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Differential-algebraic Riccati Decoupling for Linear-quadratic Optimal Control Problems for Semi-explicit Index-2 DAEs

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Abstract

We investigate existence and structure of solutions to quadratic control problems with semi-explicit differential algebraic constraints. By means of an equivalent index-1 formulation we identify conditions for the unique existence of optimal solutions. Knowing of the existence of an optimal input we provide a representation of the associated feedback-law via a Riccati-like decoupling that is formulated for the original index-2 equations.

Keywords Optimal Control, DAEs, Differential Riccati Equation, Euler-Lagrange equations
AMS subject classification 34H05, 49J15

1 Introduction

A well-posed formulation of a control system for differential-algebraic equations (DAEs) has to relate the algebraic constraints to the controls, since it may happen that some components of the input are not free. Consider the example system

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot x_1\\ \dot x_2\end{bmatrix} - \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} - \begin{bmatrix}u_1\\ u_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}, \quad x_1(0) = 0. \tag{1}$$

An arbitrarily defined input can make the model inconsistent, as in (1) the existence of solutions is only possible if u_2 is either zero or a function of the state x_1 or x_2. On the other hand the definition of the input can change the characteristics of the model. With the assignment $u_2 := \dot x_2$, System (1) can be interpreted as an ODE for x_1 and x_2. Assigning $u_2 := x_1$ the system can be reformulated to give only algebraic equations for x_1 and x_2. A general approach to this issue bases on the behavior formulation, cf. [25], which considers the control problem as an underdetermined system in the augmented variable $z := [x, u]$. For the behavior system one can identify free components of z that are then defined as the new controls, cf. [17]. This approach is natural, since if the chosen controls are not free variables in the behavior formulation, then the problem is ill-posed.

If it comes to applications, however, one cannot freely redefine the controls and variables since they are prescribed by the physical setup. In this case one can only hope that the problem is well-posed and if necessary consider a remodeling that ensures well-posedness in terms of the original inputs. We will investigate optimal control problems, where the control acts only in the differential part of the differential-algebraic equations such that the inputs will always be free variables.

In optimal control one tackles the problem of determining an input u such that

$$\mathcal{J}(x, u) \to \min \text{ subject to } F(t, \dot x, x, u) = 0,$$

where x is the state of the system, $\mathcal{J}$ is a cost functional and F stands for the constraints given by the state equations. For the solution of optimal control problems there are basically two approaches, the value function and the variational approach, cf. [27].

In order to extend these approaches that are well-understood for ODE constrained optimization problems to DAE constraints, one has to cope with the *strangeness* of the DAEs. The strangeness describes to which extent differential and algebraic equations are tied together and is quantified by means of various index concepts [28]. Generally spoken, in strangeness-free or index-1 DAEs, algebraic and differential equations are well separated and higher indices mean higher interlocking.

If one uses a variational or Lagrange-multiplier approach one ends up with variants of the so-called Euler-Lagrange equations. The structure of these equations suggests that the solution to optimal control problems with DAE constraints is a function of the state, i.e. a feedback control. Unlike the ODE case, for DAE constraints the existence and uniqueness of solutions to the involved adjoint equations and thus to the optimality system is in general not guaranteed, cf. [4, 6, 10, 22] and in particular [15] for the linear quadratic case. To provide necessary and sufficient conditions for the existence of optimal controls one can for example exploit the special structure of semi-explicit equations, cf. [10–12], or consider linear DAEs with properly stated leading term, cf. [3–5, 7, 22, 24]. The special case of Riccati-feedback solutions was investigated in [23]. The more general way to regularize the DAEs and formulate the conditions for the resulting strangeness-free system is taken in [18]. In [16, 17] conditions and procedures for the construction of state feedbacks are presented such that the behavior system is strangeness-free.

The formulations of the optimality conditions in the references listed above involve index reduction procedures, except from the contributions in [3, 4]. However, from the numerical point of view, a formulation in the unreduced equations is preferable in two respects. First, solving the original equations, one can control the modelled constraints, while solving an index reduced system may lead to a drift off the constraint manifold unseen by the solver. Second, most index reduction techniques use projectors or implicit functions which may be expensive to compute. Thus for an efficient implementation an index reduction, if necessary, should be tailored to the solution of the specific problem rather than to the derivation of theoretical results.

In our approach we make use of the structure of an optimality system that is stated in the original variables without any explicit or implicit index reduction. The obtained results regarding optimality of the solutions to the Euler-Lagrange equations are already covered by [3, 4]. The innovations we propose base on the specific structure of the semi-explicit index-2 formulation, as it arises in linearized Navier-Stokes equations. We use the structure to prove the existence of an optimal solution directly. Thereto we introduce a novel differential-algebraic matrix Riccati equation that seems suitable for numerical computations, as it is stated in the original system matrices. Finally, we show how the necessity gap between the considered formal and the *true* [19] optimality conditions can be closed in applications.

2 Semi-explicit Semi-linear DAEs of Index 2

In this section, we introduce a decoupling of the index-2 DAEs that identifies the differential and algebraic parts to read off necessary conditions for consistency and regularity of the data.

consider a semi-explicit semi-linear DAE of the form:

$$M(t)\dot{v} - A(t,v)v - J_1(t)^T p - B_1(t)u = f_v, \quad v(0) = v^0 \in \mathbb{R}^{n_v}, \tag{2a}$$

$$-J_2(t)v - B_2(t)u = f_p. \tag{2b}$$

We assume $M(t)$ invertible, so that (2) can be reformulated as a semi-explicit system.

For systems of the form (2) the differentiation index, cf. [9], is defined as follows:

Definition 2.1. A semi-explicit DAE as given by (2) is of differentiation index k if it takes $k-1$ differentiations in t of the algebraic constraints (2b) to determine the algebraic variable p in terms of the differential variable v.

In order to guarantee existence of solutions $(v,p) \in \mathcal{C}^1(\mathbb{I},\mathbb{R}^{n_v}) \times \mathcal{C}(\mathbb{I},\mathbb{R}^{n_p})$ of (2), we make the following assumption:

Assumption 2.2. For the DAE (2) with coefficients $M, A(\cdot,v) \in \mathcal{C}(\mathbb{I},\mathbb{R}^{n_v,n_v})$, $J_1, J_2 \in \mathcal{C}(\mathbb{I},\mathbb{R}^{n_v,n_p})$, $B_1 \in \mathcal{C}(\mathbb{I},\mathbb{R}^{n_u,n_v})$, $B_2 \in \mathcal{C}(\mathbb{I},\mathbb{R}^{n_u,n_u})$, right-hand sides $f_v \in \mathcal{C}(\mathbb{I},\mathbb{R}^{n_v})$, $f_p \in \mathcal{C}^1(\mathbb{I},\mathbb{R}^{n_p})$, an initial condition $v^0 \in \mathbb{R}^{n_v}$ and inputs $u \in \mathcal{C}(\mathbb{I},\mathbb{R}^{n_u})$ we assume

(A1) differentiation index 2, i.e. $S := J_2 M^{-1} J_1^T$ is invertible,

(A2) sufficient regularity of the data and the input, i.e. f_p, $B_2 u$, $M^{-1} J_1^T S^{-1}$ and J_2 are differentiable and

(A3) consistency of the data and the input, i.e. $J_2(0)v(0) = f_p(0) - B_2(0)u(0)$.

We will not investigate existence of solutions here. The following Theorem 2.3 gives a solution representation by means of the inherent ODE that will be used to ensure existence and uniqueness in the linear case of System (2).

Theorem 2.3. *Each solution (v, p) of (2) can be represented as $(v_\mathcal{P} + \mathcal{Q}v, p)$, where*

$$\mathcal{Q}v = -M^{-1} J_1^T S^{-1}[B_2 u + f_p], \tag{3a}$$

$$p = -\mathcal{Q}^-[M^{-1}[A(\mathcal{Q}v + v_\mathcal{P})[\mathcal{Q}v + v_\mathcal{P}] + B_1 u + f_v] + \dot{\mathcal{Q}v}], \tag{3b}$$

and $v_\mathcal{P} := \mathcal{P}v$ solves the ODE

$$\begin{aligned} \dot{v_\mathcal{P}} - \big[\tfrac{d}{dt}\mathcal{P} + \mathcal{P}M^{-1}A(\mathcal{Q}v + v_\mathcal{P})\big][\mathcal{Q}v + v_\mathcal{P}] - \mathcal{P}M^{-1}[B_1 u + f_v] = 0, \\ v_\mathcal{P}(0) = \mathcal{P}v^0. \end{aligned} \tag{3c}$$

with $\mathcal{P} := I - \mathcal{Q}$, $\mathcal{Q} := M^{-1} J_1^T S^{-1} J_2$ and $\mathcal{Q}^- := S^{-1} J_2$.

