Optimal boundary control of a system of semilinear parabolic equations

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Abstract

In this work, boundary control problems governed by a system of semilinear parabolic PDEs with pointwise control constraints are considered. This class of problems is related to applications in the chemical catalysis. After discussing existence and uniqueness of the state equation with both linear and nonlinear boundary conditions, the existence of an optimal solution is shown. Necessary and sufficient optimality conditions are derived to deal with numerical examples, which conclude the paper.

1 Introduction

We consider a class of optimal boundary control problems governed by a system of semilinear parabolic PDEs with application in chemical reaction. Very similar optimal control problems were dicussed first in the PhD thesis [5] by Griesse and later extended by Griesse and Volkwein [6] to a more general setting. The origin of our research on this topic is a problem of catalysis that was introduced in the colloquium talk [9].

Let us briefly explain this application to motivate our setting: In a catalyst, two substances are contained with concentrations u and v. One of them is a harmful substance v controlled by d, which we want to neutralize with the other substance u controlled by c. In our examples, we consider d as given and c as the only control function. This is a substrate with concentration c, which influences the catalyst at its boundary. The diffusion process is modelled by $(E1)$ below. The term uv in $(E1)$ describes the chemical reaction in a low order approximation. For more accuracy, it is possible to replace this term by a stronger nonlinear coupling, see [1]. The catalyst will operate more efficiently, if the substances are inserted separately in alternating intervals of time. This alternative is considered in a variant of $(E1)$ dicussed in Section 2.3. In the reality, the constants are depending on the temperature, the pressure and u and v. We ignore these dependencies.

The process of neutralization is modeled by our cost functional. We assume that the ratio u/v should be equal to k in order to neutralize the substance v. These assumptions lead to the following optimal control problem [1], [9],

$$(P1)\ \ \min J(u,v,c,d) := \frac{1}{2}\iint\limits_Q (u(x,t)-kv(x,t))^2\,dxdt + \frac{\lambda_1}{2}\int\limits_0^T c^2(t)\,dt + \frac{\lambda_2}{2}\int\limits_0^T d^2(t)\,dt$$

subject to the system of semilinear parabolic PDE's

$$
(E1)\begin{cases}
u_t - k_1 u_{xx} + \alpha_1 u & = & -\gamma_1 uv & \text{in } Q, \\
v_t - k_2 v_{xx} + \alpha_2 v & = & -\gamma_2 uv & \text{in } Q, \\
u(0,t) - k_1 u_x(0,t) & = & c(t) & \text{in } (0,T), \\
k_1 u_x(l,t) & = & 0 & \text{in } (0,T), \\
v(0,t) - k_2 v_x(0,t) & = & d(t) & \text{in } (0,T), \\
k_2 v_x(l,t) & = & 0 & \text{in } (0,T) \\
u(x,0) & = & u_0(x) & \text{in } \Omega, \\
v(x,0) & = & v_0(x) & \text{in } \Omega
\end{cases}
$$

and the box constraints

$$c \in C_{ad} = \{c \in L^2(0,T) | c_a(t) \leq c(t) \leq c_b(t) \text{ a.e. in } [0,T]\} \subset L^\infty(0,T),$$

$$d \in D_{ad} = \{d \in L^2(0,T) | d_a(t) \leq d(t) \leq d_b(t) \text{ a.e. in } [0,T]\} \subset L^\infty(0,T)$$

for a final time $T > 0$. In this setting, Ω denotes the open interval $(0,l)$ and $Q = \Omega \times (0,T)$ is the space-time cylinder. The functions c_a,c_b, d_a and d_b are given of $L^\infty(0,T)$, such that $c_a \leq c_b$, $d_a \leq d_b$ holds almost everywhere in $[0,T]$. We denote by λ_1, λ_2, k_1, k_2 positive and by α_1, α_2, γ_1 and γ_2 nonnegative constants. The control functions c and d are considered as elements of the space $L^\infty(0,T)$ and the fixed initial values u_0 and v_0 are elements of the space $L^2(\Omega)$. The control functions are operating in Robin boundary conditions with point-wise control constraints on both sides. A similar problem with almost identical state equations but a different cost functional was considered in [6] and [5]. In these papers, necessary and sufficient optimality conditions are derived and numerical techniques for this class of optimal control problem were suggested.
Our paper is organized as follows: In Section 2, following the lines of [6] and [5], the analysis of the problem $(P1)$ is briefly sketched and numerical results are presented for some cases of catalysis.
In Section 3, we consider the following more general class of systems with non-linear boundary condition in a bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, with Lipschitz-continuous boundary $\Gamma = \partial\Omega$:

$$
\begin{aligned}
(P2) \qquad \min J(u,v,c) \quad = \quad & \frac{\alpha_u}{2} \|u - u_Q\|^2_{L^2(Q)} + \frac{\alpha_v}{2} \|v - v_Q\|^2_{L^2(Q)} + \\
& \frac{\alpha_{TU}}{2} \|u(T) - u_\Omega\|^2_{L^2(\Omega)} + \frac{\alpha_{TV}}{2} \|v(T) - v_\Omega\|^2_{L^2(\Omega)} + \\
& \frac{\alpha_c}{2} \|c\|^2_{L^2(\Sigma)}
\end{aligned}
$$

subject to

$$
(E2)\begin{cases}
u_t - k_1 \Delta u + \alpha_1 u & = & -\gamma_1 uv & \text{in } Q, \\
v_t - k_2 \Delta v + \alpha_2 v & = & -\gamma_2 uv & \text{in } Q, \\
k_1 \partial_\nu u + b(x,t,u) & = & c & \text{in } \Sigma, \\
k_2 \partial_\nu v + \alpha v & = & 0 & \text{in } \Sigma, \\
u(x,0) & = & u_0(x) & \text{in } \Omega, \\
v(x,0) & = & v_0(x) & \text{in } \Omega
\end{cases}
$$

and

$$c \in C_{ad} = \{c \in L^\infty(\Sigma) : c_a \leq c \leq c_b \; a.e. \; in \; \Sigma\},$$

where $\alpha, \alpha_u, \alpha_v, \alpha_{TU}, \alpha_{TV}$ and α_c are nonnegative constants, $c_a \leq c_b$ holds almost everywhere in $\Sigma = \Gamma \times (0,T)$ with $c_a, c_b \in L^\infty(\Sigma)$, u_Q, $v_Q \in L^2(Q)$, u_Ω, $v_\Omega \in L^2(\Omega)$, u_0, $v_0 \in C(\overline{\Omega})$. For convenience, we denote by ν the outward normal vector. We do not have a real background of application for this model. However, in our opinion, the extension of the semilinear equation of $(P1)$ to nonlinear boundary conditions is mathematically interesting. This problem belongs to the class of optimal control problems for semilinear parabolic equations, where quite a number of publications were devoted to. We mention, for instance, [12], where a nonlinear boundary condition of Stefan-Boltzmann type was considered first, the contributions [4], [8], [7] to a nonlinear phase field model, and the papers [3], [11] on the Pontryagin principle for parabolic control problems. Further references on the control of nonlinear parabolic equations can be found in the monography [13].
The main emphasis in this problem is laid on existence and uniqueness of a solution to the system of the state equations for $(P2)$, existence of a solution to the optimal control problem and necessary and sufficient optimality conditions. To find upper and lower solutions in Theorem 3.5, we make the following assumption **(A1)** on b.

Assumption (**A1**): The nonlinear function $b : \Sigma \times \mathbb{R} \to \mathbb{R}$ is continuous in $\overline{Q} \times \mathbb{R}$, locally Lipschitz in u, monotone nondecreasing with respect to u for almost all $(x,t) \in \Sigma$ and fullifies $b(x,t,0) \leq c_a(x,t)$ for all $(x,t) \in \Sigma$ and $\lim\limits_{u \to \pm\infty} b(x,t,u) = \pm\infty$.

2 The problem with linear boundary conditions

2.1 Existence of an optimal control

As pointed out in the introduction, the problem with linear boundary condition is similar to one with almost identical state equations but different cost functional discussed by R. Griesse and S. Volkwein in [6]. Therefore, we can rely on their results to discuss our slightly different control problem. We briefly recall some theorems of [6] for later applications to $(P2)$. In the sequel, we follow in principle their approach.

For a Banach space V, the space $W(0,T)$ is defined by

$$W(0,T) := \{y \in L^2(0,T;V) : y' \in L^2(0,T;V^*)\}$$

with norm

$$\|y\|_{W(0,T)} = \left(\int_0^T (\|y(t)\|_V^2 + \left\|y'(t)\right\|_{V^*}^2) dt \right)^{\frac{1}{2}},$$

where V^* is the dual space of V, y' denotes the distributional derivative of y with respect to t and $L^2(0,T;V)$ is the space of all (equivalence classes of) measurable abstract functions $y : [0,T] \to V$ with

$$\int_0^T \|y(t)\|_V^2 \, dt < \infty$$

and norm

$$\|y\|_{L^2(0,T;V)} = \left(\int_0^T \|y(t)\|_V^2 \, dt \right)^{\frac{1}{2}}.$$

First, we recall existence and uniqueness of a solution to the system of the state equations. We start by the definition of a weak solution.

Definition 2.1. *A pair of functions (u,v) of $W(0,T)\times W(0,T)$ is called weak solution of the state equation (E1), if the initial conditions*

$$u(x,0) = u_0 \text{ and } v(x,0) = v_0, \tag{1}$$

as well as

$$\begin{aligned} &(u_t(\cdot,t),\varphi)_{H^1(\Omega)^*,H^1(\Omega)} + u(0,t)\varphi(0) + \\ &\int_\Omega \Big(k_1 u_x(\cdot,t)\varphi_x + \alpha_1 u(\cdot,t)\varphi + \gamma_1 u(\cdot,t)v(\cdot,t)\varphi \Big)\, dx = c(t)\varphi(0) \end{aligned} \tag{2}$$

and

$$\begin{aligned} &(v_t(\cdot,t),\varphi)_{H^1(\Omega)^*,H^1(\Omega)} + v(0,t)\varphi(0) + \\ &\int_\Omega \Big(k_2 v_x(\cdot,t)\varphi_x + \alpha_2 v(\cdot,t)\varphi + \gamma_2 u(\cdot,t)v(\cdot,t)\varphi \Big)\, dx = d(t)\varphi(0) \end{aligned} \tag{3}$$

are satisfied for all $\varphi \in H^1(\Omega)$ and almost all $t \in [0,T]$. Here, the duality pairing between $H^1(\Omega)^*$ and $H^1(\Omega)$ is denoted by $(\cdot,\cdot)_{H^1(\Omega)^*,H^1(\Omega)}$ and u_x is defined by $u_x := \frac{\partial u}{\partial x}$.

The following theorem guarantees existence and uniqueness of a solution to (E1) (see [6], Theorem 2.3).

Theorem 2.2. *For each given pair of controls $(c,d) \in L^2(0,T)\times L^2(0,T)$, there exists a unique solution $(u,v) \in W(0,T)\times W(0,T)$ to (E1).*

For the standard proof of the next theorem we refer also to [6], Theorem 2.6.

