcal{S}^d_{\mu_j} \oplus \mathcal{U}^d_{\mu_j} = \mathbb{R}^d$. Since $U(0)$ consists of orthonormal columns, we have
$$U(0)\mathcal{S}^d_{\mu_j} \oplus U(0)\mathcal{U}^d_{\mu_j} = \operatorname{range} U(0) = \mathbb{S}_0,$$
from which the first equality immediately follows. The second equality is obvious from (i) and the definition of $\mathcal{N}_j$.

(iii) The equality follows from the relation between $\mathcal{N}_j$ and $\mathcal{N}^d_j$ and the result for ODEs [15, 37]. ◻

Example 27 Consider the DAE in Example 20 with $\lambda < 1$. The distinct upper Lyapunov exponents are λ and 1 and $d = p = 2$. The subspaces of consistent initial vectors associated with the upper Lyapunov exponents are $\mathbb{S}_0 = W_1 = \{[a, b, a+b]^T,\ a, b \in \mathbb{R}\}$, $W_2 = \{[a, 0, a]^T,\ a \in \mathbb{R}\}$, and $Y_1 = \{[0, b, b]^T,\ b \in \mathbb{R}\}$, $Y_2 = W_2$. The case $\lambda > 1$ is similar by exchanging the roles of Y_1 and Y_2. For $\lambda = 1$, we have $p = 1$, $\mathbb{S}_0 = W_1$ as above and $W_2 = \{[0, 0, 0]^T\}$.

Now taking the concrete value $\lambda = -3$, then the endpoints of the spectral intervals are $a_1 = b_1 = -3$, $a_2 = -\sqrt{2}$, $b_2 = \sqrt{2}$. Choosing any triplet $\mu_0 < -3, \mu_1 \in (-3, -\sqrt{2})$ and $\mu_2 > \sqrt{2}$, then $\mathcal{S}_{\mu_0} = \{[0, 0, 0]^T\}$, $\mathcal{S}_{\mu_1} = \{[a, 0, a]^T,\ a \in \mathbb{R}\}$, $\mathcal{S}_{\mu_2} = \{[a, b, a+b]^T,\ a, b \in \mathbb{R}\}$, $\mathcal{N}_1 = \{[a, 0, a]^T,\ a \in \mathbb{R}\}$, and $\mathcal{N}_2 = \{[0, b, b]^T,\ b \in \mathbb{R}\}$.

6 Discussion

We have summarized the spectral theory for linear time-varying DAEs which was recently given in [36, 37, 39, 40]. It is shown that most spectral notions and results can be extended from ODEs to strangeness-free reformulations of general regular DAEs, but with some extra conditions posed on the coefficients. Based on suitable construction of EUODEs and the presented analysis, numerical methods for computing spectral intervals can be constructed as in [37, 40]. As future work, an analysis for quasi-linear and/or nonlinear DAEs, together with efficient methods for approximating spectral intervals of linearized DAEs, is of interest. Another challenging problem is an extension of the spectral theory from finite dimensional systems to infinite dimensional ones, i. e., for partial differential-algebraic equations (PDAEs).

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