The infinite rate symbiotic branching model: from discrete to continuous space

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Abstract

The symbiotic branching model describes a spatial population consisting of two types that are allowed to migrate in space and branch locally only if both types are present. We continue our investigation of the large scale behaviour of the system started in [BHO15], where we showed that the continuum system converges after diffusive rescaling. Inspired by a scaling property of the continuum model, a series of earlier works initiated by Klenke and Mytnik [KM12a, KM12b] studied the model on a discrete space, but with infinite branching rate. In this paper, we bridge the gap between the two models by showing that by diffusively rescaling the discrete space infinite rate model we obtain our continuum model.

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1 Introduction

1.1 The symbiotic branching model and its interface

In [EF04] Etheridge and Fleischmann introduce a spatial population model that describes the evolution of two interacting types. On the level of a particle approximation, the dynamics follows locally a branching process, where each type branches with a rate proportional to the frequency of the other type. Additionally, types are allowed to migrate to neighbouring colonies. In the continuum space and large population limit, the frequencies $u_t^{[\gamma]}(x)$ and $v_t^{[\gamma]}(x)$ of the respective types are given by the nonnegative solutions of the stochastic partial differential equations

$$\mathrm{cSBM}(\varrho,\gamma)_{u_0,v_0}: \quad \begin{cases} \frac{\partial}{\partial t}u_t^{[\gamma]}(x) = \frac{\Delta}{2}u_t^{[\gamma]}(x) + \sqrt{\gamma u_t^{[\gamma]}(x)v_t^{[\gamma]}(x)}\,\dot{W}_t^{(1)}(x), \\ \frac{\partial}{\partial t}v_t^{[\gamma]}(x) = \frac{\Delta}{2}v_t^{[\gamma]}(x) + \sqrt{\gamma u_t^{[\gamma]}(x)v_t^{[\gamma]}(x)}\,\dot{W}_t^{(2)}(x), \end{cases} \tag{1}$$

with suitable nonnegative initial conditions $u_0(x)\geq 0, v_0(x)\geq 0, x\in\mathbb{R}$. Here, $\gamma>0$ is the branching rate and $(\dot{W}^{(1)},\dot{W}^{(2)})$ is a pair of correlated standard Gaussian white noises on

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$\mathbb{R}_+ \times \mathbb{R}$ with correlation governed by a parameter $\varrho \in [-1,1]$. Existence (for $\varrho \in [-1,1]$) and uniqueness (for $\varrho \in [-1,1)$) was proved in [EF04] for a large class of initial conditions.

The model generalizes several well-known examples of spatial populations dynamics. Indeed, for $\varrho = -1$ and $u_0 = 1 - v_0$, we have $u_t = 1 - v_t$ for all $t \geq 0$ and the system reduces to the continuous-space *stepping stone model* analysed in [Tri95], while for $\varrho = 0$, the system is known as the *mutually catalytic model* due to Dawson and Perkins [DP98]. Finally, for $\varrho = 1$ and the extra assumption $u_0 = v_0$, we find that $u_t = v_t$ for all $t \geq 0$, so that the model is an instance of the *parabolic Anderson model*, see for example [Mue91].

One of the central question is how the local dynamics, where one type will eventually dominate over the other, interacts with the migration to shape the global picture. A particularly interesting situation is when initially both types are spatially separated and one would like to know how one type 'invades' the other, in other words we would like to understand the interface between the two types. Mathematically, this corresponds to 'complementary Heaviside initial conditions', i.e.

$$u_0(x) = \mathbf{1}_{\mathbb{R}^-}(x) \quad \text{and} \quad v_0(x) = \mathbf{1}_{\mathbb{R}^+}(x), \quad x \in \mathbb{R}.$$

Definition 1.1. The interface at time t of a solution $(u_t^{[\gamma]}, v_t^{[\gamma]})_{t\geq 0}$ of the symbiotic branching model $\mathrm{cSBM}(\varrho,\gamma)_{u_0,v_0}$ with $\varrho \in [-1,1]$, $\gamma > 0$ is defined as

$$\mathrm{Ifc}_t = \mathrm{cl}\big\{x \in \mathbb{R} : u_t^{[\gamma]}(x) v_t^{[\gamma]}(x) > 0\big\},$$

where $\mathrm{cl}(A)$ denotes the closure of the set A in $\mathbb{R}$.

The first question that arises is whether this interface is non-trivial. Indeed, in [EF04] it is shown that the interface is a compact set and moreover that the width of the interface growths at most linearly in t. This result is strengthened in [BDE11, Thm. 2.11] for all ϱ close to -1 by showing that the width is at most of order $\sqrt{t\log(t)}$.

Especially the latter bound on the interface seems to suggest diffusive behaviour for the interface. This belief is supported by the following *scaling property*, see [EF04, Lemma 8]: If $(u_t, v_t)_{t\geq 0}$ is a solution to $\mathrm{cSBM}(\varrho,\gamma)_{u_0,v_0}$, then if we rescale time/space diffusively, i.e. if given $K > 0$ we define $\big(u_t^{(K)}(x), v_t^{(K)}(x)\big) := \big(u_{K^2t}^{[\gamma]}(Kx), v_{K^2t}^{[\gamma]}(Kx)\big)$ for $x \in \mathbb{R}, t \geq 0$, then we have that

$$(u_t^{(K)}, v_t^{(K)})_{t\geq 0} \stackrel{d}{=} (u_t^{[K\gamma]}, v^{[K\gamma]})_{t\geq 0}, \tag{2}$$

where the RHS is a solution to $\mathrm{cSBM}(\varrho, K\gamma)_{u_0^{(K)},v_0^{(K)}}$ with correspondingly transformed initial states $(u_0^{(K)}, v_0^{(K)})$.

Provided that the initial conditions are invariant under diffusive rescaling, then a diffusive rescaling of the system is equivalent (in law) to rescaling just the branching rate. Since the complementary Heaviside initial conditions are invariant, we will in the following always consider the limit $\gamma \to \infty$. This scaling then includes the diffusive rescaling, while also giving us the flexibility to consider more general initial conditions.

For the continuous space model this programme has been carried out in [BHO15]. We define the measure-valued processes

$$\mu_t^{[\gamma]}(dx) := u_t^{[\gamma]}(x)\,dx, \qquad \nu_t^{[\gamma]}(dx) = v_t^{[\gamma]}(x)\,dx \tag{3}$$

obtained by taking the solutions of $\mathrm{cSBM}(\varrho,\gamma)_{u_0,v_0}$ as densities, where the initial conditions remain fixed.

The following result was proved in [BHO15, Thm. 1.10]. We denote by $\mathcal{M}_{\mathrm{tem}}(\mathcal{S})$ the space of tempered measures on a space $\mathcal{S}$, and by $\mathcal{M}_{\mathrm{rap}}(\mathcal{S})$ the space of rapidly decreasing measures. Similarly, $\mathcal{B}^+_{\mathrm{tem}}(\mathcal{S})$ (resp. $\mathcal{B}^+_{\mathrm{rap}}(\mathcal{S})$) denotes the space of nonnegative, tempered (resp. rapidly decreasing) measurable functions on $\mathcal{S}$. We collect all the relevant formal definitions in Appendix A.1.

Theorem 1.2 ([BHO15])**.** *Let $\varrho \in (-1,0)$. Suppose the initial conditions satisfy $(\mu_0,\nu_0) \in (\mathcal{B}^+_{\mathrm{tem}}(\mathbb{R}))^2$ resp. $(\mu_0,\nu_0) \in (\mathcal{B}^+_{\mathrm{rap}}(\mathbb{R}))^2$ and for each $\gamma > 0$ we let $(u^{[\gamma]}_t, v^{[\gamma]}_t)_{t\geq 0}$ be a solution to $\mathrm{cSBM}(\varrho,\gamma)_{\mu_0,\nu_0}$. Then the measure-valued process $(\mu^{[\gamma]}_t, \nu^{[\gamma]}_t)_{t\geq 0}$ defined by (3) converges as $\gamma \to \infty$ in law in $D_{[0,\infty)}(\mathcal{M}_{\mathrm{tem}}(\mathbb{R})^2)$ resp. in $D_{[0,\infty)}(\mathcal{M}_{\mathrm{rap}}(\mathbb{R})^2)$ equipped with the Meyer-Zheng "pseudo-path" topology to a measure-valued process $(\mu_t,\nu_t)_{t\geq 0}$ satisfying the following separation-of-types condition: for any $x \in \mathbb{R}$, $t \in (0,\infty)$*

$$\mathbb{E}_{\mu_0,\nu_0}[S_\varepsilon \mu_t(x) S_\varepsilon \nu_t(x)] \to 0, \quad \text{as } \varepsilon \to 0, \tag{4}$$

where $(S_t)_{t\geq 0}$ denotes the heat semigroup.

Remark 1.3. (a) We call the limit $(\mu_t,\nu_t)_{t\geq 0}$ the *continuous-space infinite rate symbiotic branching model* $\mathrm{cSBM}(\varrho,\infty)$.

(b) We recall the definition of the Meyer-Zheng "pseudo-path" topology in the appendix A.3. This topology is strictly weaker than the standard Skorokhod topology on $D_{[0,\infty)}$. Under the more restrictive condition that $(\mu_0,\nu_0) = (\mathbb{1}_{\mathbb{R}^-}, \mathbb{1}_{\mathbb{R}^+})$ and $\varrho \in (-1, -\frac{1}{\sqrt{2}})$, we can also show tightness in the stronger Skorokhod topology, so that then in particular $(\mu^{[\gamma]}, \nu^{[\gamma]})$ converges as $\gamma \to \infty$ in $\mathcal{C}_{[0,\infty)}(\mathcal{M}^2_{\mathrm{tem}})$, cf. Theorem 1.5 in [BHO15]. Also, we show that in this case, the limiting measures μ_t,ν_t are absolutely continuous with respect to Lebesgue measure and if we denote the densities also by μ_t and ν_t, we can derive the more intuitive separation-of-types condition:

$$\mu_t(\cdot)\nu_t(\cdot) = 0 \qquad \mathbb{P}\otimes\ell\text{-a.s.} \tag{5}$$

For $\varrho = -1$ and complementary Heaviside initial conditions, the analogue of Theorem 1.2 was already proved in Tribe [Tri95] for the continuum stepping stone model, as one of the steps of understanding the diffusively rescaled interface. Under these assumptions it was shown that the process $(\mu^{[\gamma]}_t, \nu^{[\gamma]}_t)_{t\geq 0}$ converges weakly for $\gamma \to \infty$ to

$$(\mathbb{1}_{\{x\leq B_t\}}\,dx, \mathbb{1}_{\{x\geq B_t\}}\,dx)_{t\geq 0}, \tag{6}$$

for $(B_t)_{t\geq 0}$ a standard Brownian motion. Unfortunately, our previous work does not give such a truly explicit characterization of the infinite rate system for $\varrho > -1$. However, we do have a characterization in terms of a martingale problem (which we will recall below). This allows us to show that the limit is not of the form (6), see Remark 1.14 in [BHO15], even if we allow the position to be a general diffusion rather than a Brownian motion. Still we do not yet have an explicit description of the dynamics of the position of the interface and we leave open the question whether the rescaled interface shrinks down to a single point. In fact, even the case $\varrho = -1$ with general initial conditions is not covered by [Tri95] and remains open.

In order to take a first step towards a more explicit characterization of the limit in Theorem 1.2, our aim in this paper is to make the connection to related results on the discrete lattice $\mathbb{Z}$. We first recall that the *discrete-space* finite rate symbiotic branching model is given by the nonnegative solutions $((u_t^{[\gamma]}(x), v_t^{[\gamma]}(x)), x \in \mathbb{Z}, t \geq 0)$ of

$$\mathrm{dSBM}(\varrho,\gamma)_{u_0,v_0}: \quad \begin{cases} \frac{\partial}{\partial t} u_t^{[\gamma]}(x) = \frac{\Delta}{2} u_t^{[\gamma]}(x) + \sqrt{\gamma u_t^{[\gamma]}(x) v_t^{[\gamma]}(x)}\, \dot{W}_t^{(1)}(x), \\ \frac{\partial}{\partial t} v_t^{[\gamma]}(x) = \frac{\Delta}{2} v_t^{[\gamma]}(x) + \sqrt{\gamma u_t^{[\gamma]}(x) v_t^{[\gamma]}(x)}\, \dot{W}_t^{(2)}(x), \end{cases} , \tag{7}$$

with suitable nonnegative initial conditions $u_0(x) \geq 0, v_0(x) \geq 0, x \in \mathbb{Z}$. Here, $\gamma > 0$ is the branching rate, Δ is the discrete Laplace operator, defined for any $f : \mathbb{Z} \to \mathbb{R}$ as

$$\Delta f(x) = f(x+1) + f(x-1) - 2f(x), \quad x \in \mathbb{Z}, \tag{8}$$

and the pair $(W^1(x), W^2(x))$ is a ϱ-correlated two-dimensional Brownian motion which is independent for each $x \in \mathbb{Z}$.

Prior to our work, but also inspired by the scaling property (2) for the continuous model, Klenke and Mytnik consider this discrete space model, where the branching rate is sent to ∞. Indeed, in a series of papers [KM10, KM12a, KM12b] show that a non-trivial limiting process exists for $\gamma \to \infty$ (on the lattice) and study its long-term properties. Moreover, Klenke and Oeler [KO10] give a Trotter type approximation. Their results concentrate on the case $\varrho = 0$, i.e. the mutually catalytic model, however analogous results have been derived by Döring and Mytnik in the case $\varrho \in (-1, 1)$ in [DM13, DM12], We will refer to the limit as the *discrete-space infinite rate symbiotic branching model*, abbreviated as $\mathrm{dSBM}(\varrho, \infty)$.

What makes the results on the lattice especially interesting for our purpose of identifying the continuous infinite rate model is that there is a very explicit description of the limit $\mathrm{dSBM}(\varrho, \infty)$ in terms of a infinite system of jump-type stochastic differential equations (SDEs).

As noted in [EF04] the continuous symbiotic branching model can be obtained as a diffusive time/space rescaling of the discrete model. Therefore, it seems natural to expect that by rescaling the discrete system with infinite branching rate diffusively we obtain the infinite rate continuous space system of Theorem 1.2. In other words, we expect that the following diagram (Figure 1) commutes.

Indeed, this will be one of our main results in this note. In future work, we will attempt to exploit this commutativity to give a more explicit description of the limiting object in Theorem 1.2 by rescaling the jump-type SDEs of [KM12a].

1.2 Main results

In order to state our main result, we first recall the martingale problem that characterizes the limit in Theorem 1.2. This martingale problem is very much related to the martingale problem for the discrete space model $\mathrm{dSBM}(\varrho, \infty)$ of Klenke and Mytnik [KM12a].