Theorem 2.3. *Problem $(P1)$ admits at least one optimal solution.*

2.2 Necessary and sufficient optimality conditions

To discuss optimality conditions, we introduce the control-to-state operator S (solution operator) of (E1) by

$$S : L^2(0,T)\times L^2(0,T) \to W(0,T)^2, \; (c,d) \mapsto (u,v)$$

with $W(0,T)^2 := W(0,T)\times W(0,T)$. Notice that, by the nonlinear coupling through $\gamma_i uv$, $i = 1,2$, the system of state equations is nonlinear.

Theorem 2.4. *The control-to-state operator S is twice continuously Fréchet-differentiable.*

For the proof, we refer to the more general Theorem 3.10 in our paper. It is important that the space $W(0,T)$ is continuously imbedded in $L^4(Q)$, so that the product of two elements of $W(0,T)$ is in $L^2(0,T)$. The next theorems are devoted to the form of the derivatives of S. For the standard technique of the proofs, we refer to [13].

Theorem 2.5. *The first derivative of S at $(\bar{c}, \bar{d})$ in direction (c, d) is given by*

$$S'(\bar{c}, \bar{d})(c, d) = (u, v),$$

where (u, v) is the weak solution of the state equation linearized at $(\bar{u}, \bar{v})$

$$\begin{array}{rcll} u_t - k_1 u_{xx} + \alpha_1 u &=& -\gamma_1(\bar{u}v + u\bar{v}) & \text{in } Q, \\ v_t - k_2 v_{xx} + \alpha_2 v &=& -\gamma_2(\bar{u}v + u\bar{v}) & \text{in } Q, \\ u(0,t) - k_1 u_x(0,t) &=& c(t) & \text{in } (0,T), \\ k_1 u_x(l,t) &=& 0 & \text{in } (0,T), \\ v(0,t) - k_2 v_x(0,t) &=& d(t) & \text{in } (0,T), \\ k_2 v_x(l,t) &=& 0 & \text{in } (0,T), \\ u(x,0) &=& 0 & \text{in } \Omega, \\ v(x,0) &=& 0 & \text{in } \Omega \end{array} \tag{4}$$

and $(\bar{u}, \bar{v}) = S(\bar{c}, \bar{d})$.

Moreover, for second-order sufficient conditions, we need the second derivative of S. Let us briefly comment on it in a formal way to motivate the next result on the second derivative. The system $(E1)$ is of the form

$$A\, S(e) = B(S(e)) + D(e) + w_0,$$

where $e = (c, d)$, $S(e) = (u(e), v(e))$, $B(u, v) = -(\gamma_1 uv, \gamma_2 uv)$, A stands for the linear differential operator on the lefthand side of $(E1)$, w_0 for the initial conditions and D is a linear continuous operator. Therefore,

$$A\, S'(e)e_1 = B'(S(e))S'(e)e_1 + D(e_1). \tag{5}$$

The result is obtained with $e = (\bar{c}, \bar{d})$, $e_1 = (c, d)$, $(u, v) = S'(e)e_1$, $S(e) = (\bar{u}, \bar{v})$, since

$$B'(\bar{u}, \bar{v})(u, v) = -(\gamma_1(\bar{u}v + u\bar{v}), \gamma_2(\bar{u}v + u\bar{v})).$$

Differentiating (5) with respect to e in direction e_2 yields

$$A\, S''(e)(e_1, e_2) = B'(S(e))S''(e)(e_1, e_2) + B''(S(e))(S'(e)e_1, S'(e)e_2)$$

since $D'' = 0$ by linearity. Define $e_1 = (\hat{c}, \hat{d})$, $e_2 = (\tilde{c}, \tilde{d})$, $(u, v) = S''(e)(e_1, e_2)$, $(\hat{u}, \hat{v}) = S'(e)e_1$, $(\tilde{u}, \tilde{v}) = S'(e)e_2$, then the next result on the second derivative of S is obtained. Notice that $B''(u, v)[(\hat{u}, \hat{v}), (\tilde{u}, \tilde{v})] = -(\gamma_1(\hat{u}\tilde{v} + \tilde{u}\hat{v}), \gamma_2(\hat{u}\tilde{v} + \tilde{u}\hat{v}))$. The existence of S' and S'' can be shown by a standard application of the implicit function theorem that is explained, for instance, in [13].

Theorem 2.6. *The second derivative of S at $(\bar{c}, \bar{d})$ in direction $[(\hat{c}, \hat{d}), (\tilde{c}, \tilde{d})]$ is given by*

$$S''(\bar{c}, \bar{d})[(\hat{c}, \hat{d}), (\tilde{c}, \tilde{d})] = (u, v),$$

where (u, v) is the unique weak solution of the system

$$\begin{array}{rll} u_t - k_1 u_{xx} + \alpha_1 u + \gamma_1(\bar{u}v + u\bar{v}) &= -\gamma_1(\hat{u}\tilde{v} + \tilde{u}\hat{v}) & \text{in } Q, \\ v_t - k_2 v_{xx} + \alpha_2 v + \gamma_2(\bar{u}v + u\bar{v}) &= -\gamma_2(\hat{u}\tilde{v} + \tilde{u}\hat{v}) & \text{in } Q, \\ u(0,t) - k_1 u_x(0,t) &= 0 & \text{in } (0,T), \\ k_1 u_x(l,t) &= 0 & \text{in } (0,T), \\ v(0,t) - k_2 v_x(0,t) &= 0 & \text{in } (0,T), \\ k_2 v_x(l,t) &= 0 & \text{in } (0,T), \\ u(x,0) &= 0 & \text{in } \Omega, \\ v(x,0) &= 0 & \text{in } \Omega \end{array}$$

where $(\overline{u},\overline{v}) = S(\overline{c},\overline{d})$ *and* $(\hat{u},\hat{v}) = S'(\overline{c},\overline{d})(\hat{c},\hat{d})$ *is the solution of*

$$\begin{array}{rcll}
\hat{u}_t - k_1\hat{u}_{xx} + \alpha_1\hat{u} & = & -\gamma_1(\overline{u}\hat{v} + \hat{u}\overline{v}) & \textit{in } Q, \\
\hat{v}_t - k_2\hat{v}_{xx} + \alpha_2\hat{v} & = & -\gamma_2(\overline{u}\hat{v} + \hat{u}\overline{v}) & \textit{in } Q, \\
\hat{u}(0,t) - k_1\hat{u}_x(0,t) & = & \hat{c}(t) & \textit{in } (0,T), \\
k_1\hat{u}_x(l,t) & = & 0 & \textit{in } (0,T), \\
\hat{v}(0,t) - k_2\hat{v}_x(0,t) & = & \hat{d}(t) & \textit{in } (0,T), \\
k_2\hat{v}_x(l,t) & = & 0 & \textit{in } (0,T), \\
\hat{u}(x,0) & = & 0 & \textit{in } \Omega, \\
\hat{v}(x,0) & = & 0 & \textit{in } \Omega.
\end{array}$$

Analogously, $(\tilde{u},\tilde{v}) = S'(\overline{c},\overline{d})(\tilde{c},\tilde{d})$ *is defined.*

Because of the nonconvexity of problem $(P1)$, several different global solutions might exist. Moreover, additional local minima can occur. In the next part of this section, we derive the necessary optimality conditions for any pair of locally optimal control functions $(\overline{c},\overline{d})$. We first focus on the control c and fix $\overline{d}$, hence we redefine

$$S(c) := S(c,\overline{d})$$

and consider the Fréchet-differentiable functional

$$f(c) = J(S(c),c,\overline{d}) = J(u,v,c,\overline{d}).$$

Let us write for short $y = (u,v)$, $\overline{y} = (\overline{u},\overline{v})$. Then $J(u,v,c,\overline{d}) = J(y,c,\overline{d})$.

Because the set of admissible controls C_{ad} is convex, we obtain the following standard result:

Lemma 2.7. *Every locally optimal control function* $\overline{c}$ *of a locally optimal pair* $(\overline{c},\overline{d})$ *of the problem (P1) satisfies the variational inequality*

$$f'(\overline{c})(c-\overline{c}) \geq 0 \qquad \forall c \in C_{ad}.$$

By using the chain rule we calculate f':

$$\begin{aligned}
f'(\overline{c})(c-\overline{c}) &= D_yJ(\overline{y},\overline{c})S'(\overline{c})(c-\overline{c}) + D_cJ(\overline{y},\overline{c})(c-\overline{c}) \\
&= \iint\limits_Q ((\overline{u}-k\overline{v})u - k(\overline{u}-k\overline{v})v)\,dx\,dt + \int_0^T \lambda_1\overline{c}(c-\overline{c})\,dt, \quad (6)
\end{aligned}$$

where $(u,v) = S'(\overline{c})(c-\overline{c})$ is the weak solution of the linearized problem

$$\begin{array}{rcll}
u_t - k_1u_{xx} + \alpha_1 u & = & -\gamma_1(\overline{u}v + u\overline{v}) & \text{in } Q, \\
v_t - k_2v_{xx} + \alpha_2 v & = & -\gamma_2(\overline{u}v + u\overline{v}) & \text{in } Q, \\
u(0,t) - k_1u_x(0,t) & = & c(t) - \overline{c}(t) & \text{in } (0,T), \\
k_1u_x(l,t) & = & 0 & \text{in } (0,T), \\
v(0,t) - k_2v_x(0,t) & = & 0 & \text{in } (0,T), \\
k_2v_x(l,t) & = & 0 & \text{in } (0,T), \\
u(x,0) & = & 0 & \text{in } \Omega, \\
v(x,0) & = & 0 & \text{in } \Omega.
\end{array} \qquad (7)$$

The functions u and v in (6) can be expressed in terms of $(c-\overline{c})$ by introducing adjoint states p und q. Utilizing a Lagrangian function, we obtain p und q as solutions of the system

$$\begin{array}{rcll} -p_t - k_1 p_{xx} + \alpha_1 p + \gamma_1 \overline{v} p + \gamma_2 \overline{v} q & = & \overline{u} - k\overline{v} & \text{in } Q, \\ p(0,t) - k_1 p_x(0,t) & = & 0 & \text{in } (0,T), \\ k_1 p_x(l,t) & = & 0 & \text{in } (0,T), \\ p(x,T) & = & 0 & \text{in } \Omega, \\ -q_t - k_2 q_{xx} + \alpha_2 q + \gamma_1 \overline{u} p + \gamma_2 \overline{u} q & = & -k(\overline{u} - k\overline{v}) & \text{in } Q, \\ q(0,t) - k_2 q_x(0,t) & = & 0 & \text{in } (0,T), \\ k_2 q_x(l,t) & = & 0 & \text{in } (0,T), \\ q(x,T) & = & 0 & \text{in } \Omega. \end{array} \tag{8}$$

Theorem 2.8. *If (u,v) is the weak solution of (7) and (p,q) is the solution of (8), than it holds for all $c \in L^2(0,T)$*

$$\iint\limits_Q ((\overline{u} - k\overline{v})u - k(\overline{u} - k\overline{v})v)\; dx\; dt = \int\limits_0^T p(0,t)(c(t) - \overline{c}(t))\; dt.$$

For the technique of proof we refer the reader to [13], Lemma 5.12.
It follows that (6) is equal to

$$f'(\overline{c})(c - \overline{c}) = \int\limits_0^T (p(0,t) + \lambda_1 \overline{c}(t))(c(t) - \overline{c}(t))\; dt. \tag{9}$$

This leads to the following theorem.

