Technische Universität Berlin
Institut für Mathematik

Weak formulations of linear differential-algebraic systems

P. Kunkel, V. Mehrmann, M. Schmidt, I. Seufer, and A. Steinbrecher

Preprint 2006/16

Preprint-Reihe des Instituts für Mathematik
Technische Universität Berlin

Report 2006/16 October 2006

We discuss different weak formulations for systems of linear differential-algebraic equations with variable coefficients. These weak formulations include systems with factored or properly stated leading term, strangeness-free systems, a distributional formulation and a weak formulation in terms of Sobolev spaces.

AMS(MOS) subject classification: 65L05, 34H05

Keywords: differential-algebraic equation, weak formulation, index reduction, properly stated leading term, strangeness index, distributional formulation

Weak formulations of linear differential-algebraic systems

Peter Kunkel * Volker Mehrmann†‡ Michael Schmidt†§
Ingo Seufer †‡ Andreas Steinbrecher†¶

October 5, 2006

Abstract

We discuss different weak formulations for systems of linear differential-algebraic equations with variable coefficients. These weak formulations include systems with factored or properly stated leading term, strangeness-free systems, a distributional formulation and a weak formulation in terms of Sobolev spaces.

Keywords: differential-algebraic equation, weak formulation, index reduction, properly stated leading term, strangeness index, distributional formulation

AMS(MOS) subject classification: 65L05, 34H05

1 Introduction

Differential-algebraic equations (DAEs) are currently the standard modeling concept in many applications such as circuit simulation, multibody dynamics, chemical process engineering, and computational fluid dynamics, see [3, 5, 9, 16, 19, 20, 27, 36, 37] and the references therein. They have a particular advantage for the treatment of multi-physics models arising from modern automatic modeling tools such as [8, 32].

In this paper we will focus our analysis on linear systems with variable coefficients which describe also the local behaviour of general nonlinear DAEs, when the nonlinear system is linearized along a trajectory [7]. We will discuss systems of the form

$$E\dot{x} = Ax + f \tag{1}$$

*Mathematisches Institut, Universität Leipzig, Augustusplatz 10-11, D-04109 Leipzig, Fed. Rep. Germany; kunkel@mathematik.uni-leipzig.de.

†Institut für Mathematik, TU Berlin, Straße des 17. Juni 136, D-10623 Berlin, Fed. Rep. Germany; {mehrmann,mschmidt,seufer,anst}@math.tu-berlin.de.

‡These authors were partially supported by DFG Research Center Matheon 'Mathematics for key technologies' in Berlin.

§This author was supported by DFG Sonderforschungsbereich 557 'Beeinflussung von komplexen turbulenten Scherströmungen' at TU Berlin.

¶This author was supported by *Deutsche Forschungsgemeinschaft*, through Research Grant ME13-2

in a real interval $\mathbb{I} = [t_0, t_f]$, where $E, A \in C^k(\mathbb{I}, \mathbb{C}^{m,n})$ and $f \in C^k(\mathbb{I}, \mathbb{C}^m)$ are sufficiently often differentiable and $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$. Here for $k \in \mathbb{N}_0 \cup \{\infty\}$, $C^k(\mathbb{I}, \mathbb{C}^{m,n})$ denotes the k times continuously differentiable functions from the interval $\mathbb{I}$ to the real $m \times n$ matrices $\mathbb{C}^{m,n}$ (we leave off the superscript if $k = 0$).

The aim of this paper is to study different formulations of (1) that allow to weaken the classical solvability concept given by the following definition.

Definition 1 A function $x : \mathbb{I} \to \mathbb{C}^n$ is called a *classical solution* of (1) if $x \in C^1(\mathbb{I}, \mathbb{C}^n)$ and x satisfies (1) pointwise. It is called a *classical solution of the initial value problem* consisting of (1) and

$$x(t_0) = x_0, \tag{2}$$

if x is a solution of (1) and satisfies (2). An initial condition (2) is called *consistent* if the corresponding initial value problem has at least one solution.

Considering system (1) one can immediately see that the concept of classical solutions can be weakend by observing that in the kernel of the coefficient function E the derivative $\dot{x}$ does not occur.

Definition 2 Consider system (1) with sufficiently smooth coefficient functions E, A, and f. A function $x : \mathbb{I} \to \mathbb{C}^n$ is called *strong solution* of (1) if $\dot{x}$ exists in the cokernel of E, x is continuous and satisfies (1) pointwise.

A very general approach to classify existence and uniqueness of solutions and consistency of initial conditions uses a reformulation of the original differential-algebraic equation via derivative arrays or by means of equivalence transformations and allows a reformulation of the differential-algebraic equation to one where the minimal smoothness requirements can be read off directly. This approach is described in detail in the recent textbook [27]. We will discuss this concept in a different setting in Section 3.

In some practical applications, in particular in circuit simulation, see e.g., [17, 18], and mechanical multibody systems [9] the differential-algebraic equation has a specific structure that can be used to weaken the solvability requirements. Often also the DAE has the factored form

$$F\frac{\mathrm{d}}{\mathrm{d}t}(Dx) = Gx + f,\ t \in \mathbb{I} \tag{3}$$

with $F \in C^k(\mathbb{I}, \mathbb{C}^{m,l})$, $D \in C^k(\mathbb{I}, \mathbb{C}^{l,n})$, $G \in C^k(\mathbb{I}, \mathbb{C}^{m,n})$. We say that the differential-algebraic system has *factored leading term* if F, D^T have pointwise full column rank r_F, r_D, respectively, with $r_D = r_F = l$.

A different formulation for systems, where the leading coefficient is factored and satisfies further requirements, was studied in [4, 29, 30, 31], where a linear differential-algebraic system of the form (3) is called *DAE with properly stated leading term* if $F \in C(\mathbb{I}, \mathbb{C}^{m,l})$, $D \in C(\mathbb{I}, \mathbb{C}^{l,n})$ with $m = n$,

$$\operatorname{kernel} F(t) \oplus \operatorname{range} D(t) = \mathbb{C}^l \text{ for all } t \in \mathbb{I} \tag{4}$$

and there exists a projector $R \in C^1(\mathbb{I}, \mathbb{C}^{l,l})$ such that

$$\operatorname{range} R(t) = \operatorname{range} D(t),\ \operatorname{kernel} R(t) = \operatorname{kernel} F(t),\ \text{for all } t \in \mathbb{I}. \tag{5}$$

If a differential-algebraic equation has factored or properly stated leading term, then clearly only Dx has to be differentiable and thus differentiability of x is not necessary in the kernel of D, i.e., one can consider the larger solution space

$$C^1_D(\mathbb{I},\mathbb{C}^n) = \{x \in C(\mathbb{I},\mathbb{C}^n) \mid Dx \in C^1(\mathbb{I},\mathbb{C}^l)\}. \tag{6}$$

In general this solution space can be further enlarged. We will discuss this for systems with factored or properly stated leading term in Section 3.

The term *weak formulation* in numerical analysis is usually associated with the concept of finding solutions by integrating against appropriately chosen test functions. We will discuss this topic in Section 4.

Another weak formulation can be obtained for linear differential-algebraic equations via a *distributional setting*. Such an approach based on the space of impulsive smooth distributions was introduced for differential-algebraic systems in [13, 14, 21] and then extended in [34, 35], see also [27]. We will discuss this approach in Section 5.

There are also other weak formulations of differential-algebraic equations that are used in conjunction with computer algebra methods, see [42, 43] or the differential-geometric approach via jet spaces [46]. We do not discuss these formulations here.

2 Preliminaries

Throughout the paper we will make use of several smooth factorizations of matrix valued functions.

The first result, which can be viewed as a smooth version of the singular value decomposition, see [15], has been discussed in different places in the literature, e.g., see [27, 38].

Lemma 3 *Let* $E \in C^k(\mathbb{I},\mathbb{C}^{m,n})$*,* $k \in \mathbb{N}_0 \cup \{\infty\}$*, with* $\operatorname{rank} E(t) = r_E$ *for all* $t \in \mathbb{I}$*. Then there exist pointwise unitary (and therefore nonsingular) functions* $U \in C^k(\mathbb{I},\mathbb{C}^{m,m})$ *and* $V \in C^k(\mathbb{I},\mathbb{C}^{n,n})$*, such that*

$$U^H E V = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \tag{7}$$

with pointwise nonsingular $\Sigma \in C^k(\mathbb{I},\mathbb{C}^{r_E,r_E})$*.*

Lemma 3 immediately leads to one-sided factorizations

$$EV = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} E_1 & 0 \end{bmatrix}, \tag{8}$$

$$U^H E = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^H = \begin{bmatrix} \tilde{E}_1 \\ 0 \end{bmatrix} \tag{9}$$

with E_1 of full column rank and $\tilde{E}_1$ of full row rank. This implies that for a matrix $E \in C^k(\mathbb{I},\mathbb{C}^{m,n})$ there exist bases, which behave as smoothly as E, of the subspaces kernel and cokernel, spanned by the last $n - r_E$, first r_E columns of V, respectively, as well as of the subspaces range and corange, which are spanned by first r_E, last $m - r_E$ columns of U, respectively.

In the concept of systems with properly stated leading term, it is not required that the matrix functions F, D^H have full column rank. This property, however, can be easily achieved via a matrix factorization.