Proof. Rewriting and abbreviating (2) by

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} - \begin{bmatrix} M^{-1}A(v) & M^{-1}J_1^T \\ J_2 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} M^{-1}(B_1 u + f_v) \\ B_2 u + f_p \end{bmatrix} \tag{4}$$

and $\mathcal{E}\dot{x} - \mathcal{A}(x)x = q$, respectively, with $x := (v, p)$, we compute the operator chain as described e.g. in [7] but with a slightly different notation. It will turn out that the nonlinear part in $\mathcal{A}$ does not interfere with the definitions of the projections and subspaces such that the linear theory of [7] is applicable here. The operator chain is given by

$$\mathcal{E}_0 := \mathcal{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{A}_0 := \mathcal{A} = \begin{bmatrix} M^{-1}A(v) & M^{-1}J_1^T \\ J_2 & 0 \end{bmatrix}, \tag{5a}$$

$$\mathscr{Q}_0 = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \text{ (projector onto } \ker \mathcal{E}_0), \quad \mathscr{P}_0 = I - \mathscr{Q}_0 = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \tag{5b}$$

$$\mathcal{E}_1 = \mathcal{E}_0 + \mathcal{A}_0\mathscr{Q}_0 = \begin{bmatrix} I & M^{-1}J_1^T \\ 0 & 0 \end{bmatrix}, \quad \mathcal{A}_1 = \mathcal{A}_0\mathscr{P}_0 = \begin{bmatrix} M^{-1}A(v) & 0 \\ J_2 & 0 \end{bmatrix}, \tag{5c}$$

$$\mathscr{Q}_1 = \begin{bmatrix} M^{-1}J_1^T(J_2M^{-1}J_1^T)^{-1}J_2 & 0 \\ -(J_2M^{-1}J_1^T)^{-1}J_2 & 0 \end{bmatrix} =: \begin{bmatrix} \mathcal{Q} & 0 \\ -\mathcal{Q}^- & 0 \end{bmatrix}, \quad \mathscr{P}_1 = \begin{bmatrix} \mathcal{P} & 0 \\ \mathcal{Q}^- & I \end{bmatrix}. \tag{5d}$$

$$\mathcal{E}_2 = \begin{bmatrix} I + M^{-1}A\mathcal{Q} & M^{-1}J_1^T \\ J_2 & 0 \end{bmatrix}. \tag{5e}$$

With the projectors $\mathcal{Q} = M^{-1}J_1^T S^{-1} J_2$ which satisfies

$$\mathcal{Q}^2 = \mathcal{Q}, \quad J_2\mathcal{Q} = J_2, \quad \mathcal{Q}M^{-1}J_1^T = M^{-1}J_1^T \quad \text{and} \quad \mathcal{Q}^-\mathcal{Q} = \mathcal{Q}^-,$$

and $\mathcal{P} = I - \mathcal{Q}$ one can verify that

$$\mathcal{E}_2^{-1} = \begin{bmatrix} \mathcal{P} & [I - \mathcal{P}M^{-1}A]M^{-1}J_1^T S^{-1} \\ \mathcal{Q}^- & -[I + \mathcal{Q}^-M^{-1}AM^{-1}J_1^T]S^{-1} \end{bmatrix} \tag{6}$$

for any $A = A(v)$. Scaling the state equations (4) by $\mathcal{E}_2^{-1}$ we get

$$\begin{bmatrix} \mathcal{P} & 0 \\ \mathcal{Q}^- & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} - \left[\begin{bmatrix} \mathcal{P}M^{-1}A\mathcal{P} & 0 \\ \mathcal{Q}^- M^{-1}A\mathcal{P} & 0 \end{bmatrix} + \begin{bmatrix} \mathcal{Q} & 0 \\ -\mathcal{Q}^- & I \end{bmatrix} \right] \begin{bmatrix} v \\ p \end{bmatrix} = \mathcal{E}_2^{-1} \begin{bmatrix} M^{-1}[B_1u + f_v] \\ B_2u + f_p \end{bmatrix}. \tag{7}$$

Having applied the projectors $\mathscr{Q}_1$, $\mathscr{Q}_0\mathscr{P}_1$ and $\mathscr{P}_0\mathscr{P}_1$, cf. (5), to (7) we obtain the three subsystems

$$-\begin{bmatrix} \mathcal{Q} & 0 \\ -\mathcal{Q}^- & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} = \mathscr{Q}_1\mathcal{E}_2^{-1}q = \begin{bmatrix} M^{-1}J_1^T S^{-1}[B_2u + f_p] \\ -S^{-1}[B_2u + f_p] \end{bmatrix}, \tag{8a}$$

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ \mathcal{Q}^- & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \mathcal{Q}^- M^{-1}A\mathcal{P} & I \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} &= \mathscr{Q}_0\mathscr{P}_1\mathcal{E}_2^{-1}q \\ &= \begin{bmatrix} 0 \\ \mathcal{Q}^- M^{-1}[B_1u + f_v - AM^{-1}J_1^T S^{-1}[B_2u + f_p]] \end{bmatrix} \end{aligned} \tag{8b}$$

and

$$\begin{aligned} \begin{bmatrix} \mathcal{P} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} - \begin{bmatrix} \mathcal{P}M^{-1}A\mathcal{P} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} &= \mathscr{P}_0\mathscr{P}_1\mathcal{E}_2^{-1}q \\ &= \begin{bmatrix} \mathcal{P}M^{-1}[B_1u + f_v - AM^{-1}J_1^T S^{-1}[B_2u + f_p]] \\ 0 \end{bmatrix}, \end{aligned} \tag{8c}$$

respectively, where q denotes the rightmost vector in (4). Since $\mathscr{Q}_1 + \mathscr{P}_0\mathscr{P}_1 + \mathscr{Q}_0\mathscr{P}_1 = I$, Equations (8) contain all information of (7) and vice versa. We decompose $v = v_\mathcal{P} + \mathcal{Q}v$, where $v_\mathcal{P} := \mathcal{P}v$ so that from (8a) we can deduce that

$$\mathcal{Q}v = -M^{-1}J_1^T S^{-1}[B_2u + f_p] \tag{9}$$

and that $\mathcal{Q}v$ is differentiable by assumption. With $\dot{v} = \dot{\mathcal{Q}v} + \dot{v_\mathcal{P}}$ and $\mathcal{Q}^-\dot{v_\mathcal{P}} = 0$, Equation (8b) gives

$$p = -\mathcal{Q}^-[M^{-1}A(\mathcal{Q}v + v_\mathcal{P})[\mathcal{Q}v + v_\mathcal{P}] + M^{-1}B_1u + M^{-1}f_v] + \mathcal{Q}^-\dot{\mathcal{Q}v}, \tag{10}$$

while (8c) defines the inherent ODE for $v_\mathcal{P} := \mathcal{P}v$ via

$$\dot{v_\mathcal{P}} - \left[\tfrac{d}{dt}\mathcal{P} + \mathcal{P}M^{-1}A(\mathcal{Q}v + v_\mathcal{P})\right][\mathcal{Q}v + v_\mathcal{P}] = \mathcal{P}[M^{-1}B_1u + M^{-1}f_v], \quad v_\mathcal{P}(0) = \mathcal{P}v^0. \tag{11}$$

Note the necessity of the consistency condition (A3) in Assumption 2.2, since by (9) the condition

$$J_2v(0) = J_2[\mathcal{Q}v(0) + \mathcal{P}v(0)] = J_2\mathcal{Q}v(0) = -B_2u(0) - f_p(0),$$

must hold and note, that an initial condition for p would have to fulfill (10) at $t = 0$. □

Remark 2.4. In the setting of the Navier-Stokes equations, the projector $\mathcal{Q}$ realizes the discrete Helmholtz-decomposition that splits a vector field into a divergence free part and a part that can be expressed as the gradient of a scalar potential, cf. [13, Cor. 3.4]. If J_2 is the discrete divergence operator, then the decomposition $v = \mathcal{Q}v + \mathcal{P}v =: \mathcal{Q}v + v_\mathcal{P}$ delivers that $J_2v_\mathcal{P} = 0$ and $\mathcal{Q}v$ is in the range of $M^{-1}J_1^T$, which is the discrete gradient operator in many discretization schemes. The matrix $\mathcal{Q}^-$ is a generalized left inverse of $M^{-1}J_1^T$ and can be seen as the operator that maps the potential field $\mathcal{Q}v = M^{-1}J_1^T\rho$ onto its potential ρ. Accordingly, (3b) is the discrete Pressure-Poisson equation, cf. [14].