Theorem 2.9. *To every locally optimal control function $\overline{c}$ of a locally optimal pair $(\overline{c}, \overline{d})$ of $(P1)$ there exist adjoint states $(p,q) \in W(0,T) \times W(0,T)$ defined by (8) such that it holds*

$$\int\limits_0^T (p(0,t) + \lambda_1 \overline{c}(t))(c(t) - \overline{c}(t))\; dt \;\geq\; 0 \qquad \forall c \in C_{ad}.$$

In the case $\lambda_1 > 0$ we obtain (see [13], (2.54) on page 56)

$$\overline{c}(t) = P_{[c_a(t), c_b(t)]}\{-\frac{1}{\lambda_1} p(0,t)\}$$

for almost all $t \in [0,T]$ where $P_{[a,b]}$ denotes the projection of $\mathbb{R}$ to $[a,b]$. Doing the same for the control function $\overline{d}$ we derive:

Theorem 2.10. *Every locally optimal pair of control functions $(\overline{c}, \overline{d})$ of $(P1)$, satisfies, with a pair of adjoint states (p,q) defined by (8) the variational inequalities*

$$\begin{array}{rcll} \displaystyle\int\limits_0^T (p(0,t) + \lambda_1 \overline{c}(t))(c(t) - \overline{c}(t))\; dt & \geq & 0 & \qquad \forall c \in C_{ad}, \\ \displaystyle\int\limits_0^T (q(0,t) + \lambda_2 \overline{d}(t))(d(t) - \overline{d}(t))\; dt & \geq & 0 & \qquad \forall d \in D_{ad}. \end{array}$$

If λ_1 and λ_2 are positive, then the inequalities are equivalent to the pointwise projection formulas

$$\begin{aligned}\overline{c}(t) &= P_{[c_a(t),c_b(t)]}\{-\frac{1}{\lambda_1}p(0,t)\},\\ \overline{d}(t) &= P_{[d_a(t),d_b(t)]}\{-\frac{1}{\lambda_2}q(0,t)\}\end{aligned}$$

for almost all $t \in [0,T]$.

In the following, we consider also sufficient second order optimality conditions for $(P1)$. Let $(\overline{c},\overline{d}) \in S_{ad} := C_{ad} \times D_{ad}$ together with $(\overline{u},\overline{v}) = S(\overline{c},\overline{d})$ and (p,q) satisfy the first-order necessary optimality conditions, presented in Theorem 2.10. We want to introduce sufficient conditions, so that $(\overline{c},\overline{d},\overline{u},\overline{v})$ is a local optimum.
Because the cost functional J and the control-to-state operator S are twice continuously Fréchet-differentiable,

$$f(c,d) = J(S(c,d),c,d)$$

is also twice continuously Fréchet-differentiable. By using the chain rule, we derive

$$f'(c,d)(\hat{c},\hat{d}) = D_yJ(S(c,d),c,d)S'(c,d)(\hat{c},\hat{d}) + D_{(c,d)}J(S(c,d),c,d)(\hat{c},\hat{d}).$$

The derivative of $f'(c,d)(\hat{c},\hat{d})$ with respect to (c,d) in direction $(\tilde{c},\tilde{d})$ is

$$\begin{aligned}f''(c,d)[(\hat{c},\hat{d}),(\tilde{c},\tilde{d})] &= D_y^2J(S(c,d),c,d)[S'(c,d)(\hat{c},\hat{d}),S'(c,d)(\tilde{c},\tilde{d})]\\ &\quad +D_{(c,d)}D_yJ(S(c,d),c,d)[S'(c,d)(\hat{c},\hat{d}),(\tilde{c},\tilde{d})]\\ &\quad +D_yJ(S(c,d),c,d)S''(c,d)[(\hat{c},\hat{d}),(\tilde{c},\tilde{d})]\\ &\quad +D_yD_{(c,d)}J(S(c,d),c,d)[(\hat{c},\hat{d}),S'(c,d)(\tilde{c},\tilde{d})]\\ &\quad +D^2_{(c,d)}J(S(c,d),c,d)[(\hat{c},\hat{d}),(\tilde{c},\tilde{d})]\\ &= J''(y,c,d)[(\hat{y},\hat{c},\hat{d}),(\tilde{y},\tilde{c},\tilde{d})] + D_yJ(y,c,d)z.\end{aligned}$$

with states z, $\hat{y}$ and $\tilde{y}$, $z := (\omega_1,\omega_2) = S''(c,d)[(\hat{c},\hat{d}),(\tilde{c},\tilde{d})]$, $\hat{y} := S'(c,d)(\hat{c},\hat{d})$ and $\tilde{y} := S'(c,d)(\tilde{c},\tilde{d})$, defined by cf. also (5). The last expression,

$$D_yJ(y,c,d)z = \iint_Q (u-kv)\omega_1 - k(u-kv)\omega_2 \; dx\, dt \quad \text{with } y=(u,v),$$

can be transformed analogously to Theorem 2.8 to

$$D_yJ(y,c,d)z = \iint_Q \gamma_1(\hat{u}\tilde{v}+\tilde{u}\hat{v})p + \gamma_2(\hat{u}\tilde{v}+\tilde{u}\hat{v})q \; dxdt$$

by using the adjoints (p,q) of (8). We derive

$$\begin{aligned}f''(c,d)[(\hat{c},\hat{d}),(\tilde{c},\tilde{d})] =& J''(y,c,d)[(\hat{y},\hat{c},\hat{d}),(\tilde{y},\tilde{c},\tilde{d})]\\ &+\iint_Q \gamma_1(\hat{u}\tilde{v}+\tilde{u}\hat{v})p + \gamma_2(\hat{u}\tilde{v}+\tilde{u}\hat{v})q \; dxdt. \qquad (10)\end{aligned}$$

To formulate our sufficient optimality conditions, we introduce the Lagrange function:

$$\begin{aligned}\mathcal{L}(u,v,c,d,p,q) = \quad & J(u,v,c,d) + \iint_Q (u_t - k_1 u_{xx} + \alpha_1 u + \gamma_1 uv)p \; dxdt \\ & + \int_0^T (u(0,t) - k_1 u_x(0,t) - c(t) + k_1 u_x(l,t))p(0,t) \; dt \\ & + \int_\Omega (u(x,0) - u_0)p(x,0) \; dx \\ & + \iint_Q (v_t - k_2 v_{xx} + \alpha_2 v + \gamma_2 uv)q \; dxdt \\ & + \int_0^T (v(0,t) - k_2 v_x(0,t) - d(t) + k_2 v_x(l,t))q(0,t) \; dt \\ & + \int_\Omega (v(x,0) - v_0)q(x,0) \; dx.\end{aligned} \tag{11}$$

We obtain

$$\begin{aligned}&\mathcal{L}''(u,v,c,d,p,q)[(\hat{u},\hat{v},\hat{c},\hat{d}),(\tilde{u},\tilde{v},\tilde{c},\tilde{d})] \\ &= J''(u,v,c,d)[(\hat{u},\hat{v},\hat{c},\hat{d}),(\tilde{u},\tilde{v},\tilde{c},\tilde{d})] + \iint_Q \gamma_1(\hat{u}\tilde{v} + \tilde{u}\hat{v})p + \gamma_2(\hat{u}\tilde{v} + \tilde{u}\hat{v})q \; dxdt \\ &= f''(c,d)[(\hat{u},\hat{v},\hat{c},\hat{d}),(\tilde{u},\tilde{v},\tilde{c},\tilde{d})]\end{aligned}$$

in view of (10). This result was to be expected, since the second derivative f'' of the reduced objective functional can be expressed in general by $\mathcal{L}''$ defined with the associated adjoint states, cf. [13]. The derivative J'' is given by

$$\begin{aligned}&J''(u,v,c,d)[(\hat{u},\hat{v},\hat{c},\hat{d}),(\tilde{u},\tilde{v},\tilde{c},\tilde{d})] \\ &= \iint_Q \hat{u}\tilde{u} - k\hat{u}\tilde{v} - k\tilde{u}\hat{v} + k^2\hat{v}\tilde{v} \; dx \; dt + \lambda_1 \int_0^T \hat{c}\tilde{c} \; dt + \lambda_2 \int_0^T \hat{d}\tilde{d} \; dt.\end{aligned}$$

By the variational inequality, see Theorem 2.10, for the optimal solution $(\overline{c},\overline{d})$ of (P1), we obtain

$$\overline{c}(t) = \begin{cases} c_a, & \text{if } p(0,t) + \lambda_1\overline{c}(t) > 0 \\ c_b, & \text{if } p(0,t) + \lambda_1\overline{c}(t) < 0 \end{cases} \tag{12}$$

and

$$\overline{d}(t) = \begin{cases} d_a, & \text{if } q(0,t) + \lambda_2\overline{d}(t) > 0 \\ d_b, & \text{if } q(0,t) + \lambda_2\overline{d}(t) < 0. \end{cases} \tag{13}$$

By the first-order conditions, the control function $\overline{c}$ is defined in the set $\{t \in (0,T) : |p(0,t) + \lambda_1\overline{c}| > 0\}$ and $\overline{d}$ in $\{t \in (0,T) : |q(0,t) + \lambda_2\overline{d}| > 0\}$. Therefor, second-order sufficient conditions should be required on the remaining sets.

Definition 2.11. *For a given τ and controls $\overline{c}$, $\overline{d}$, we define by*

$$A_\tau(\overline{c}) := \{t \in (0,T) : |p(0,t) + \lambda_1\overline{c}(t)| > \tau\},$$
$$A_\tau(\overline{d}) := \{t \in (0,T) : |q(0,t) + \lambda_1\overline{d}(t)| > \tau\}$$

the sets of strongly active restrictions for $\overline{c}$ respectivly $\overline{d}$.

Definition 2.12. *The* <u>τ*-critical cone*</u> $C_\tau(\overline{c})$ *is the set of all* $c \in L^\infty(0,T)$ *with*

$$c(t) \left\{ \begin{array}{lll} = & 0\ , & t \in A_\tau(\overline{c}) \\ \geq & 0\ , & \overline{c}(t) = c_a \text{ und } t \notin A_\tau(\overline{c}) \\ \leq & 0\ , & \overline{c}(t) = c_b \text{ und } t \notin A_\tau(\overline{c}). \end{array} \right.$$

Analogously, we define $C_\tau(\overline{d})$.