We can formulate the martingale problem in both discrete and continuous space simultaneously and throughout we use the notation define in Appendix A.1. Therefore, let $\mathcal{S}$ be

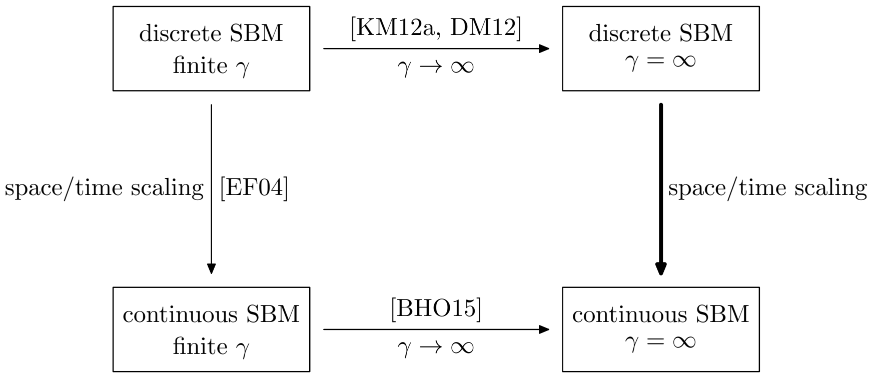

Figure 1: A commuting diagram.

either $\mathbb{Z}$ or $\mathbb{R}$. We recall the self-duality function employed in [EF04]: Let $\varrho \in (-1,1)$ and if either $(\mu,\nu,\phi,\psi) \in \mathcal{M}_{\text{tem}}(\mathcal{S})^2 \times \mathcal{B}_{\text{rap}}(\mathcal{S})^2$ or $(\mu,\nu,\phi,\psi) \in \mathcal{M}_{\text{rap}}(\mathcal{S})^2 \times \mathcal{B}_{\text{tem}}(\mathcal{S})^2$, denote

$$\langle\langle \mu,\nu,\phi,\psi\rangle\rangle_\varrho := -\sqrt{1-\varrho}\,\langle \mu+\nu, \phi+\psi\rangle + i\sqrt{1+\varrho}\,\langle \mu-\nu,\phi-\psi\rangle, \tag{9}$$

where $\langle \mu,\phi\rangle$ denotes the integral $\int_{\mathcal{S}} \phi(x)\mu(dx)$, for μ a measure and ϕ a measurable function. Then, we define the *self-duality function* F as

$$F(\mu,\nu,\phi,\psi) := \exp\langle\langle \mu,\nu,\phi,\psi\rangle\rangle_\varrho. \tag{10}$$

With this notation, we define a martingale problem, which in the continuous setting was called $\mathbf{MP}'$ in [BHO15].

Definition 1.4 (Martingale Problem $(\mathbf{MP}_F)^\varrho_{\mu_0,\nu_0}$)**.** Fix $\varrho \in (-1,1)$ and (possibly random) initial conditions $(\mu_0,\nu_0) \in \mathcal{M}_{\text{tem}}(\mathcal{S})^2$ (resp. $\mathcal{M}_{\text{rap}}(\mathcal{S})^2$). A càdlàg $\mathcal{M}_{\text{tem}}(\mathcal{S})^2$-valued (resp. $\mathcal{M}_{\text{rap}}(\mathcal{S})^2$-valued) stochastic process $(\mu_t,\nu_t)_{t\ge 0}$ is called a solution to the martingale problem $(\mathbf{MP}_F(\mathcal{S}))^\varrho_{\mu_0,\nu_0}$ if the following holds: There exists an increasing càdlàg $\mathcal{M}_{\text{tem}}(\mathcal{S})$-valued (resp. $\mathcal{M}_{\text{rap}}(\mathcal{S})$-valued) process $(\Lambda_t)_{t\ge0}$ with $\Lambda_0 = 0$ and

$$\mathbb{E}_{\mu_0,\nu_0}\big[\Lambda_t(dx)\big] \in \mathcal{M}_{\text{tem}}(\mathcal{S}) \quad (\text{resp. } \mathbb{E}_{\mu_0,\nu_0}\big[\Lambda_t(dx)\big] \in \mathcal{M}_{\text{rap}}(\mathcal{S})) \tag{11}$$

for all $t > 0$, such that for all test functions ϕ,ψ the process

$$\begin{aligned} &F(\mu_t,\nu_t,\phi,\psi) - F(\mu_0,\nu_0,\phi,\psi)\\ &\qquad - \frac{1}{2}\int_0^t F(\mu_s,\nu_s,\phi,\psi)\,\langle\langle \mu_s,\nu_s,\Delta\phi,\Delta\psi\rangle\rangle_\varrho\,ds\\ &\qquad - 4(1-\varrho^2)\int_{[0,t]\times\mathcal{S}} F(\mu_s,\nu_s,\phi,\psi)\,\phi(x)\psi(x)\,\Lambda(ds,dx) \end{aligned} \tag{12}$$

is a martingale, where if $\mathcal{S} = \mathbb{R}$ then $\phi,\psi \in \big(\mathcal{C}^{(2)}_{\text{rap}}\big)^+$ (resp. $\phi,\psi \in \big(\mathcal{C}^{(2)}_{\text{tem}}\big)^+$) and Δ is the continuum Laplace operator, while if $\mathcal{S} = \mathbb{Z}$ then $\phi,\psi \in \mathcal{B}_{\text{rap}}(\mathbb{Z})^+$ (resp. $\phi,\psi \in \mathcal{B}_{\text{tem}}(\mathbb{Z})^+$) and Δ is the discrete Laplace operator.

In (12) we have interpreted the right-continuous and increasing process $t \mapsto \Lambda_t(dx)$ as a (locally finite) measure $\Lambda(ds,dx)$ on $\mathbb{R}^+ \times \mathcal{S}$, via

$$\Lambda([0,t]\times B) := \Lambda_t(B).$$

In order to characterize $\mathrm{cSBM}(\varrho,\infty)$, it does not suffice to require that the martingale problem $(\mathbf{MP}_F)^{\varrho}_{u_0,v_0}$ is satisfied, since it holds for $\mathrm{cSBM}(\varrho,\gamma)$ for arbitrary $\gamma<\infty$, see Proposition A.5 in [BHO15]. However, we do get uniqueness if we require additionally that a suitable separation of types condition is satisfied, as we recall from [BHO15] (where the martingale problem $\mathbf{MP}_F(\mathbb{R})$ was denoted by $\mathbf{MP'}$):

Theorem 1.5 ([BHO15])**.** *Let $\varrho\in(-1,0)$. The solution to* $\mathrm{cSBM}(\varrho,\infty)$ *in Theorem 1.2 is characterized as the unique solution to* $(\mathbf{MP}_F(\mathbb{R}))^{\varrho}_{\mu_0,\nu_0}$ *satisfying the separation-of-types condition: for all $t\in(0,\infty)$, $x\in\mathbb{R}$ and $\varepsilon>0$,*

$$S_{t+\varepsilon}\mu_0(x)S_{t+\varepsilon}\nu_0(x)\geq\mathbb{E}_{\mu_0,\nu_0}[S_\varepsilon\mu_t(x)S_\varepsilon\nu_t(x)]\stackrel{\varepsilon\to0}{\longrightarrow}0. \tag{13}$$

We note that in the discrete context our martingale problem is not exactly the same as the martingale problem in [KM12a, Theorem 1.1]. Indeed, the main difference is the appearance of the measure Λ, which, in some sense that can be made precise, characterizes the correlations. The reason why we need this extra term in the continuous case can be understood if we recall that the martingale problem $\mathbf{MP}_F$ is tailored to an application of a self-duality (introduced in this context by Mytnik [Myt98]), which characterizes the finite-dimensional distributions. In the discrete context it suffices to consider test functions ϕ,ψ that satisfy $\phi(x)\psi(x)=0$ for all $x\in\mathbb{Z}$, see Corollary 2.4 in [KM10]. However, the same arguments do not carry over to the continuous space, where we need arbitrary test functions $\phi,\psi\in\big(\mathcal{C}^{(2)}_{\mathrm{rap}}(\mathcal{S})\big)^+$ (resp. $\phi,\psi\in\big(\mathcal{C}^{(2)}_{\mathrm{tem}}(\mathcal{S})\big)^+$).

But obviously we note that any solution of our martingale problem $\mathbf{MP}_F(\mathbb{Z})$ (together with a separation-of-types) satisfies the martingale problem of Theorem 1.1 in [KM12a] (respectively Theorem 4.4 in [DM13] for general ϱ). So as a first preliminary result, we show that the converse is also true and that there is a unique solution to the discrete analogue of the martingale problem in [BHO15]. Moreover, we allow for more general initial conditions. As for measures on $\mathbb{R}$, we will freely use $\nu(k)$ instead of $\nu(\{k\})$ for a measure ν on $\mathbb{Z}$ (and vice versa).

Theorem 1.6. *Assume that $\varrho\in(-1,0)$. Consider initial conditions $(u_0,v_0)\in\mathcal{M}_{\mathrm{tem}}(\mathbb{Z})^2$, resp. $(u_0,v_0)\in\mathcal{M}_{\mathrm{rap}}(\mathbb{Z})^2$.*

(i) There exists a unique solution $(u_t,v_t)_{t\geq0}$ to the martingale problem $(\mathbf{MP}_F(\mathbb{Z}))^{\varrho}_{u_0,v_0}$ such that almost surely for all $t>0$ and $k\in\mathbb{Z}$

$$u_t(k)v_t(k)=0. \tag{14}$$

(ii) Moreover, for each $\gamma>0$ denote by $(u^{[\gamma]}_t,v^{[\gamma]}_t)_{t\geq0}$ the solution to $\mathrm{dSBM}(\varrho,\gamma)_{u_0,v_0}$ *given in* (7)*, considered as measure-valued processes. Then, as $\gamma\uparrow\infty$, the sequence of processes $(u^{[\gamma]}_t,v^{[\gamma]}_t)_{t\geq0}$ converges in law in $D_{[0,\infty)}(\mathcal{M}_{\mathrm{tem}}(\mathbb{Z})^2)$ resp. in $D_{[0,\infty)}(\mathcal{M}_{\mathrm{rap}}(\mathbb{Z})^2)$ equipped with the Meyer-Zheng "pseudo-path" topology to the unique solution of the martingale problem $(\mathbf{MP}_F(\mathbb{Z}))^{\varrho}_{u_0,v_0}$ satisfying* (14)*.*

Following [KM12a, DM13], we call the unique solution to the martingale problem $(\mathbf{MP}_F)^{\varrho}_{u_0,v_0}$ satisfying (14) the *discrete-space infinite rate symbiotic branching process* and denote it by $\mathrm{dSBM}(\varrho,\infty)$.

Remark 1.7. As noted above our martingale problem is more restrictive than the version of [KM12a, DM13], since we require the martingale problem to hold for a larger class of test functions. Thus, our theorem generalizes their results in two ways: we show that their solution also satisfies our stronger martingale problem. Further we allow for more general initial conditions, while [KM12a, DM13] require separation-of-types also for the initial conditions, i.e. $\mu_0(x)\nu_0(x) = 0$ for all $x \in \mathbb{Z}$. If this holds, by uniqueness our solution coincides of course with the infinite rate process constructed in [KM12a] and [DM13].

Nevertheless, the work in [KM12a] goes beyond what we claim here in the sense that they are also able to show that the solution of $\mathrm{dSBM}(\varrho,\infty)$ can be characterized as a solution to a jump-type SDE, see [KM12a, Thm 1.3] for $\varrho = 0$ and [DM13, Prop. 4.14]. Moreover, [KM12a] considers more general operators than the discrete Laplacian. Also, they define solutions as taking values in a Liggett-Spitzer space (characterized by a suitable test function $\beta : \mathbb{Z} \to \mathbb{R}^+$), whereas we follow [DP98] in using tempered measures as state space. By choosing β in a suitable way, one can show that for initial conditions that satisfy (14) our solution agrees with theirs.

The method of proof for Theorem 1.6 is in fact very similar to the continuous-space case considered in [BHO15], and also to the proof of the convergence of the discrete to the continuous model. Therefore we have decided to give only a sketch of the proof, see Section 3.

Now, we can finally state the main result of our paper, which states states that for $\varrho \in (-1, 0)$ the discrete-space infinite rate model $\mathrm{dSBM}(\varrho,\infty)$ converges under diffusive rescaling (in the Meyer-Zheng sense) to the continuous-space model $\mathrm{cSBM}(\varrho,\infty)$ introduced in [BHO15]. More precisely, given initial conditions $(\mu_0, \nu_0) \in \mathcal{B}^+_{\mathrm{tem}}(\mathbb{R})^2$ for $\mathrm{cSBM}(\varrho,\infty)$, for each $n \in \mathbb{N}$ define $(u_0^{(n)}, v_0^{(n)}) \in \mathcal{M}_{\mathrm{tem}}(\mathbb{Z})^2$ by

$$u_0^{(n)}(k) := n \int_{k/n}^{(k+1)/n} \mu_0(x)\, dx, \qquad v_0^{(n)}(k) := n \int_{k/n}^{(k+1)/n} \nu_0(x)\, dx \tag{15}$$

and denote by $(u_t^{(n)}, v_t^{(n)})_{t\geq 0}$ the solution to $\mathrm{dSBM}(\varrho,\infty)_{u_0^{(n)},v_0^{(n)}}$. Now define approximating processes by diffusive rescaling, as follows:

$$\mu_t^{(n)}(x) := u_{n^2t}^{(n)}(\lfloor nx \rfloor), \qquad \nu_t^{(n)}(x) := v_{n^2t}^{(n)}(\lfloor nx \rfloor), \qquad t \geq 0,\ x \in \mathbb{R}. \tag{16}$$

We will consider $\mu_t^{(n)}$ and $\nu_t^{(n)}$ as densities w.r.t. Lebesgue measure and use the same symbols to denote the corresponding measures on $\mathbb{R}$. From (15), it is clear that

$$(\mu_0^{(n)}, \nu_0^{(n)}) \to (\mu_0, \nu_0)$$

in $\mathcal{M}_{\mathrm{tem}}(\mathbb{R})^2$ as $n \to \infty$.