Corollary 2.5. *If $B_2 = 0$, then the solutions of* (2) *do not depend on the time derivative of the input. The condition $B_2 = 0$ is also necessary for the existence of solutions for all continuous inputs.*

The first fact of Corollary 2.5 follows from the representation of the solution as given in Theorem 2.3. For the converse direction, one concludes that the solution component $\frac{d}{dt}(B_2 u)$ can only exist for all continuous u if $B_2 = 0$.

For the results of the next sections we will always require $B_2 = 0$ which by Corollary 2.5 is necessary and sufficient for the admissibility of inputs that are only continuous. This is what in [3, 4] and [26] is also assumed and referred to as *causality*.

3 Linear Quadratic Optimal Control

In this section, we formulate an optimality system and determine necessary and sufficient conditions for optimal solutions without resorting to index-1 formulations.

We investigate a linearized version of (2), i.e. $A(t, v) = A(t)$, and a quadratic cost functional of type

$$\mathcal{J}(v,p,u) = \frac{1}{2}\begin{bmatrix} v \\ p \end{bmatrix}^T \begin{bmatrix} V_1 & V_{12} \\ V_{21} & V_2 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix}\Bigg|_{t=T} + \frac{1}{2}\int_0^T \begin{bmatrix} v \\ p \\ u \end{bmatrix}^T \begin{bmatrix} W_1 & W_{12} & S_{vu} \\ W_{21} & W_2 & S_{pu} \\ S_{uv} & S_{up} & R \end{bmatrix} \begin{bmatrix} v \\ p \\ u \end{bmatrix} \,\mathrm{d}t, \tag{12}$$

with R invertible and symmetric positive semi-definite weighting matrices $\begin{bmatrix} V_1 & V_{12} \\ V_{21} & V_2 \end{bmatrix}$ and $\begin{bmatrix} W_1 & W_{12} & S_{vu} \\ W_{21} & W_2 & S_{pu} \\ S_{uv} & S_{up} & R \end{bmatrix}$, that is appropriate for driving the system into the zero state. In this setting the formal Euler-Lagrange equations, cf. [18], are given by

$$M\dot{v} - Av - J_1^T p - B_1 u = f_p, \quad v(0) = v^0 \tag{13a}$$

$$-J_2 v - B_2 u = f_p \tag{13b}$$

$$-\tfrac{d}{dt}(M^T\lambda_1) - A^T\lambda_1 - J_2^T\lambda_2 + W_1 v + W_{12} p + S_{vu} u = 0,$$
$$M^T\lambda_1(T) = -V_1 v\Big|_{t=T} - V_{12} p\Big|_{t=T}, \tag{13c}$$

$$-J_1\lambda_1 + W_{21} v + W_2 p + S_{pu} u = 0, \quad 0 = V_{21} v\Big|_{t=T} + V_2 p\Big|_{t=T}, \tag{13d}$$

$$-B_1^T\lambda_1 - B_2^T\lambda_2 + S_{uv} v + S_{up} p + Ru = 0. \tag{13e}$$

If system (13) possesses a solution, then it provides necessary and sufficient conditions for an optimal input u, cf. [3, 24]. Thus, we will establish conditions for existence of solutions of (13). Note, that the optimal control problem can be solvable also if (13) is not well posed, cf. [19]

Since we consider state solutions $(v, p) \in \mathcal{C}^1 \times \mathcal{C}$ and inputs $u \in \mathcal{C}$ candidate solutions of (13) must not contain $\dot{u}$ or $\dot{p}$. Thus, by Corollary 2.5 it is necessary that

$$B_2 = 0, \quad W_2 = 0 \quad \text{and} \quad S_{pu} = S_{up}^T = 0. \tag{14}$$

The other possibility that solutions of DAEs fail to exist, is the inconsistency of the initial data. The true optimality conditions, cf. [19], necessitate that span $\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \subset$ span $\begin{bmatrix} M^T & 0 \\ 0 & 0 \end{bmatrix}$, i.e. V_{22} and $V_{12} = V_{21}^T$ must be zero. By combining (13d) and the terminal condition for λ_1, we find

$$-J_1\lambda_1(T) = -W_{21}v(T) = J_1 M^{-T} V_1 v(T).$$

We will ensure this condition by requiring

$$J_1 M^{-T} V_1 = 0, \quad \text{and} \quad W_{21} = W_{12}^T = 0. \tag{15}$$

The latter condition means that V_1 acts only on the dynamical part of v as it is given by (3c). Note that these conditions are equivalent to the assumptions that were made in [3].

Remark 3.1. In theory, setting W_2, $W_{12} = W_{21}^T$ to zero, does not cause a loss of generality, as p is an affine linear function of v and u, cf. Theorem 2.3. Thus, in the cost functional, all terms in p can be replaced by terms in v and u. Furthermore, the cross terms of v and u can be formally eliminated by an input shift, cf. Section 5. However, for applications, the exclusion of p from the cost functional is a restriction.

The next Lemma will show that the assumptions for smooth solutions (14), that were derived for the single equations (13a-b) and (13c-d), are necessary also for the coupled system. In particular, we will confirm that if $B_2 = 0$, then there is no hope for a lower index of the optimality system that may weaken the regularity conditions.

Lemma 3.2. *The Euler-Lagrange equation system* (13) *is of differentiation index* $\nu_d = 1$ *if and only if*

$$\begin{bmatrix} B_2R^{-1}B_2^T & -B_2R^{-1}S_{up} \\ -S_{pu}R^{-1}B_2^T & -W_2 + S_{pu}R^{-1}S_{up} \end{bmatrix} \text{ is invertible.}$$

In particular, if B_2 *does not have full row rank, then* (13) *has differentiation index* $\nu_d \geq 2$.

Assuming (14), *then* (13) *is a semi-explicit DAE of differentiation index* $\nu_d = 2$.

Proof. We use the invertibility of R to express u via

$$u = R^{-1}[B_1^T\lambda_1 + B_2^T\lambda_2 - S_{uv}v - S_{up}p]$$

and write (13) in matrix vector form

$$\begin{bmatrix} f_v \\ f_p \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} M\dot{v} \\ 0 \\ -\frac{d}{dt}(M^T\lambda_1) \\ 0 \end{bmatrix} -$$
$$\begin{bmatrix} B_1R^{-1}B_1^T & B_1R^{-1}B_2^T & A - B_1R^{-1}S_{uv} & J_1^T - B_1R^{-1}S_{up} \\ B_2R^{-1}B_1^T & B_2R^{-1}B_2^T & J_2 - B_2R^{-1}S_{uv} & -B_2R^{-1}S_{up} \\ A^T - S_{vu}R^{-1}B_1^T & J_2^T - S_{vu}R^{-1}B_2^T & -W_1 + S_{vu}R^{-1}S_{uv} & -W_{12} + S_{uv}R^{-1}S_{up} \\ J_1 - S_{pu}R^{-1}B_1^T & -S_{pu}R^{-1}B_2^T & -W_{21} + S_{pu}R^{-1}S_{uv} & -W_2 + S_{up}R^{-1}S_{up} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ v \\ p \end{bmatrix} \tag{16}$$

If now the submatrix $\mathscr{H}_{22} := \begin{bmatrix} B_2R^{-1}B_2^T & -B_2R^{-1}S_{up} \\ -S_{pu}R^{-1}B_2^T & -W_2 + S_{pu}R^{-1}S_{up} \end{bmatrix}$ is invertible, then one can solve algebraically for λ_2 and p in (16) and end up with an boundary value problem for λ_1 and v, which is the characterization of a DAE of differentiation index $\nu_d = 1$. If, however, B_2 does not have full row rank, then $\mathscr{H}_{22}$ is singular and (16) has index $\nu_d \geq 2$.