Theorem 2.13. *Suppose that the control functions* $(\overline{c}, \overline{d})$ *satisfy the first-order necessary optimality conditions of Theorem 2.10. If there exist psitive constants* δ *and* τ *such that*

$$\mathcal{L}''(\overline{u}, \overline{v}, \overline{c}, \overline{d}, p, q)(u, v, c, d)^2 \geq \delta(\|c\|^2_{L^2(0,T)} + \|d\|^2_{L^2(0,T)})$$

holds for all $(c, d) \in C_\tau(\overline{c}) \times C_\tau(\overline{d})$ *and all* $(u, v) \in W(0,T) \times W(0,T)$ *satisfying (4), then we find positive constants* ε *and* σ *such that*

$$J(u, v, c, d) \geq J(\overline{u}, \overline{v}, \overline{c}, \overline{d}) + \sigma(\|c - \overline{c}\|_{L^2(Q)} + \|d - \overline{d}\|_{L^2(Q)})$$

holds for all $(c, d) \in S_{ad}$ *with* $\|c - \overline{c}\|_{L^\infty(Q)} + \|d - \overline{d}\|_{L^\infty(Q)} \leq \varepsilon$. *Hence, the control functions* $\overline{c}$ *and* $\overline{d}$ *are locally optimal.*

2.3 Numerical examples

Here, we consider examples related to the catalysis problem explained in the introduction. We consider d as a periodic piecewise constant function that is given fixed. This means that the harmful substance is feeded into the catalyst periodically by a certain quantity d_0 where d has the form

$$d(t) = \left\{ \begin{array}{l} d_0 \text{ on } [T/4, T/2[\cup [3T/4, T[\\ 0 \text{ on } [0, T/4[\cup [T/2, 3T/4[. \end{array} \right.$$

We assume that we are able to insert the harmless substance only when the harmful substance is not be inserted. Hence, we choose the bounds c_a und c_b as functions presented in Figure 1 where

$$c_i(t) = \left\{ \begin{array}{l} 0 \text{ on } [T/4, T/2[\cup [3T/4, T[\\ \tilde{c}_i \text{ on } [0, T/4[\cup [T/2, 3T/4[\end{array} \right.$$

with $i = a, b$ with $c_a \leq c_b$. They are periodic and piecewise constant functions with the only possible values 0 and $\tilde{c}_i$, $i = a, b$.

Example 1:
Setting $l = 1$, $T = 10$, $k = k_1 = k_2 = 1$, $\gamma_1 = \gamma_2 = 0.5$ $\alpha_1 = \alpha_2 = 0.3$, $\lambda_1 = \lambda_2 = 0.001$, $d_0 = 7$, $u_0 = v_0 \equiv 0$ and for the control constraints c

$$c_a(t) = \left\{ \begin{array}{l} 0 \text{ on } [T/4, T/2[\cup [3T/4, T[\\ 1 \text{ on } [0, T/4[\cup [T/2, 3T/4[\end{array} \right.$$

and

$$c_b(t) = \left\{ \begin{array}{l} 0 \text{ on } [T/4, T/2[\cup [3T/4, T[\\ 10 \text{ on } [0, T/4[\cup [T/2, 3T/4[. \end{array} \right.$$

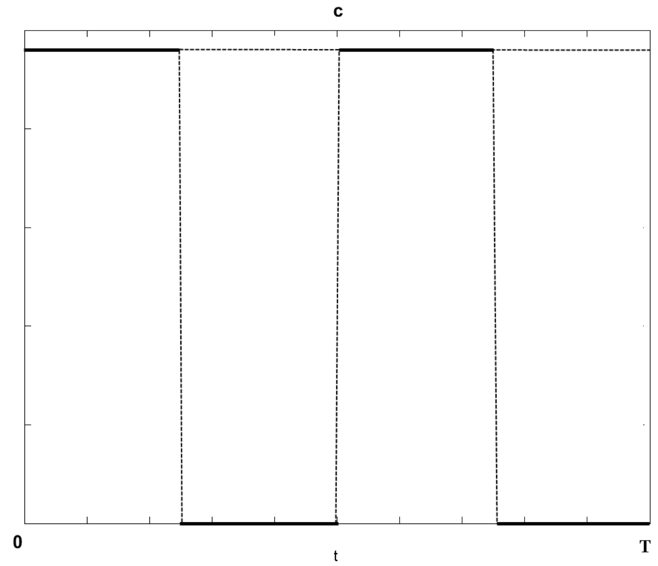

Figure 1: Periodic, piecewise constant function with a period of length $\frac{T}{2}$.

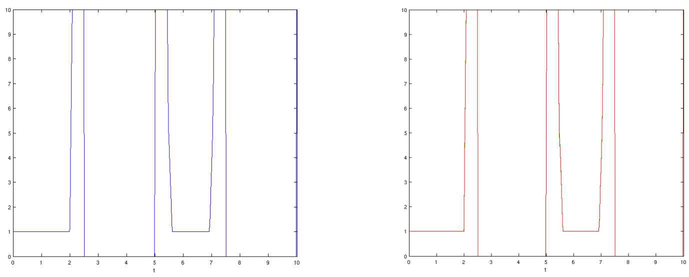

Figure 2: Optimal control $\bar{c}$ and Projection $P_{[c_a,c_b]}\{-\frac{1}{\lambda_1}p(0,t)\}$ for Example1

we obtain the results, presented in the figures 2,4 and 3. Figure 2 shows the compatibility of the control function c and the Projection $P_{[c_a,c_b]}\{-\frac{1}{\lambda_1}p(0,t)\}$ with respect to the necessary optimality conditions.

Next, we consider the same data as in example 1, but we choose $d_0 = 12$ in Figure 5, while we set $\lambda_1 = \lambda_2 = 1$ in Figure 6. Figure 7 is based on the choice

$$c_b(t) = \begin{cases} 0 \text{ on } [T/4, T/2[\cup[3T/4, T[\\ 1.7 \text{ on } [0, T/4[\cup[T/2, 3T/4[\end{cases}$$

and $\lambda_1 = \lambda_2 = 1$.

Example 2: Next, we choose the control function c as a periodic and piecewise

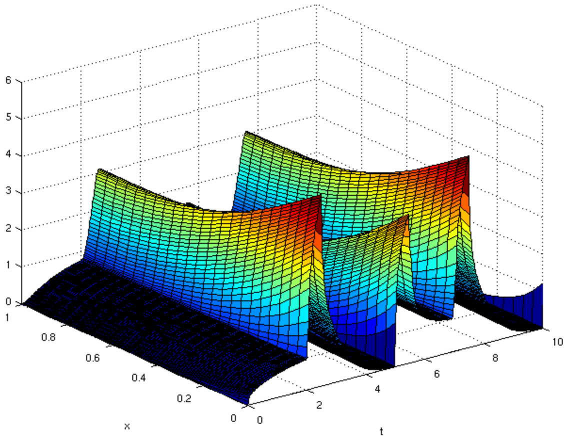

Figure 3: State u for Example1

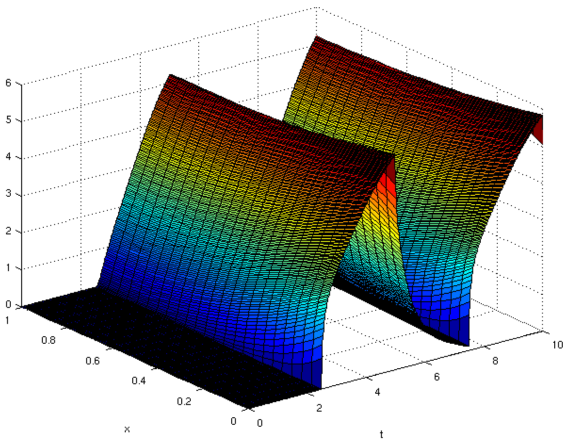

Figure 4: State v for Example1

constant. In view of this, we reformulate our control problem. We consider the control function as a function described in Figure 1 and optimize only the height c_0. This leads to:

$$\min \ J(c_0, u, v) = \frac{1}{2} \iint\limits_Q (u - kv)^2 \, dx \, dt + c_0^2 \frac{\lambda_1}{2} \int\limits_0^T e \, dt,$$

where e has the form

$$e(t) = \begin{cases} 0 \text{ on } [T/4, T/2[\cup [3T/4, T[\\ 1 \text{ on } [0, T/4[\cup [T/2, 3T/4[. \end{cases}$$

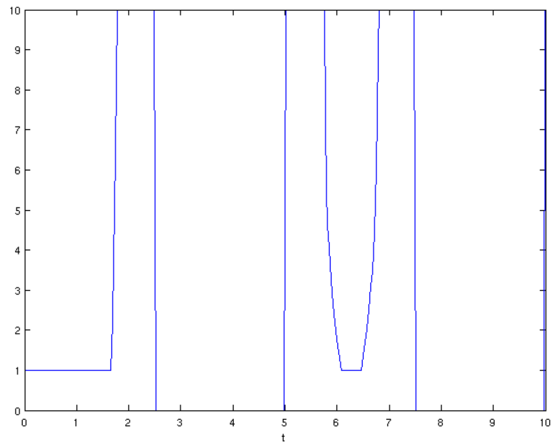

Figure 5: control function $\overline{c}$

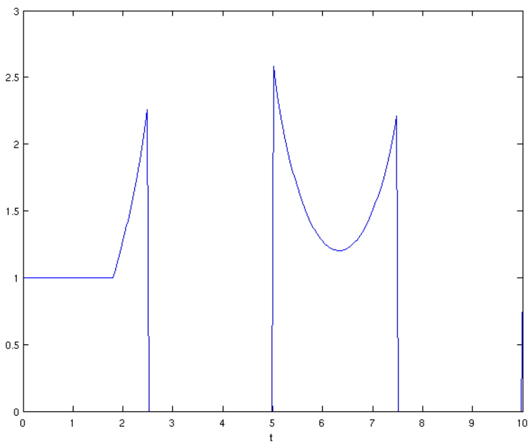

Figure 6: Control function $\overline{c}$

We obtain

$$f'(c_0) = \int\limits_0^T (p(0,t)e(t) + \lambda_1 c_0 e(t)) \; dt$$

and the results, presented in Figure 8 for the same data as in Example 1 with $d_0 = 2.4$.
It turned out that the optimal value (4.69) of the objective functional for this restricted class of controls was (up to a relative error of 0.04) equal to the first one. However, the computation needed only half the time (6 seconds instead of 12 seconds), since the degrees of freedom of the control is much smaller in this case. This shows that, in our concrete application, it is justified to work with controls that are constant in each period of time.