Lemma 4 *Consider a product of matrix functions*

$$E = FD$$

with $F \in C^k(\mathbb{I}, \mathbb{C}^{m,l})$, $D \in C^k(\mathbb{I}, \mathbb{C}^{l,n})$, *where* $r_E = \operatorname{rank} E = \operatorname{rank} D = r_D$ *is constant. Then there exist unitary matrix functions* $V \in C^k(\mathbb{I}, \mathbb{C}^{l,l})$, $U \in C^k(\mathbb{I}, \mathbb{C}^{m,m})$, *and* $W \in C^k(\mathbb{I}, \mathbb{C}^{n,n})$ *such that*

$$U^H F V = \begin{bmatrix} F_{1,1} & F_{1,2} \\ 0 & F_{2,2} \end{bmatrix}, \; V^H D W = \begin{bmatrix} D_{1,1} & 0 \\ 0 & 0 \end{bmatrix} \tag{10}$$

with $F_{1,1}, D_{1,1} \in C^k(\mathbb{I}, \mathbb{C}^{r_E, r_E})$ *pointwise nonsingular. Furthermore, the matrix function* E *can be transformed as*

$$U^H E W = \begin{bmatrix} F_{1,1} & F_{1,2} \\ 0 & F_{2,2} \end{bmatrix} \begin{bmatrix} D_{1,1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} F_{1,1} \\ 0 \end{bmatrix} \begin{bmatrix} D_{1,1} & 0 \end{bmatrix}. \tag{11}$$

Proof. By Lemma 3 there exist unitary matrix functions $V \in C^k(\mathbb{I}, \mathbb{C}^{l,l})$, and $W \in C^k(\mathbb{I}, \mathbb{C}^{n,n})$ such that

$$V^H D W = \begin{bmatrix} D_{1,1} & 0 \\ 0 & 0 \end{bmatrix}$$

with $D_{1,1} \in C^k(\mathbb{I}, \mathbb{C}^{r_D, r_D})$ pointwise nonsingular. If we partition the matrix function FV as

$$FV = \begin{bmatrix} \tilde{F}_{1,1} & \tilde{F}_{1,2} \\ \tilde{F}_{2,1} & \tilde{F}_{2,2} \end{bmatrix}$$

with respect to the block structure of $V^H D W$, then we get

$$\begin{aligned} EW &= FVV^H DW = \begin{bmatrix} \tilde{F}_{1,1} & \tilde{F}_{1,2} \\ \tilde{F}_{2,1} & \tilde{F}_{2,2} \end{bmatrix} \begin{bmatrix} D_{1,1} & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \tilde{F}_{1,1} D_{1,1} & 0 \\ \tilde{F}_{2,1} D_{1,1} & 0 \end{bmatrix}. \end{aligned}$$

With the assumption $r_E = r_D$ it follows that the first block column of FV has pointwise full rank r_E. Hence, according to Lemma 3 there exists a unitary matrix function $U \in C^k(\mathbb{I}, \mathbb{C}^{m,m})$ such that

$$U^H F V = U^H \begin{bmatrix} \tilde{F}_{1,1} & \tilde{F}_{1,2} \\ \tilde{F}_{2,1} & \tilde{F}_{2,2} \end{bmatrix} = \begin{bmatrix} F_{1,1} & F_{1,2} \\ 0 & F_{2,2} \end{bmatrix}.$$

□

Remark 5 In Lemma 4 we have assumed that $\operatorname{rank} E = \operatorname{rank} D$, but analogously we may assume that $\operatorname{rank} E = \operatorname{rank} F$ and obtain in a similar way a factorization

$$U^H FV = \begin{bmatrix} F_{1,1} & 0 \\ 0 & 0 \end{bmatrix}, \; V^H DW = \begin{bmatrix} D_{1,1} & 0 \\ D_{2,1} & D_{2,2} \end{bmatrix}, \tag{12}$$

which again implies the factorization (11).

If a differential-algebraic system is given with factored or properly stated leading term, then it is often useful to analyze its properties directly from the *factored matrix triple* (F, D, G) without bringing it to the form (1). To do this, we make use of equivalence transformations that operate on this triple directly.

Given pointwise nonsingular matrix functions U, V, and W of appropriate size, we can scale the system (3) by multiplying with U from the left and by transforming the solution space by setting $x = W\tilde{x}$. We can also apply a transformation $D = V\tilde{D}$, but in this case we have to consider that

$$\frac{\mathrm{d}}{\mathrm{d}t}(Dx) = \frac{\mathrm{d}}{\mathrm{d}t}(V\tilde{D}x) = \dot{V}\tilde{D}x + V\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{D}x),$$

and hence we need that V is differentiable.

These transformations lead to the following definition of an equivalence transformation for the matrix triple (F, D, G).

Definition 6 *Two matrix triples* (F, D, G) *and* $(\tilde{F}, \tilde{D}, \tilde{G})$ *of matrix functions with* $F \in C(\mathbb{I}, \mathbb{C}^{m,l})$*,* $D \in C(\mathbb{I}, \mathbb{C}^{l,n})$ *and* $G \in C(\mathbb{I}, \mathbb{C}^{m,n})$ *are called* (globally) equivalent *if there exist pointwise nonsingular matrix functions* $U \in C(\mathbb{I}, \mathbb{C}^{m,m})$*,* $V \in C^1(\mathbb{I}, \mathbb{C}^{l,l})$ *and* $W \in C(\mathbb{I}, \mathbb{C}^{n,n})$ *such that*

$$\tilde{F} = U^{-1}FV, \; \tilde{D} = V^{-1}DW, \; \tilde{G} = U^{-1}(G + FV\frac{\mathrm{d}}{\mathrm{d}t}(V^{-1})D)W$$

as equality of functions. We write this as

$$(F, D, G) \sim (\tilde{F}, \tilde{D}, \tilde{G}).$$

Using Lemma 3, we can show that a large class of linear systems can be easily expressed as differential-algebraic systems with factored leading term.

Lemma 7 *Consider a linear system of the form (1). If* $\operatorname{rank} E = r_E$ *is constant in* $\mathbb{I}$ *and if there exists a differentiable matrix function whose columns form a basis of* $\operatorname{cokernel} E$*, then there exist matrix functions*

$$F \in C^k(\mathbb{I}, \mathbb{C}^{m,r_E}), \; D \in C^k(\mathbb{I}, \mathbb{C}^{r_E,n}), \; G \in C^k(\mathbb{I}, \mathbb{C}^{m,n})$$

such that $E = FD$*,* $A = G - F\dot{D}$ *and (1) is equivalent to*

$$F\frac{\mathrm{d}}{\mathrm{d}t}(Dx) = Gx + f. \tag{13}$$

Furthermore, D *can be chosen to have orthonormal rows.*

Conversely, if a system of the form (1) can be expressed as (13), then $E = FD$ *has constant rank and* $\operatorname{cokernel} E$ *is spanned by continuously differentiable basis functions.*

Proof. A differential-algebraic system of the form (1) can be represented by the matrix triple (E, I_n, A). According to Lemma 3 there exist unitary matrix valued functions $U \in C^k(\mathbb{I}, \mathbb{C}^{m,m})$ and $W \in C^k(\mathbb{I}, \mathbb{C}^{n,n})$ such that

$$U^H E W = \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \tag{14}$$

where $E_{11} \in C^k(\mathbb{I}, \mathbb{C}^{r_E, r_E})$ is pointwise nonsingular. If we partition

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}$$

according to the block structure of $U^H E W$, then we get

$$\begin{aligned} (E, I_n, A) \ &\sim \ \left(EW, W^H, A + EW\dot{W}^H\right) \\ &= \ \left(U \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} W_{11}^H & W_{21}^H \\ W_{12}^H & W_{22}^H \end{bmatrix}, \right. \\ &\qquad \left. A + U \begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{W}_{11}^H & \dot{W}_{21}^H \\ \dot{W}_{12}^H & \dot{W}_{22}^H \end{bmatrix}\right). \end{aligned} \tag{15}$$

Because

$$\begin{bmatrix} E_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{W}_{11}^H & \dot{W}_{21}^H \\ \dot{W}_{12}^H & \dot{W}_{22}^H \end{bmatrix} = \begin{bmatrix} E_{11}\dot{W}_{11}^H & E_{11}\dot{W}_{21}^H \\ 0 & 0 \end{bmatrix},$$

we see that only the matrix function $\begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix}$, which represents the cokernel of E, has to be differentiable. Furthermore, using the zero structure, we can replace the matrix triple (15) by the triple (F, D, G) with

$$F = U \begin{bmatrix} E_{11} \\ 0 \end{bmatrix}, \ D = \begin{bmatrix} W_{11}^H & W_{21}^H \end{bmatrix}, \ G = A + U \begin{bmatrix} E_{11}\dot{W}_{11}^H & E_{11}\dot{W}_{21}^H \\ 0 & 0 \end{bmatrix},$$

which represents a system of the form (13) with factored leading term.

For the converse, suppose that the system (1) can be expressed as (13). It is obvious that the matrix function E has pointwise constant rank. To show that $\operatorname{range} E^H = \operatorname{cokernel} E$ is spanned by continuously differentiable basis functions, note that $E^H = D^H F^H$, $\operatorname{rank} E^H(t) = \operatorname{rank} D^H(t)$ on $\mathbb{I}$. Then $\operatorname{range} E^H = \operatorname{range} D^H$ and the columns $\eta_1(t), \ldots, \eta_{r_E}(t)$ of $D^H(t)$ are linear independent and continuously differentiable on $\mathbb{I}$. It follows that $\operatorname{range} D^H(t) = \operatorname{span}\{\eta_1(t), \ldots, \eta_{r_E}(t)\}$. □

Remark 8 If a system has properly stated leading term, then by Lemma 4, it can be reformulated in the form (13) by transformed matrix functions $\tilde{F} \in C(\mathbb{I}, \mathbb{C}^{m,r_E})$ and $\tilde{D} \in (\mathbb{I}, \mathbb{C}^{r_E,n})$ of pointwise full rank $r_E = \operatorname{rank} F = \operatorname{rank} D \leq l$. In particular, if $\operatorname{rank} F = \operatorname{rank} D = r_E$, then the projector function R can be chosen as $R = I_{r_E}$, i.e., the system has factored leading term.

Conversely, every square system with factored leading term has also a properly stated leading term, while for rectangular systems with $m \neq n$ a properly stated leading term is not defined.

After presenting some preliminary results, in the next section we will now discuss strong solutions.