Theorem 1.8. *Let $\varrho \in (-1, 0)$. Consider absolutely continuous and tempered initial conditions $(\mu_0, \nu_0) \in (\mathcal{B}^+_{\mathrm{tem}})^2$. Define $(u_0^{(n)}, v_0^{(n)})$ by (15) and denote by $(u_t^{(n)}, v_t^{(n)})_{t\geq 0}$ the solution to $\mathrm{dSBM}(\varrho,\infty)_{u_0^{(n)},v_0^{(n)}}$ from Theorem 1.6. Then as $n \to \infty$, the sequence of measure-valued processes $(\mu_t^{(n)}, \nu_t^{(n)})_{t\geq 0}$ from (16) converges weakly in $D_{[0,\infty)}(\mathcal{M}_{\mathrm{tem}}(\mathbb{R})^2)$ equipped with the Meyer-Zheng "pseudo-path" topology to the unique solution $(\mu_t, \nu_t)_{t\geq 0}$ of $\mathrm{cSBM}(\varrho,\infty)_{\mu_0,\nu_0}$ from Theorem 1.2.*

For the proof of the convergence of the discrete to the continuous model, we need some moment properties of the solution, which might be of independent interest. These estimates have direct continuous-space analogues that are implicit in [BHO15], but were not stated explicitly there. In order to formulate them, it is convenient to introduce the following notation:

Again let $\mathcal{S}$ be either $\mathbb{Z}$ or $\mathbb{R}$, and let $(S_t)_{t\geq 0}$ denote the usual heat semigroup on $\mathcal{S}$, i.e. the semigroup of simple symmetric (continuous-time) random walk if $\mathcal{S}=\mathbb{Z}$ and the semigroup of standard Brownian motion if $\mathcal{S}=\mathbb{R}$. Further, we write $(S_t^{(2)})_{t\geq 0}$ for the corresponding two-dimensional semigroup on $\mathcal{S}^2$. Finally, we define a semigroup $(\tilde{S}_t)_{t\geq 0}$ of the respective process killed upon hitting the diagonal in $\mathcal{S}^2$, i.e.

$$\tilde{S}_t f(x,y) := \mathbb{E}_{x,y}\left[f(X_t^{(1)}, X_t^{(2)})\mathbb{1}_{t<\tau^{1,2}}\right], \qquad f:\mathcal{S}^2\to\mathbb{R},\ (x,y)\in\mathcal{S}^2, \tag{17}$$

where $\tau^{1,2} := \inf\{t>0 : X_t^{(1)} = X_t^{(2)}\}$ denotes the first hitting time of the diagonal. Here obviously $(X^{(1)}, X^{(2)})$ denotes simple symmetric (continuous-time) random walk if $\mathcal{S}=\mathbb{Z}$ and standard Brownian motion if $\mathcal{S}=\mathbb{R}$.

Proposition 1.9 (Moments)**.** *Assume that* $\varrho\in(-1,0)$*.*

For $\mathcal{S}=\mathbb{Z}$*, consider initial conditions* $(\mu_0,\nu_0)\in\mathcal{M}_{\text{tem}}(\mathcal{S})^2$ *(resp.* $(\mu_0,\nu_0)\in\mathcal{M}_{\text{rap}}(\mathcal{S})^2$*), and let* $(\mu_t,\nu_t)_{t\geq 0}$ *denote the solution of* $\mathrm{dSBM}(\varrho,\infty)_{\mu_0,\nu_0}$*. Then we have the following estimate on the second mixed moment:*

$$\mathbb{E}_{\mu_0,\nu_0}\left[\langle\mu_t,\phi\rangle\,\langle\nu_t,\psi\rangle\right] \leq \left\langle \phi\otimes\psi, \tilde{S}_t(\mu_0\otimes\nu_0)\right\rangle_{\mathcal{S}^2} \tag{18}$$

for all $t>0$ *and test functions* $\phi,\psi\in\bigcup_{\lambda>0}\mathcal{B}^+_{-\lambda}(\mathcal{S})$ *(resp.* $\bigcup_{\lambda>0}\mathcal{B}^+_{\lambda}(\mathcal{S})$*). Further, the process* $(\Lambda_t)_{t\geq 0}$ *in the martingale problem* $(\mathbf{MP}_F(\mathcal{S}))^{\varrho}_{u_0,v_0}$ *can be chosen such that for all such test functions and* $t>0$ *we have the following first moment estimate:*

$$\begin{aligned}\mathbb{E}_{\mu_0,\nu_0}\Big[\int_{[0,t]\times\mathcal{S}} S_{t-s}\phi(x)S_{t-s}\psi(x)\,\Lambda(ds,dx)\Big]\\ \leq \frac{1}{|\varrho|}\big\langle\phi\otimes\psi, (S_t^{(2)}-\tilde{S}_t)(\mu_0\otimes\nu_0)\big\rangle_{\mathcal{S}^2}.\end{aligned} \tag{19}$$

If $\mathcal{S}=\mathbb{R}$*, the bounds* (18) *and* (19) *hold for initial conditions* $(\mu_0,\nu_0)\in\mathcal{B}^+_{\text{tem}}(\mathcal{S})^2$ *(resp.* $(\mu_0,\nu_0)\in\mathcal{B}^+_{\text{rap}}(\mathcal{S})^2$*), with* $(\mu_t,\nu_t)_{t\geq 0}$ *denoting the solution of* $\mathrm{cSBM}(\varrho,\infty)_{\mu_0,\nu_0}$*.*

Remark 1.10. In the discrete-space case $\mathcal{S}=\mathbb{Z}$, and for integrable initial conditions with disjoint support, the second mixed moment bound in (18) is already known, see [DM12, Theorem 1.2]. For the continuous-space setting, it is implicitly shown in the proof of [BHO15, Lemma 4.4], see in particular the derivation of inequality (51). Moreover the bound (19) follows by taking the limit $\gamma\to\infty$ in [BHO15, Lemma 3.1], see (35) there. These arguments carry over to the discrete case, see Prop. 3.4 below.

The remaining paper is structured as follows: In Section 2, we prove Theorem 1.8 by showing that the diffusively rescaled solutions of the discrete model are tight and that the corresponding martingale problem converges to the continuous version. Finally, in Section 3 we sketch the proof of Theorems 1.6 and 1.9 by showing how to adapt the corresponding arguments from [BHO15].

Notation: We have collected some of the standard facts and notations about measure-valued processes in Appendix A.1. In Appendix A.2, we recall some standard results for the (killed) heat semigroup and in Appendix A.3 we recall the Meyer-Zheng "pseudo-path" topology. Throughout this paper, we will denote by c, C generic constants whose value may change from line to line. If the dependence on parameters is essential we will indicate this correspondingly.

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2 Convergence of the discrete to the continuous model

In this section, we prove Theorem 1.8. Recall that given initial conditions $(\mu_0, \nu_0) \in (\mathcal{B}^+_{\text{tem}}(\mathbb{R}))^2$ for cSBM(ϱ, ∞), we define $(\mu^{(n)}, \nu^{(n)})$ by (15)-(16), and our goal is to show that $(\mu^{(n)}_t, \nu^{(n)}_t)_{t\geq 0} \xrightarrow{n\to\infty} (\mu_t, \nu_t)_{t\geq 0}$ as measure-valued processes, where (μ, ν) denotes the (unique) solution to cSBM$(\varrho, \infty)_{\mu_0,\nu_0}$ introduced in [BHO15]. The strategy of proof is familiar: First we prove tightness in the next subsection, then we show that limit points solve the martingale problem $(\mathbf{MP}_F(\mathbb{R}))^\varrho$ from Definition 1.4 and the 'separation of types'-property (13).

We begin with some preliminaries. Note that if $\phi \in \bigcup_{\lambda>0} \mathcal{C}^+_\lambda$ is a test function and for each $n \in \mathbb{N}$ we define a function $\phi^{(n)} : \mathbb{Z} \to \mathbb{R}^+$ by

$$\phi^{(n)}(k) := \int_{k/n}^{(k+1)/n} \phi(x)\, dx, \qquad k \in \mathbb{Z}, \tag{20}$$

then $\phi^{(n)} \in \bigcup_{\lambda>0} \mathcal{B}^+_\lambda(\mathbb{Z})$ and

$$\langle \mu^{(n)}_t, \phi\rangle = \int_{\mathbb{R}} \mu^{(n)}_t(x)\phi(x)\, dx = \sum_{k\in\mathbb{Z}} u^{(n)}_{n^2 t}(k)\phi^{(n)}(k) = \langle u^{(n)}_{n^2 t}, \phi^{(n)}\rangle, \tag{21}$$

and analogously for $\nu^{(n)}$.

For each $n \in \mathbb{N}$, since $(u^{(n)}_t, v^{(n)}_t)_{t\geq 0}$ is a solution to the martingale problem $(\mathbf{MP}_F(\mathbb{Z}))^\varrho_{u^{(n)}_0, v^{(n)}_0}$, there exists an increasing càdlàg $\mathcal{M}_{\text{tem}}(\mathbb{Z})$-valued process $(L^{(n)}_t)_{t\geq 0}$ fulfilling the requirements of Definition 1.4. We define an increasing càdlàg process $(\Lambda^{(n)}_t)_{t\geq 0}$ taking values in $\mathcal{M}_{\text{tem}}(\mathbb{R})$ by

$$\langle \Lambda^{(n)}_t, \phi\rangle := \frac{1}{n^2} \left\langle L^{(n)}_{n^2 t}, \phi(\cdot/n)\right\rangle \tag{22}$$

for each $\phi \in \bigcup_{\lambda>0} \mathcal{C}^+_\lambda$.

In the following, we will need to distinguish the discrete-space versions of the semigroups and generators introduced in Section 1 from their continuous-space counterparts. Therefore, from now on we shall use the notations $({}^d S_t)_{t\geq 0}$ and ${}^d\Delta$ for the discrete heat semigroup on the lattice $\mathbb{Z}$ and its generator, the discrete Laplacian. Moreover, we will write $({}^d S^{(2)}_t)_{t\geq 0}$ for the corresponding two-dimensional semigroup on $\mathbb{Z}^2$ and $({}^d \tilde{S}_t)_{t\geq 0}$ for the discrete version of the killed semigroup introduced in (17). The symbols S_t, Δ, $S^{(2)}_t$ and $\tilde{S}_t$ will be reserved for the continuous-space versions of the above.

The following easy lemma will be used several times in the sequel:

Lemma 2.1. *Given* $\mu_0 \in \mathcal{B}^+_{\mathrm{tem}}(\mathbb{R})$*, define* $u_0^{(n)}$ *by* (15)*. Then we have for all* $T > 0$ *and* $\lambda > 0$ *that*

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \frac{1}{n} \left\langle \phi_\lambda(\cdot/n), {}^dS_{n^2t} u_0^{(n)} \right\rangle < \infty. \tag{23}$$

Proof. Choose $\tilde{\lambda} \in (0, \lambda)$. Since by assumption $\mu_0 \in \mathcal{B}^+_{\mathrm{tem}}(\mathbb{R})$, there is a constant $\tilde{C} = \tilde{C}(\tilde{\lambda})$ such that $\mu_0(x) \leq \tilde{C}\phi_{-\tilde{\lambda}}(x)$ for all $x \in \mathbb{R}$. By (15), it follows that for some constant $C' = C'(\tilde{\lambda})$

$$u_0^{(n)}(k) \leq C' \phi_{-\tilde{\lambda}}(k/n)$$

for all $k \in \mathbb{Z}$. Now we use [DEF+02, Cor. A3(a)] (see also Lemma A.2 b) for a reformulation to our context) to obtain

$${}^dS_{n^2t} u_0^{(n)}(k) \leq C' \, {}^dS_{n^2t}(\phi_{-\tilde{\lambda}}(\cdot/n))(k) \leq C(\tilde{\lambda}, t)\, \phi_{-\tilde{\lambda}}(k/n)$$

for all $k \in \mathbb{Z}$, where the constant $C(\tilde{\lambda}, t) := C' \, 2 \exp(t \, \tilde{\lambda}^2 \, e^{\tilde{\lambda}^2})$ is independent of n and bounded on compact time intervals. This shows that

$$\begin{aligned}
\sup_{t\in[0,T]} \frac{1}{n} \left\langle \phi_\lambda(\cdot/n), {}^dS_{n^2t} u_0^{(n)} \right\rangle &\leq C(\mu_0, \lambda, T)\, \frac{1}{n} \sum_{k\in\mathbb{Z}} \phi_\lambda(k/n)\, \phi_{-\tilde{\lambda}}(k/n) \\
&= C(\mu_0, \lambda, T)\, \frac{1}{n} \sum_{k\in\mathbb{Z}} e^{-(\lambda-\tilde{\lambda})|k|/n} \\
&\to C(\mu_0, \lambda, T) \int_{\mathbb{R}} e^{-(\lambda-\tilde{\lambda})|x|} < \infty
\end{aligned}$$

as $n \to \infty$, since $\tilde{\lambda} < \lambda$. □

2.1 Tightness

Lemma 2.2. *Suppose* $\varrho < 0$ *and* $(\mu_0, \nu_0) \in (\mathcal{B}^+_{\mathrm{tem}}(\mathbb{R}))^2$*. Then for all* $T > 0$ *and* $\phi \in \bigcup_{\lambda>0} \mathcal{C}^+_\lambda$ *we have*

$$\sup_{n\in\mathbb{N}} \mathbb{E}_{\mu_0^{(n)},\nu_0^{(n)}} \Big[\sup_{t\in[0,T]} \langle \phi, \mu_t^{(n)} \rangle^2 \Big] < \infty, \qquad \sup_{n\in\mathbb{N}} \mathbb{E}_{\mu_0^{(n)},\nu_0^{(n)}} \Big[\sup_{t\in[0,T]} \langle \phi, \nu_t^{(n)} \rangle^2 \Big] < \infty, \tag{24}$$

and

$$\sup_{n\in\mathbb{N}} \mathbb{E}_{\mu_0^{(n)},\nu_0^{(n)}} \Big[\sup_{t\in[0,T]} \langle \phi, \Lambda_t^{(n)} \rangle \Big] < \infty. \tag{25}$$

Proof. Fix $T > 0$ and assume w.l.o.g. that $\phi = \phi_\lambda$ for a suitable $\lambda > 0$. Evidently, we have $\phi_\lambda^{(n)}(k) \leq \frac{1}{n} e^\lambda \phi_\lambda(k/n)$ for all $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. Further, by Lemma A.2(b) there is a constant $C = C(\lambda, T)$ independent of n such that

$$\phi_\lambda(k/n) \leq C \, {}^dS_{n^2T-s}(\phi_\lambda(\cdot/n))(k) \tag{26}$$

for all $s \in [0,n^2T]$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. Using this together with the Green function representation for the discrete model (see Lemma 3.7 below, with $[0,n^2T]$ in place of $[0,T]$), we get

$$
\begin{aligned}
\langle \mu_t^{(n)}, \phi_\lambda \rangle = \langle u_{n^2t}^{(n)}, \phi_\lambda^{(n)} \rangle &\le \frac{C}{n} \left\langle u_{n^2t}^{(n)}, {}^dS_{n^2(T-t)}(\phi_\lambda(\cdot/n)) \right\rangle \\
&= \frac{C}{n} \left(\langle u_0^{(n)}, {}^dS_{n^2T}\phi_\lambda(\cdot/n) \rangle + M_{n^2t}^{n^2T}(\phi_\lambda(\cdot/n)) \right),
\end{aligned} \tag{27}
$$

where the second moment of the martingale term is bounded by

$$
\begin{aligned}
&\mathbb{E}_{u_0^{(n)},v_0^{(n)}} \Big[\sup_{t\in[0,T]} \big| \tfrac{1}{n} M_{n^2t}^{n^2T}(\phi_\lambda(\cdot/n)) \big|^2 \Big] \\
&\qquad \le \frac{4}{|\varrho|} \frac{1}{n^2} \left\langle \phi_\lambda(\cdot/n) \otimes \phi_\lambda(\cdot/n), ({}^dS^{(2)}_{n^2T} - {}^d\tilde{S}_{n^2T})(u_0^{(n)} \otimes v_0^{(n)}) \right\rangle_{\mathbb{Z}^2} \\
&\qquad \le \frac{4}{|\varrho|} \frac{1}{n^2} \left\langle \phi_\lambda(\cdot/n), {}^dS_{n^2T} u_0^{(n)} \right\rangle \left\langle \phi_\lambda(\cdot/n), {}^dS_{n^2T} v_0^{(n)} \right\rangle
\end{aligned} \tag{28}
$$

for all $n \in \mathbb{N}$ (see the estimate (52)). Combining (27)-(28) with Lemma 2.1, the first inequality in (24) follows easily, and the proof of the second one is analogous.