Assuming now that (14) holds, the corresponding terms in (16) vanish:

$$\begin{bmatrix} M\dot{v} \\ 0 \\ -\frac{d}{dt}(M^T\lambda_1) \\ 0 \end{bmatrix} - \begin{bmatrix} B_1R^{-1}B_1^T & 0 & A - B_1R^{-1}S_{uv} & J_1^T \\ 0 & 0 & J_2 & 0 \\ A^T - S_{vu}R^{-1}B_1^T & J_2^T & -W_1 + S_{vu}R^{-1}S_{uv} & -W_{12} \\ J_1 & 0 & -W_{21} & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ v \\ p \end{bmatrix}$$
$$= \begin{bmatrix} f_v^T & f_p^T & 0 & 0 \end{bmatrix}^T. \tag{17}$$

By inverting the mass matrices and permuting the rows and the columns, System (17) can be brought into the form of (2). Then the differentiation index 2 property, cf. Definition 2.1, follows from

$$\begin{bmatrix} 0 & J_2 \\ J_1 & -W_{21} \end{bmatrix} \begin{bmatrix} 0 & M \\ -M^T & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & J_1^T \\ J_2^T & -W_{12} \end{bmatrix} =$$
$$\begin{bmatrix} 0 & J_2M^{-1}J_1^T \\ -J_1M^{-T}J_2^T & -W_{21}M^{-1}J_1^T - J_1M^{-T}W_{12} \end{bmatrix}$$

being invertible by Assumption 2.2 (A1). □

Assuming further that $W_{21} = 0$, cf. (15), we can write the system as $u = R^{-1}B_1^T\lambda_1$,

$$\begin{bmatrix} -\frac{d}{dt}(M^T\lambda_1) \\ 0 \\ M\dot{v} \\ 0 \end{bmatrix} - \begin{bmatrix} G & 0 & F & J_1^T \\ 0 & 0 & J_2 & 0 \\ F^T & J_2^T & H & 0 \\ J_1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ v \\ p \end{bmatrix} = \begin{bmatrix} f_v \\ f_p \\ 0 \\ 0 \end{bmatrix}, \tag{18a}$$
$$v(0) = v^0 \quad \text{and} \quad M^T\lambda_1(T) = -V_1v(T), \tag{18b}$$

with $F := A - B_1R^{-1}S_{uv}$, symmetric matrices $G := B_1R^{-1}B_1^T$ and $H := -W_1 + S_{vu}R^{-1}S_{uv}$.

4 Existence and Representations of Optimal Solutions

In this section, we introduce a Riccati-decoupling for the optimality system. Using the projectors from Section 2, we determine differential and algebraic parts of the obtained differential-algebraic matrix Riccati equation and prove well-posedness. As a side-product we establish the unique solvability of the corresponding optimality system.

One outcome of the proof of Lemma 3.2 is that under the assumption (14) the considered Euler-Lagrange equations are in the form (2). Therefore, one may apply Theorem 2.3 to identify the inherent ODE (11). If then the data is consistent, one may use the theory for ODEs to state the existence of solutions to the obtained linear boundary value problem, cf. [2, Thm. 3.26]. However, the reformulation as used in Theorem 2.3 will not preserve the symmetry of (18) and thus make it more difficult to investigate whether the boundary values admit the existence of a solution. We will use a reformulation that preserves the structure such that the existence of a solution can be obtained via a standard differential Riccati equation.

Lemma 4.1. *Consider the semi-explicit linear DAE of index 2*

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} - \begin{bmatrix} A & J_1^T \\ J_2 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u = \begin{bmatrix} f_v \\ f_p \end{bmatrix}, \quad v(0) = v^0 \tag{19}$$

and a cost functional

$$\mathcal{J}(v,u) = \frac{1}{2}v^T(T)V_1v(T) + \frac{1}{2}\int_0^T \begin{bmatrix} v \\ u \end{bmatrix}^T \begin{bmatrix} W_1 & S_{vu} \\ S_{uv} & R \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix} \, dt, \tag{20}$$

which does not act onto the algebraic variable p *and with symmetric positive semi-definite weighting matrices and* R *symmetric positive definite. Define the matrix functions* $F := A - B_1R^{-1}S_{uv}$, $G := B_1R^{-1}B_1^T$ *and* $H := -W_1 + S_{vu}R^{-1}S_{uv}$.

1.) *Each solution* $(v, p, \lambda_1, \lambda_2)$ *of the associated Euler-Lagrange equations as given by* (18) *has a representation with* $(v, p) = (v_{\mathcal{P}} + \mathcal{Q}v, p)$ *and* $(M^T\lambda_1, \lambda_2) = (\lambda_{\mathcal{P}} + \mathcal{Q}^TM^T\lambda_1, \lambda_2)$ *given by the decoupled system*

$$\mathcal{Q}v = -M^{-1}J_1^TS^{-1}f_p, \tag{21a}$$

$$\mathcal{Q}^TM^T\lambda_1 = 0, \tag{21b}$$

$$\lambda_2 = -S^{-T}J_1M^{-T}\big[H[\mathcal{Q}v + v_{\mathcal{P}}] + F^TM^{-T}\lambda_{\mathcal{P}}\big], \tag{21c}$$

$$\begin{aligned} p = &-\mathcal{Q}^-[M^{-1}[F[\mathcal{Q}v + v_{\mathcal{P}}] + f_v] - \tfrac{d}{dt}(\mathcal{Q}v)]- \\ &- \mathcal{Q}^-M^{-1}GM^{-T}[\lambda_{\mathcal{P}} + \mathcal{Q}^TFM^{-T}\lambda_{\mathcal{P}} + J_2^T\lambda_2], \end{aligned} \tag{21d}$$

and

$$\begin{gathered} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} \dot{\lambda}_{\mathcal{P}} \\ \dot{v}_{\mathcal{P}} \end{bmatrix} - \begin{bmatrix} G_0 & F_0 \\ F_0^T & H_0 \end{bmatrix} \begin{bmatrix} \lambda_{\mathcal{P}} \\ v_{\mathcal{P}} \end{bmatrix} = \begin{bmatrix} \mathcal{P}M^{-1}[f_v - FM^{-1}J_1^TS^{-1}f_p] \\ \mathcal{P}^THM^{-1}J_1^TS^{-1}f_p \end{bmatrix}, \\ v_{\mathcal{P}}(0) = \mathcal{P}v^0 \quad \text{and} \quad \lambda_{\mathcal{P}}(T) = -\mathcal{P}^TV_1v(T), \end{gathered} \tag{21e}$$

where $F_0 := \frac{d}{dt}\mathcal{P} + \mathcal{P}M^{-1}F\mathcal{P}$, $G_0 = G_0^T := \mathcal{P}M^{-1}GM^{-T}\mathcal{P}^T$, $H_0 = H_0^T := \mathcal{P}^TH\mathcal{P}$ *and* $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}^-$ *and* S *as defined in Theorem 2.3.*

2.) *If in addition*

$$J_2v^0 = f_p(0) \quad \text{and} \quad J_1M^{-T}V_1 = 0, \tag{22}$$

then the Euler-Lagrange equations (18) *possess a unique solution.*

3.) *If in addition* f_v *and* f_p *are zero, then* (18) *can be decoupled via*

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} X_1 & X_2^T \\ X_2 & 0 \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix}, \tag{23}$$

where $X_1 = X_1^T$ *and* X_2 *fulfill the differential-algebraic matrix Riccati equation*

$$\begin{aligned} \tfrac{d}{dt}M^TX_1M + M^TX_1F + F^TX_1M + M^TX_1GX_1M + H+ \\ +M^TX_2^TJ_2 + J_2^TX_2M = 0, \\ M^TX_1(T)M = -V_1, \end{aligned} \tag{24a}$$

$$M^TJ_1X_1 = 0 \quad \text{and} \quad J_1X_1M = 0. \tag{24b}$$

Equations (24) *uniquely define a symmetric negative semi-definite* X_1.