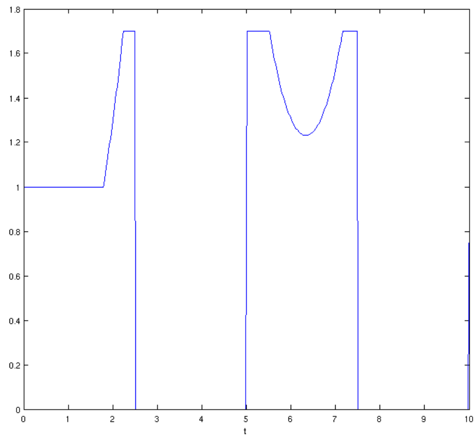

Figure 7: Control function $\bar{c}$

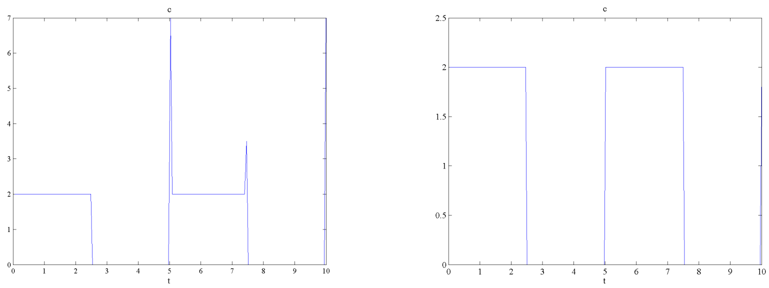

Figure 8: Optimal control function $\bar{c}$ for the original problem with the data of example 2 and for example 2

3 The problem with nonlinear boundary conditions

3.1 Well-posedness of the state equation

In this section, we consider the problem $(P2)$ with nonlinear boundary conditions. Systems of this type might be interesting for the applications. For instance, the equations might model the diffusion of a substance with concentrations v and temperature u, where a Stefan-Boltzmann type boundary condition for u is given. However, we do not aim at discussing specific applications. We think that the system is interesting from a mathematical point of view. To show an existence and uniqueness theorem for the new nonlinear system, we invoke the method of sub- and super-solutions. Moreover, we need higher regularity of u and v to make the nonlinearities well defined and to ensure the diffentiability

of the control-to-state mapping.
We assume that $u_0 \in L^\infty(\Omega)$ and $v_0 \in L^\infty(\Omega)$ are given.

Definition 3.1. *A pair of functions $(u, v) \in (W(0,T) \cap L^\infty(Q))^2$ is called weak solution of the system System $(E2)$, if the equations*

$$
\begin{aligned}
&u(\cdot,0) = u_0, \;\; v(\cdot,0) = v_0, \\
&\int_0^T (u_t, \varphi)_{H^1(\Omega)^*,H^1(\Omega)} \, dt + \iint_\Sigma b(x,t,u)\varphi \, dx \, dt \\
&\quad + \iint_Q (k_1 \nabla u \cdot \nabla \varphi + \alpha_1 u \varphi + \gamma_1 u v \varphi) \, dx \, dt = \iint_\Sigma c \, \varphi \, dx \, dt
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^T (v_t, \varphi)_{H^1(\Omega)^*,H^1(\Omega)} \, dt + \iint_\Sigma \alpha \, v \, \varphi \, dx \, dt \\
&\quad + \iint_Q (k_2 \nabla v \cdot \nabla \varphi + \alpha_2 v \varphi + \gamma_2 u v \varphi) \, dx \, dt = 0
\end{aligned}
$$

are satisfied for all $\varphi \in L^2(0,T;H^1(\Omega))$, where ∇ denotes the gradient with respect to x.

To prove the existence and uniqueness of a weak solution for (E2), we apply the method of upper and lower solutions and follow the arguments of Pao [10], pp. 459-470. We first show the existence of classical solutions for given smooth data.

Definition 3.2. *Two pairs of functions $(\tilde{u}, \tilde{v})$ and $(\hat{u}, \hat{v})$ in $C(\overline{Q}) \cap C^{1,2}(Q)$ are called ordered upper and lower solutions of (E2), if $(\tilde{u}, \tilde{v})(x,t) \geq (\hat{u}, \hat{v})(x,t)$ in $\overline{Q}$ and the following inequalities are satisfied:*

$$
\begin{array}{rclcll}
\tilde{u}(x,0) & \geq & u_0(x) & \geq & \hat{u}(x,0) & \text{in } \Omega \\
\tilde{v}(x,0) & \geq & v_0(x) & \geq & \hat{v}(x,0) & \text{in } \Omega \\
\partial_\nu \tilde{u} - c + b(x,t,\tilde{u}) & \geq & 0 & \geq & \partial_\nu \hat{u} - c + b(x,t,\hat{u}) & \text{in } \Sigma \\
\partial_\nu \tilde{v} + \alpha \tilde{v} & \geq & 0 & \geq & \partial_\nu \hat{v} + \alpha \hat{v} & \text{in } \Sigma \\
\tilde{u}_t - k_1 \Delta \tilde{u} + \alpha_1 \tilde{u} + \gamma_1 \tilde{u} \hat{v} & \geq & 0 & \geq & \hat{u}_t - k_1 \Delta \hat{u} + \alpha_1 \hat{u} + \gamma_1 \hat{u} \tilde{v} & \text{in } Q \\
\tilde{v}_t - k_2 \Delta \tilde{v} + \alpha_2 \tilde{v} + \gamma_2 \hat{u} \tilde{v} & \geq & 0 & \geq & \hat{v}_t - k_2 \Delta \hat{v} + \alpha_2 \hat{v} + \gamma_2 \tilde{v} \hat{v} & \text{in } Q.
\end{array}
$$

To prove the solvability of the system $(E2)$, we invoke a Theorem 3.2 of [10], cf. Theorem 3.4 below. For his theorem, we have to define a quasimonotone nonincreasing function.

Definition 3.3. *A function $f : \mathbb{R}^n \to \mathbb{R}^n$ is called quasimonotone nonincreasing, if for fixed x_i, f_i, $i \in \{1, \ldots, n\}$ of f is monotone nonincreasing in x_k for all k with $k \neq i$.*

In contrast to our paper, this theorem considers the coupling of the two states through the boundary and concerns the following system of parabolic equations:

$$\left\{\begin{array}{rcll} (u_1)_t - L_1 u_1 & = & f_1(x,t,u_1) & \text{in } Q, \\ (u_2)_t - L_2 u_2 & = & f_2(x,t,u_2) & \text{in } Q, \\ k_1\partial_\nu u_1 + \beta_1 u_1 & = & g_1(x,t,u_1,u_2) & \text{in } \Sigma, \\ k_2\partial_\nu u_2 + \beta_2 u_2 & = & g_2(x,t,u_1,u_2) & \text{in } \Sigma, \\ u_1(x,0) & = & u_{1,0}(x) & \text{in } \Omega, \\ u_2(x,0) & = & u_{2,0}(x) & \text{in } \Omega, \end{array}\right. \tag{14}$$

where L_i, defined by

$$L_i u = \sum_{i,j=1}^{n} a_{ij}(x,t)\partial^2 u/\partial x_i \partial x_j + \sum_{j=1}^{n} b_j(x,t)\partial u/\partial x_j,$$

is a uniformly elliptic operator in the sense that the matrix (a_{ij}) is positive definite. The coefficients of L_i are assumed to be Hölder continuous in Q, $k_i, \beta_i \geq 0$, $k_i + \beta_i > 0$, f_i is Hölder continuous, g_i is continuous , $i = 1, 2$, and the following property is satisfied

$$\partial_{u_2} f_1 \geq 0 \text{ and } \partial_{u_1} f_2 \geq 0, \tag{15}$$

$$\partial_{u_2} f_1 \leq 0 \text{ and } \partial_{u_1} f_2 \leq 0, \tag{16}$$

$$\partial_{u_2} f_1 \leq 0 \text{ and } \partial_{u_1} f_2 \geq 0. \tag{17}$$

Furthermore, the functions f_i and g_i, $i = 1, 2$, are supposed to be locally Lipschitz, that means for all $M > 0$ there exists a Lipschitz constant $L(M) > 0$ such that

$$|f_i(x,t,u_1) - f_i(x,t,u_2)| \leq L_{f_i}(M)|u_1 - v_2| \tag{18}$$

and

$$|g_i(x,t,u_1,v_1) - g_i(x,t,u_2,v_2)| \leq L_{g_i}(M)(|u_1 - v_1| + |u_2 - v_2|) \tag{19}$$

for all $(x,t) \in \overline{Q}$ and all $|u_i| \leq M$, $|v_i| \leq M$, $i = 1, 2$, and f_2, g_i, $i = 1, 2$, satisfy this assumption analogously.

Theorem 3.4 ([10]). *Let $(\tilde{u}_1, \tilde{u}_2)$ und $(\hat{u}_1, \hat{u}_2)$ be ordered upper and lower solutions of (14), (g_1, g_2) be quasimonotone nonincreasing in $[(\tilde{u_1}, \tilde{u_2}), (\hat{u_1}, \hat{u_2})] := \{(u,v) \in C(\overline{Q}) \times C(\overline{Q}) : \tilde{u} \leq u \leq \hat{u} \text{ and } \tilde{v} \leq v \leq \hat{v}\}$, and satisfy (18). Then the system (14) has a unique solution (u_1, u_2) in $[(\tilde{u_1}, \tilde{u_2}), (\hat{u_1}, \hat{u_2})]$. Moreover, the sequences $(\overline{u_1}^k, \underline{u_2}^k), (\underline{u_1}^k, \overline{u_2}^k)$ obtained from 20, with $(\overline{u_1}^0, \underline{u_2}^0) = (\tilde{u_1}, \hat{u_2})$ and $(\underline{u_1}^0, \overline{u_2}^0) = (\hat{u_1}, \tilde{u_2})$ converge monotonically to (u_1, u_2) and satisfy the relation*

$$(\hat{u_1}, \hat{u_2}) \leq (\underline{u_1}^k, \underline{u_2}^k) \leq (u_1, u_2) \leq (\overline{u_1}^k, \overline{u_2}^k) \leq (\tilde{u_1}, \tilde{u_2}) \qquad \forall k = 1, 2, ... on\ Q.$$

$$\begin{array}{rcll} (u_1^k)_t - L_1 u_1^k + L_{f_1} u^k & = & f_1(x,t,u_1^{k-1}) + L_{f_1} u^{k-1} & \text{in } Q, \\ (u_2^k)_t - L_2 u_2^k + L_{f_2} u^k & = & f_2(x,t,u_2^{k-1}) + L_{f_2} u^{k-1} & \text{in } Q, \\ k_1\partial_\nu u_1^k + \beta_1 u_1^k + L_{g_1} u^k & = & g_1(x,t,u_1^{k-1},u_2^{k-1}) + L_{g_1} u^{k-1} & \text{in } \Sigma, \\ k_2\partial_\nu u_2^k + \beta_2 u_2^k + L_{g_2} u^k & = & g_2(x,t,u_1^{k-1},u_2^{k-1}) + L_{g_2} u^{k-1} & \text{in } \Sigma, \\ u_1^k(x,0) & = & u_{1,0}(x) & \text{in } \Omega, \\ u_2^k(x,0) & = & u_{2,0}(x) & \text{in } \Omega, \end{array} \tag{20}$$

Theorem 3.5. *For every function $c \in C(\Sigma)$ with $c(x,t) \geq c_a(x,t)$ for all $(x,t) \in \Sigma$ and $u_0, v_0 \in C(\overline{\Omega})$, there exists an unique solution $(u,v) \in \big(C(\overline{Q}) \cap C^{1,2}(Q)\big)^2$ for the system (E2).*

Proof. Following [10], we first have to find pairs of ordered upper and lower solution. As a lower solution we take

$$(\hat{u}, \hat{v}) = (0,0),$$

and for the upper solution we choose

$$(\tilde{u}, \tilde{v}) = (\delta, \delta)$$

where $\delta := \max(\delta_v, \delta_u)$ and δ_u, δ_v are positive constants satisfying

$$\delta_u \geq u_0(x), \ \delta_v \geq v_0(x) \quad \forall x \in \Omega, \tag{21}$$

and

$$b(x,t,\delta_u) \geq c(x,t) \quad \forall (x,t) \in \Sigma. \tag{22}$$

We derive the existence of these constants, because of Assumption **A1** and the assumptions on c in this theorem. The conditions (21) and (22) can be satisfied since $u_0, v_0 \in C(\overline{\Omega})$ and the assumptions **A1** on b hold.
In the application of Theorem 3.4, we take $L_1 = -\Delta$, $L_2 = -\Delta$, $f_1(x,t,u,v) = -\alpha_1 u - \gamma_1 uv$, $f_2(x,t,u,v) = -\alpha_2 v - \gamma_2 uv$, $g_1(x,t,u) = c - b(x,t,u)$, $g_2(x,t,v) = 0$, $\beta_1 = 0$ and $\beta_2 = \alpha$. In [10], the nonlinear coupling occurs in the boundary conditions, while it is n the right-hand side of the equation in our case. Therefore, to apply the theory of [10], we only have to interchange the roles of f_i and g_i, $i = 1,2$. It is easy to see that these functions satify the assumptions (16) and (18), because of Assmption **A1**. For instance, f_1 and f_2 are monotone nonincreasing. Thanks to Theorem 3.4, there exists a unique solution $(u,v) \in \big(C(\overline{Q}) \cap C^{1,2}(Q)\big)^2$ to (E2). □

Our next goal is to generalize this result to $c \in L^\infty(\Sigma)$ instead of $c \in C(\Sigma)$.