For the increasing process $\Lambda^{(n)}$, we observe that

$$
\begin{aligned}
\mathbb{E}_{\mu_0^{(n)},\nu_0^{(n)}} \Big[\sup_{t\in[0,T]} \langle \phi_\lambda, \Lambda_t^{(n)} \rangle \Big] &= \frac{1}{n^2} \mathbb{E}_{u_0^{(n)},v_0^{(n)}} \left[\langle \phi_\lambda(\cdot/n), L_{n^2T}^{(n)} \rangle \right] \\
&= \frac{1}{n^2} \mathbb{E}_{u_0^{(n)},v_0^{(n)}} \left[\langle \phi_{\lambda/2}(\cdot/n)^2, L_{n^2T}^{(n)} \rangle \right] \\
&\le \frac{C}{n^2} \mathbb{E}_{u_0^{(n)},v_0^{(n)}} \left[\int_{[0,n^2T]\times\mathbb{Z}} \left({}^dS_{n^2T-s} \phi_{\lambda/2}(k/n) \right)^2 L^{(n)}(ds, dk) \right] \\
&\le \frac{C}{|\varrho|} \frac{1}{n^2} \left\langle \phi_{\lambda/2}(\cdot/n) \otimes \phi_{\lambda/2}(\cdot/n), ({}^dS^{(2)}_{n^2T} - {}^d\tilde{S}_{n^2T})(u_0^{(n)} \otimes v_0^{(n)}) \right\rangle_{\mathbb{Z}^2},
\end{aligned}
$$

where we used again (26) for the first inequality and estimate (46) for the second one. Now we can argue as before to conclude that the RHS of the previous display is bounded uniformly in $n \in \mathbb{N}$. □

Corollary 2.3 (Compact Containment)**.** *Suppose* $\varrho < 0$ *and* $(\mu_0, \nu_0) \in (\mathcal{B}^+_{\text{tem}}(\mathbb{R}))^2$*. Then the* compact containment condition *holds for the family* $(\mu_t^{(n)}, \nu_t^{(n)}, \Lambda_t^{(n)})_{t\ge0}$*, i.e. for every* $\varepsilon > 0$ *and* $T > 0$ *there exists a compact subset* $K = K_{\varepsilon,T} \subseteq \mathcal{M}_{\text{tem}}(\mathbb{R})$ *such that*

$$
\inf_{n\in\mathbb{N}} \mathbb{P}\big\{ \mu_t^{(n)} \in K_{\varepsilon,T} \text{ for all } t \in [0,T] \big\} \ge 1 - \varepsilon,
$$

and similarly for $\nu_t^{(n)}$ *and* $\Lambda_t^{(n)}$*.*

Proof. Given the uniform first moment bounds from Lemma 2.2, the proof is virtually identical to that of Corollary 3.3 in [BHO15]. □

Proposition 2.4. *Suppose* $\varrho < 0$ *and* $(\mu_0, \nu_0) \in (\mathcal{B}^+_{\text{tem}}(\mathbb{R}))^2$*. Then the family of processes* $(\mu_t^{(n)}, \nu_t^{(n)}, \Lambda_t^{(n)})_{t\ge0}$ *is tight with respect to the Meyer-Zheng topology on* $D_{[0,\infty)}(\mathcal{M}_{\text{tem}}(\mathbb{R})^3)$*.*

Proof. Suppose $(\mu_0, \nu_0) \in (\mathcal{B}^+_{\text{tem}}(\mathbb{R}))^2$. We aim at applying [Kur91, Cor. 1.4], which requires us to check the Meyer-Zheng tightness condition (see e.g. (66) in the appendix) for the coordinate processes plus a compact containment condition. Let $\phi \in (\mathcal{C}^{(2)}_{\text{rap}})^+$ and fix $T > 0$. By Lemma 3.6, we know that

$$\langle \phi, \mu^{(n)}_t \rangle = \langle u^{(n)}_{n^2 t}, \phi^{(n)} \rangle = \langle u^{(n)}_0, \phi^{(n)} \rangle + \frac{1}{2} \int_0^{n^2 t} \langle u^{(n)}_s, {}^d\Delta(\phi^{(n)}) \rangle \, ds \; + M_{n^2 t}(\phi^{(n)}), \tag{29}$$

where $M_{n^2t}(\phi^{(n)})$ is a martingale with second moment bounded by

$$\mathbb{E}_{u_0^{(n)}, v_0^{(n)}} \left[|M_{n^2 t}(\phi^{(n)})|^2 \right] \leq \sup_{\gamma > 0} \mathbb{E}_{u_0^{(n)}, v_0^{(n)}} \left[\langle L^{[\gamma]}_{n^2 t}, (\phi^{(n)})^2 \rangle \right].$$

Now fixing $T > 0$, choosing a suitable $\lambda > 0$ and using the lower bound from Lemma A.2 b), we see that there is a constant $C = C(\lambda, T)$ such that the previous display is bounded by

$$\begin{aligned} &\frac{C}{n^2} \sup_{\gamma > 0} \mathbb{E}_{u_0^{(n)}, v_0^{(n)}} \left[\int_{[0, n^2 t] \times \mathbb{Z}} \left({}^d S_{n^2 t - s} \phi_\lambda(\cdot/n)(k) \right)^2 \, L^{[\gamma]}(ds, dk) \right] \\ &\leq \frac{C}{n^2} \frac{1}{|\varrho|} \left\langle \phi_\lambda(\cdot/n) \otimes \phi_\lambda(\cdot/n), ({}^d S^{(2)}_{n^2 t} - {}^d \tilde{S}_{n^2 t})(u_0^{(n)} \otimes v_0^{(n)}) \right\rangle_{\mathbb{Z}^2} \end{aligned}$$

for all $t \in [0, T]$, where we have also used estimate (40). But the last display is bounded uniformly in $t \in [0, T]$ and $n \in \mathbb{N}$ by Lemma 2.1 (see also the proof of Lemma 2.2), hence we get

$$\sup_{n \in \mathbb{N}} \sup_{t \in [0, T]} \mathbb{E}_{u_0^{(n)}, v_0^{(n)}} \left[|M_{n^2 t}(\phi^{(n)})|^2 \right] < \infty$$

for all $T > 0$. This implies immediately the Meyer-Zheng tightness condition (66) for the sequence of martingales in (29).

In view of (29), it remains to show tightness of the term

$$X^{(n)}_t := \int_0^{n^2 t} \langle u^{(n)}_s, {}^d\Delta(\phi^{(n)}) \rangle \, ds = n^2 \int_0^t \langle u^{(n)}_{n^2 s}, {}^d\Delta(\phi^{(n)}) \rangle \, ds, \quad t \in [0, T].$$

But Lemma 2.2 implies that this term is tight in the stronger Skorokhod topology, as follows: Since $\phi \in (\mathcal{C}^{(2)}_{\text{rap}})^+$, there is a suitable $\lambda > 0$ and some constant $C = C(\phi)$ such that $|{}^d\Delta(\phi^{(n)})(k)| \leq \frac{C}{n^3} \phi_\lambda(k/n)$ for all $k \in \mathbb{Z}$, $n \in \mathbb{N}$, thus we get for $s < t$, $s, t \in [0, T]$ that

$$\begin{aligned} \mathbb{E}_{u_0^{(n)}, v_0^{(n)}} \left[|X^{(n)}_t - X^{(n)}_s|^2 \right] &\leq C(\phi) \, \mathbb{E}_{u_0^{(n)}, v_0^{(n)}} \left[\left(\int_s^t \frac{1}{n} \langle u^{(n)}_{n^2 r}, \phi_\lambda(\cdot/n) \rangle \, dr \right)^2 \right] \\ &\leq C \, (t - s) \int_s^t \mathbb{E}_{u_0^{(n)}, v_0^{(n)}} \left[\frac{1}{n^2} \langle u^{(n)}_{n^2 r}, \phi_\lambda(\cdot/n) \rangle^2 \right] dr, \end{aligned} \tag{30}$$

where we have used Jensen's inequality. But again by (the proof of) Lemma 2.2, the integrand in the above display is bounded uniformly in $n \in \mathbb{N}$ and $s, t \in [0, T]$, whence we get

$$\mathbb{E}_{u_0^{(n)}, v_0^{(n)}} \left[|X^{(n)}_t - X^{(n)}_s|^2 \right] \leq C \, (t - s)^2,$$

confirming Kolmogorov's tightness criterion for the Laplace term.

This shows that the sequence of coordinate processes $(\langle \phi, \mu_t^{(n)} \rangle)_{t \geq 0}$, $n \in \mathbb{N}$, is tight w.r.t the Meyer-Zheng topology. The same argument works for $(\langle \phi, \nu_t^{(n)} \rangle)_{t \geq 0}$. For the increasing process $t \mapsto \langle \phi, \Lambda_t^{(n)} \rangle$, condition (66) reduces to

$$\sup_{n \in \mathbb{N}} \mathbb{E}_{\mu_0^{(n)}, \nu_0^{(n)}} \left[\langle \phi, \Lambda_T^{(n)} \rangle \right] < \infty$$

which is also ensured by Lemma 2.2.

The above argument shows that the Meyer-Zheng tightness criterion is satisfied for the coordinate processes. The compact containment condition has already been checked in Corollary 2.3. Applying [Kur91, Cor. 1.4], we are done. □

2.2 Properties of Limit Points

In this subsection, we check that limit points $(\mu_t, \nu_t, \Lambda_t)_{t \geq 0}$ of the sequence $(\mu_t^{(n)}, \nu_t^{(n)}, \Lambda_t^{(n)})_{t \geq 0}$ satisfy the martingale problem $(\mathbf{MP}_F(\mathbb{R}))^{\varrho}_{\mu_0, \nu_0}$ and the separation of types-property (13). By Theorem 1.5, this implies that the rescaled discrete processes converge indeed to the unique solution of cSBM(ϱ, ∞).

Proposition 2.5. *Let $\varrho < 0$ and $(\mu_0, \nu_0) \in (\mathcal{B}^+_{\text{tem}}(\mathbb{R}))^2$. If $(\mu_t, \nu_t, \Lambda_t)_{t \geq 0} \in D_{[0,\infty)}(\mathcal{M}_{\text{tem}}(\mathbb{R})^3)$ is any limit point with respect to the Meyer-Zheng topology of the sequence $(\mu_t^{(n)}, \nu_t^{(n)}, \Lambda_t^{(n)})_{t \geq 0}$, $n \in \mathbb{N}$, then $(\mu_t, \nu_t)_{t \geq 0}$ solves the martingale problem $(\mathbf{MP}_F(\mathbb{R}))^{\varrho}_{\mu_0, \nu_0}$, where the process $(\Lambda_t)_{t \geq 0}$ satisfies the requirements of Definition 1.4.*

Proof. First of all, the limit point $(\Lambda_t)_{t \geq 0}$ of the sequence $(\Lambda_t^{(n)})_{t \geq 0}$, $n \in \mathbb{N}$, has the properties required in Def. 1.4: It is clear that $(\Lambda_t)_{t \geq 0}$ is increasing with $\Lambda_0 = 0$, and from the first moment estimate (25) we see by an application of Fatou's lemma that for all $\phi \in \bigcup_{\lambda > 0} \mathcal{C}^+_{\lambda}$

$$\mathbb{E}_{\mu_0, \nu_0}[\langle \Lambda_t, \phi \rangle] < \infty,$$

thus also condition (11) is satisfied.

It remains to check that for all test functions $\phi, \psi \in \left(\mathcal{C}^{(2)}_{\text{rap}}\right)^+$ the process

$$\begin{aligned} M_t(\phi, \psi) := {} & F(\mu_t, \nu_t, \phi, \psi) - F(\mu_0, \nu_0, \phi, \psi) \\ & - \frac{1}{2} \int_0^t F(\mu_s, \nu_s, \phi, \psi) \, \langle\langle \mu_s, \nu_s, \Delta\phi, \Delta\psi \rangle\rangle_{\varrho} \, ds \\ & - 4(1 - \varrho^2) \int_{[0,t] \times \mathbb{R}} F(\mu_s, \nu_s, \phi, \psi) \, \phi(x) \psi(x) \, \Lambda(ds, dx), \qquad t \geq 0 \end{aligned} \tag{31}$$

is a martingale.

Since $(u^{(n)}, v^{(n)})$ solves the discrete martingale problem $(\mathbf{MP}_F(\mathbb{Z}))^{\varrho}_{u_0^{(n)}, v_0^{(n)}}$, we know that

$$\begin{aligned} \tilde{M}_{n^2 t}(\phi^{(n)}, \psi^{(n)}) := {} & F(u^{(n)}_{n^2 t}, v^{(n)}_{n^2 t}, \phi^{(n)}, \psi^{(n)}) - F(u^{(n)}_0, v^{(n)}_0, \phi^{(n)}, \psi^{(n)}) \\ & - \frac{1}{2} \int_0^{n^2 t} F(u^{(n)}_s, v^{(n)}_s, \phi^{(n)}, \psi^{(n)}) \, \langle\langle u^{(n)}_s, v^{(n)}_s, {}^d\Delta(\phi^{(n)}), {}^d\Delta(\psi^{(n)}) \rangle\rangle_{\varrho} \, ds \\ & - 4(1 - \varrho^2) \int_{[0, n^2 t] \times \mathbb{Z}} F(u^{(n)}_s, v^{(n)}_s, \phi^{(n)}, \psi^{(n)}) \, \phi^{(n)}(k) \psi^{(n)}(k) \, L^{(n)}(ds, dk) \end{aligned}$$

is a martingale for each $n \in \mathbb{N}$.