Proof. ad *1.)* We write the Euler-Lagrange system, cf. (18), as

$$\begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \frac{d}{dt} \begin{bmatrix} M^T\lambda_1 \\ \lambda_2 \\ v \\ p \end{bmatrix} - \begin{bmatrix} M^{-1}GM^{-T} & 0 & M^{-1}F & M^{-1}J_1^T \\ 0 & 0 & J_2 & 0 \\ F^TM^{-T} & J_2^T & H & 0 \\ J_1M^{-T} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M^T\lambda_1 \\ \lambda_2 \\ v \\ p \end{bmatrix}$$
$$= \begin{bmatrix} f_v^TM^{-T} & f_p^T & 0 & 0 \end{bmatrix}^T,$$
$$v(0) = v^0 \quad \text{and} \quad M^T\lambda_1(T) = -V_1v(T).$$

In order to preserve the self-adjoint structure, cf. [20], only congruence transformations should be applied, i.e. a scaling of the equations must be accompanied by the transpose inverse scaling of the variables. In accordance to (7) we congruently transform the system by

$$S_2 := \begin{bmatrix} \mathcal{E}_2^{-1} & & \\ & I & \\ & & I \end{bmatrix} = \begin{bmatrix} \mathcal{P} & [I - \mathcal{P}M^{-1}F]M^{-1}J_1^TS^{-1} & 0 & 0 \\ \mathcal{Q}^- & -[I + \mathcal{Q}^-M^{-1}FM^{-1}J_1^T]S^{-1} & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

where $\mathcal{E}_2 = \begin{bmatrix} I + M^{-1}F\mathcal{Q} & M^{-1}J_1^T \\ J_2 & 0 \end{bmatrix}$ as defined in (5) with the inverse given in (6). The summand that comes from the time-dependency in the variable transformation S_2^T is given by

$$S_2 \begin{bmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ -I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dot{S}_2^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{d}{dt}\mathcal{P}^T & -\dot{\mathcal{Q}}^{T-} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

With this we get the scaled and transformed system

$$\tilde{f} = \begin{bmatrix} 0 & 0 & \mathcal{P} & 0 \\ 0 & 0 & \mathcal{Q}^- & 0 \\ -\mathcal{P}^T & -\mathcal{Q}^{T-} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{\lambda}}_1 \\ \dot{\tilde{\lambda}}_2 \\ \dot{v} \\ \dot{p} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{d}{dt}\mathcal{P}^T & -\dot{\mathcal{Q}}^{T-} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \\ v \\ p \end{bmatrix} -$$
$$\begin{bmatrix} \mathcal{P}M^{-1}GM^{-T}\mathcal{P}^T & M^{-1}GM^{-T}\mathcal{Q}^{T-} & \mathcal{P}M^{-1}F\mathcal{P} + \mathcal{Q} & 0 \\ \mathcal{Q}^-M^{-1}GM^{-T} & 0 & \mathcal{Q}^-M^{-1}F\mathcal{P} - \mathcal{Q}^- & I \\ \mathcal{P}^TF^TM^{-T}\mathcal{P}^T + \mathcal{Q}^T & \mathcal{P}^TF^TM^{-T}\mathcal{Q}^{T-} - \mathcal{Q}^{T-} & H & 0 \\ 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \\ v \\ p \end{bmatrix} \tag{26}$$

with the transformed state and scaled right hand side

$$\begin{bmatrix} \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \\ v \\ p \end{bmatrix} := S_2^{-T} \begin{bmatrix} M^T\lambda_1 \\ \lambda_2 \\ v \\ p \end{bmatrix} = \begin{bmatrix} [I + \mathcal{Q}^TF^TM^{-T}]M^T\lambda_1 + J_2^T\lambda_2 \\ J_1\lambda_1 \\ v \\ p \end{bmatrix}$$

and $\tilde{f} := S_2 \begin{bmatrix} f_v^TM^{-T} & f_p^T & 0 & 0 \end{bmatrix}^T$, respectively. From the last line in (26) we find that $\tilde{\lambda}_2 = 0$ so that we can rewrite the equations for $(\tilde{\lambda}_1, v, p)$ as

$$\begin{bmatrix} \mathcal{P} & 0 \\ \mathcal{Q}^- & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} - \begin{bmatrix} \mathcal{P}M^{-1}GM^{-T}\mathcal{P}^T & 0 \\ \mathcal{Q}^-M^{-1}GM^{-T} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \end{bmatrix} - \begin{bmatrix} \mathcal{P}M^{-1}F\mathcal{P} + \mathcal{Q} & 0 \\ \mathcal{Q}^-M^{-1}F\mathcal{P} - \mathcal{Q}^- & I \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} =$$
$$\mathcal{E}_2^{-1} \begin{bmatrix} M^{-1}f_v \\ f_p \end{bmatrix} \tag{27a}$$

and

$$-\tfrac{d}{dt}(\mathcal{P}^T\tilde{\lambda}_1) - [\mathcal{P}^T F^T M^{-T}\mathcal{P}^T + \mathcal{Q}^T]\tilde{\lambda}_1 - Hv = 0. \tag{27b}$$

Analogously to (8) we apply the projectors

$$\mathscr{Q}_1 = \begin{bmatrix} \mathcal{Q} & 0 \\ -\mathcal{Q}^- & 0 \end{bmatrix}, \quad \mathscr{Q}_0\mathscr{P}_1 = \begin{bmatrix} 0 & 0 \\ \mathcal{Q}^- & I \end{bmatrix} \quad \text{and} \quad \mathscr{P}_0\mathscr{P}_1 = \begin{bmatrix} \mathcal{P} & 0 \\ 0 & 0 \end{bmatrix}$$

to (27a) to obtain the three subsystems

$$-\begin{bmatrix} \mathcal{Q} & 0 \\ -\mathcal{Q}^- & 0 \end{bmatrix}\begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} M^{-1}J_1^T S^{-1} f_p \\ -S^{-1} f_p \end{bmatrix}, \tag{28a}$$

$$\begin{bmatrix} 0 & 0 \\ \mathcal{Q}^- & 0 \end{bmatrix}\begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \mathcal{Q}^- M^{-1} G M^{-T} & 0 \end{bmatrix}\begin{bmatrix} \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \mathcal{Q}^- M^{-1} F\mathcal{P} & I \end{bmatrix}\begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ \mathcal{Q}^- M^{-1}[f_v - FM^{-1}J_1^T S^{-1} f_p] \end{bmatrix} \tag{28b}$$

and

$$\begin{bmatrix} \mathcal{P} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} - \begin{bmatrix} \mathcal{P}M^{-1}GM^{-T}\mathcal{P}^T & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \end{bmatrix} - \begin{bmatrix} \mathcal{P}M^{-1}F\mathcal{P} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} \mathcal{P}M^{-1}[f_v - FM^{-1}J_1^T S^{-1} f_p] \\ 0 \end{bmatrix}, \tag{28c}$$

respectively. Using the projector property $\mathcal{P}^T = \mathcal{P}^T\mathcal{P}^T$ to obtain the relation

$$\tfrac{d}{dt}(\mathcal{P}^T\tilde{\lambda}_1) = \dot{\mathcal{P}}^T\mathcal{P}^T\tilde{\lambda}_1 + \mathcal{P}^T\tfrac{d}{dt}(\mathcal{P}^T\tilde{\lambda}_1) = \tfrac{d}{dt}(\mathcal{P}^T\tilde{\lambda}_1) - \mathcal{Q}^T\tfrac{d}{dt}(\mathcal{P}^T\tilde{\lambda}_1) + \dot{\mathcal{P}}^T\mathcal{P}^T\tilde{\lambda}_1$$

we split (27b) into the two subsystems

$$\mathcal{Q}^T\tfrac{d}{dt}(\mathcal{P}^T\tilde{\lambda}_1) \; - \mathcal{Q}^T\tilde{\lambda}_1 - \mathcal{Q}^T Hv = 0 \tag{29a}$$

and

$$-\tfrac{d}{dt}(\mathcal{P}^T\tilde{\lambda}_1) - \tfrac{d}{dt}(\mathcal{P}^T)\mathcal{P}^T\tilde{\lambda}_1 - \mathcal{P}^T F^T M^{-T}\mathcal{P}^T\tilde{\lambda}_1 - \mathcal{P}^T Hv = 0. \tag{29b}$$