Theorem 3.6. *For every given $c \in C_{ad}$, there exists a unique weak solution $(u,v) \in Y^2 := \big(W(0,T) \cap C(\overline{Q})\big)^2$ of (E2).*

Proof. (i) Existence: Let $c \in C_{ad} \subset L^\infty(\Sigma)$ given and c_n a sequence with $c_n \in C(\Sigma)$, $c_n(x,t) \geq c_a(x,t)$ for all $(x,t) \in \Sigma$ and $c_n \to c$ in $L^s(\Sigma)$, $s > N+1$. Such a sequence exists because of the density of $C(\Sigma)$ in $L^s(\Sigma)$. This sequence is uniformly bounded. By Theorem 3.5, for every $c_n \in C_{ad}$ there exists a unique solution $(u_n, v_n) \in C^1(0,T; C^2(\overline{\Omega}))^2$ of $(E2)$. For every c_n we choose the same upper and lower solution like in Theorem 3.5 as initial values for the monotone sequences. From [10], page 465, it follows

$$(\hat{u}, \hat{v}) \leq (u_n, v_n) \leq (\tilde{u}, \tilde{v}) \qquad \forall n = 1,2,... \text{ in } Q.$$

We can take the same upper and lower solutions $\tilde{u} = \tilde{v} = \delta$, $\hat{u} = \hat{v} = 0$, since all c_n are uniformly bounded. So an $M > 0$ exists with

$$\|u_n\|_{C(\overline{Q})} + \|v_n\|_{C(\overline{Q})} \leq M$$

for all states (u_n, v_n), belonging to the control functions c_n. We define

$$h_n = c_n - b(\cdot, u_n) \qquad \text{and} \qquad g_n = g(u_n, v_n) = -\gamma_1 u_n v_n.$$

Because of $\|u_n\|_{C(\overline{Q})} + \|v_n\|_{C(\overline{Q})} \leq M$ it follows that h_n and g_n are bounded in $L^\infty(Q)$, hence also in $L^s(\Sigma)$ and $L^s(Q)$, respectively, with $s > N+1$. Summarized we obtain

$$\begin{aligned} (u_n)_t - k_1 \Delta u_n + \alpha_1 u_n &= g_n \\ \partial_\nu u_n &= h_n \\ u_n(x,0) &= u_0(x). \end{aligned} \tag{23}$$

This linear boundary value problem (23) possess for all $(g_n, h_n) \in L^r(Q) \times L^s(\Sigma)$ a unique solution u_n in Y. For a proof, we refer the reader to [13] on page 203, Theorem 5.6. The Nemyzki operator $h(u)$ is on Σ locally Lipschitz-continuous. From Casas [3] or Raymond and Zidani [11], we infer that the control-to-state mapping $S_1 : L^s(Q) \cap L^s(\Sigma) \to Y$, $(g_n, h_n) \mapsto u_n$ is continuous. Every linear continuous mapping is also weakly continuous, so the weak convergence tranfers from h_n and g_n to u_n, i.e. $u_n \rightharpoonup u$ in Y. We know that, for a homogeneous initial value $u_0 = 0$, the mapping S_1 is continuous from $L^r(Q) \times L^s(\Sigma)$ to the space of Hölder-continuous functions $C^\kappa(\overline{Q})$, $\kappa \in (0,1)$ (see [2] Theorem 4). Let $\overline{u} \in C(\overline{Q})$ denote the fixed part of the solution u_n with inhomogeneous initial value u_0, homogeneous right-hand side and homogeneous boundary condition. The sequence $(u_n - \overline{u})$ converges weakly in $C^\kappa(\overline{Q})$ and strongly in $C(\overline{Q})$, because $C^\kappa(\overline{Q})$ is compactly imbedded in $C(\overline{Q})$. Because of $\overline{u} \in C(\overline{Q})$, it follows

$$u_n \to u \text{ for } n \to \infty \quad \text{with} \quad u \in C(\overline{Q}).$$

Analogously, we find that $v_n \to v$ in $C(\overline{Q})$. This implies also $b(x,t,u_n) \to b(x,t,u)$ and $u_n v_n \to uv$ uniformly in $\overline{Q}$. Passing to the limit, we confirm that (u,v) satisfies the system $(E2)$.

(ii) Uniqueness: Let $(u,v), (\hat{u}, \hat{v}) \in Y^2$ be two pairs of weak solutions to $(E2)$. Then, $\tilde{u} := u - \hat{u} \in Y$ and $\tilde{v} := v - \hat{v} \in Y$ satisfy

$$\tilde{u}(0) = 0, \ \tilde{v}(0) = 0, \tag{24}$$

$$\begin{aligned} &(\tilde{u}_t(t), \varphi)_{H^1(\Omega)^*, H^1(\Omega)} + \int_\Gamma (b(x,t,u) - b(x,t,\hat{u}))\varphi \, dx + \int_\Omega k_1 \nabla \tilde{u}(t) \cdot \nabla \varphi \, dx \\ &+ \int_\Omega \alpha_1 \tilde{u}(t) \varphi \, dx + \int_\Omega \gamma_1 (\tilde{u}(t) v(t) + \hat{u}(t) \tilde{v}(t)) \varphi \, dx = 0, \end{aligned} \tag{25}$$

$$\begin{aligned} &(\tilde{v}_t(t), \varphi)_{H^1(\Omega)^*, H^1(\Omega)} + \alpha \int_\Sigma \tilde{v} \varphi \, dx + \int_\Omega k_2 \nabla \tilde{v}(t) \cdot \nabla \varphi \, dx \\ &+ \int_\Omega \alpha_2 \tilde{v}(x,t) \varphi dx + \int_\Omega \gamma_2 (\tilde{u}(t) v(t) + \hat{u}(t) \tilde{v}(t)) \varphi \, dx = 0 \end{aligned} \tag{26}$$

for all $\varphi \in H^1(\Omega)$ and almost all $t \in [0,T]$. Choosing $\varphi = \tilde{u}$ in (25), $\varphi = \tilde{v}$ in (26) and adding both equations we obtain

$$\int_\Gamma (b(x,t,u) - b(x,t,\hat{u}))(u(x,t) - \hat{u}(x,t))\, dx \geq 0$$

$$\int_\Omega \alpha_1 \tilde{u}^2(x,t) + \alpha_2 \tilde{v}^2(x,t)\, dx \geq 0$$

for almost all $t \in [0,T]$. This yields the following inequality

$$\begin{array}{ll} \frac{1}{2}\frac{d}{dt} & (\|\tilde{u}(t)\|^2_{L^2(\Omega)} + \|\tilde{v}(t)\|^2_{L^2(\Omega)}) + k_1 \|\tilde{u}(t)\|^2_{H^1(\Omega)} + k_2 \|\tilde{v}(t)\|^2_{H^1(\Omega)} \\ \leq & k_1 \|\tilde{u}(t)\|^2_{L^2(\Omega)} + k_2 \|\tilde{v}(t)\|^2_{L^2(\Omega)} + \\ & \int_\Omega \gamma_1(|\tilde{u}^2(t)v(t)| + |\hat{u}(t)\tilde{v}(t)\tilde{u}(t)|\, dx + \\ & \int_\Omega \gamma_2(|\tilde{u}(t)v(t)\tilde{v}(t)| + |\hat{u}(t)\tilde{v}^2(t)|\, dx \end{array}$$

which is equivalent to [6], (A.2). Now, we continue as in the proof [6], A.1.1, to show $\tilde{u} = 0$ and $\tilde{v} = 0$. □

Remark 3.7. *A study of the proof reveals that $u \geq 0$ and $v \geq 0$, because we have choosen $(0,0)$ as lower solution.*

Theorem 3.8. *Problem $(P2)$ admits at least one optimal control $\bar{c}$.*

Let us briefly sketch the proof. The technique is along the lines of [13], Theorem 5.8. We obtain the uniform boundedness of the states through the proof of Theorem 3.6, because u_n and v_n converge in $W(0,T) \cap C(\overline{Q})$ to u and v and are uniform bounded. Hence the cost functional is bounded from below and we find a weakly convergent minimizing sequence for the control function. One shows in a standard way that this limit is optimal and the associated pair of states fulfills the system $(E2)$.

3.2 Necessary and sufficient optimality conditions

Let us define the control-to-state operator

$$S : L^\infty(\Sigma) \to Y,\ c \mapsto (u,v)$$

with $S_1(c) := u$ and $S_2(c) := v$. In view of [13], Lemma 4.10, we obtain the next theorem.