3 Weak formulations with strong solutions

Using the factorization in Lemma 3, it was shown in [22] that under some constant rank assumptions for the system (1), there exist nonsingular matrix valued functions $U \in C(\mathbb{I}, \mathbb{C}^{m,m})$ and $W \in C^1(\mathbb{I}, \mathbb{C}^{n,n})$ such that the transformed system (with $x = Wy$) given by

$$U^{-1}EW\dot{y} = (U^{-1}AW - U^{-1}E\dot{W})y + U^{-1}f$$

has the (normal) form (without arguments)

$$\begin{bmatrix} I_{s_0} & 0 & 0 & 0 \\ 0 & I_{d_0} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \\ \dot{y}_4 \end{bmatrix} = \begin{bmatrix} 0 & A_{12} & 0 & A_{14} \\ 0 & 0 & 0 & A_{24} \\ 0 & 0 & I_{a_0} & 0 \\ I_{s_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \\ g_5 \end{bmatrix}, \quad (16)$$

where the fourth block column has size u_0 and the fifth block row has size v_0. Subtracting the derivative of the fourth block equation from the first equation reduces the rank of the left hand side coefficient.

The resulting system can again be transformed to a system of the form (16) with new block sizes s_1, d_1, a_1, u_1, v_1. If this process is iterated, then (under the assumption that the block sizes s_i, d_i, a_i, u_i, v_i are constant) it follows that after a finite number of μ steps one obtains $s_\mu = 0$. This integer μ is called the *strangeness index* of the system.

The analysis in [22] shows that under the described constant rank assumptions the differential-algebraic system (1) is equivalent (in the sense that there is a one-to-one correspondence between the solution spaces via a pointwise nonsingular matrix function) to a differential-algebraic system of the form

$$\begin{array}{lll} (a) & \dot{x}_1 = A_{13}x_3 + f_1, & d_\mu \\ (b) & 0 = x_2 + f_2, & a_\mu \\ (c) & 0 = f_3, & v_\mu, \end{array} \quad (17)$$

where $A_{13} \in C(\mathbb{I}, \mathbb{C}^{d_\mu, u_\mu})$ and the inhomogeneities f_1, f_2, f_3 are determined from $f^{(0)}, \dots, f^{(\mu)}$, see also [27].

From the form (17) then existence and uniqueness of solutions as well as consistency of initial conditions can be read off. Furthermore, this form allows to identify exactly the minimal smoothness requirements for a strong solution, since in (17) only the variable x_1 has to be differentiable. It is, however, not easy to express these requirements in a simple way in terms of the original data.

Moreover, the form (17) is mainly good for theoretical purposes. For numerical computations it is not feasible to use derivatives of locally computed matrix functions, since this may lead to highly erroneous results. Typically, the only quantities that may be differentiated in a safe way are the original coefficent functions E, A, f or F, D, G, f if the system is given in factored form. For this reason in [25, 26], based on so-called derivative arrays as introduced in [6], a numerically computable analogue to (17) was developed, see also [27]. For the system (1) one introduces the derivative array of order $\ell \in \mathbb{N}$

$$M_\ell(t)\dot{z}_\ell = N_\ell(t)z_\ell + g_\ell(t), \tag{18}$$

where

$$\begin{aligned}
(M_\ell)_{i,j} &= \binom{i}{j}E^{(i-j)} - \binom{i}{j+1}A^{(i-j-1)},\ i,j = 0,\ldots,\ell,\\
(N_\ell)_{i,j} &= \begin{cases} A^{(i)} & \text{for } i = 0,\ldots,\ell,\ j = 0,\\ 0 & \text{otherwise,}\end{cases}\\
(z_\ell)_j &= x^{(j)},\ j = 0,\ldots,\ell,\\
(g_\ell)_i &= f^{(i)},\ i = 0,\ldots,\ell,
\end{aligned} \tag{19}$$

using the convention that $\binom{i}{j} = 0$ for $i < 0$, $j < 0$ or $j > i$.

Theorem 9 *[28] Let the strangeness index μ be well defined for the system (1) and let (M_μ, N_μ) be the associated derivative array. Setting*

$$\hat{a} = a_\mu,\ \hat{d} = d_\mu,\ \hat{v} = v_0 + \ldots + v_\mu, \tag{20}$$

the inflated pair (M_μ, N_μ) associated with (E, A) has the following properties:

1. *For all $t \in [t_0, t_f]$ the matrix function M_μ satisfies $\operatorname{rank} M_\mu(t) = (\mu+1)m - \hat{a} - \hat{v}$. This implies the existence of a smooth matrix function Z with orthonormal columns and size $((\mu+1)m, \hat{a}+\hat{v})$ satisfying $Z^H M_{\hat{\mu}} = 0$.*

2. *For all $t \in [t_0, t_f]$ the projected matrix function $\operatorname{rank} Z^H N_\mu$ satisfies $\operatorname{rank} Z^H N_\mu [I_n\ 0\ \cdots\ 0]^H = \hat{a}$ and without loss of generality Z can be partitioned as $[Z_2, Z_3]$ with Z_2 of size $((\mu+1)m, \hat{a})$ and Z_3 of size $((\mu+1)m, \hat{v})$, such that $\hat{A}_2 = Z_2^H N_\mu [I_n\, 0\ \cdots\ 0]^H$ has full row rank $\hat{a}$ and that $Z_3^H N_\mu [I_n\, 0\ \cdots\ 0]^H = 0$. Furthermore there exists a smooth matrix function T_2 with orthonormal columns and size $(n, \hat{d})$, $\hat{d} = m - \hat{a} - \hat{v}$ satisfying $\hat{A}_2 T_2 = 0$.*

3. *For all $t \in [t_0, t_f]$ then $\operatorname{rank} E(t)T_2(t) = \hat{d}$. This implies the existence of a smooth matrix function Z_1 with orthonormal columns and size $(m, \hat{d})$ so that $\hat{E}_1 = Z_1^H E$ has constant rank $\hat{d}$.*

Furthermore, system (1) has the same solution set as the strangeness-free (i.e., with vanishing strangeness index) system

$$\begin{bmatrix} \hat{E}_1(t) \\ 0 \\ 0 \end{bmatrix} \dot{x} = \begin{bmatrix} \hat{A}_1(t) \\ \hat{A}_2(t) \\ 0 \end{bmatrix} x + \begin{bmatrix} \hat{f}_1(t) \\ \hat{f}_2(t) \\ \hat{f}_3(t) \end{bmatrix}, \tag{21}$$

where $\hat{A}_1 = Z_1^H A$, $\hat{f}_1 = Z_1^H f$, $\hat{f}_i = Z_i^H g_\mu$ for $i = 2, 3$.

The properties described in parts 1.-3. of Theorem 9 guarantee the existence of the reformulation of (1) in the strangeness-free form (21). To achieve the form (21), however, the requirement that the strangeness index is well-defined is a sufficient condition, but not necessary. For this reason in [28] and also for general nonlinear systems [25, 26], the properties described in parts 1.-3. of Theorem 9 are formulated as a hypothesis and the integer μ which is then guaranteed to exist is still called the strangeness index despite the fact that the constant rank assumptions on the intermediate ranks s_i, d_i, a_i, u_i, v_i may not hold.

This hypothesis allows the reformulation of the original differential-algebraic equation in the form (21) without changing the solution set. In this way then the strangeness index concept is extended to a more general class of problems and it has been shown in [27] that all regular problems with sufficiently smooth coefficient matrices have a well-defined strangeness-index defined in this way.

Using this extended concept of a strangeness index it has been shown in [27] that for linear systems the strangeness index concept generalizes the *differentiation index*, see [5], and the *perturbation index*, see [20], to nonsquare systems. It is also closely related to the *tractability index* as defined in [16] and modified for differential-algebraic systems with properly stated leading terms in [4, 29], see [31, 40].

The condensed form (21) immediately allows the characterization of strong solutions.

Corollary 10 *Suppose that a differential-algebraic equation of the form (1) satisfies conditions 1.-3. of Theorem 9 with integers $\hat{a}$, $\hat{d}$, and $\hat{v}$ and that it has been reformulated in the form (21). Then every strong solution x lies in the space*

$$\hat{\mathbb{S}} = \{x \in C(\mathbb{I}, \mathbb{C}^n) \mid \hat{E}_1(t)x \in C^1(\mathbb{I}, \mathbb{C}^{\hat{d}})\}. \tag{22}$$

Furthermore, the space $\hat{\mathbb{S}}$ is the largest function space of strong solutions, i.e, it cannot be enlarged within the set of continuous functions.

We will now show that similar results can be obtained for systems in factored form. We only present the result analogous to (17), the corresponding result for derivative arrays may be obtained by forming the derivative array from the equation in factored form, computing the projection matrices Z_1, Z_2, Z_3 and (using a factorization) reformulating the system again in factored form.

Theorem 11 *Let $F \in C(\mathbb{I}, \mathbb{C}^{m,l})$, $D \in C(\mathbb{I}, \mathbb{C}^{l,n})$ and $G \in C(\mathbb{I}, \mathbb{C}^{m,n})$ be sufficiently smooth and* $\operatorname{rank} F = \operatorname{rank} D = l$ *for all $t \in \mathbb{I}$. Let*

$$\begin{array}{lll} (a) & T & \textit{basis of } \operatorname{kernel} D, \\ (b) & Z & \textit{basis of } \operatorname{corange} F = \operatorname{kernel} F^H, \\ (c) & T' & \textit{basis of } \operatorname{cokernel} D = \operatorname{range} D^H, \\ (d) & V & \textit{basis of } \operatorname{corange}(Z^H G T) \end{array} \tag{23}$$

and let

$$\begin{array}{llll} (a) & a & = \operatorname{rank}(Z^H A T) & \textit{(algebraic part)} \\ (b) & s & = \operatorname{rank}(V^H Z^H A T') & \textit{(strangeness)} \end{array} \tag{24}$$

be constant in $\mathbb{I}$. Then the factored triple (F, D, G) is globally equivalent to the factored triple

$$\left(\begin{bmatrix} F_1 \\ F_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} D_1 & D_2 & 0 & 0 \end{bmatrix}, \begin{bmatrix} G_{1,1} & G_{1,2} & G_{1,3} & G_{1,4} \\ G_{2,1} & G_{2,2} & G_{2,3} & G_{2,4} \\ G_{3,1} & G_{3,2} & G_{3,3} & 0 \\ 0 & G_{4,2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right), \qquad (25)$$

where all block entries are again matrix functions on $\mathbb{I}$ and the blocks $G_{3,3}$ and $G_{4,2}$ are pointwise nonsingular.