If we then define $v_{\mathcal{P}} := \mathcal{P}v$ and $\tilde{\lambda}_{\mathcal{P}} := \mathcal{P}^T\tilde{\lambda}_1$ and decompose $\tilde{\lambda}_1 = \tilde{\lambda}_{\mathcal{P}} + \mathcal{Q}^T\tilde{\lambda}_1$ and $v = v_{\mathcal{P}} + \mathcal{Q}v$ we find that (28a-b) and (29a) define algebraic relations for

$$\mathcal{Q}v = -M^{-1}J_1^T S^{-1} f_p, \tag{30a}$$

$$\mathcal{Q}^T\tilde{\lambda}_1 = -\mathcal{Q}^T[H\mathcal{Q}v + Hv_{\mathcal{P}}] + \mathcal{Q}^T\dot{\tilde{\lambda}}_{\mathcal{P}} \tag{30b}$$

and, with $\mathcal{Q}^-\dot{v_{\mathcal{P}}}$

$$p = -\mathcal{Q}^-[M^{-1}F[\mathcal{Q}v + v_{\mathcal{P}}] + M^{-1}f_v + M^{-1}GM^{-T}\tilde{\lambda}_1 - \tfrac{d}{dt}(\mathcal{Q}v)], \tag{30c}$$

while $\tilde{\lambda}_{\mathcal{P}}$ and $v_{\mathcal{P}}$ are defined by the coupled ODEs given by (29b) and (28c):

$$-\dot{\tilde{\lambda}}_{\mathcal{P}} - \left[\tfrac{d}{dt}\mathcal{P}^T + \mathcal{P}^T F^T M^{-T}\mathcal{P}^T\right]\tilde{\lambda}_{\mathcal{P}} - \mathcal{P}^T H\mathcal{P}v_{\mathcal{P}} = \mathcal{P}^T HM^{-1}J_1^T S^{-1} f_p \tag{31a}$$

and

$$\dot{v}_{\mathcal{P}} - \mathcal{P}M^{-1}GM^{-T}\mathcal{P}^T\tilde{\lambda}_{\mathcal{P}} - \big[\tfrac{d}{dt}\mathcal{P} + \mathcal{P}M^{-1}F\mathcal{P}\big]v_{\mathcal{P}} = \\ \mathcal{P}M^{-1}[f_v - FM^{-1}J_1^TS^{-1}f_p]. \tag{31b}$$

Note that we have used the projector property $\mathcal{P} = \mathcal{P}^2$ to keep the symmetry in (31) obvious.

In view of expressing the obtained relations in terms of the original variables (λ_1, λ_2) we observe that

$$\tilde{\lambda}_{\mathcal{P}} = \mathcal{P}^T\tilde{\lambda}_1 = \mathcal{P}^T[M^T\lambda_1 + \mathcal{Q}^TF^T\lambda_1 + J_2^T\lambda_2] = \mathcal{P}^TM^T\lambda_1 =: \lambda_{\mathcal{P}}.$$

From $\tilde{\lambda}_2 = J_1\lambda_1 = 0$ we confer

$$\mathcal{Q}^TM^T\lambda_1 = J_2^TS^{-T}J_1\lambda_1 = 0.$$

For λ_2 we use $\mathcal{Q}^T\tilde{\lambda}_1 = \mathcal{Q}^T[I + \mathcal{Q}^TFM^{-T}]M^T\lambda_1 + \mathcal{Q}^TJ_2^T\lambda_2 = \mathcal{Q}^TFM^{-T}\lambda_{\mathcal{P}} + J_2^T\lambda_2$, relation (30b), and the invertibility of $S^T = J_1M^{-1}J_2^T$ to get

$$\lambda_2 = -S^{-T}J_1M^{-T}[H[\mathcal{Q}v + v_{\mathcal{P}}] + FM^{-T}\lambda_{\mathcal{P}}].$$

Note that $J_1^TM^{-T}\mathcal{P}^T = 0$, so that $\dot{\lambda}_{\mathcal{P}} \in \operatorname{span}\mathcal{P}^T$ does not appear.

Similarly, one can express the equation for p in terms of (λ_1, λ_2) which completes the derivation of Equations (21).

ad *2.)*

First, we show that for any v^0 and $\mathcal{P}^TV_1$ symmetric positive semi-definite the decoupled system (21) has a unique solution $(v_{\mathcal{P}}, \mathcal{Q}v, p, \lambda_{\mathcal{P}}, \mathcal{Q}^TM^T\lambda_1, \lambda_2)$. Second, we confer that under the consistency conditions (22) the solution of (21) provides a solution of the Euler-Lagrange equations (18). Finally, by *1.)* every solution of (18) has a representation in (21), such that in summary the Euler-Lagrange equations must possess a unique solution.

We first consider in (21e-f) the case with a zero right hand side. With the Riccati ansatz $\lambda_{\mathcal{P}} = X_0(t)v_{\mathcal{P}}(t)$ these equations can be rewritten as the differential matrix Riccati equation

$$\dot{X}_0 = -X_0G_0X_0 - X_0F_0 - F_0^TX_0 - H_0, \quad X_0(T) = -\mathcal{P}^TV_1, \tag{32}$$

which has a unique solution, cf. [1, Thm. 4.1.6], since $\mathcal{P}^TV_1$, G_0, and $-H_0$ are symmetric positive semi-definite. With this X_0 we get $v_{\mathcal{P}}$ and $\lambda_{\mathcal{P}}$ as the solution of $\dot{v}_{\mathcal{P}} - [G_0X_0 + F_0]v_{\mathcal{P}} = 0, v_{\mathcal{P}}(0) = \mathcal{P}v^0$ and $\lambda_{\mathcal{P}} = X_0v_{\mathcal{P}}$, respectively.

One can show that if there exists a solution to (21e-f) with a zero right hand side, then it is unique. This is equivalent to the fact that the linear part of the affine boundary conditions are stated such, that (21e-f) with $\mathcal{P}^TV_1$ symmetric positive semi-definite, has a unique solution, cf. [2, Thm. 3.26], for any continuous right hand side.

By construction a solution of (18) uniquely defines a solution to (21). The converse is true if and only if the algebraic variables fulfill the initial and terminal conditions, i.e.,

$$\mathcal{Q}v(0) = \mathcal{Q}v_{\mathcal{P}} = M^{-1}J_1^TS^{-1}J_2v_{\mathcal{P}} \quad \text{and} \tag{33a}$$

$$\mathcal{Q}^TM^T\lambda_1(T) = -\mathcal{Q}^TV_1v(T) = -J_2^TS^{-T}J_1M^{-T}V_1v(T). \tag{33b}$$

By (21a) we have that $\mathcal{Q}v(0) = M^{-1}J_1^TS^{-1}f_p(0)$ such that $J_2v(0) = f_p(0)$ is necessary and sufficient for (33a). By (21b) we have that $\mathcal{Q}^TM^T\lambda_1 = 0$ such that $J_1M^{-T}V_1 = 0$ is sufficient but not necessarily necessary for (33b). Note, however, that in this case we can infer that

$$J_1^TM^{-T}V_1 = 0 \quad \Rightarrow \quad V_1M^{-1}J_1 = 0 \quad \Rightarrow \quad V_1\mathcal{Q} = 0 \quad \Rightarrow \quad V_1v = V_1\mathcal{P}v$$

such that in (21f) $\mathcal{P}^T V_1$ can be replaced by $\mathcal{P}^T V_1 \mathcal{P}$. Thus, condition (22) implies the symmetry in the terminal condition that was sufficient for the existence of X_0 in (32).

ad *3.)*

With the ansatz

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} X_1 & X_2^T \\ X_2 & 0 \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} \tag{34}$$

we obtain that

$$\tfrac{d}{dt}\left(\begin{bmatrix} M^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}\right) = \begin{bmatrix} \frac{d}{dt} M^T X_1 M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} + \begin{bmatrix} M^T X_1 & M^T X_2^T \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix}. \tag{35}$$