Choose a sequence $n_k \uparrow \infty$ such that $(\mu_t^{(n_k)}, \nu_t^{(n_k)}, \Lambda_t^{(n_k)})_{t\geq 0}$ converges to $(\mu_t, \nu_t, \Lambda_t)_{t\geq 0}$ w.r.t. the Meyer-Zheng topology on $D_{[0,\infty)}(\mathcal{M}_{\text{tem}}(\mathbb{R})^3)$. In view of (21) and (22) (and also using the usual approximation of the Laplace operator by its rescaled discrete counterpart), we get that $(\tilde{M}_{n^2t}(\phi^{(n)}, \psi^{(n)}))_{t\geq 0}$ converges to $(M_t(\phi, \psi))_{t\geq 0}$ w.r.t. the Meyer-Zheng topology on $D_{[0,\infty)}(\mathbb{R})$ as $k \to \infty$. In addition, by estimate (44) in Prop. 3.3 we know that

$$\mathbb{E}_{u_0^{(n)}, v_0^{(n)}}\left[|\tilde{M}_{n^2t}(\phi^{(n)}, \psi^{(n)})|^2\right] \leq 8(1-\varrho^2) \sup_{\gamma>0} \mathbb{E}_{u_0^{(n)}, v_0^{(n)}}\left[\langle L_{n^2t}^{[\gamma]}, \phi^{(n)}\psi^{(n)}\rangle\right].$$

Now we fix $T > 0$ and argue as in the proof of Prop. 2.4: Choosing a suitable $\lambda > 0$ and combining the lower bound from Lemma A.2 b) with estimate (40), we see that there is a constant $C = C(\lambda, T)$ such that the previous display is bounded by

$$\begin{aligned} &8(1-\varrho^2)\,\frac{C}{n^2} \sup_{\gamma>0} \mathbb{E}_{u_0^{(n)}, v_0^{(n)}}\left[\int_{[0,n^2t]\times\mathbb{Z}} \left({}^dS_{n^2t-s}\phi_\lambda(\cdot/n)(k)\right)^2 L^{[\gamma]}(ds, dk)\right] \\ &\leq 8(1-\varrho^2)\,\frac{C}{n^2}\,\frac{1}{|\varrho|} \left\langle \phi_\lambda(\cdot/n) \otimes \phi_\lambda(\cdot/n), ({}^dS_{n^2t}^{(2)} - {}^d\tilde{S}_{n^2t})(u_0^{(n)} \otimes v_0^{(n)}) \right\rangle_{\mathbb{Z}^2} \end{aligned}$$

for all $t \in [0, T]$. As in the proof of Lemma 2.2, the last display is bounded uniformly in $t \in [0, T]$ and $n \in \mathbb{N}$ by Lemma 2.1, hence we get

$$\sup_{n\in\mathbb{N}} \sup_{t\in[0,T]} \mathbb{E}_{u_0^{(n)}, v_0^{(n)}}\left[|\tilde{M}_{n^2t}(\phi^{(n)}, \psi^{(n)})|^2\right] < \infty$$

for all $T > 0$. Applying [MZ84, Thm. 11], we infer that the Meyer-Zheng limit $(M_t(\phi, \psi))_{t\geq 0}$ is again a martingale, which completes our argument. □

Proposition 2.6 (Second mixed moments)**.** *Let $\varrho < 0$ and $(\mu_0, \nu_0) \in (\mathcal{B}_{\text{tem}}^+(\mathbb{R}))^2$. Suppose $(\mu_t, \nu_t)_{t\geq 0} \in D_{[0,\infty)}(\mathcal{M}_{\text{tem}}(\mathbb{R})^2)$ is any limit point with respect to the Meyer-Zheng topology of the sequence $(\mu_t^{(n)}, \nu_t^{(n)})_{t\geq 0}$, $n \in \mathbb{N}$, from (16). Then we have for the second mixed moment of (μ, ν)*

$$\mathbb{E}_{\mu_0,\nu_0}\left[\langle \mu_t, \phi\rangle \langle \nu_t, \psi\rangle\right] \leq \left\langle \phi \otimes \psi, \tilde{S}_t(\mu_0 \otimes \nu_0)\right\rangle \tag{32}$$

for all $t > 0$ and $\phi, \psi \in \bigcup_{\lambda>0} \mathcal{B}_\lambda^+(\mathbb{R})$.

Proof. By [MZ84, Thm. 5] (see also [Kur91, Thm. 1.1(b)]) we can find a sequence $n_k \uparrow \infty$ and a set $I \subseteq (0, \infty)$ of full Lebesgue measure such that the finite dimensional distributions of $(\mu_t^{(n_k)}, \nu_t^{(n_k)})_{t\in I}$ converge weakly to those of $(\mu_t, \nu_t)_{t\in I}$ as $k \to \infty$. Fix $t \in I$. Then for all test functions ϕ, ψ we have weak convergence

$$\langle \mu_t^{(n_k)}, \phi\rangle \langle \nu_t^{(n_k)}, \psi\rangle \xrightarrow{k\uparrow\infty} \langle \mu_t, \phi\rangle \langle \nu_t, \psi\rangle \tag{33}$$

in $\mathbb{R}$. Using Fatou's lemma, we get

$$\mathbb{E}_{\mu_0,\nu_0}\left[\langle \mu_t, \phi\rangle \langle \nu_t, \psi\rangle\right] \leq \liminf_{k\to\infty} \mathbb{E}_{\mu_0^{(n_k)}, \nu_0^{(n_k)}}\left[\langle \mu_t^{(n_k)}, \phi\rangle \langle \nu_t^{(n_k)}, \psi\rangle\right].$$

But for all $n \in \mathbb{N}$ we have by estimate (45) in Prop. 3.4 that

$$\begin{aligned}\mathbb{E}_{\mu_0^{(n)},\nu_0^{(n)}}\left[\langle\mu_t^{(n)},\phi\rangle\langle\nu_t^{(n)},\psi\rangle\right] &= \mathbb{E}_{u_0^{(n)},v_0^{(n)}}\left[\langle u_{n^2t}^{(n)},\phi^{(n)}\rangle\,\langle v_{n^2t}^{(n)},\psi^{(n)}\rangle\right] \\ &\leq \left\langle \phi^{(n)}\otimes\psi^{(n)}, {}^d\tilde{S}_{n^2t}(u_0^{(n)}\otimes v_0^{(n)})\right\rangle_{\mathbb{Z}^2}.\end{aligned} \tag{34}$$

As the usual discrete heat semigroup converges to its continuous counterpart under diffusive rescaling, the same holds for the killed semigroup $({}^d\tilde{S}_t)_{t\geq 0}$, see e.g. Lemma A.3 for details. Thus the RHS of the above display converges to the corresponding continuous quantity, namely to the RHS of (32), which is thus shown for all $t \in I$. Using the fact that I has full Lebesgue measure together with right-continuity of the paths of $(\mu_t,\nu_t)_{t\geq 0}$ and Fatou's lemma, we get the same estimate for *all* $t > 0$. □

Corollary 2.7 ("Separation of types")**.** *Let $\varrho < 0$ and $(\mu_0,\nu_0) \in (\mathcal{B}^+_{\text{tem}}(\mathbb{R}))^2$. Suppose that $(\mu_t,\nu_t)_{t\geq 0} \in D_{[0,\infty)}(\mathcal{M}^2_{\text{tem}})$ is any limit point with respect to the Meyer-Zheng topology of the tight sequence of measure-valued processes $(\mu_t^{(n)},\nu_t^{(n)})_{t\geq 0}$ from* (16)*. Then for each $t > 0$, $x \in \mathbb{R}$ and $\varepsilon > 0$ we have*

$$S_{t+\varepsilon}\mu_0(x)\,S_{t+\varepsilon}\nu_0(x) \geq \mathbb{E}_{\mu_0,\nu_0}\left[S_\varepsilon\mu_t(x)\,S_\varepsilon\nu_t(x)\right] \xrightarrow{\varepsilon\downarrow 0} 0. \tag{35}$$

Proof. Having shown the upper bound (32) for the second mixed moment, the proof of the 'separation of types'-property is basically the same as that of Lemma 4.4 in [BHO15]: For each $x \in \mathbb{R}$ and $\varepsilon > 0$ fixed, letting $\phi(\cdot) := \psi(\cdot) := p_\varepsilon(x-\cdot)$ in (32) gives

$$\begin{aligned}&\mathbb{E}_{\mu_0,\nu_0}\left[S_\varepsilon\mu_t(x)\,S_\varepsilon\nu_t(x)\right] \\ &\leq \iint dydz\, p_\varepsilon(x-y)p_\varepsilon(x-z)\,\tilde{S}_t(\mu_0\otimes\nu_0)(y,z) \\ &\leq S_{t+\varepsilon}\mu_0(x)\,S_{t+\varepsilon}\nu_0(x),\end{aligned} \tag{36}$$

for all $x \in \mathbb{R}$ and $t > 0$. Now we can argue as in [BHO15, Proof of Lemma 4.4] to complete the proof. □

3 Existence and uniqueness of solution to $(\mathbf{MP}_F(\mathbb{Z}))^\varrho_{u_0,v_0}$

In this section, we sketch a proof for existence and uniqueness (subject to the separation of types-property) of the solution to the martingale problem $(\mathbf{MP}_F(\mathbb{Z}))^\varrho_{u_0,v_0}$ for $\varrho \in (-1,0)$ and *general* initial conditions (with possibly non-disjoint support). As for the continuous-space case considered in [BHO15], the solution is given as the $\gamma\uparrow\infty$-limit in the Meyer-Zheng topology of the finite rate processes dSBM(ϱ,γ). Since most of the steps in the existence and uniqueness proof are analogous, we mostly only state the results, referring the reader to [BHO15] for the details. Also, the general method is very similar to the approach in Section 2 above.

In this section, for initial conditions $(u_0,v_0) \in \mathcal{M}_{\text{rap}}(\mathbb{Z})^2$ resp. $\mathcal{M}_{\text{tem}}(\mathbb{Z})^2$ we denote by $(u_t^{[\gamma]},v_t^{[\gamma]})_{t\geq 0} \in \mathcal{C}_{[0,\infty)}(\mathcal{M}_{\text{rap}}(\mathbb{Z}))^2$ resp. $\mathcal{C}_{[0,\infty)}(\mathcal{M}_{\text{tem}}(\mathbb{Z}))^2$ the solution to dSBM$(\varrho,\gamma)_{u_0,v_0}$

with these initial conditions and finite branching rate $\gamma > 0$. Further, we define another continuous $\mathcal{M}_{\mathrm{rap}}(\mathbb{Z})$- resp. $\mathcal{M}_{\mathrm{tem}}(\mathbb{Z})$-valued increasing process $(L_t^{[\gamma]})_t$ by

$$L_t^{[\gamma]}(k) := \gamma \int_0^t u_s^{[\gamma]}(k) v_s^{[\gamma]}(k)\, ds, \qquad t \geq 0,\ k \in \mathbb{Z}. \tag{37}$$

Now fix $T > 0$. By the Green function representation for $\mathrm{dSBM}(\varrho, \gamma)_{u_0,v_0}$ (see e.g. [DP98, Thm. 2.2(b)(ii)] for the case $\varrho = 0$, cp. also [EF04, Lemma 18/Cor. 19] for $\mathrm{cSBM}(\varrho, \gamma)$) we have for every $\gamma > 0$ and $\phi \in \bigcup_{\lambda>0} \mathcal{B}_{-\lambda}(\mathbb{Z})$ (resp. $\phi \in \bigcup_{\lambda>0} \mathcal{B}_{\lambda}(\mathbb{Z})$) that

$$\begin{aligned} M_t^{[\gamma,T]}(\phi) &:= \left\langle u_t^{[\gamma]}, {}^d S_{T-t}\phi \right\rangle - \left\langle u_0, {}^d S_T \phi \right\rangle, \qquad t \in [0,T], \\ N_t^{[\gamma,T]}(\phi) &:= \left\langle v_t^{[\gamma]}, {}^d S_{T-t}\phi \right\rangle - \left\langle v_0, {}^d S_T \phi \right\rangle, \qquad t \in [0,T] \end{aligned} \tag{38}$$

are martingales with quadratic (co-)variation

$$\begin{aligned} &[M^{[\gamma,T]}(\phi), M^{[\gamma,T]}(\phi)]_t \\ &\qquad = [N^{[\gamma,T]}(\phi), N^{[\gamma,T]}(\phi)]_t = \int_{[0,t]\times\mathbb{Z}} \left({}^d S_{T-r}\phi(k)\right)^2 L^{[\gamma]}(dr, dk), \\ &[M^{[\gamma,T]}(\phi), N^{[\gamma,T]}(\psi)]_t = \varrho \int_{[0,t]\times\mathbb{Z}} {}^d S_{T-r}\phi(k)\, {}^d S_{T-r}\psi(k)\, L^{[\gamma]}(dr, dk). \end{aligned} \tag{39}$$

with $L^{[\gamma]}$ from (37).

Proposition 3.1. *Suppose $\varrho < 0$ and $(u_0, v_0) \in \mathcal{M}_{\mathrm{rap}}(\mathbb{Z})^2$ (resp. $\mathcal{M}_{\mathrm{tem}}(\mathbb{Z})^2$). Then the family of processes $(u_t^{[\gamma]}, v_t^{[\gamma]}, L_t^{[\gamma]})_{t\geq 0}$ is tight with respect to the Meyer-Zheng topology on $D_{[0,\infty)}(\mathcal{M}_{\mathrm{rap}}(\mathbb{Z})^3)$ (resp. $D_{[0,\infty)}(\mathcal{M}_{\mathrm{tem}}(\mathbb{Z})^3)$).*

As in the continuous-space case, the key step in the proof of the Meyer-Zheng tightness is the following lemma which relies crucially on the colored particle moment duality for finite rate symbiotic branching, see [EF04, Prop. 9]. The estimate shows that (39) is bounded in expectation, uniformly in $\gamma > 0$. We omit its proof since it is virtually identical to that of the corresponding Lemma 3.1 in [BHO15], replacing the Brownian motions by simple symmetric random walks and the corresponding local times. Recall that $({}^d S_t)_{t\geq 0}$ resp. $({}^d S_t^{(2)})_{t\geq 0}$ denotes the one- resp. two-dimensional discrete heat semigroup, and that $({}^d \tilde{S}_t)_t$ denotes the discrete version of the killed semigroup from (17).

Lemma 3.2. *Suppose $\varrho < 0$ and $(u_0, v_0) \in \mathcal{M}_{\mathrm{rap}}(\mathbb{Z})^2$ (resp. $\mathcal{M}_{\mathrm{tem}}(\mathbb{Z})^2$). Then for all $t > 0$, $\gamma > 0$ and $\phi, \psi \in \bigcup_{\lambda>0} \mathcal{B}^+_{-\lambda}(\mathbb{Z})$ (resp. $\bigcup_{\lambda>0} \mathcal{B}^+_{\lambda}(\mathbb{Z})$) we have monotone convergence*

$$\mathbb{E}_{u_0,v_0}\left[\int_{[0,t]\times\mathbb{Z}} {}^d S_{t-r}\phi(k)\, {}^d S_{t-r}\psi(k)\, L^{[\gamma]}(dr, dk)\right] \uparrow \frac{1}{|\varrho|} \left\langle \phi\otimes\psi, ({}^d S_t^{(2)} - {}^d \tilde{S}_t)(u_0 \otimes v_0)\right\rangle_{\mathbb{Z}^2} \tag{40}$$

as $\gamma \uparrow \infty$.

Observe that in view of the definition of the semigroup $({}^d \tilde{S}_t)_{t\geq 0}$, the RHS of (40) is indeed an exact discrete-space analogue of (35) in [BHO15]. Also note that the RHS is finite since it is bounded by

$$\frac{1}{|\varrho|} \langle \phi, {}^d S_t u_0\rangle\, \langle \psi, {}^d S_t v_0\rangle < \infty.$$

With estimate (40) at hand, Prop. 3.1 is proved along the same lines as Lemma 2.2 above: First use the Green function representation (38)-(39) combined with the lower bound from (61) (for $n = 1$), the Burkholder-Davis-Gundy inequality and the upper bound (40) to derive uniform moment estimates

$$\sup_{\gamma>0} \mathbb{E}_{u_0,v_0}\Big[\sup_{0\le t\le T} \big\langle u_t^{[\gamma]}, \phi\big\rangle^2\Big] < \infty, \quad \sup_{\gamma>0} \mathbb{E}_{u_0,v_0}\Big[\sup_{0\le t\le T} \big\langle v_t^{[\gamma]}, \phi\big\rangle^2\Big] < \infty, \tag{41}$$

$$\sup_{\gamma>0} \mathbb{E}_{u_0,v_0}\Big[\sup_{0\le t\le T} \big\langle L_t^{[\gamma]}, \phi\big\rangle\Big] < \infty. \tag{42}$$

As in [BHO15, Prop. 3.3], these estimates in turn imply the compact containment condition for the family of processes $(u_t^{[\gamma]}, v_t^{[\gamma]}, L_t^{[\gamma]})_{t\ge0}$, $\gamma > 0$. Tightness in the Meyer-Zheng topology is then proved similarly to Prop. 2.4 above, using the martingale problem formulation of $\mathrm{dSBM}(\varrho,\gamma)$ together with the bounds (41)-(42).