Theorem 3.9. *The cost functional J is continuously Fréchet-differentiable from $Y^2 \times L^\infty(\Sigma)$ to $\mathbb{R}$.*

Let us show instead:

Theorem 3.10. *The control-to-state operator S is twice continuously Fréchet-differentiable from $L^\infty(\Sigma)$ to Y^2.*

Proof. First, we derive an operator equation for $(u, v) = S(c)$. To this aim, shifting the nonlinearities to the right-hand sides, we transform the statesystem of (E2) to

$$\begin{array}{rcll} u_t - k_1\Delta u + \alpha_1 u &=& -\gamma_1 uv & \text{in } Q, \\ v_t - k_2\Delta v + \alpha_2 v &=& -\gamma_2 uv & \text{in } Q, \\ k_1\partial_\nu u &=& c - b(x,t,u) & \text{in } \Sigma, \\ k_2\partial_\nu v + \alpha v &=& 0 & \text{in } \Sigma, \\ u(x,0) &=& u_0(x) & \text{in } \Omega, \\ v(x,0) &=& v_0(x) & \text{in } \Omega. \end{array} \tag{27}$$

For the left linear part we establish linear and continuous solution operators $S_Q, G_Q : L^\infty(Q) \to Y$, $S_\Sigma : L^\infty(\Sigma) \to Y$ and $S_0, G_0 : C(\overline{\Omega}) \to Y$. $S_Q : d \mapsto u$ is associated with the linear problem

$$\begin{array}{rcll} u_t - k_1\Delta u + \alpha_1 u &=& d & \text{in } Q, \\ k_1\partial_\nu u &=& 0 & \text{in } \Sigma, \\ u(x,0) &=& 0 & \text{in } \Omega, \end{array}$$

$S_\Sigma : c \mapsto u$ with

$$\begin{array}{rcll} u_t - k_1\Delta u + \alpha_1 u &=& 0 & \text{in } Q, \\ k_1\partial_\nu u &=& c & \text{in } \Sigma, \\ u(x,0) &=& 0 & \text{in } \Omega \end{array}$$

and $S_0 : e \mapsto u$ with

$$\begin{array}{rcll} u_t - k_1\Delta u + \alpha_1 u &=& 0 & \text{in } Q, \\ k_1\partial_\nu u &=& 0 & \text{in } \Sigma, \\ u(x,0) &=& e(x) & \text{in } \Omega. \end{array}$$

Analogously, $G_Q : d \mapsto v$ belongs to the linear equation

$$\begin{array}{rcll} v_t - k_2\Delta v + \alpha_2 v &=& d & \text{in } Q, \\ k_2\partial_\nu v + \alpha v &=& 0 & \text{in } \Sigma, \\ v(x,0) &=& 0 & \text{in } \Omega \end{array}$$

and $G_0 : e \mapsto v$ to

$$\begin{array}{rcll} v_t - k_2\Delta v + \alpha_2 v &=& 0 & \text{in } Q, \\ k_2\partial_\nu v + \alpha v &=& 0 & \text{in } \Sigma, \\ v(x,0) &=& e(x) & \text{in } \Omega. \end{array}$$

We consider these operators with image in $C(\overline{Q})$ and reformulate the nonlinear equation (27) as

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -S_Q\gamma_1 uv + S_\Sigma(c - b(\cdot,\cdot,u)) + S_0 u_0 \\ -G_Q\gamma_2 uv + G_0 v_0 \end{pmatrix}, \tag{28}$$

which is equivalent to

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} u + S_Q\gamma_1 uv - S_\Sigma(c - b(\cdot,\cdot,u)) - S_0 u_0 \\ v + G_Q\gamma_2 uv - G_0 v_0 \end{pmatrix} =: F(u,v,c).$$

Because S_Q, G_Q, S_Σ, S_0 and G_0 are linear and continuous and $-\gamma_1 uv, -\gamma_2 uv$, $b(\cdot,\cdot,u)$ are twice continuously Fréchet-differentiable from $C(\overline{Q})$ to $L^\infty(Q)$ respectively $L^\infty(\Sigma)$, F is a twice continuously Fréchet-differentiable mapping from $C(\overline{Q}) \times C(\overline{Q}) \times L^\infty(\Sigma)$ to $C(\overline{Q}) \times C(\overline{Q})$, since F is the composition of linear and twice continuosly Fréchet-differentiable mappings.
To use the implicit function theorem, we have to show the invertibility of the partial Fréchet-derivative $F_{(u,v)}(u,v,c)$. This applies, because

$$F_{(u,v)}(u,v,c)w = z$$

is equivalent to

$$\begin{pmatrix} w_1 + S_Q\gamma_1(vw_1 + uw_2) + S_\Sigma b_u(x,t,u)w_1 \\ w_2 + G_Q\gamma_2(vw_1 + uw_2) \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

and, after the substitution of $r = z - w$, equivalent to

$$\begin{array}{rcll} (r_1)_t - k_1\Delta r_1 + \alpha_1 r_1 + \gamma_1(vr_1 + ur_2) &=& \gamma_1(vz_1 + uz_2) & \text{in } Q, \\ (r_2)_t - k_2\Delta r_2 + \alpha_2 r_2 + \gamma_2(vr_1 + ur_2) &=& \gamma_2(vz_1 + uz_2) & \text{in } Q, \\ k_1\partial_\nu r_1 + b_u(x,t,u)r_1 &=& c + b_u(x,t,u)z_1 & \text{in } \Sigma, \\ k_2\partial_\nu r_2 + \alpha r_2 &=& 0 & \text{in } \Sigma, \\ u(x,0) &=& 0 & \text{in } \Omega, \\ v(x,0) &=& 0 & \text{in } \Omega. \end{array}$$

For every $(z_1, z_2) \in C(\overline{Q})^2$, this boundary problem has a unique solution $(r_1, r_2) \in Y^2$. So we can invoke the implicit function theorem and obtain that the control-to-state operator S is twice continuously Fréchet-differentiable. □

Theorem 3.11. *The derivative of the control-to-state operator S at $\overline{c}$ in direction c is given by*

$$S'(\overline{c})c = (u,v),$$

where (u,v) is the weak solution of the in $(\overline{u},\overline{v})$ linearized problem (E2)

$$\begin{array}{rll} u_t - k_1\Delta u + \alpha_1 u &= -\gamma_1(\overline{u}v + u\overline{v}) & \textit{in } Q, \\ v_t - k_2\Delta v + \alpha_2 v &= -\gamma_2(\overline{u}v + u\overline{v}) & \textit{in } Q, \\ k_1\partial_\nu u + b_u(x,t,\overline{u})u &= c & \textit{in } \Sigma, \\ k_2\partial_\nu v + \alpha v &= 0 & \textit{in } \Sigma, \\ u(x,0) &= 0 & \textit{in } \Omega, \\ v(x,0) &= 0 & \textit{in } \Omega \end{array} \tag{29}$$

and $(\overline{u},\overline{v}) = S(\overline{c})$.

Proof. The proof is along the lines of [13], Theorem 5.17, p. 218. To derive the form of $S'(\overline{c})$, we use (28):

$$\begin{pmatrix} u \\ v \end{pmatrix} = S(c) = \begin{pmatrix} -S_Q\gamma_1 S_1(c)S_2(c) + S_\Sigma(c - b(\cdot,\cdot,S_1(c)) + S_0 u_0 \\ -G_Q\gamma_2 S_1(c)S_2(c) + G_0 v_0 \end{pmatrix}.$$

Differentiating both sides, we obtain

$$S'(\overline{c})c = \begin{pmatrix} -S_Q\gamma_1(S_1'(\overline{c})cS_2(\overline{c}) + S_1(\overline{c})S_2'(\overline{c})c) + S_\Sigma(c - b_u(\cdot,\cdot,S_1(\overline{c}))S_1'(\overline{c})c) \\ -G_Q\gamma_2(S_1'(\overline{c}c)S_2(\overline{c}) + S_1(\overline{c})S_2'(\overline{c})c) \end{pmatrix}.$$

In view of $S_1'(\overline{c})c = u$ and $S_2'(\overline{c})c = v$, we get

$$S'(\overline{c})c = \begin{pmatrix} -S_Q\gamma_1(u\,\overline{v} + \overline{u}\,v) + S_\Sigma(c - b_u(\cdot,\cdot,\overline{u})\,u) \\ -G_Q\gamma_2(u\,\overline{v} + \overline{u}\,v) \end{pmatrix}$$

which implies the statement. □

Analogously to Section 2, we obtain the following theorem, see also [13], Theorem 5.17.

Theorem 3.12. *The second derivative of S at $\overline{c}$ in direction $(\hat{c}, \tilde{c})$ is given by*

$$S''(\overline{c})(\hat{c}, \tilde{c}) = (u, v),$$

where (u, v) is the weak solution of the system

$$\begin{array}{rll} u_t - k_1\Delta u + \alpha_1 u + \gamma_1(\overline{u}v + u\overline{v}) & = -\gamma_1(\hat{u}\tilde{v} + \tilde{u}\hat{v}) & \text{in } Q, \\ v_t - k_2\Delta v + \alpha_2 v + \gamma_2(\overline{u}v + u\overline{v}) & = -\gamma_2(\hat{u}\tilde{v} + \tilde{u}\hat{v}) & \text{in } Q, \\ k_1\partial_\nu u + b_u(x,t,\overline{u})u & = -b_{uu}(x,t,\overline{u})[\hat{u} + \tilde{u}] & \text{on } \Sigma, \\ k_2\partial_\nu v + \alpha v & = 0 & \text{on } \Sigma, \\ u(x,0) & = 0 & \text{in } \Omega, \\ v(x,0) & = 0 & \text{in } \Omega \end{array}$$

where $(\overline{u}, \overline{v}) = S(\overline{c})$ and $(\hat{u}, \hat{v}) = S'(\overline{c})(\hat{c})$ is the solution of

$$\begin{array}{rcll} \hat{u}_t - k_1\hat{u}_{xx} + \alpha_1\hat{u} & = & -\gamma_1(\overline{u}\hat{v} + \hat{u}\overline{v}) & \text{in } Q, \\ \hat{v}_t - k_2\hat{v}_{xx} + \alpha_2\hat{v} & = & -\gamma_2(\overline{u}\hat{v} + \hat{u}\overline{v}) & \text{in } Q, \\ k_1\partial_\nu\hat{u} + b_u(x,t,\overline{u})\hat{u} & = & \hat{c} & \text{on } \Sigma, \\ k_2\partial_\nu\hat{v} + \alpha\hat{v} & = & 0 & \text{on } \Sigma, \\ \hat{u}(x,0) & = & 0 & \text{in } \Omega, \\ \hat{v}(x,0) & = & 0 & \text{in } \Omega. \end{array}$$

Analogously, $(\tilde{u}, \tilde{v}) = S'(\overline{c})(\tilde{c})$ is defined.