Proof. By Lemma 3 there exist smooth unitary matrix valued functions U_1 and W_1 such that

$$U_1^H F = \begin{bmatrix} F_1^{(1)} \\ 0 \end{bmatrix} \text{ and } DW_1 = \begin{bmatrix} D_1^{(1)} & 0 \end{bmatrix},$$

where $F_1^{(1)}$ and $D_1^{(1)}$ are pointwise nonsingular matrix functions of size $r \times r$. If we partition

$$U_1 = \begin{bmatrix} Z' & Z \end{bmatrix} \text{ and } W_1 = \begin{bmatrix} T' & T \end{bmatrix}$$

according to the above block structure then the columns of T and T' are bases of the kernel and the cokernel of D and the columns of Z are a basis of the corange of F. Thus we get

$$\begin{aligned} (F, D, G) &\sim \left(\begin{bmatrix} F_1^{(1)} \\ 0 \end{bmatrix}, \begin{bmatrix} D_1^{(1)} & 0 \end{bmatrix}, \begin{bmatrix} Z'^H GT' & Z'^H GT \\ Z^H GT' & Z^H GT \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} F_1^{(1)} \\ 0 \end{bmatrix}, \begin{bmatrix} D_1^{(1)} & 0 \end{bmatrix}, \begin{bmatrix} G_{1,1}^{(1)} & G_{1,2}^{(1)} \\ G_{2,1}^{(1)} & G_{2,2}^{(1)} \end{bmatrix} \right), \end{aligned}$$

where $\mathrm{rank} G_{2,2}^{(1)} = \mathrm{rank}(Z^H GT) = a$ is constant on $\mathbb{I}$ by assumption. By Lemma 3 there exist unitary matrix functions U_2 and W_2 such that

$$U_2^H G_{2,2}^{(1)} W_2 = \begin{bmatrix} G_{2,2}^{(2)} & 0 \\ 0 & 0 \end{bmatrix}$$

with $G_{2,2}^{(2)}$ pointwise nonsingular on $\mathbb{I}$ and we get

$$\begin{aligned} &\left(\begin{bmatrix} F_1^{(1)} \\ 0 \end{bmatrix}, \begin{bmatrix} D_1^{(1)} & 0 \end{bmatrix}, \begin{bmatrix} G_{1,1}^{(1)} & G_{1,2}^{(1)} \\ G_{2,1}^{(1)} & G_{2,2}^{(1)} \end{bmatrix} \right) \\ \sim &\left(\begin{bmatrix} F_1^{(2)} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} D_1^{(2)} & 0 & 0 \end{bmatrix}, \begin{bmatrix} G_{1,1}^{(2)} & G_{1,2}^{(2)} & G_{1,3}^{(2)} \\ G_{2,1}^{(2)} & G_{2,2}^{(2)} & 0 \\ G_{3,1}^{(2)} & 0 & 0 \end{bmatrix} \right), \end{aligned}$$

where $F_1^{(2)} = F_1^{(1)}$ and $D_1^{(2)} = D_1^{(1)}$ and the matrix functions $G_{i,j}^{(2)}$ denote the blocks of the transformed function G. If we partition

$$U_2 = \begin{bmatrix} V' & V \end{bmatrix}$$

such that the columns of V are a basis of the corange of $G_{2,2}^{(1)}$ then the function $G_{3,1}^{(2)}$ can be written as

$$G_{3,1}^{(2)} = V^H Z^H G T'$$

and thus $\operatorname{rank} G_{3,1}^{(2)} = s$ is constant on $\mathbb{I}$ by assumption. By Lemma 3 there exist unitary matrix functions U_3 and W_3 such that

$$U_3^H G_{3,1}^{(2)} W_3 = \begin{bmatrix} 0 & G_{4,2} \\ 0 & 0 \end{bmatrix}$$

with $G_{4,2}$ pointwise nonsingular. Applying these transformations to the factored triple finally gives the factored triple (25), where

$$D_1^{(2)} W_3 = \begin{bmatrix} D_1 & D_2 \end{bmatrix} \quad \text{and} \quad F_1^{(2)} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

are partitioned such that F_1 and D_1 are of size $d \times l$ and $l \times d$ respectively with $d = \dim(\operatorname{kernel}(G_{3,1}^{(2)}))$. □

The factored triple (25) represents a differential-algebraic equation with factored leading term of the form

$$\begin{array}{lrcl} \text{(a)} & F_1 \frac{\mathrm{d}}{\mathrm{d}t}(D_1 x_1 + D_2 x_2) & = & G_{1,1} x_1 + G_{1,2} x_2 + G_{1,3} x_3 + G_{1,4} x_4 + f_1, \\ \text{(b)} & F_2 \frac{\mathrm{d}}{\mathrm{d}t}(D_1 x_1 + D_2 x_2) & = & G_{2,1} x_1 + G_{2,2} x_2 + G_{2,3} x_3 + G_{2,4} x_4 + f_2, \\ \text{(c)} & 0 & = & G_{3,1} x_1 + G_{3,2} x_2 + G_{3,3} x_3 + f_3, \\ \text{(d)} & 0 & = & G_{4,2} x_2 + f_4, \\ \text{(e)} & 0 & = & f_5. \end{array} \tag{26}$$

Equation (26c) represents an algebraic condition and (26e) gives a solvability condition for the differential-algebraic system. Equation (26d) can be solved for x_2 and the result can be inserted into the equations (26a) and (26b) if the *smoothness condition*

$$D_2 x_2 = -D_2 G_{4,2}^{-1} f_4 \in C^1(\mathbb{I}, \mathbb{C}^l) \tag{27}$$

is satisfied. We can then replace the equations (26a) and (26b) by

$$\begin{array}{lrcl} \text{(a)} & F_1 \frac{\mathrm{d}}{\mathrm{d}t}(D_1 x_1) & = & G_{1,1} x_1 + G_{1,2} x_2 + G_{1,3} x_3 + G_{1,4} x_4 + f_1 \\ & & & + F_1 \frac{\mathrm{d}}{\mathrm{d}t}(D_2 G_{4,2}^{-1} f_4), \\ \text{(b)} & F_2 \frac{\mathrm{d}}{\mathrm{d}t} dt(D_1 x_1) & = & G_{2,1} x_1 + G_{2,2} x_2 + G_{2,3} x_3 + G_{2,4} x_4 + f_2 \\ & & & + F_2 \frac{\mathrm{d}}{\mathrm{d}t}(D_2 G_{4,2}^{-1} f_4). \end{array} \tag{28}$$

This leads to a system which does not have a factored leading term, because the function D_1 does not have pointwise full row rank. However, we have

$$\operatorname{rank} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} D_1 = d,$$

and thus according to Lemma 4 there exist unitary matrix functions U, V and W such that

$$U^H \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} V V^H D_1 W = \begin{bmatrix} F_{1,1} & F_{1,2} \\ 0 & F_{2,2} \end{bmatrix} \begin{bmatrix} D_{1,1} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} F_{1,1} \\ 0 \end{bmatrix} \begin{bmatrix} D_{1,1} & 0 \end{bmatrix},$$

where $F_{1,1}$ and $D_{1,1}$ are pointwise nonsingular matrix functions of size $d \times d$. After applying these transformations to the modified system consisting of the equations (28a,b) and (26c,d,e) we end up with a system with factored leading term which can be represented by a factored triple of the form

$$\left(\begin{bmatrix} F_{1,1} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} D_{1,1} & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} G_{1,1} & G_{1,2} & G_{1,3} & G_{1,4} \\ G_{2,1} & G_{2,2} & G_{2,3} & G_{2,4} \\ G_{3,1} & G_{3,2} & G_{3,3} & 0 \\ 0 & G_{4,2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right). \tag{29}$$

If the assumptions of Theorem 11 are satisfied for the triple (29), then we can again transform it to the form (25) and repeat the above process. This allows for an inductive procedure analogous to that for general linear system of differential-algebraic equations. We start with the factored triple $(F_0, D_0, G_0) = (F, D, G)$ and obtain a sequence (F_i, D_i, G_i), $i \in \mathbb{N}$, of factored triples by transforming (F_i, D_i, G_i) to the form (25). According to the above statements the corresponding system can be represented by a factored triple $(F_{i+1}, D_{i+1}, G_{i+1})$ of the form (29). In each step of this procedure the assumptions (24) and (27) have to be satisfied. This defines the sequence (r_i, a_i, s_i) of characteristic values of the triples (F_i, D_i, G_i). This sequence becomes stationary if $s_i = 0$ for some $i \in \mathbb{N}$. Because $r_{i+1} = r_i - s_i$, this is always the case after a finite number μ of steps. The corresponding index μ is a characteristic value of the tuple (F, D, G) and we call $\mu = \min\{i \in \mathbb{N}_0 : s_i = 0\}$ the *strangeness index* of the triple (F, D, G) and of (13). With the described construction we have shown the following Theorem.