Next, one has to check that limit points of the family $(u_t^{[\gamma]}, v_t^{[\gamma]}, L_t^{[\gamma]})$ solve the martingale problem $(\mathbf{MP}_F(\mathbb{Z}))_{u_0,v_0}^{\varrho}$ and satisfy the separation of types-property for positive times. The following corresponds to [BHO15, Prop. 4.3]:

Proposition 3.3. *Let $\varrho < 0$ and $(u_0, v_0) \in (\mathcal{M}_{\mathrm{rap}}(\mathbb{Z}))^2$ (resp. $(\mathcal{M}_{\mathrm{tem}}(\mathbb{Z}))^2$). Suppose that $(u_t, v_t, L_t)_{t\ge0} \in D_{[0,\infty)}(\mathcal{M}_{\mathrm{rap}}(\mathbb{Z})^3)$ (resp. $D_{[0,\infty)}(\mathcal{M}_{\mathrm{tem}}(\mathbb{Z})^3)$) is any limit point with respect to the Meyer-Zheng topology of the family $(u_t^{[\gamma]}, v_t^{[\gamma]}, L_t^{[\gamma]})_{t\ge0}$, $\gamma > 0$. Then for all test functions $\phi, \psi \in \mathcal{B}^+_{\mathrm{tem}}(\mathbb{Z})$ (resp. $\in \mathcal{B}^+_{\mathrm{rap}}(\mathbb{Z})$), the process*

$$\begin{aligned}\tilde{M}_t(\phi,\psi) := {} & F(u_t, v_t, \phi, \psi) - F(u_0, v_0, \phi, \psi) \\ & - \frac{1}{2}\int_0^t F(u_s, v_s, \phi, \psi)\, \langle\langle u_s, v_s, {}^d\Delta\phi, {}^d\Delta\psi\rangle\rangle_\varrho \, ds \\ & - 4(1-\varrho^2)\int_{[0,t]\times\mathbb{Z}} F(u_s, v_s, \phi, \psi)\, \phi(k)\psi(k)\, L(ds, dk)\end{aligned} \tag{43}$$

is a martingale with second moments bounded by

$$\mathbb{E}_{u_0,v_0}\Big[|\tilde{M}_t(\phi,\psi)|^2\Big] \le 8(1-\varrho^2) \sup_{\gamma>0} \mathbb{E}_{u_0,v_0}\Big[\big\langle L_t^{[\gamma]}, \phi\psi\big\rangle\Big] < \infty. \tag{44}$$

In particular, $(u_t, v_t)_{t\ge0}$ solves the martingale problem $(\mathbf{MP}_F(\mathbb{Z}))_{u_0,v_0}^{\varrho}$, where the process $(L_t)_{t\ge0}$ satisfies the requirements of Definition 1.4.

As in the proof of [BHO15, Prop. 4.3], this follows from the fact (43)-(44) hold for the finite rate model $\mathrm{dSBM}(\varrho,\gamma)$, by taking the limit $\gamma\to\infty$. Note that finiteness of the RHS of (44) follows by combining estimate (40) with the lower bound from (61).

We now turn to the separation of types. This property is easier to state in the discrete-space than in the continuous-space context, since here it means just mutual singularity of the measures. But as in the continuous-space case, we will derive it from a bound for second mixed moments. Therefore we first restate the discrete-space version of Prop. 1.9:

Proposition 3.4 (Moments)**.** *Let $\varrho < 0$ and $(u_0, v_0) \in (\mathcal{M}_{\mathrm{rap}}(\mathbb{Z}))^2$ (resp. $(\mathcal{M}_{\mathrm{tem}}(\mathbb{Z}))^2$). Suppose that $(u_t, v_t, L_t)_{t\ge0} \in D_{[0,\infty)}(\mathcal{M}_{\mathrm{rap}}(\mathbb{Z})^3)$ (resp. $D_{[0,\infty)}(\mathcal{M}_{\mathrm{tem}}(\mathbb{Z})^3)$) is any limit point* *with respect to the Meyer-Zheng topology of the family* $(u_t^{[\gamma]}, v_t^{[\gamma]}, L_t^{[\gamma]})_{t\geq 0}$, $\gamma > 0$*. Then for all* $\phi, \psi \in \bigcup_{\lambda>0} \mathcal{B}^+_{-\lambda}(\mathbb{Z})$ *(resp.* $\bigcup_{\lambda>0} \mathcal{B}^+_{\lambda}(\mathbb{Z})$*) we have*

$$\mathbb{E}_{u_0,v_0}\left[\langle u_t, \phi\rangle \langle v_t, \psi\rangle\right] \leq \left\langle \phi \otimes \psi, {}^d\tilde{S}_t(u_0 \otimes v_0)\right\rangle_{\mathbb{Z}^2} \tag{45}$$

and

$$\mathbb{E}_{u_0,v_0}\left[\int_{[0,t]\times\mathbb{Z}} {}^dS_{t-s}\phi(k)\, {}^dS_{t-s}\psi(k)\, L(ds, dk)\right] \leq \frac{1}{|\varrho|}\left\langle \phi \otimes \psi, ({}^dS_t^{(2)} - {}^d\tilde{S}_t)(u_0 \otimes v_0)\right\rangle_{\mathbb{Z}^2}. \tag{46}$$

The second mixed moment estimate (45) is again a consequence of the colored particle moment duality and is proved exactly as in [BHO15, Prop. 4.4] (see in particular ineq. (51)). The first moment bound (46) for L follows by an application of Fatou's lemma by taking $\gamma \uparrow \infty$ in estimate (40).

Choosing $\phi := \psi := \mathbb{1}_{\{k\}}$ in (45), we get immediately

Corollary 3.5 (Separation of Types)**.** *Under the assumptions of Theorem 3.4 we have for all* $t > 0$

$$u_t(k)v_t(k) = 0 \qquad \text{for all } k \in \mathbb{Z} \tag{47}$$

$\mathbb{P}_{u_0,v_0}$*-almost surely.*

$$M_t^{(n)}(\phi) := \langle u^{(n)}_{n^2t}, \phi^{(n)}\rangle - \langle u^{(n)}_0, \phi^{(n)}\rangle - \frac{1}{2}\int_0^t \langle u^{(n)}_{n^2s}, {}^d\Delta(\phi^{(n)})\rangle\, ds,$$

$$\mathbb{E}_{u_0,v_0}\left[\sup_{t\in[0,T]} |M_t^{(n)}(\phi)|^2\right] \leq \frac{C}{|\varrho|}\left\langle \phi \otimes \phi, ({}^dS_T^{(2)} - {}^d\tilde{S}_T)(u_0 \otimes v_0)\right\rangle_{\mathbb{Z}^2}. \tag{48}$$

We also note the following two lemmas, showing that for limit points we have analogues of the martingale problem formulation of (7) and of the Green function representation (38). The difference to the finite rate case is that we cannot (but also need not) identify the quadratic (co-)variation structure. However, we can still estimate the second moments:

Lemma 3.6. *Let* $\varrho < 0$ *and* $(u_0, v_0) \in \mathcal{M}_{\text{rap}}(\mathbb{Z})^2$ *(resp.* $\mathcal{M}_{\text{tem}}(\mathbb{Z})^2$*). Suppose that* $(u_t, v_t)_{t\geq 0} \in D_{[0,\infty)}(\mathcal{M}_{\text{rap}}(\mathbb{Z})^2)$ *(resp.* $D_{[0,\infty)}(\mathcal{M}_{\text{tem}}(\mathbb{Z})^2)$*) is any limit point with respect to the Meyer-Zheng topology of the family* $(u_t^{[\gamma]}, v_t^{[\gamma]})_{t\geq 0}$, $\gamma > 0$*. Then for all* $\phi, \psi \in \bigcup_{\lambda>0} \mathcal{B}^+_{-\lambda}(\mathbb{Z})$ *(resp.* $\bigcup_{\lambda>0} \mathcal{B}^+_{\lambda}(\mathbb{Z})$*) we have that*

$$\begin{aligned} M_t(\phi) &:= \langle u_t, \phi\rangle - \langle u_0, \phi\rangle - \frac{1}{2}\int_0^t \langle u_s, {}^d\Delta\phi\rangle\, ds, \\ N_t(\psi) &:= \langle v_t, \psi\rangle - \langle v_0, \psi\rangle - \frac{1}{2}\int_0^t \langle v_s, {}^d\Delta\psi\rangle\, ds \end{aligned} \tag{49}$$

are square-integrable martingales with second moments bounded by

$$\begin{aligned} \mathbb{E}_{u_0,v_0}\left[|M_t(\phi)|^2\right] &\leq \sup_{\gamma>0} \mathbb{E}_{u_0,v_0}\left[\langle L_t^{[\gamma]}, \phi^2\rangle\right] < \infty, \\ \mathbb{E}_{u_0,v_0}\left[|N_t(\phi)|^2\right] &\leq \sup_{\gamma>0} \mathbb{E}_{u_0,v_0}\left[\langle L_t^{[\gamma]}, \psi^2\rangle\right] < \infty \end{aligned} \tag{50}$$

for all $t > 0$*.*

Proof. If $\gamma_k \uparrow \infty$ is a sequence such that $(u_t^{[\gamma_k]}, v_t^{[\gamma_k]})_{t\geq 0}$ converges to $(u_t, v_t)_{t\geq 0}$ w.r.t. the Meyer-Zheng topology as $k \to \infty$, then also $(M_t^{[\gamma_k]}(\phi))_{t\geq 0}$ converges to $(M_t(\phi))_{t\geq 0}$, where $M^{[\gamma]}(\phi)$ is defined as in (49) but with u replaced by the finite rate process $u^{[\gamma]}$. By (7), it is clear that $M^{[\gamma]}(\phi)$ is a martingale for each $\gamma > 0$ with quadratic variation

$$\left[M^{[\gamma]}(\phi)\right]_t = \gamma \int_0^t \langle u_s^{[\gamma]} v_s^{[\gamma]}, \phi^2 \rangle \, ds = \langle L_t^{[\gamma]}, \phi^2 \rangle,$$

thus

$$\mathbb{E}_{u_0,v_0}\left[|M_t^{[\gamma]}(\phi)|^2\right] = \mathbb{E}_{u_0,v_0}\left[\langle L_t^{[\gamma]}, \phi^2 \rangle\right].$$

But combining the lower bound in (61) with the estimate (40), we see that the previous display is bounded uniformly in $\gamma > 0$ (and also uniformly on compact time intervals). Applying [MZ84, Thm. 11], we conclude that the Meyer-Zheng limit point $M(\phi)$ is again a martingale, and an application of Fatou's lemma yields

$$\mathbb{E}_{u_0,v_0}\left[|M_t(\phi)|^2\right] \leq \liminf_{k\to\infty} \mathbb{E}_{u_0,v_0}\left[|M_t^{[\gamma_k]}(\phi)|^2\right] \leq \sup_{\gamma>0} \mathbb{E}_{u_0,v_0}\left[\langle L_t^{[\gamma]}, \phi^2 \rangle\right] < \infty.$$

□

Lemma 3.7 ('Green function representation')**.** *Under the assumptions of Lemma 3.6, we have for all $T > 0$ and $\phi, \psi \in \bigcup_{\lambda>0} \mathcal{B}_{-\lambda}^+(\mathbb{Z})$ (resp. $\bigcup_{\lambda>0} \mathcal{B}_{\lambda}^+(\mathbb{Z})$) that*

$$\langle u_t, {}^dS_{T-t}\phi \rangle = \langle u_0, {}^dS_T\phi \rangle + M_t^T(\phi), \qquad \langle v_t, {}^dS_{T-t}\psi \rangle = \langle v_0, {}^dS_T\psi \rangle + N_t^T(\psi), \tag{51}$$

for $t \in [0, T]$, where $(M_t^T(\phi))_{t\in[0,T]}$ and $(N_t^T(\psi))_{t\in[0,T]}$ are square-integrable martingales with second moments bounded uniformly by

$$\begin{aligned} \mathbb{E}_{u_0,v_0}\left[\sup_{t\in[0,T]} |M_t^T(\phi)|^2\right] &\leq \frac{4}{|\varrho|} \left\langle \phi\otimes\phi, ({}^dS_T^{(2)} - {}^d\tilde{S}_T)(u_0\otimes v_0) \right\rangle_{\mathbb{Z}^2}, \\ \mathbb{E}_{u_0,v_0}\left[\sup_{t\in[0,T]} |N_t^T(\psi)|^2\right] &\leq \frac{4}{|\varrho|} \left\langle \psi\otimes\psi, ({}^dS_T^{(2)} - {}^d\tilde{S}_T)(u_0\otimes v_0) \right\rangle_{\mathbb{Z}^2}. \end{aligned} \tag{52}$$

Proof. Let $\gamma_k \uparrow \infty$ be a sequence such that $(u_t^{[\gamma_k]}, v_t^{[\gamma_k]})_{t\geq 0}$ converges to $(u_t, v_t)_{t\geq 0}$ w.r.t. the Meyer-Zheng topology as $k \to \infty$. Fix $T > 0$. From the Green function representation (38) for the finite rate model, we get that the sequence of martingales $(M_t^{[\gamma_k,T]}(\phi))_{t\in[0,T]}$ converges to

$$M_t^T(\phi) := \langle u_t, {}^dS_{T-t}\phi \rangle - \langle u_0, {}^dS_T\phi \rangle, \qquad t \in [0,T]$$

w.r.t. the Meyer-Zheng topology as $k \to \infty$. Further, by (39) combined with the Burkholder-Davis-Gundy inequality and estimate (40) we have

$$\begin{aligned} \mathbb{E}_{u_0,v_0}\left[\sup_{t\in[0,T]} |M_t^{[\gamma_k,T]}(\phi)|^2\right] &\leq 4\, \mathbb{E}_{u_0,v_0}\left[\int_{[0,T]\times\mathbb{Z}} \left({}^dS_{T-r}\phi(k)\right)^2 L^{[\gamma_k]}(dr, dk)\right] \\ &\leq \frac{4}{|\varrho|} \left\langle \phi\otimes\phi, ({}^dS_T^{(2)} - {}^d\tilde{S}_T)(u_0\otimes v_0) \right\rangle_{\mathbb{Z}^2} \end{aligned} \tag{53}$$

uniformly in $k \in \mathbb{N}$. Applying [MZ84, Thm. 11], we deduce that $(M_t^T(\phi))_{t\in[0,T]}$ is a martingale.