In (35) we replace $\frac{d}{dt}\left(\begin{bmatrix} M^T & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}\right)$ and $\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix}$ via the relations given in (18) and every occurrence of $\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$ by the ansatz (34) to obtain $\mathscr{X} \begin{bmatrix} v \\ p \end{bmatrix} = 0$, where

$$\mathscr{X} := \begin{bmatrix} \begin{matrix} \frac{d}{dt}(M^T X_1 M) + F^T X_1 M + M^T X_1 F + \\ M^T X_1 G X_1 M + H + J_2^T X_2 M + M^T X_2^T J_2 \end{matrix} & M^T X_1 J_1^T \\ J_1 X_1 M & 0 \end{bmatrix}. \tag{36}$$

Since $\mathscr{X} \begin{bmatrix} v \\ p \end{bmatrix} = 0$ must hold for every state trajectory, one requires $\mathscr{X} = 0$ which gives the equations for X_1 and X_2:

$$\begin{aligned} \tfrac{d}{dt} M^T X_1 M + M^T X_1 F + F^T X_1 M + M^T X_1 G X_1 M + H + & \\ + M^T X_2^T J_2 + J_2^T X_2 M &= 0, \\ M^T X_1(T) M &= -V_1, \end{aligned} \tag{37a}$$

$$M^T J_1 X_1 = 0 \quad \text{and} \quad J_1 X_1 M = 0. \tag{37b}$$

The terminal condition in (37a) is defined via (18b) and (23):

$$M^T \lambda_1(T) = M^T X_1(T) M v(T) = -V_1 v(T) \quad \Rightarrow \quad M^T X_1(T) M = -V_1.$$

To show that (37) has a solution we consider Equation (37a) in the transformed variables $X := -M^T X_1 M$ and $Y := X_2 M$:

$$\begin{aligned} -\dot{X} - F^T M^{-T} X - X M^{-1} F + X M^{-1} G^{-T} M^{-T} X + H + J_2^T Y + Y^T J_2 &= 0, \\ X(T) &= V_1. \end{aligned} \tag{38}$$

By means of the projector $\mathcal{Q} := M^{-1} J_1^T [J_2 M^{-1} J_1^T]^{-1} J_2$ we write $X = [\mathcal{Q}^T + \mathcal{P}^T] X [\mathcal{P} + \mathcal{Q}]$. From (37b) we obtain that $\mathcal{Q}^T X = X \mathcal{Q} = 0$ and thus X is completely defined via $X_0 := \mathcal{P}^T X \mathcal{P}$. Applying $\mathcal{P}^T$ and $\mathcal{P}$ to (38) from the left and the right, respectively, we get a standard differential Riccati equation

$$\begin{aligned} -\dot{X}_0 - F_0^T X_0 - X_0 F_0 + X_0 M^{-1} G M^{-T} X_0 + \mathcal{P}^T H \mathcal{P} &= 0, \\ X_0(T) &= \mathcal{P}^T V_1 \mathcal{P}, \end{aligned} \tag{39}$$

which has a unique and symmetric positive semi-definite solution, cf. [1, Thm. 4.1.6], since V_1, G and $-H$ are symmetric positive semi-definite. Again, the consistency condition (22) ensures that

$X_0(T)$ also satisfies the initial condition and the algebraic constraints in (37). Since $\mathcal{Q}^T X = 0$ and $X\mathcal{Q} = 0$, we have $X_1 = -M^{-T} X M^{-1}$ is unique and symmetric negative semi-definite.

Application of $\mathcal{P}^T$ from the left and $\mathcal{Q}$ from the right to (38) gives

$$-X_0\dot{\mathcal{Q}} - X_0 M^{-1} F\mathcal{Q} + \mathcal{P}^T H\mathcal{Q} = -\mathcal{P}^T Y^T J_2 \mathcal{Q} = -\mathcal{P}^T Y^T J_2,$$

which is uniquely solvable for $\mathcal{P}^T Y^T$. The projected equation obtained via $\mathcal{Q}^T$ and $\mathcal{P}$ is the transpose of the above equation and bears no additional information.

Finally, one can determine $\mathcal{Q}^T Y^T$ from the projection of (38) onto the range of $\mathcal{Q}^T$ and $\mathcal{Q}$ which reads

$$\mathcal{Q}^T H\mathcal{Q} + \mathcal{Q}^T Y^T J_2 \mathcal{Q} + \mathcal{Q}^T J_2^T Y \mathcal{Q} = 0. \tag{40}$$

With $J_2\mathcal{Q} = J_2$, we find that (40) is of the form $[Y\mathcal{Q}]^T J_2 + J_2^T [Y\mathcal{Q}] = -\mathcal{Q}^T H Q$ that was investigated in [8]. With $\mathcal{Q}^- := M^{-1} J_1^T [J_2 M^{-1} J_1^T]^{-1}$ being a generalized inverse to J_2, we obtain the projectors $P_1 := \mathcal{Q}^- J_2 = \mathcal{Q}$ and $P_2 := J_2 \mathcal{Q}^- = I$ and the existence of solutions to (40) follows by [8, Thm. 1], since $\mathcal{Q}^T H\mathcal{Q}$ is symmetric and $[I - P_1]^T \mathcal{Q}^T H\mathcal{Q}[I - P_1] = 0$.

The general solution to (40) is given by

$$Y\mathcal{Q} = \frac{1}{2}[J_1 M^{-T} J_2^T]^{-1} J_1 M^{-T} H\mathcal{Q} + Z J_2,$$

where Z is arbitrary with $Z^T = -Z$. Thus existence of $M^T X_1 M$ and $M^T X_2^T = Y^T = \mathcal{P}^T Y^T + \mathcal{Q}^T Y^T$ and therefore X_1 and X_2 is proved.

By construction, with X_1 and X_2 as determined above, the solution of

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} - \left(\begin{bmatrix} G & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 & X_2^T \\ X_2 & 0 \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} F & J_1^T \\ J_2 & 0 \end{bmatrix} \right) \begin{bmatrix} v \\ p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$v(0) = v^0,$$

and

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} X_1 & X_2^T \\ X_2 & 0 \end{bmatrix} \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix}$$

gives the solution of (18) with a zero right-hand side.

□

Remark 4.2. The solution of (24) is unique up to an additive term ZJ_2M^{-1} in X_2, with an arbitrary matrix Z, that fulfills $Z^T = -Z$. However, this does not contradict the unique solvability of the Euler-Lagrange equations, since λ_1 and λ_2 as defined via (23) are independent of any choice of Z.

In view of optimal control, the above results can be summarized as follows. To obtain an optimal input u for (19) with respect to a cost functional of type (20) it is sufficient to have a solution of the associated Euler-Lagrange equations (18), cf. [24]. By Lemma 4.1 it follows that for the considered state equations and cost functionals this solution exists, that it is unique, that it can be obtained via the separation ansatz (23), and that an optimal u is obtained via expression (18c). For the nonhomogenous and for the trajectory tracking case one can use an affine linear Riccati-ansatz, cf. [21]. Thus, we can state the following theorem:

Theorem 4.3. *Let $T > 0$, $\mathbb{I} = (0, T]$ a time interval, $n_u, n_v, n_p \in \mathbb{N}$, $n_v > n_p$, $M \in \mathcal{C}(\mathbb{I}, \mathbb{R}^{n_v,n_v})$ pointwise invertible, $A \in \mathcal{C}(\mathbb{I}, \mathbb{R}^{n_v,n_v})$, and let $J_1, J_2 \in \mathcal{C}(\mathbb{I}, \mathbb{R}^{n_p,n_v})$ such that $J_2M^{-1}J_1^T$ is invertible and that $M^{-1}J_1^T S^{-1}J_2$ is differentiable. Let $W_1, V_1 \in \mathbb{R}^{n_v,n_v}$ symmetric positive semi-definite, $S_{uv} = S_{vu}^T \in \mathbb{R}^{n_u,n_v}$ an arbitrary matrix and let $R \in \mathbb{R}^{n_u,n_u}$ symmetric positive definite.*