Theorem 12 *Consider a system with factored leading term in the form (3). Suppose that (24) and (27) hold for each triple (F_i, D_i, G_i) of the above sequence, and let $f \in C^{\mu}(\mathbb{I}, \mathbb{C}^m)$. Then the differential-algebraic equation (3) is equivalent (in the sense that there is a one-to-one correspondence between the solution spaces via a pointwise nonsingular matrix function) to a strangeness free differential-algebraic system with factored leading term of the form*

$$\begin{bmatrix} F_1 \\ 0 \\ 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \left(\begin{bmatrix} D_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) \tag{30}$$
$$= \begin{bmatrix} G_{1,1} & G_{1,2} & G_{1,3} \\ G_{2,1} & G_{2,2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix},$$

where F_1, D_1 are pointwise nonsingular of size $d_\mu \times d_\mu$, $d_\mu = r_\mu$, and $G_{2,2}$ is pointwise nonsingular of size $a_\mu \times a_\mu$. The inhomogeneity is determined from $f, \dot{f}, \ldots, f^{(\mu)}$.

Remark 13 It has been shown in [22, 23] that the block sizes in (30) are invariants of the differential-algebraic equation. If we would multiply out the factors at every step of the reduction procedure and transform the system to the standard from (1), then after some further non-unitary transformations this procedure would lead directly to the canonical form (17). For this reason the notation that uses the same quantities r_i, a_i, s_i and μ is justified.

Remark 14 The points where any of the constant rank assumptions (24) are violated are candidates for critical points, where the solution behavior changes. However, even if a rank change happens, then this may be a removable singularity, like in the example,

$$\begin{aligned} t\dot{x}(t) &= tx(t), \qquad t \in (-1,1), \\ x(-1) &= 0, \end{aligned}$$

where the solution is 0 everywhere except at $t = 0$ where it is not characterized. But clearly it can be continued at $t = 0$ if we require a classical solution, i.e., that x is at least continuous.

In summary, we obtain the following existence and uniqueness result for systems with factored leading term.

Corollary 15 *Consider a system of the form (3) and suppose that the transformation to the normal form (30) exists, i.e., the constant rank and differentiability conditions (24) and (27) are satisfied. Then*

1. *the problem (3) is solvable if and only if the $v_\mu = m - d_\mu - a_\mu$ functional consistency conditions*
$$f_3 = 0 \tag{31}$$
are fulfilled,

2. *an initial condition $x(t_0) = \xi$ is consistent if and only if in addition the a_μ conditions*
$$G_{2,1}x_1(t_0) + G_{2,2}x_2(t_0) + f_2(t_0) = 0 \tag{32}$$
are implied by the initial condition,

3. *the corresponding initial value problem is uniquely solvable if and only if in addition*
$$u_\mu = n - d_\mu - a_\mu = 0 \tag{33}$$
holds.

As we have seen in the previous analysis, it is not necessary to require the solution to be differentiable in all components. If the system is in one of the normal forms (17) or (30) or in the condensed form (21) then we can determine the minimal smoothness requirements for classical solutions. For systems of the form (3) with properly stated leading term therefore in [29] the space

$$C^1_D(\mathbb{I}, \mathbb{C}^n) = \{x \in C(\mathbb{I}, \mathbb{C}^n) \mid Dx \in C^1(\mathbb{I}, \mathbb{C}^l)\}. \tag{34}$$

was introduced and the minimal smoothness requirement in this solution space has been characterized using a projector chain in [30].

For general linear differential-algebraic equation systems of the form (1) we can alternatively consider the projector function E^+E which projects onto the cokernel of E. Here, E^+ denotes the Moore-Penrose pseudo-inverse of E that can be computed using the decomposition of Lemma 3. With this, a weaker solution space for (strangeness-free) systems of the form (1) has been defined in [24] as

$$C^1_{E^+E}(\mathbb{I}, \mathbb{C}^n) = \{x \in C(\mathbb{I}, \mathbb{C}^n) \mid E^+Ex \in C^1(\mathbb{I}, \mathbb{C}^n)\}. \tag{35}$$

The following Lemma shows that for systems with factored leading term, both solution spaces are equal provided that the matrix function D is continuously differentiable.

Lemma 16 *Let $F \in C(\mathbb{I}, \mathbb{C}^{m,l})$ and $D \in C^1(\mathbb{I}, \mathbb{C}^{l,n})$ be two matrix functions with $\operatorname{rank} F = \operatorname{rank} D = \operatorname{rank} FD = r$ for all $t \in \mathbb{I}$. Then for $E = FD$ we get*

$$C^1_D(\mathbb{I}, \mathbb{C}^n) = C^1_{E^+E}(\mathbb{I}, \mathbb{C}^n).$$

Proof. The Moore-Penrose pseudo-inverse of $E = FD$ can be computed using the decomposition of Lemma 4 which gives

$$E = U \begin{bmatrix} F_{1,1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} D_{1,1} & 0 \\ 0 & 0 \end{bmatrix} W^H$$

with pointwise nonsingular matrix functions $F_{1,1}$ and $D_{1,1}$ of size $r \times r$ and unitary matrix functions U and W. The pseudo-inverse can then be computed as

$$E^+ = W \begin{bmatrix} D_{1,1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_{1,1}^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^H$$

and we get

$$E^+E = W \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} W^H = D^+D$$

and thus

$$\begin{aligned} \{x \in C(\mathbb{I}, \mathbb{C}^n) \mid Dx \in C^1(\mathbb{I}, \mathbb{C}^l)\} &\subseteq \{x \in C(\mathbb{I}, \mathbb{C}^n) \mid D^+Dx \in C^1(\mathbb{I}, \mathbb{C}^n)\} \\ &= \{x \in C(\mathbb{I}, \mathbb{C}^n) \mid E^+Ex \in C^1(\mathbb{I}, \mathbb{C}^n)\} \\ &\subseteq \{x \in C(\mathbb{I}, \mathbb{C}^n) \mid DD^+Dx \in C^1(\mathbb{I}, \mathbb{C}^l)\} \\ &= \{x \in C(\mathbb{I}, \mathbb{C}^n) \mid Dx \in C^1(\mathbb{I}, \mathbb{C}^l)\}. \end{aligned}$$

□

Note that for systems with a strangeness index $\mu > 0$ these spaces can still be enlarged by studying the equivalent strangeness-free normal form (30). For this let W be the product of all transformation matrix functions determined in each step of the transformation procedure leading to the form (30). Then in terms of the partitioned vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

in (30) we have for the original solution vector x in (1) that

$$x = W \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} W_{1,1} & W_{1,2} & W_{1,3} \\ W_{2,1} & W_{2,2} & W_{2,3} \\ W_{3,1} & W_{3,2} & W_{3,3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

and the minimal smoothness requirement for a strong solution is that

$$D_1 \begin{bmatrix} W_{1,1} & W_{1,2} & W_{1,3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in C^1(\mathbb{I}, \mathbb{C}^{d_\mu}). \tag{36}$$

Thus, we immediately have the following Corollary.

Corollary 17 *Consider a system with factored leading term in the form (3). Suppose that the constant rank conditions (24) and (27) hold for each triple* (F_i, D_i, G_i) *of the above sequence and that* $f \in C^\mu(\mathbb{I}, \mathbb{C}^m)$*. If* W *is the product of all transformation matrix functions determined in each step of the transformation procedure leading to the form (30), then every strong solution* x *lies in the space*

$$\mathbb{S} = \{x \in C(\mathbb{I}, \mathbb{C}^n) \mid D_1 \begin{bmatrix} W_{1,1} & W_{1,2} & W_{1,3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in C^1(\mathbb{I}, \mathbb{C}^{d_\mu})\}. \tag{37}$$

Thus, for systems with factored leading term for which the constant rank conditions (24) and (27) hold, we have

$$C^1_D(\mathbb{I}, \mathbb{C}^n) = C^1_{E^+E}(\mathbb{I}, \mathbb{C}^n) \subseteq \mathbb{S} \tag{38}$$

and the inclusion is proper whenever $\mu > 0$.

In a similar way it has been shown via projector chains in [29, 30] how the solution space may be enlarged for systems with properly stated leading term. The major difference between the resulting solution spaces is that using (30) the space is characterized in terms of a transformed system and the back-transformation has to be carried out to get the solution space in the original variables, while in the characterization of [29, 30] the space is characterized in terms of a projector which is constructed via a matrix chain.

However, as we have already noted above, the normal forms (17) and (30) and also the matrix chains of [29, 30] need further assumptions than (21) and cannot be computed well numerically. Thus, we have

$$C^1_D(\mathbb{I}, \mathbb{C}^n) = C^1_{E^+E}(\mathbb{I}, \mathbb{C}^n) \subseteq \mathbb{S} \subseteq \hat{\mathbb{S}}, \tag{39}$$

where $\hat{\mathbb{S}}$ is defined in (22).
The last inclusion here is proper in the sense that there exist examples such as

$$\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}, \quad \mathbb{I} = [-1, 1],$$

see, e. g., [34], where the form (17) and the projector chains of [29] do not exist, while the form (21) can be determined and is

$$0 = x + \hat{f}, \quad \hat{f}(t) = \begin{bmatrix} f_1(t) + t\dot{f}_2(t) \\ f_2(t) \end{bmatrix}$$

and hence here $\hat{\mathbb{S}} = C(\mathbb{I}, \mathbb{C}^2)$.

4 Weak formulations and weak solutions

In numerical analysis the concept of weak formulations or weak solutions is used in a different way than in the previous section.

Let $C_0^\infty(\mathbb{I}, \mathbb{C}^n)$ be the space of infinite times differentiable functions $\phi : \mathbb{I} \to \mathbb{C}^n$ which have compact support in the open interval (t_0, t_f). $L^2(\mathbb{I}, \mathbb{C}^n)$ denotes the usual Lebesgue space of square integrable functions $x : \mathbb{I} \to \mathbb{C}^n$, equipped with the scalar product

$$(x, y) = \int_{\mathbb{I}} y(t)^H x(t) \, \mathrm{d}t.$$

We say that a function $x \in L^2(\mathbb{I}, \mathbb{C}^n)$ has a (weak) time derivative $y =: \partial_t x$, if $y \in L^2(\mathbb{I}, \mathbb{C}^n)$ and

$$(\phi, y) = -(\frac{\mathrm{d}}{\mathrm{d}t}\phi, x) \quad \text{for all } \phi \in C_0^\infty(\mathbb{I}, \mathbb{C}^n).$$

The subspace of functions $x \in L^2(\mathbb{I}, \mathbb{C}^n)$ possessing such a weak derivative $\partial_t x$ is the Sobolev space $H^1(\mathbb{I}, \mathbb{C}^n)$, e.g., see [1] for details.