To formulate necessary optimality conditions, let $\overline{c}$ be an optimal control of $(P2)$ with states $(\overline{u}, \overline{v})$.
We have $(u, v) = S(c)$ with the control-to-state operator $S : L^\infty(\Sigma) \to Y^2$, hence we obtain the reduced functional f,

$$f(c) := J(u, v, c) = J(S(c), c).$$

The functional f is Fréchet-differentiable, because S is differentiable by Theorem 3.10 and J is differentiable by Theorem 3.9. Analogously to the last section, we obtain the following standard result:

Lemma 3.13. *Every locally optimal control function $\overline{c}$ of (P2) satisfies the variational inequality*

$$f'(\overline{c})(c - \overline{c}) \geq 0 \qquad \forall c \in C_{ad}.$$

We determine f' by the chain rule and obtain

$$\begin{aligned} f'(\overline{c})(c-\overline{c}) &= J_y(\overline{y},\overline{c})\, S'(\overline{c})(c-\overline{c}) + J_c(\overline{y},\overline{c})(c-\overline{c}) \\ &= \iint\limits_Q \Big(\alpha_u(\overline{u}-u_Q)u + \alpha_v(\overline{v}-v_Q)v \Big)\, dx\, dt \\ &+ \int\limits_\Omega \Big(\alpha_{TU}(\overline{u}(T)-u_{QT})u(T) + \alpha_{TV}(\overline{v}(T)-v_{QT})v(T) \Big)\, dx \\ &+ \int\limits_0^T \alpha_c\, \overline{c}(c-\overline{c})\, dt, \end{aligned} \tag{30}$$

where, by Theorem 3.11, $y=(u,v)=S'(\overline{c})(c-\overline{c})$ is the weak solution of the linearized problem

$$\begin{array}{rll} u_t(t,x) - k_1\Delta u + \alpha_1 u &= -\gamma_1(\overline{u}v + u\overline{v}) & \text{in } Q, \\ v_t(t,x) - k_2\Delta v + \alpha_2 v &= -\gamma_2(\overline{u}v + u\overline{v}) & \text{in } Q, \\ k_1\partial_\nu u + b_u(x,t,\overline{u})u &= c-\overline{c} & \text{in } \Sigma, \\ k_2\partial_\nu v + \alpha v &= 0 & \text{in } \Sigma, \\ u(x,0) &= 0 & \text{in } \Omega, \\ v(x,0) &= 0 & \text{in } \Omega. \end{array} \tag{31}$$

By Lemma 3.13 $f'(\overline{c})(c-\overline{c})$ is nonnegative. We can eliminate the states u and v in (30) by the adjoint states p und q, the solutions of the adjoint system

$$(A2) \left\{ \begin{array}{rcll} -p_t - k_1\Delta p + \alpha_1 p + \gamma_1 vp + \gamma_2 vq &=& \alpha_u(u-u_Q) & \text{in } Q, \\ -q_t - k_2\Delta q + \alpha_2 q + \gamma_1 up + \gamma_2 uq &=& \alpha_v(v-v_Q) & \text{in } Q, \\ k_1\partial_\nu p + b_u(x,t,u)p &=& 0 & \text{in } \Sigma, \\ k_2\partial_\nu q + \alpha q &=& 0 & \text{in } \Sigma, \\ p(x,T) = \alpha_{TU}(u(x,T) - u_\Omega(x,T)) &&& \text{in } \Omega, \\ q(x,T) = \alpha_{TV}(v(x,T) - v_\Omega(x,T)) &&& \text{in } \Omega. \end{array} \right.$$

Theorem 3.14. *If (u,v) is the weak solution of the linearized system (29) and (p,q) is the solution of the adjoint system (A2), then it holds for all $c \in L^s(\Sigma)$ that*

$$\begin{aligned} &\iint\limits_Q \Big(\alpha_u(\overline{u}-u_Q)u + \alpha_v(\overline{v}-v_Q)v \Big)\, dx\, dt \\ &\quad + \int\limits_\Omega \Big(\alpha_{TU}(\overline{u}(T)-u_{QT})u(T) + \alpha_{TV}(\overline{v}(T)-v_{QT})v(T) \Big)\, dx \\ &= \int\limits_\Sigma p(c-\overline{c})\, dx\, dt. \end{aligned}$$

The proof is analogous to the one of Theorem 2.8 at page 7. In this way, (30) leads to

$$f'(\overline{c})c = \int\limits_\Sigma (p + \alpha_c\overline{c})c\, dx\, dt. \tag{32}$$

Theorem 3.15. *Every locally optimal solution $\overline{c}$ of $(P2)$ satisfies, together with the adjoint states (p,q) of (A2), the variational inequality*

$$\int_{\Sigma} (p+\alpha_c \overline{c})(c-\overline{c})\ dt\ \geq\ 0 \qquad \forall c \in C_{ad}.$$

The following theorem furnishs an equivalent pointwise expression of the variational inequality.

Theorem 3.16. *If $\overline{c}$ is locally optimal for (P2) and (p,q) are the adjoint states, then*

$$\min_{c_a(x,t)\leq c\leq c_b(x,t)} (p(x,t)+\alpha_c\overline{c}(x,t))c = (p(x,t)+\alpha_c\overline{c}(x,t))\overline{c}(x,t)$$

will be attained almost everywhere in Σ by $c=\overline{c}(x,t)$.

The proof is well known.
For $\alpha_c > 0$, this leads to the projection formula

$$\overline{c}(x,t) = P_{[c_a(x,t),c_b(x,t)]}\{-\frac{1}{\alpha_c}p(x,t)\}$$

for almost all $(x,t)\in\Sigma$. Because of similarity to the second section, we state the sufficient second-order optimality conditions for the problem $(P2)$ without proof. We define the Lagrangian function analogously to (11) with the second derivative

$$\mathcal{L}''(u,v,c,p,q)[(\hat{u},\hat{v},\hat{c}),(\tilde{u},\tilde{v},\tilde{c})] = J''(u,v,c)[(\hat{u},\hat{v},\hat{c}),(\tilde{u},\tilde{v},\tilde{c})]$$
$$+\iint_Q \gamma_1(\hat{u}\tilde{v}+\tilde{u}\hat{v})p+\gamma_2(\hat{u}\tilde{v}+\tilde{u}\hat{v})q\ dxdt - \iint_\Sigma b_{uu}(x,t,u)[\hat{u},\tilde{u}]p\ dxdt.$$

For given $\tau>0$, we define

$$A_\tau(\overline{c}) := \{(x,t)\in\Sigma : |p+\alpha_c\overline{c}| > \tau\}$$

as the set of strong active restrictions for $\overline{c}$. The τ-citical cone $C_\tau(\overline{c})$ is made up of all $c\in L^\infty(\Sigma)$ with

$$c(x,t)\begin{cases} =0 & \text{for } (x,t)\in A_\tau(\overline{c}) \\ \geq 0 & \text{for } \overline{c}(x,t)=c_a \text{ and } (x,t)\notin A_\tau(\overline{c}) \\ \leq 0 & \text{for } \overline{c}(x,t)=c_b \text{ and } (x,t)\notin A_\tau(\overline{c}). \end{cases}$$

Theorem 3.17. *Suppose that the control function $\overline{c}$ satisfies the first-order necessary optimality conditions of Theorem 3.15. If there exist positive constants δ and τ such that*

$$\mathcal{L}''(\overline{u},\overline{v},\overline{c},p,q)(u,v,c)^2 \geq \delta\|c\|^2_{L^2(0,T)}$$

holds for all $c\in C_\tau(\overline{c})$ and all $(u,v)\in Y\times Y$ satisfying (29), then we find positive constants ε and σ such that

$$J(u,v,c)\geq J(\overline{u},\overline{v},\overline{c})+\sigma\|c-\overline{c}\|^2_{L^2(Q)}$$

holds for all $c\in C_{ad}$ with $\|c-\overline{c}\|_{L^\infty(Q)}\leq\varepsilon$. Therefore, the control functions $\overline{c}$ is locally optimal.

3.3 Numerical examples

Example 3: We investigate the problem (P2) with the following data: $l = 1, T = 2, k_1 = k_2 = \epsilon = \alpha_u = \alpha_v = 1$, $\gamma_1 = \gamma_2 = \alpha_1 = \alpha_2 = \alpha_{TU} = \alpha_{TV} = 0$, $\alpha_c = \alpha = 0.01$, $u_0(x) = \cos(\frac{\pi}{2}x)$, $v_0 \equiv 0$, $u_Q(x,t) = \cos(\frac{\pi}{2}x)$, $v_Q \equiv 0$. For the constraints of the control function c we choose $c_a \equiv -10$ and $c_b \equiv 10$ and as the initial value $c \equiv 1$. By using the *gradient-projection-method*, we obtain the control functions and states, presented in Figures 9-12. We use an equidistant mesh with 15 supporting points in the x-direction and 150 node points in the t-direction.

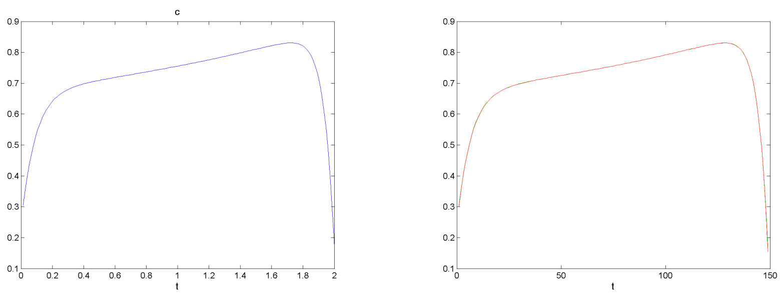

Figure 9: Optimal control $\overline{c}(0,t)$ and projection $P_{[c_a,c_b]}\{-\frac{1}{\alpha_c}p(0,t)\}$ for Example 3

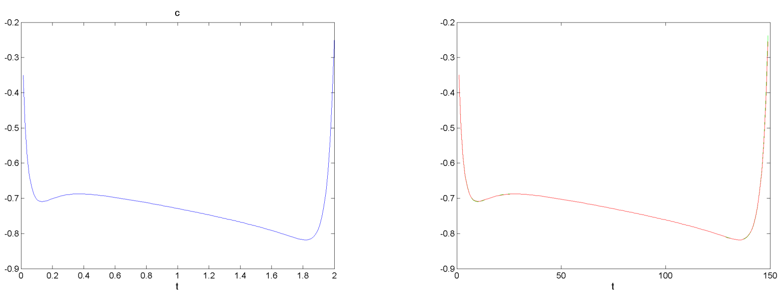

Figure 10: Optimal control $\overline{c}(l,t)$ and projection $P_{[c_a,c_b]}\{-\frac{1}{\alpha_c}p(l,t)\}$ for Example 3

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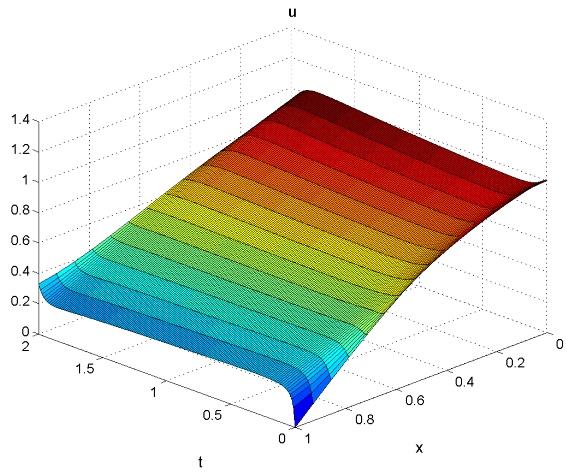

Figure 11: State u for Example 3

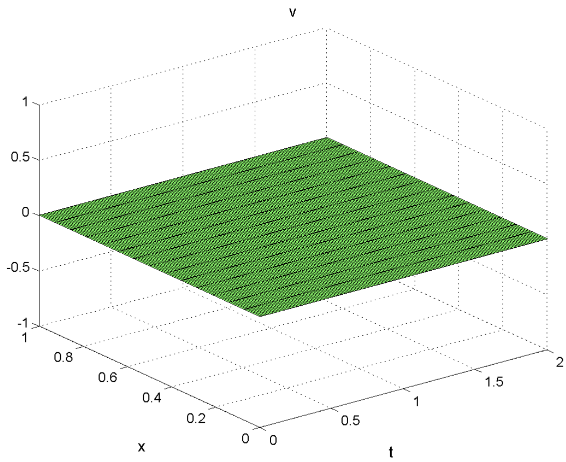

Figure 12: State v for Example 3

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