For a given $v^* \in \mathcal{C}^1(\mathbb{I}, \mathbb{R}^{n_v})$ consider the linear-quadratic optimal control problem of finding $u \in \mathcal{C}(\mathbb{I}, \mathbb{R}^{n_u})$ such that

$$\frac{1}{2}\left[v - v^*\right]^T V_1 \left[v - v^*\right]\Bigg|_{t=T} + \frac{1}{2}\int_0^T \begin{bmatrix} v - v^* \\ u \end{bmatrix}^T \begin{bmatrix} W_1 & S_{vu} \\ S_{uv} & R \end{bmatrix} \begin{bmatrix} v - v^* \\ u \end{bmatrix}\, dt,$$

is minimal, where v on $\mathbb{I}$ satisfies the state equations

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot v \\ \dot p \end{bmatrix} - \begin{bmatrix} A & J_1^T \\ J_2 & 0 \end{bmatrix}\begin{bmatrix} v \\ p \end{bmatrix} - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u = \begin{bmatrix} f_v \\ f_p \end{bmatrix}, \quad v(0) = v^0.$$

If $f_v \in \mathcal{C}(\mathbb{I}, \mathbb{R}^{n_v})$, $f_p \in \mathcal{C}^1(\mathbb{I}, \mathbb{R}^{n_p})$ and if

$$J_2 v^0 = f_p(0) \quad \text{and} \quad J_1 M^{-T} V_1 = 0,$$

then the optimal control problem is solvable and an optimal input u is given via the feedback-law

$$u = R^{-1}[B_1^T[X_1 M v + w_1] - S_{uv}(v - v^*)],$$

where $X_1 = X_1^T$, negative semi-definite, and w_1 are the unique solutions of

$$\begin{aligned} \tfrac{d}{dt} M^T X_1 M + F^T X_1 M + M^T X_1 F + M^T X_1 G X_1 M + H + &\\ + J_2^T X_2 M + M^T X_2^T J_2 &= 0, \\ M^T X_1(T) M &= -V_1, \\ J_1 X_1 M &= 0, \end{aligned}$$

and

$$\begin{aligned} \tfrac{d}{dt}(M^T w_1) - [M^T X_1 G + F^T] w_1 - J_2^T w_2 &= f_{\lambda_1} + M^T X_1 \tilde f_v + M^T X_2 f_p, \\ M^T w_1(T) &= V_1 v^*(T), \\ J_1 w_1 &= 0, \end{aligned}$$

respectively, where $F := A - B_1 R^{-1} S_{uv}$, $G := B_1 R^{-1} B_1^T$ and $H := -W_1 + S_{vu} R^{-1} S_{uv}$ and with $\tilde f_v := f_v + B_1 R^{-1} S_{uv} v^*$ and $f_{\lambda_1} := [W_1 - S_{vu} R^{-1} S_{uv}] v^*$.

Theorem 4.3 gives – in particular – sufficient optimality conditions without resorting to any index reduction. The conditions are not necessarily necessary, as an inconsistent V_1 renders them ill-posed, although for well-posed state equations a solution of the optimal control problem always exists, cf. the true optimality system defined in [21]. However, the true optimality system is formulated by means index-reduction.

For practical applications the following modification that closes the gap between sufficiency and necessity of the optimality conditions in Theorem 4.3 may be considered.

Remark 4.4. By Theorem 2.3 one has that if v solves (19), then it writes $v = \mathcal{P}v - c$, with $c := M^{-1}J_1^T S^{-1} f_p$ independent of u and v and that the end point penalization in the cost functional (20) can be replaced like

$$\frac{1}{2} v^T(T) V_1 v(t) \quad \leftarrow \quad \frac{1}{2}[\mathcal{P}v(t) - c(t)]^T V_1 [\mathcal{P}v(T) - c(T)].$$

With this equivalent formulation, the end condition on $M^T\lambda_1$ in (18b) coming from the variation of the cost functional with respect to v reads $M^T\lambda_1(T) = -\mathcal{P}^T V_1[\mathcal{P}v(T) - c(T)]$. Then the end condition for the gain matrix X_1 is given via $\mathcal{P}^T V_1 \mathcal{P}$ and for the affine part w_1 via $M^T w_1(T) = \mathcal{P}^T V_1[v^*(T) + M^{-1}J_1^T S^{-1} f_p(T)$. Both conditions are consistent as $J_1 M^{-T}\mathcal{P}^T = 0$. With this modification, in Theorem 4.3, the restriction $J_1 M^{-T} V_1 = 0$ is obsolete and the given optimality conditions are equivalent to the true optimality conditions.

5 Pressure Terms and Crossterms in the Cost Functional

As mentioned in Remark 3.1, in the linear case, one can theoretically reformulate any costfunctional of type (12) as an equivalent cost weighting without the algebraic variable p and without cross terms in the integral part.

To illustrate this, we assume that the right hand sides of the state equations f_v and f_p are zero. Then by Theorem 2.3 we have

$$p = -\mathcal{Q}^- M^{-1} A v - \mathcal{Q}^- M^{-1} B_1 u$$

and the trajectory weighting $\begin{bmatrix} v \\ p \\ u \end{bmatrix}^T \begin{bmatrix} W_1 & W_{12} & S_{vu} \\ W_{21} & W_2 & S_{pu} \\ S_{uv} & S_{up} & R \end{bmatrix} \begin{bmatrix} v \\ p \\ u \end{bmatrix}$ is equivalent to $\begin{bmatrix} v \\ u \end{bmatrix}^T \begin{bmatrix} \tilde{W}_1 & \tilde{S}_{vu} \\ \tilde{S}_{uv} & \tilde{R} \end{bmatrix} \begin{bmatrix} v \\ u \end{bmatrix}$ with $\begin{bmatrix} \tilde{W}_1 & \tilde{S}_{vu} \\ \tilde{S}_{uv} & \tilde{R} \end{bmatrix} :=$

$$\begin{bmatrix} I & 0 \\ -\mathcal{Q}^- M^{-1} A & -\mathcal{Q}^- M^{-1} B_1 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} W_1 & W_{12} & S_{vu} \\ W_{21} & W_2 & S_{pu} \\ S_{uv} & S_{up} & R \end{bmatrix} \begin{bmatrix} I & 0 \\ -\mathcal{Q}^- M^{-1} A & -\mathcal{Q}^- M^{-1} B_1 \\ 0 & I \end{bmatrix}.$$

Also, because of the linearity of the problem, one can formally eliminate the crossterms given by $\tilde{S}_{uv}$ in the costfunctional by shifting the input and considering $\tilde{u} = u + \tilde{R}^{-1}\tilde{S}_{uv}^T v$. This gives an equivalent formulation of the optimal control problem:

$$\mathcal{J}(v, p, \tilde{u}) = \cdots + \frac{1}{2}\int_0^T v^T[\tilde{W} - \tilde{S}_{uv}\tilde{R}^{-1}\tilde{S}_{uv}^T]v + \tilde{u}^T\tilde{R}\tilde{u} \, \mathrm{d}t \to \min \tag{42a}$$

subject to

$$\begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{p} \end{bmatrix} - \begin{bmatrix} A - B_1\tilde{R}^{-1}\tilde{S}_{uv}^T & J_1^T \\ J_2 & 0 \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix} - \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \tilde{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{42b}$$
$$v(0) = v^0.$$

Finally, if f_v and f_g are not zero, the same approach leads to a linear in v and u term in the cost functional, which then appears in the right hand side of the adjoint equation as does $W_1 v^*$ in the affine linear formulation in Theorem 4.3.

6 Discussion and Outlook

The results of Section 3 for the linear case can serve as a basis for the numerical treatment of semi-linear equations that sooner or later will carry out linearizations. Another crucial point is the incorporation of constraints for the control as well as lower regularity.

The latter has been investigated in [26] for a large class of optimal control problems subject to semi-explicit DAEs that also contains cases that are considered here. The formulation of a maximum principle in [26] bases on an equivalent index 1 representation of the state equations that can be formally obtained for the semi-explicit case. Because of the index-reduction the results of [26] are not the natural extension of our results to input constrained problems.

In view of solving the optimal control problem numerically it may be worth investigating, whether the Riccati decoupling can be exploited for efficient numerical routines.

The type of system considered here was chosen to fit spatially discretized PDEs as the Navier-Stokes Equations. For a system-theoretical insight, one may consider similar manipulations on the original infinite-dimensional system.

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