In order to consider the strangeness-free system (21) in a weaker sense, we test (21) against all test functions $\phi \in C_0^\infty(\mathbb{I}, \mathbb{C}^{\hat{d}+\hat{a}+\hat{v}})$, or equivalently,

$$(\hat{E}_1 \partial_t x, \phi_1) = (\hat{A}_1 x, \phi_1) + (\hat{f}_1, \phi_1), \qquad \phi_1 \in C_0^\infty(\mathbb{I}, \mathbb{C}^{\hat{d}}), \tag{40a}$$
$$0 = (\hat{A}_2 x, \phi_2) + (\hat{f}_2, \phi_2), \qquad \phi_2 \in C_0^\infty(\mathbb{I}, \mathbb{C}^{\hat{a}}), \tag{40b}$$
$$0 = (\hat{f}_3, \phi_3), \qquad \phi_3 \in C_0^\infty(\mathbb{I}, \mathbb{C}^{\hat{v}}). \tag{40c}$$

We see that the DAE (40) now only has to hold in an L^2-sense, i.e., the differential equations, the algebraic constraints, and the consistency conditions for the inhomogeneity may be violated on a set of measure zero.

In order to interpret (40a) in an even weaker sense, we make the assumption that $\hat{E}_1 \in C^\infty(\mathbb{I}, \mathbb{C}^{\hat{d},n})$ and introduce the restricted space of test functions

$$C^\infty_{0,\hat{E}_1}(\mathbb{I}, \mathbb{C}^n) = \{\hat{\phi} \in C_0^\infty(\mathbb{I}, \mathbb{C}^n) \mid \hat{\phi} = \hat{E}_1^H \phi_1 \text{ for } \phi_1 \in C_0^\infty(\mathbb{I}, \mathbb{C}^{\hat{d}})\}.$$

We say that a function $x \in L^2(\mathbb{I}, \mathbb{C}^n)$ has a derivative $y =: \partial_t^{(\hat{E}_1)} x$, which is *weak in the DAE sense*, if $y \in L^2(\mathbb{I}, \mathbb{C}^n)$ and

$$(\hat{\phi}, y) = -(\frac{\mathrm{d}}{\mathrm{d}t}\hat{\phi}, x) \quad \text{for all } \hat{\phi} \in C^\infty_{0,\hat{E}_1}(\mathbb{I}, \mathbb{C}^n).$$

The derivative becomes unique by requiring in addition that $\partial_t^{(\hat{E}_1)} x(t) \in \operatorname{cokernel}(\hat{E}_1(t))$ for almost every $t \in \mathbb{I}$. The subspace of functions which are weakly differentiable in the DAE sense is denoted by

$$\check{\mathbb{S}} = \{x \in L^2(\mathbb{I}, \mathbb{C}^n) \mid \partial_t^{(\hat{E}_1)} x \text{ exists in } L^2(\mathbb{I}, \mathbb{C}^n)\}. \tag{41}$$

Replacing $\partial_t x$ in (40a) by the (in the DAE sense) weak derivative $\partial_t^{(\hat{E}_1)} x$, equation (40a) makes sense not only for functions $x \in H^1(\mathbb{I}, \mathbb{C}^n)$ but also for $x \in \check{\mathbb{S}} \supseteq H^1(\mathbb{I}, \mathbb{C}^n)$.

We note that for $x \in \hat{\mathbb{S}}$, we have

$$g := \frac{\mathrm{d}}{\mathrm{d}t}(\hat{E}_1 x) - (\frac{\mathrm{d}}{\mathrm{d}t}\hat{E}_1)x \in C(\mathbb{I}, \mathbb{C}^{\hat{d}}),$$

and since $\hat{E}_1$ is smooth and has full rank, there exists a unique function $y \in C(\mathbb{I}, \mathbb{C}^n)$ with $y(t) \in \operatorname{cokernel}(\hat{E}_1(t))$ for $t \in \mathbb{I}$ satisfying $\hat{E}_1 y = g$. Hence, $\partial_t^{(\hat{E}_1)} x = y \in L^2(\mathbb{I}, \mathbb{C}^n)$ and $\hat{\mathbb{S}} \subseteq \check{\mathbb{S}}$ whenever $\mathbb{I}$ is a compact interval.

It is a classical result that $H^1(\mathbb{I}, \mathbb{C}^n)$ can be embedded into $C(\mathbb{I}, \mathbb{C}^n)$, e.g., see [10]. For $x \in \check{\mathbb{S}}$ we only have that $\hat{E}_1 x \in H^1(\mathbb{I}, \mathbb{C}^{\hat{d}})$ and thus there is a continuous representative of $\hat{E}_1 x$. Since $\hat{E}_1$ is smooth and has full rank, the dynamical components of x (lying in $\operatorname{cokernel}(\hat{E}_1)$) are also continuous, such that the prescription of initial values for the underlying ODE still makes sense.

5 Weak formulations and distributional solutions

Another way to relax the smoothness requirements on the inhomogeneity f is to allow generalized functions (or distributions) as solutions of (1) or (3). Such an approach, which uses a particular class of distributions as solutions of differential-algebraic equations, allows to include non-differentiable and even discontinuous inhomogeneities as well as non-consistent initial values and was introduced in the context of constant coefficient control problems, see [12, 13, 14, 21]. This concept was then studied and extended to differential-algebraic systems with variable coefficients in [34, 35], see also [27]. We briefly review this concept here and discuss it in the context of systems with factored leading term.

Let $\mathcal{C}$ be the space of all distributions acting on the set $\mathcal{D}$ of test functions, see [41]. The *Dirac delta distribution* $\delta \in \mathcal{C}$ is defined by

$$\delta(\phi) = \phi(0) \text{ for all } \phi \in \mathcal{D}.$$

We can then define a subspace of $\mathcal{C}$ which is appropriate for solutions of differential-algebraic equations. The basic idea is to restrict the nonsmooth behaviour of these solutions to a single point $t^* \in \mathbb{I}$ such that we can assign values to these distributions away from t^*. For simplicity we set $t^* = 0$ here. It has been discussed in [27], how one can extend this to the case where the nonsmooth behaviour happens at a countable number of points.

Definition 18 *[21, 41] A generalized function $x \in \mathcal{C}$ is called* impulsive smooth *if it can be written in the form*

$$x = x_- + x_+ + x_{\text{imp}}, \tag{42}$$

where $x_- \in C^\infty((-\infty, 0], \mathbb{C})$, $x_+ \in C^\infty([0, \infty), \mathbb{C})$ and the impulsive *part x_{imp} has the form*

$$x_{\text{imp}} = \sum_{i=0}^{q} c_i \delta^{(i)}, \quad c_i \in \mathbb{C},\ i = 0, \dots, q, \tag{43}$$

with some $q \in \mathbb{N}_0$. The set of impulsive smooth distributions is denoted by $\mathcal{C}_{\text{imp}}$.

A distribution $x \in \mathcal{C}_{\text{imp}}$ uniquely determines the decomposition (42). The set $\mathcal{C}_{\text{imp}}$ is a complex vector space and it is closed under multiplications with functions $A \in C^\infty(\mathbb{R}, \mathbb{C})$. In particular, we have

$$Ax = Ax_- + Ax_+ + \sum_{i=0}^{q} \sum_{j=0}^{q-i} (-1)^j \tbinom{j+i}{j} A^{(j)}(0) c_{i+j} \delta^{(i)} \tag{44}$$

for x with (42). These definitions can easily be extended to the n-dimensional case. A vector $x \in \mathcal{C}^n_{\text{imp}}$ can then be multiplied with a matrix function $A \in C^\infty(\mathbb{R}, \mathbb{C}^{m,n})$ by decomposing x according to (42), where we replace $\mathbb{C}$ by $\mathbb{C}^n$, and computing the distribution $Ax \in \mathcal{C}^m_{\text{imp}}$ by (44).

Finally, we will need a measure for the smoothness of impulsive smooth distributions.

Definition 19 *[34, 35] Let the impulsive part of $x \in \mathcal{C}^n_{\text{imp}}$ have the form*

$$x_{\text{imp}} = \sum_{i=0}^{q} c_i \delta^{(i)}, \quad c_i \in \mathbb{C}^n,\ i = 0, \dots, q. \tag{45}$$

The impulse order *of x is defined as $\operatorname{iord} x = -q - 2$ if x can be associated with a continuous function and q with $0 \le q \le \infty$ is the largest integer such that $x \in C^q(\mathbb{R}, \mathbb{C})$. It is defined as $\operatorname{iord} x = -1$ if x can be associated with a function that is continuous everywhere except at $t = 0$ and it is defined as*

$$\operatorname{iord} x = \max\{i \in \mathbb{N}_0 \mid 0 \le i \le q,\ c_i \ne 0\}$$

otherwise.

Lemma 20 *Let $x \in \mathcal{C}^n_{\text{imp}}$ and $A \in C^\infty(\mathbb{R}, \mathbb{C}^{m,n})$. Then*

$$\operatorname{iord} Ax \le \operatorname{iord} x$$

with equality for $m = n$ and $A(0)$ invertible.

Proof. This is a direct consequence of (44). □

For a detailed analysis of the distributional formulation, see [34, 35] or [27].