It remains to show the upper bound (52). (Note again that we do *not* know here that the quadratic variation converges along with the martingale.) Since we are interested in proving only a distributional property of the limit, we may assume that the convergence $M^{[\gamma_k,T]}(\phi) \to M^T(\phi)$ takes place not only in law but almost surely. Moreover, using [MZ84, Thm. 5] and arguing as in the proof of Prop. 2.6, we may assume that there is a set of time points $I \subseteq [0,T]$ of full Lebesgue measure such that $M_t^{[\gamma_k,T]}(\phi) \to M_t^T(\phi)$ for all $t \in I$, almost surely. Now observe that for all $t \in I$ we have

$$|M_t^T(\phi)|^2 = \liminf_{k\to\infty} |M_t^{[\gamma_k,T]}(\phi)|^2 \le \liminf_{k\to\infty} \sup_{t\in[0,T]} |M_t^{[\gamma_k,T]}(\phi)|^2$$

almost surely and thus also

$$\sup_{t\in[0,T]} |M_t^T(\phi)|^2 = \sup_{t\in I} |M_t^T(\phi)|^2 \le \liminf_{k\to\infty} \sup_{t\in[0,T]} |M_t^{[\gamma_k,T]}(\phi)|^2,$$

where we have also used the right-continuity of the paths of $M^T(\phi)$. Now we obtain by an application of Fatou's lemma and (53) that

$$\begin{aligned} \mathbb{E}_{u_0,v_0}\left[\sup_{t\in[0,T]} |M_t^T(\phi)|^2\right] &\le \liminf_{k\to\infty} \mathbb{E}_{u_0,v_0}\left[\sup_{t\in[0,T]} |M_t^{[\gamma_k,T]}(\phi)|^2\right] \\ &\le \frac{4}{|\varrho|}\left\langle \phi\otimes\phi, ({}^dS_T^{(2)} - {}^d\tilde{S}_T)(u_0\otimes v_0)\right\rangle_{\mathbb{Z}^2}. \end{aligned} \tag{54}$$

□

Finally, we have uniqueness in our martingale problem under the separation of types-condition, which as in the continuous-space case follows from self-duality:

Proposition 3.8 (Uniqueness). *Fix $\varrho \in (-1,0)$ and (possibly random) initial conditions $(u_0,v_0) \in \mathcal{M}_{\mathrm{tem}}(\mathbb{Z})^2$ or $\mathcal{M}_{\mathrm{rap}}(\mathbb{Z})^2$. Then there is at most one solution $(u_t,v_t)_{t\ge0}$ to the martingale problem $(\mathbf{MP}_F(\mathbb{Z}))^{\varrho}_{u_0,v_0}$ satisfying the separation of types-property* (14).

Proof. As in [BHO15, Prop. 5.1], one shows that any two solutions to $(\mathbf{MP}_F(\mathbb{Z}))^{\varrho}_{u_0,v_0}$ satisfying the separation of types are self-dual w.r.t. the function F from (10). In fact, the proof simplifies considerably, since we can apply the discrete Laplace operator directly to the solution (u,v) and do not need to perform a spatial smoothing via the heat kernel S_ε. See also the proof of [KM12a, Prop. 4.7] for the slightly different martingale problem employed in that paper, or the proof of [DP98, Thm. 2.4(b)] for the discrete finite rate model. With the self-duality at hand, uniqueness follows by the usual arguments, see e.g. [KM12a, proof of Prop. 4.1] or [DP98, proof of Thm. 2.4(a)] □

A Appendix

A.1 Notation and spaces of functions and measures

In this appendix, for the convenience of the reader, we have collected our notation and we recall some well-known facts concerning the spaces of functions and measures employed throughout the paper. Most of the material in this subsection can be found e.g.

in [DEF+02], [DFM+03] or [EF04]. We can develop the notation for both the discrete and the continuous setting simultaneously, so throughout we let $\mathcal{S}$ be either $\mathbb{Z}$ or $\mathbb{R}$.

For $\lambda \in \mathbb{R}$, let

$$\phi_\lambda(x) := e^{-\lambda|x|}, \quad x \in \mathcal{S},$$

and for $f : \mathcal{S} \to \mathbb{R}$ define

$$|f|_\lambda := ||f/\phi_\lambda||_\infty,$$

where $||\cdot||_\infty$ is the supremum norm. Let $\mathcal{B}_\lambda(\mathcal{S})$ denote the space of all measurable functions $f : \mathcal{S} \to \mathbb{R}$ such that $|f|_\lambda < \infty$ and with the property that $f(x)/\phi_\lambda(x)$ has a finite limit as $|x| \to \infty$. Next, introduce the spaces

$$\mathcal{B}_{\text{rap}}(\mathcal{S}) := \bigcap_{\lambda>0} \mathcal{B}_\lambda(\mathcal{S}) \quad \text{and} \quad \mathcal{B}_{\text{tem}} := \bigcap_{\lambda>0} \mathcal{B}_{-\lambda}(\mathcal{S}) \tag{55}$$

of *rapidly decreasing* and *tempered* measurable functions, respectively.

For $\mathcal{S} = \mathbb{R}$, we write $\mathcal{C}_\lambda, \mathcal{C}_{\text{rap}}, \mathcal{C}_{\text{tem}}$ for the subspaces of continuous functions in $\mathcal{B}_\lambda(\mathbb{R})$, $\mathcal{B}_{\text{rap}}(\mathbb{R}), \mathcal{B}_{\text{tem}}(\mathbb{R})$ respectively. If we additionally require that all partial derivatives up to order $k \in \mathbb{N}$ exist and belong to $\mathcal{C}_\lambda, \mathcal{C}_{\text{rap}}, \mathcal{C}_{\text{tem}}$, we write $\mathcal{C}_\lambda^{(k)}, \mathcal{C}_{\text{rap}}^{(k)}, \mathcal{C}_{\text{tem}}^{(k)}$. We will also use the space $\mathcal{C}_c^\infty$ of infinitely differentiable functions with compact support.

For each $\lambda \in \mathbb{R}$, the linear space $\mathcal{C}_\lambda$ endowed with the norm $|\cdot|_\lambda$ is a separable Banach space, and the spaces $\mathcal{C}_{\text{rap}}, \mathcal{C}_{\text{tem}}$ can be toplogized by a suitable metric to turn them into a Polish spaces, for the details see Appendix A.1 in [BHO15].

If $\mathcal{F}$ is any of the above spaces of functions, the notation $\mathcal{F}^+$ will refer to the subset of nonnegative elements of $\mathcal{F}$.

Let $\mathcal{M}(\mathcal{S})$ denote the space of (nonnegative) Radon measures on $\mathcal{S}$. For $\mu \in \mathcal{M}(\mathcal{S})$ and a measurable function f, we will use any of the notations

$$\langle \mu, f \rangle, \quad \int_{\mathcal{S}} \mu(dx)\, f(x), \quad \int_{\mathcal{S}} f(x)\, \mu(dx)$$

to denote the integral of f with respect to the measure μ (if it exists). For integrals with respect to the Lebesgue measure ℓ on $\mathbb{R}$, we will simply write dx in place of $\ell(dx)$. If $\mu \in \mathcal{M}(\mathbb{R})$ is absolutely continuous w.r.t. ℓ, we will identify μ with its density, writing

$$\mu(dx) = \mu(x)\, dx.$$

Similarly, for $\mu \in \mathcal{M}(\mathbb{Z})$, we will often write $\mu(k) := \mu(\{k\})$.

For $\lambda \in \mathbb{R}$, define

$$\mathcal{M}_\lambda(\mathcal{S}) := \{\mu \in \mathcal{M}(\mathcal{S}) : \langle \mu, \phi_\lambda \rangle < \infty\}$$

and introduce the spaces

$$\mathcal{M}_{\text{tem}}(\mathcal{S}) := \bigcap_{\lambda>0} \mathcal{M}_\lambda(\mathcal{S}), \qquad \mathcal{M}_{\text{rap}}(\mathcal{S}) := \bigcap_{\lambda>0} \mathcal{M}_{-\lambda}(\mathcal{S})$$

of *tempered* and *rapidly decreasing measures* on $\mathcal{S}$, respectively. Again by defining suitable metrics it can be seen that these spaces are Polish. Moreover, $\mu_n \to \mu$ in $\mathcal{M}_{\text{tem}}(\mathbb{R})$, resp. $\mathcal{M}_{\text{tem}}(\mathbb{Z})$, iff $\langle \mu_n, \varphi \rangle \to \langle \mu, \varphi \rangle$ for all $\varphi \in \bigcup_{\lambda>0} \mathcal{C}_\lambda$, resp. $\varphi \in \bigcup_{\lambda>0} \mathcal{B}_\lambda(\mathbb{Z})$. Denote by $\mathcal{M}_f(\mathcal{S})$

the space of finite measures on $\mathcal{S}$ endowed with the topology of weak convergence. Note that we have $\mathcal{M}_{\text{rap}}(\mathcal{S}) \subseteq \mathcal{M}_f(\mathcal{S})$. The space $\mathcal{M}_{\text{rap}}(\mathcal{S})$ is then topologized by saying that $\mu_n \to \mu$ in $\mathcal{M}_{\text{rap}}(\mathcal{S})$ iff $\mu_n \to \mu$ in $\mathcal{M}_f(\mathcal{S})$ (w.r.t. the weak topology) and $\sup_{n\in\mathbb{N}}\langle \mu_n, \phi_\lambda\rangle < \infty$ for all $\lambda < 0$ (see [DFM$^+$03], p. 140).

It is clear that $\mathcal{C}^+_{\text{tem}}$ may be viewed as a subspace of $\mathcal{M}_{\text{tem}}(\mathbb{R})$ by taking a function $u \in \mathcal{C}^+_{\text{tem}}$ as a density w.r.t. Lebesgue measure, i.e. by identifying it with the measure $u(x)dx$. It is also clear that the topology of $\mathcal{M}_{\text{tem}}(\mathbb{R})$ restricted to $\mathcal{C}^+_{\text{tem}}$ is weaker than the topology on $\mathcal{C}_{\text{tem}}$ introduced above. The same holds for the relation between $\mathcal{C}^+_{\text{rap}}$ and $\mathcal{M}_{\text{rap}}(\mathbb{R})$. Thus we have *continuous* embeddings $\mathcal{C}^+_{\text{tem}} \hookrightarrow \mathcal{M}_{\text{tem}}(\mathbb{R})$ and $\mathcal{C}^+_{\text{rap}} \hookrightarrow \mathcal{M}_{\text{rap}}(\mathbb{R})$.

A.2 Semigroup estimates

Let $(p_t)_{t\geq 0}$ denote the heat kernel in $\mathbb{R}$ corresponding to $\frac{1}{2}\Delta$,

$$p_t(x) = \frac{1}{(2\pi t)^{1/2}} \exp\left\{-\frac{|x|^2}{2t}\right\}, \qquad t > 0, x \in \mathbb{R}, \tag{56}$$

and write $(S_t)_{t\geq 0}$ for the associated heat semigroup (i.e. the transition semigroup of Brownian motion).

Similarly, let $({}^dS_t)_{t\geq 0}$ denote the semigroup corresponding to a continuous-time simple random walk $(X_t)_{t\geq 0}$ with generator $\frac{1}{2}{}^d\Delta$, the discrete Laplace operator as defined in (8).

For $\mu \in \mathcal{M}(\mathbb{R})$ and $x \in \mathbb{R}$, let $S_t\mu(x) := \int_{\mathbb{R}} p_t(x-y)\,\mu(dy)$ and similarly for ${}^dS$. The following estimates are well known and can be proved as in Appendix A of [DFM$^+$03] (see also [Shi94, Lemma 6.2 (ii)]):

Lemma A.1. *Fix $\lambda \in \mathbb{R}$ and $T > 0$.*

a) For all $\varphi \in \mathcal{B}^+_\lambda(\mathbb{R})$, we have

$$\sup_{t\in[0,T]} S_t\varphi(x) \leq C(\lambda, T)\,|\varphi|_\lambda\,\phi_\lambda(x), \qquad x \in \mathbb{R}. \tag{57}$$

Moreover, there is a positive constant $C'(\lambda, T) > 0$ such that we have a lower bound

$$\inf_{t\in[0,T]} S_t\phi_\lambda(x) \geq C'(\lambda, T)\,\phi_\lambda(x), \qquad x \in \mathbb{R}. \tag{58}$$

b) Let $0 < \varepsilon < T$. Then for all $\mu \in \mathcal{M}_\lambda(\mathbb{R})$ we have

$$\sup_{t\in[\varepsilon,T]} S_t\mu(x) \leq C(\lambda, T, \varepsilon)\,\langle \mu, \phi_\lambda\rangle\,\phi_{-\lambda}(x), \qquad x \in \mathbb{R}. \tag{59}$$

In particular, the heat semigroup preserves the space $\mathcal{B}_\lambda(\mathbb{R})$ and maps $\mathcal{M}_\lambda(\mathbb{R})$ into $\mathcal{B}_\lambda(\mathbb{R})$.

We have analogous estimates for the discrete space semigroup.

Lemma A.2. *a) Let $p^{(2)}_t$ denote the usual two-dimensional heat kernel, and ${}^dp^{(2)}_t$ its discrete counterpart. Then for all $t > 0$ we have*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{Z}^2} \left| n^2\, {}^dp^{(2)}_{n^2t}(x) - p^{(2)}_t(x/n) \right| = 0. \tag{60}$$

b) Let $\lambda \in \mathbb{R}$ and $T > 0$. Then there are constants $c(\lambda, T), C(\lambda, t) > 0$ such that for all $n \in \mathbb{N}$, $k \in \mathbb{Z}$ and all $t \in [0,T]$ we have

$$c(\lambda, T)\,\phi_\lambda(k/n) \le {}^{d}S_{n^2t}(\phi_\lambda(\cdot/n))(k) \le C(\lambda, T)\,\phi_\lambda(k/n). \tag{61}$$

Proof. Part (a) and the upper bound in (b) are just a reformulation of [DEF+02, Lemma 2(a)] and [DEF+02, Corollary A3(a)].

For the lower bound in (b), let X_t be a continuous time random walk with generator ${}^{d}\Delta$ started in 0. If we define $X_t^{(n)} := X_{n^2t}/n$, then we know from Donsker's theorem that $(X_t^{(n)})_{t\ge 0}$ converges in distribution to a standard Brownian motion B. Fix $T > 0$, by Skorokhod's representation theorem, we can choose a common probability space $\mathbb{P}$ (with expectation $\mathbb{E}$) such that $\sup_{t\in[0,T]} |X_t^{(n)} - B_t| \to 0$ almost surely.

For $\lambda \ge 0$, we can estimate using the triangle inequality

$${}^{d}S_{n^2t}(\phi_\lambda(\cdot/n))(k) = \mathbb{E}[e^{-\lambda|X_t^{(n)}+k|}] \ge e^{-\lambda|k/n|}\mathbb{E}[e^{-\lambda|X_t^{(n)}|}].$$

So it remains to show that the expectation on the LHS is bounded from below uniformly in $t \in [0,T]$ and n. Now, choose n_0 large enough such that for all $n \ge n_0$

$$\mathbb{P}\Big\{ \sup_{t\in[0,T]} |X_t^{(n)} - B_t| \ge \tfrac{1}{2} \Big\} \le \tfrac{1}{2}\mathbb{E}[e^{-\lambda|B_T|}].$$

Using the above estimate and the fact that $t \mapsto \mathbb{E}[e^{-\lambda|B_t|}]$ is decreasing, we thus obtain for $n \ge n_0$

$$\begin{aligned}\mathbb{E}\big[e^{-\lambda|X^{(n)}|}\big] &\ge e^{-\frac{1}{2}\lambda}\mathbb{E}[e^{-\lambda|B_t|}\mathbb{1}_{\{\sup_{t\in[0,T]}|X_t^{(n)}-B_t|\le\frac{1}{2}\}}] \\ &\ge e^{-\frac{1}{2}\lambda}\Big(\mathbb{E}[e^{-\lambda|B_T|}] - \mathbb{P}\Big\{ \sup_{t\in[0,T]} |X_t^{(n)} - B_t| \ge \tfrac{1}{2} \Big\}\Big) \ge \tfrac{1}{2}e^{-\frac{1}{2}\lambda}\mathbb{E}[e^{-\lambda|B_T|}].\end{aligned}$$

This proves the claim for $\lambda \ge 0$, since for any $n \le n_0$ we can use the trivial estimate

$$\mathbb{E}[e^{-\lambda|X_t^{(n)}|}] \ge \mathbb{P}\{X_s^{(n)} = 0 \text{ for all } s \in [0,t]\} \ge \mathbb{P}\{X_s^{(n)} = 0 \text{ for all } s \in [0,T]\}.$$

so that we choose the constant $c(\lambda, T)$ as claimed.

Finally, if $\lambda < 0$, we can use that

$${}^{d}S_{n^2t}(\phi_\lambda(\cdot/n))(k) = \mathbb{E}[e^{-\lambda|X_{n^2t}+k|/n}] \ge e^{-\lambda|k/n|}\mathbb{E}[e^{\lambda|X_t^{(n)}|}],$$

and the latter expectation can be bounded uniformly in n and $t \in [0,T]$ as in case $\lambda \ge 0$. □

Recall that $(\tilde{S}_t)_{t\ge 0}$ (resp. $({}^{d}\tilde{S}_t)_{t\ge 0}$) denotes the semigroup of two-dimensional standard Brownian motion (resp. simple symmetric random walk) killed upon hitting the diagonal in $\mathbb{R}^2$ (resp. $\mathbb{Z}^2$).

Lemma A.3 (Convergence of killed semigroup)**.** *Suppose $(\mu_0, \nu_0) \in (\mathcal{B}^+_{\text{tem}}(\mathbb{R}))^2$ and $\phi, \psi \in \bigcup_{\lambda>0} \mathcal{B}^+_\lambda(\mathbb{R})$. Then we have*

$$\Big\langle \phi^{(n)} \otimes \psi^{(n)}, {}^{d}\tilde{S}_{n^2t}(u_0^{(n)} \otimes v_0^{(n)}) \Big\rangle_{\mathbb{Z}^2} \xrightarrow{n\to\infty} \langle \phi \otimes \psi, \tilde{S}_t(\mu_0 \otimes \nu_0)\rangle. \tag{62}$$

for all $t > 0$.

Proof. The transition density $\tilde{p}_t$ of $\tilde{S}_t$ is given by

$$\begin{aligned}\tilde{p}_t(x,y;a,b) &= \begin{cases} \mathbb{1}_{\{a<b\}}\big(p_t^{(2)}(x-a,y-b)-p_t^{(2)}(x-b,y-a)\big) & \text{if } x<y \\ \mathbb{1}_{\{a>b\}}\big(p_t^{(2)}(x-a,y-b)-p_t^{(2)}(x-b,y-a)\big) & \text{if } x>y \end{cases} \\ &= \big(\mathbb{1}_{\{x<y,a<b\}}+\mathbb{1}_{\{x>y,a>b\}}\big)\,\big(p_t^{(2)}(x-a,y-b)-p_t^{(2)}(x-b,y-a)\big),\end{aligned} \tag{63}$$

where $p_t^{(2)}$ denotes the usual two-dimensional heat kernel.

The corresponding discrete-space transition density reads

$${}^d\tilde{p}_t(k,\ell;a,b) = \big(\mathbb{1}_{\{k<\ell,a<b\}}+\mathbb{1}_{\{k>\ell,a>b\}}\big)\,\big({}^dp_t^{(2)}(k-a,\ell-b)-{}^dp_t^{(2)}(k-b,\ell-a)\big). \tag{64}$$

In particular, by the above form of the density (and the symmetry of the usual heat kernel) it is immediately seen that these semigroups are symmetric.

Now we have

$$\begin{aligned}&\Big\langle \phi^{(n)}\otimes\psi^{(n)}, {}^d\tilde{S}_{n^2t}(u_0^{(n)}\otimes v_0^{(n)})\Big\rangle_{\mathbb{Z}^2} = \Big\langle {}^d\tilde{S}_{n^2t}(\phi^{(n)}\otimes\psi^{(n)}), u_0^{(n)}\otimes v_0^{(n)}\Big\rangle_{\mathbb{Z}^2} \\ &= \sum_{k,\ell\in\mathbb{Z}} u_0^{(n)}(k)\,v_0^{(n)}(\ell) \sum_{k',\ell'\in\mathbb{Z}} {}^d\tilde{p}_{n^2t}(k,\ell;k',\ell')\,\phi^{(n)}(k')\psi^{(n)}(\ell') \\ &= \frac{1}{n^2}\sum_{k,\ell\in\mathbb{Z}} u_0^{(n)}(k)\,v_0^{(n)}(\ell) \sum_{k',\ell'\in\mathbb{Z}} \tilde{p}_t(k/n,\ell/n;k'/n,\ell'/n)\,\phi^{(n)}(k')\psi^{(n)}(\ell') \\ &\quad + \frac{1}{n^2}\sum_{k,\ell\in\mathbb{Z}} u_0^{(n)}(k)\,v_0^{(n)}(\ell) \\ &\qquad\quad \times \sum_{k',\ell'\in\mathbb{Z}} \Big(n^2\,{}^d\tilde{p}_{n^2t}(k,\ell;k',\ell')-\tilde{p}_t(k/n,\ell/n;k'/n,\ell'/n)\Big)\,\phi^{(n)}(k')\psi^{(n)}(\ell').\end{aligned} \tag{65}$$

By definition of $\phi^{(n)}$, $\psi^{(n)}$, $u_0^{(n)}$ and $v_0^{(n)}$ (see (15) and (20)), the first term on the RHS of the above display converges to

$$\iint dxdy\,\phi(x)\psi(y)\,\tilde{S}_t(\mu_0\otimes\nu_0)(x,y),$$

as desired. We show that the second term converges to 0: By (60) combined with (63)-(64), we get that

$$n^2\,{}^d\tilde{p}_{n^2t}(k,\ell;k',\ell')-\tilde{p}_t(k/n,\ell/n;k'/n,\ell'/n) \xrightarrow{n\to\infty} 0$$

uniformly in $k,\ell,k',\ell'\in\mathbb{Z}$. Since ϕ,ψ are integrable, we can deduce that for all fixed $k,\ell\in\mathbb{Z}$, the inner sum $\sum_{k',\ell'}\cdots$ in (65) converges to 0 as $n\to\infty$. Then we use the upper bound from (61) together with the assumption that $\mu_0,\nu_0\in\mathcal{B}^+_{\text{tem}}(\mathbb{R})$ and dominated convergence to conclude that also the big sum on the RHS of (65) converges to 0. □

A.3 The topology on path space

For a Polish space E and $I\subseteq\mathbb{R}$, we denote by $D_I(E)$ resp. $\mathcal{C}_I(E)$ the space of càdlàg resp. continuous E-valued paths $t\mapsto f_t$, $t\in I$. (In our case, we will always have $I=[0,\infty)$ or $I=(0,\infty)$ and $E\in\{(\mathcal{C}^+_{\text{tem}})^m,(\mathcal{C}^+_{\text{rap}})^m,\mathcal{M}_{\text{tem}}(\mathcal{S})^m,\mathcal{M}_{\text{rap}}(\mathcal{S})^m\}$ for $\mathcal{S}$ either $\mathbb{Z}$ or $\mathbb{R}$ and some power $m \in \mathbb{N}$.) Endowed with the usual Skorokhod (J_1)-topology, $D_I(E)$ is then also Polish. In this paper, we will use the Skorokhod topology only in restriction to $\mathcal{C}_I(E)$ where it coincides with the usual topology of locally uniform convergence.

For processes which are càdlàg but not continuous, we will instead use the weaker *Meyer-Zheng 'pseudo-path' topology* on $D_{[0,\infty)}(E)$. To describe the Meyer-Zheng topology, introduced in [MZ84], let $\lambda(dt) := \exp(-t)\,dt$ and let $w(t), t \in [0,\infty)$ be an E-valued Borel function. Then, a 'pseudo-path' corresponding to w is the probability law ψ_w on $[0,\infty) \times E$ given as the image measure of λ under the mapping $t \mapsto (t, w(t))$. Note that two functions which are equal Lebesgue-a.e. give rise to the same pseudo-path. Further $w \mapsto \psi_w$ is one-to-one on the space of càdlàg paths $D_{[0,\infty)}(E)$, and thus yields an embedding of $D_{[0,\infty)}(E)$ into the space of probability measures on $[0,\infty) \times E$. The induced topology on $D_{[0,\infty)}$ is then called the pseudo-path topology. Very conveniently, convergence in this topology is equivalent to convergence in Lebesgue measure (see [MZ84, Lemma 1]).

For $E = \mathbb{R}$, [MZ84, Thm. 4] provides a rather convenient sufficient condition for relative compactness of a sequence of stochastic processes on $D_{[0,\infty)}(E)$ equipped with this topology. The condition can be stated as follows: If $(X_t^{(n)})_{t\geq 0}$, $n \in \mathbb{N}$ is a sequence of càdlàg real-valued stochastic processes, with $(X_t^{(n)})_{t\geq 0}$ adapted to a filtration $(\mathcal{F}^{(n)})_{t\geq 0}$, then Meyer and Zheng require that

$$\sup_{n\in\mathbb{N}} \left(V_T(X^{(n)}) + \sup_{t\leq T} \mathbb{E}[|X_t^{(n)}|] \right) < \infty \tag{66}$$

for all $T > 0$. Here $V_T(X^{(n)}) := \sup \mathbb{E}\left[\sum_i \left|\mathbb{E}[X_{t_{i+1}}^{(n)} - X_{t_i}^{(n)} \mid \mathcal{F}_{t_i}^{(n)}]\right|\right]$, where the sup is taken over all partitions of the interval $[0,T]$, denotes the conditional variation of $X^{(n)}$ up to time T. In [Kur91], this tightness criterion was extended to processes taking values in general separable metric spaces E, which is the version we need for our measure-valued processes. In fact, by [Kur91, Cor. 1.4] we only have to check condition (66) for the coordinate processes and in addition a compact containment condition in order to obtain tightness of our measure-valued processes in the pseudopath topology (which again is equivalent to the topology of convergence in Lebesgue measure).[3]

References

[BDE11] J. Blath, L. Döring, and A. Etheridge. On the moments and the interface of the symbiotic branching model. *Ann. Probab.*, 39(1):252–290, 2011.

[BHO15] J. Blath, M. Hammer, and M. Ortgiese. The scaling limit of the interface of the continuous-space symbiotic branching model. *To appear in Ann. Probab.*, 2015.

[DEF⁺02] D. A. Dawson, A. M. Etheridge, K. Fleischmann, L. Mytnik, E. A. Perkins, and J. Xiong. Mutually catalytic branching in the plane: infinite measure states. *Electron. J. Probab.*, 7(15), 2002.

[3] Note however that the main result in [Kur91] is much stronger than just an extension of the Meyer-Zheng tightness criterion to a general state space E. Also note that in [Kur91], eq. (1.7), there seems to be missing a term $\sup_{s\leq t} \mathbb{E}[|f_i \circ X_s^{(n)}|]$ (compare with eq. (1.2)).

[DFM+03] D. A. Dawson, K. Fleischmann, L. Mytnik, E. A. Perkins, and J. Xiong. Mutually catalytic branching in the plane: uniqueness. *Ann. Inst. H. Poincaré Probab. Statist.*, 39(1):135–191, 2003.

[DM12] L. Döring and L. Mytnik. Mutually catalytic branching processes and voter processes with strength of opinion. *ALEA Lat. Am. J. Probab. Math. Stat.*, 9:1–51, 2012.

[DM13] L. Döring and L. Mytnik. Longtime behavior for mutually catalytic branching with negative correlations. In J. Englander and B. Rider, editors, *Advances in Superprocesses and Nonlinear PDEs*, volume 38 of *Springer Proceedings in Mathematics and Statistics*, pages 93–111. Springer US, 2013.

[DP98] D. A. Dawson and E. A. Perkins. Long-time behavior and coexistence in a mutually catalytic branching model. *Ann. Probab.*, 26(3):1088–1138, 1998.

[EF04] A. M. Etheridge and K. Fleischmann. Compact interface property for symbiotic branching. *Stochastic Process. Appl.*, 114(1):127–160, 2004.

[KM10] A. Klenke and L. Mytnik. Infinite rate mutually catalytic branching. *Ann. Probab.*, 38(4):1690–1716, 2010.

[KM12a] A. Klenke and L. Mytnik. Infinite rate mutually catalytic branching in infinitely many colonies: construction, characterization and convergence. *Probab. Theory Related Fields*, 154(3-4):533–584, 2012.

[KM12b] A. Klenke and L. Mytnik. Infinite rate mutually catalytic branching in infinitely many colonies: the longtime behavior. *Ann. Probab.*, 40(1):103–129, 2012.

[KO10] A. Klenke and M. Oeler. A Trotter-type approach to infinite rate mutually catalytic branching. *Ann. Probab.*, 38(2):479–497, 2010.

[Kur91] T. G. Kurtz. Random time changes and convergence in distribution under the Meyer-Zheng conditions. *Ann. Probab.*, 19(3):1010–1034, 1991.

[Mue91] C. Mueller. On the support of solutions to the heat equation with noise. *Stochastics Stochastics Rep.*, 37(4):225–245, 1991.

[Myt98] L. Mytnik. Uniqueness for a mutually catalytic branching model. *Probab. Theory Related Fields*, 112(2):245–253, 1998.

[MZ84] P. A. Meyer and W. A. Zheng. Tightness criteria for laws of semimartingales. *Ann. Inst. H. Poincaré Probab. Statist.*, 20(4):353–372, 1984.

[Shi94] T. Shiga. Two contrasting properties of solutions for one-dimensional stochastic partial differential equations. *Canad. J. Math.*, 46(2):415–437, 1994.

[Tri95] R. Tribe. Large time behavior of interface solutions to the heat equation with Fisher-Wright white noise. *Probab. Theory Related Fields*, 102(3):289–311, 1995.