Let us now, for completeness, consider the distributional version of a differential-algebraic equation with factored leading term

$$F\frac{\mathrm{d}}{\mathrm{d}t}(Dx) = Gx + f,\ t \in \mathbb{I} \tag{46}$$

with $f \in \mathcal{C}^m_{\text{imp}}$. Looking for solutions $x \in \mathcal{C}^n_{\text{imp}}$ we must require that $F \in C^\infty(\mathbb{R}, \mathbb{C}^{m,l})$, $D \in C^\infty(\mathbb{R}, \mathbb{C}^{l,n})$ and $G \in C^\infty(\mathbb{R}, \mathbb{C}^{m,n})$ in order to have well-defined products $F\frac{\mathrm{d}}{\mathrm{d}t}(Dx)$ and Gx. Lemma 3 is also valid for infinitely often differentiable functions and thus all the techniques of the previous sections can be applied to these distributional differential-algebraic systems. In particular, we can obtain systems like (26) and (30) in the same way with infinitely often differentiable matrix functions but without the smoothness requirements for the inhomogeneity. We can merge these results into the following statement.

Theorem 21 *Let $F \in C^\infty(\mathbb{R}, \mathbb{C}^{m,l})$, $D \in C^\infty(\mathbb{R}, \mathbb{C}^{l,n})$ and $G \in C^\infty(\mathbb{R}, \mathbb{C}^{m,n})$ and let the strangeness index μ of the factored matrix tuple (F, D, G) be well-defined. Let $f \in \mathcal{C}^m_{\text{imp}}$ with $\operatorname{iord} f = q \in \mathbb{Z} \cup \{-\infty\}$. Then the differential-algebraic equation (46) is equivalent (in the sense that there is a one-to-one correspondence between the solution spaces via a pointwise nonsingular infinitely often differentiable matrix function) to a differential-algebraic equation of the form*

$$\begin{bmatrix} F_1 \\ 0 \\ 0 \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}\left(\begin{bmatrix} D_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) \tag{47}$$

$$= \begin{bmatrix} G_{1,1} & G_{1,2} & G_{1,3} \\ G_{2,1} & G_{2,2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix},$$

where $F_1, D_1 \in C^\infty(\mathbb{R}, \mathbb{C}^{d_\mu, d_\mu})$, $d_\mu = r_\mu$, and $G_{2,2} \in C^\infty(\mathbb{R}, \mathbb{C}^{a_\mu, a_\mu})$ are pointwise nonsingular and $\operatorname{iord}([f_1^T\ f_2^T\ f_3^T]^T) \leq q + \mu$.

Proof. All corresponding constructions can be executed by infinitely often differentiable matrix functions due to Theorem 3. The inhomogeneity $[f_1^T\ f_2^T\ f_3^T]^T$ is determined from $f, \dot{f}, \ldots, f^{(\mu)}$ via infinitely often differentiable matrix functions. ◻

The following solvability and uniqueness results then follow immediately.

Corollary 22 *Let $F \in C^\infty(\mathbb{R}, \mathbb{C}^{m,l})$, $D \in C^\infty(\mathbb{R}, \mathbb{C}^{l,n})$, and $G \in C^\infty(\mathbb{R}, \mathbb{C}^{m,n})$ satisfy the assumptions of Theorem 21. Then, we have:*

1. *Problem (46) has a solution in $\mathcal{C}^n_{\text{imp}}$ if and only if the $v_\mu = m - d_\mu - a_\mu$ distributional conditions*

$$f_3 = 0 \tag{48}$$

are fulfilled.

2. *Let* $t_0 \neq 0$ *and* $x_0 \in \mathbb{C}^n$. *There is a solution* $x \in \mathcal{C}^n_{\mathrm{imp}}$ *satisfying one of the initial conditions*

$$x(t_0) = x_0, \quad x(0^-) = x_0, \quad x(0^+) = x_0 \tag{49}$$

if and only if in addition to (48) the corresponding condition out of

$$x_2(t_0) = -f_2(t_0), \quad x_2(0^-) = -f_2(0^-), \quad x_2(0^+) = -f_2(0^+) \tag{50}$$

is implied by the initial condition.

3. *The corresponding initial value problem has a unique solution in* $\mathcal{C}^n_{\mathrm{imp}}$ *if and only if in addition*

$$u_\mu = n - d_\mu - a_\mu = 0 \tag{51}$$

holds.

Moreover, all solutions x *satisfy* $\operatorname{iord} x \leq \max\{q + \mu, \operatorname{iord} x_3\}$.

Inconsistent initial values can be treated within the distributional setting as well, see [34, 35] and [27]. For systems with factored leading term this can be done analogously. The basic idea is to change the inhomogeneity of the differential-algebraic system such that it satisfies a given history and to solve the modified system in the context of nonsmooth inhomogeneities.

Suppose that (46) satisfies the assumptions of Theorem 21 with $u_\mu = v_\mu = 0$ which implies that it is uniquely solvable. Consider $x_-^0 \in C^\infty((-\infty, 0], \mathbb{C}^n)$ that indicates how the system has behaved until $t = 0$. Setting

$$f_- = F\frac{\mathrm{d}}{\mathrm{d}t}(Dx_-^0) - Gx_-^0 \tag{52}$$

forces x_-^0 to be a solution for the part

$$F\frac{\mathrm{d}}{\mathrm{d}t}(Dx) = Gx + f_-, \; t \in (-\infty, 0] \tag{53}$$

of (46) thus making the initial condition consistent. Then, according to Theorem 21, the problem

$$F\frac{\mathrm{d}}{\mathrm{d}t}(Dx) = Gx + f, \quad x_- = x_-^0, \tag{54}$$

where f satisfies (52), has a unique solution $x \in \mathcal{C}^n_{\mathrm{imp}}$. Now let x be this solution and let

$$\begin{aligned} x &= x_- + x_+ + x_{\mathrm{imp}}, \\ f &= f_- + f_+ + f_{\mathrm{imp}} \end{aligned}$$

according to (42). Because

$$\dot{x} = \dot{x}_- + \dot{x}_+ + \dot{x}_{\mathrm{imp}} + (x_+(0) - x_-(0))\delta,$$

the system (54) can be written as

$$F\dot{D}x + FD(\dot{x}_- + \dot{x}_+ + \dot{x}_{\mathrm{imp}} + (x_+(0) - x_-(0))\delta) = Gx + f_- + f_+ + f_{\mathrm{imp}}$$

with $x_- = x_-^0$ and due to (52) we get

$$F\dot{D}(x - x_-) + FD(\dot{x}_+ + \dot{x}_{\mathrm{imp}} + x_+(0)\delta) = G(x - x_-) + FDx_-(0)\delta + f_+ + f_{\mathrm{imp}}.$$

Setting $\tilde{x} = x - x_-$ and $\tilde{f} = f - f_-$, this can be expressed in the form

$$F\frac{\mathrm{d}}{\mathrm{d}t}(D\tilde{x}) = G\tilde{x} + FDx_0\delta + \tilde{f}, \quad \tilde{x}_- = 0, \tag{55}$$

where $x_0 = x_-(0)$. This shows that the impulsive behaviour and the future smooth development of the system does not depend on the whole history but only on the (possibly inconsistent) initial condition. In this sense, problem (55) is the adequate form to treat inconsistent initial conditions. Observe that the initial condition does not occur as it is stated in the classical formulation (we cannot prescribe values of distributions) but as part of the inhomogeneity.

If we turn over to general (possibly nonsquare) systems of the form

$$F\frac{\mathrm{d}}{\mathrm{d}t}(Dx) = Gx + FDx_0\delta + f, \quad \tilde{x}_- = 0, \tag{56}$$

then the formulation (56) of an initial value problem suggests that for sufficiently smooth f the smoothness of x will depend on the initial condition. We therefore assume now that in the problem (56) the distribution $f \in \mathcal{C}_{\mathrm{imp}}^m$ has $\operatorname{iord} f \leq -1$ and satisfies $f_- = 0$.

Definition 23 *Let $f \in \mathcal{C}_{\mathrm{imp}}^m$ be given with $f_- = 0$ and* $\operatorname{iord} f \leq -1$. *We say that $x_0 \in \mathbb{C}^n$ is* weakly consistent *with f if there exists a solution $x \in \mathcal{C}_{\mathrm{imp}}^n$ of (56) with* $\operatorname{iord} x \leq -1$. *We say that x_0 is* consistent *with f if x_0 is weakly consistent with f and there exists a solution $x \in \mathcal{C}_{\mathrm{imp}}^n$ of (56) satisfying $x(0^+) = x_0$.*

Theorem 24 *Let the strangeness index μ of $((FD), G)$ in (55) be well-defined, let $v_\mu = 0$, and let $f \in \mathcal{C}_{\mathrm{imp}}^m$ be given with $f_- = 0$ and* $\operatorname{iord} f \leq -1$.

1. *All vectors $x_0 \in \mathbb{C}^n$ are consistent with f if and only if $\mu = 0$ and $a_0 = 0$.*
2. *All vectors $x_0 \in \mathbb{C}^n$ are weakly consistent with f if and only if $\mu = 0$.*

All results presented in this section can be extended to the case of discontinuities or initial values given at points $t_0 \neq 0$ and also to the case where the nonsmooth behaviour of the inhomogeneity or inconsistent initial conditions occur at a countable number of points, see [27].

We have seen in this section, that via the distributional setting the solution spaces can be further increased whenever the normal forms (17) or (30) exist, i.e., $\mathbb{S}$ can be interpreted as a subset of the set of distributional solutions.

6 Multibody systems

The use of derivative arrays to determine the strangeness-free formulation (30) can be significantly simplified if the system has extra structure, such as in the case of multibody systems or electrical circuits. We will discuss this topic here only briefly.

Consider first the equations of motion of a constrained multibody system in linear time dependent form

$$M\ddot{p} = Cp + K\dot{p} - H^T\lambda, \tag{57a}$$

$$0 = Hp, \tag{57b}$$

where $M, C, K \in C^1(\mathbb{I}, \mathbb{R}^{n_p,n_p})$, $H \in C^3(\mathbb{I}, \mathbb{R}^{n_\lambda,n_p})$, see [9]. Here the state of the multibody system is given by the position variables $p \in C^2(\mathbb{I}, \mathbb{R}^{n_p})$. The inertia of the different bodies of the multibody system is represented by the positive definite mass matrix M and the applied forces are given by $Cp+K\dot{p}$. We assume that the systems satisfies the holonomic constraints (57b) depending only on position and time with $Hp \in C^3(\mathbb{I}, \mathbb{R}^{n_\lambda})$. The constraint forces $-H^T\lambda$ are determined by the constraint matrix function H and the Lagrange-multipliers $\lambda \in C(\mathbb{I}, \mathbb{R}^{n_\lambda})$. To avoid redundant constraints we assume that H has full row rank in $\mathbb{I}$. Furthermore, $n_f = n_p - n_\lambda$ denotes the number of degrees of freedom of the multibody system.

Typically the system is transformed to first order form by introducing the (generalized) velocity variables $v = \dot{p} \in C^1(\mathbb{I}, \mathbb{R}^{n_p})$, one has to be careful, however, in doing this transformation, see [33, 39, 47]. The resulting first order form of the equations of motion (57) is a system of strangeness index 2, see [27], and can be easily formulated as system with factored leading term

$$\underbrace{\begin{bmatrix} I_{n_p} & 0 \\ 0 & M \\ 0 & 0 \end{bmatrix}}_{F} \frac{\mathrm{d}}{\mathrm{d}t}\left(\underbrace{\begin{bmatrix} I_{n_p} & 0 & 0 \\ 0 & I_{n_p} & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} p \\ v \\ \lambda \end{bmatrix}}_{x} \right) = \underbrace{\begin{bmatrix} 0 & I_{n_p} & 0 \\ C & K & -H^T \\ H & 0 & 0 \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} p \\ v \\ \lambda \end{bmatrix}}_{x}. \tag{58}$$

For systems in this form it is easy to obtain the strangeness-free formulation (21) without much computational effort, see [2, 44]. In matrix notation it has the form

$$\underbrace{\begin{bmatrix} S_p & 0 & 0 \\ 0 & S_vM & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\hat{E}} \underbrace{\begin{bmatrix} \dot{p} \\ \dot{v} \\ \dot{\lambda} \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} S_pv \\ S_v(Cp + Kv - H^T\lambda) \\ Hp \\ h^{I}(p, v, t) \\ h^{II}(p, v, \lambda, t) \end{bmatrix}}_{\hat{A}x} \tag{59}$$

with $h^{I}(p, v, t) = \frac{\mathrm{d}}{\mathrm{d}t}(Hp)$ and $h^{II}(p, v, \lambda, t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}(Hp)$.

Here, the so-called *selectors* S_p and S_v are chosen such that

$$\begin{bmatrix} S_p \\ H \end{bmatrix}, \begin{bmatrix} S_vM \\ H \end{bmatrix} \tag{60}$$

are nonsingular.

Introducing smooth matrix functions $H^- \in C(\mathbb{I}, \mathbb{R}^{n_p,n_f})$, S_p^-, $S_v^- \in C(\mathbb{I}, \mathbb{R}^{n_p,n_\lambda})$ that satisfy

$$\begin{aligned} \operatorname{span}(H^-) &= \operatorname{kernel}(H), \\ \operatorname{span}(S_p^-) &= \operatorname{kernel}(S_p), \\ \operatorname{span}(S_v^-) &= \operatorname{kernel}(S_v M) \end{aligned}$$

we can form the transformation matrix

$$W = \begin{bmatrix} H^- & 0 & S_p^- & 0 & 0 \\ 0 & H^- & 0 & S_v^- & 0 \\ 0 & 0 & 0 & 0 & I_{n_\lambda} \end{bmatrix}.$$

Here, the nonsingularity of $\begin{bmatrix} H^- & S_p^- \end{bmatrix}$ and $\begin{bmatrix} H^- & S_v^- \end{bmatrix}$ follows from (60) and the positive definiteness of M.

We can then factorize the leading term in (59) as $\hat{E} = (\hat{E}W)W^{-1}$ with

$$\hat{E}W = \begin{bmatrix} S_pH^- & 0 & 0 & 0 & 0 \\ 0 & S_vMH^- & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$W^{-1} = \begin{bmatrix} (S_pH^-)^{-1}S_p & 0 & 0 \\ 0 & (S_vMH^-)^{-1}S_vM & 0 \\ (HS_p^-)^{-1}H & 0 & 0 \\ 0 & (HS_v^-)^{-1}H & 0 \\ 0 & 0 & I_{n_\lambda} \end{bmatrix},$$

and employing the zero structure of $\hat{E}W$ we obtain

$$\hat{E} = \underbrace{\begin{bmatrix} S_pH^- & 0 \\ 0 & S_vMH^- \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\hat{F}} \underbrace{\begin{bmatrix} (S_pH^-)^{-1}S_p & 0 & 0 \\ 0 & (S_vMH^-)^{-1}S_vM & 0 \end{bmatrix}}_{\hat{D}},$$

leading to a strangeness-free reformulation of (59) with factored leading term

$$\hat{F}\frac{\mathrm{d}}{\mathrm{d}t}(\hat{D}x) = (\hat{A} + \hat{F}\frac{\mathrm{d}}{\mathrm{d}t}\hat{D})x \tag{61}$$

and we immediately obtain the following corollary.

Corollary 25 *Consider the equations of motion of a multibody system in the form (59) and the strangeness-free form (61). Then, the maximal solution space is given by*

$$C^1_{\hat{D}} = \{x \in C(\mathbb{I}, \mathbb{R}^n) \mid \hat{D}x \in C^1(\mathbb{I}, \mathbb{R}^{2n_f})\}.$$

For other structured systems such as electrical circuits the maximal solution spaces are characterized in an equally easy way, see [11, 45].

C^1	:	space of classical solutions, cf. Def. 1
$\subseteq$		
C^1_D	:	enlarged solution space for DAEs with p.s.l.t., cf. (6) and (34)
$=$		
$C^1_{E^+E}$	:	enlarged solution space for general DAEs, cf. (35)
$\subseteq$		
$\mathbb{S}$	:	largest space of strong solutions for DAEs with f.l.t., cf. (37)
$\subseteq$		
$\hat{\mathbb{S}}$	:	largest space of strong solutions for general DAEs, cf. (22)
$\subseteq$		
$\check{\mathbb{S}}$	:	space of solutions with in the DAE sense weak derivative, cf. (41)
$\mathcal{C}_{\text{imp}}$	:	solution space of impulsive smooth distributions, cf. (42)

Figure 1: Overview of solution concepts. 'p.s.l.t.'='properly stated leading term', 'f.l.t.'='factored leading term'. The space $\mathcal{C}_{\text{imp}}$ is in general not a subset nor a superset of the other solution spaces.

7 Conclusion

We have discussed different weak formulations for linear differential algebraic systems with variable coefficients, in particular systems with factored leading term. We have shown how to characterize the maximal solution set that has the minimal smoothness requirements for the solution x. In summary, we have

$$C^1 \subseteq C^1_D = C^1_{E^+E} \subseteq \mathbb{S} \subseteq \hat{\mathbb{S}} \subseteq \check{\mathbb{S}}.$$

An overview over the spaces and their properties is given in Figure 1.

Acknowledgement

The authors would like to thank Etienne Emmrich for his constructive help in the consideration of weak solutions.

References

[1] R. A. Adams. *Sobolev spaces.* Academic Press, New York, NY, 1975.

[2] M. Arnold, V. Mehrmann, and A. Steinbrecher. Index reduction of linear equations of motions in industrial multibody system simulation. Technical Report 146, DFG Research Center Matheon in Berlin, TU Berlin, Berlin, Germany, 2004.

[3] U. M. Ascher and L. R. Petzold. *Computer Methods for Ordinary Differential and Differential-Algebraic Equations.* SIAM Publications, Philadelphia, PA, 1998.

[4] K. Balla and R. März. Linear differential algebraic equations of index 1 and their adjoint equations. *Res. in Math.*, 37:13–35, 2000.

[5] K. E. Brenan, S. L. Campbell, and L. R. Petzold. *Numerical Solution of Initial-Value Problems in Differential Algebraic Equations*. SIAM Publications, Philadelphia, PA, 2nd edition, 1996.

[6] S. L. Campbell. Comment on controlling generalized state-space (descriptor) systems. *Internat. J. Control*, 46:2229–2230, 1987.

[7] S. L. Campbell. Linearization of DAE's along trajectories. *Z. Angew. Math. Phys.*, 46:70–84, 1995.

[8] Dynasim AB, Ideon Research Park - SE-223 70 Lund - Sweden. *Dymola,Multi-Engineering Modelling and Simulation*, 2006.

[9] E. Eich-Soellner and C. Führer. *Numerical Methods in Multibody Systems*. Teubner Verlag, Stuttgart, Germany, 1998.

[10] E. Emmrich. *Gewöhnliche und Operator-Differentialgleichungen*. Vieweg Verlag, Wiesbaden, 2004.

[11] D. Estévez-Schwarz and C. Tischendorf. Structural analysis for electrical circuits and consequences for MNA. *Internat. J. Circ. Theor. Appl.*, 28:131–162, 2000.

[12] T. Geerts. *Structure of Linear-Quadratic Contro*. Phd thesis, Eindhoven, The Netherlands, Dept. of Mathematics, Univ. Eindhoven, 1989.

[13] T. Geerts. Solvability conditions, consistency, and weak consistency for linear differential-algebraic equations and time-invariant linear systems: The general case. *Lin. Alg. Appl.*, 181:111–130, 1993.

[14] T. Geerts and V. Mehrmann. Linear differential equations with constant coefficients: A distributional approach. Technical Report SFB 343/90–073, Fakultät für Mathematik, Universität Bielefeld, Bielefeld, Germany, 1990.

[15] G. H. Golub and C. F. Van Loan. *Matrix Computations*. The Johns Hopkins University Press, Baltimore, MD, 3rd edition, 1996.

[16] E. Griepentrog and R. März. *Differential-Algebraic Equations and their Numerical Treatment*. Teubner Verlag, Leipzig, Germany, 1986.

[17] M. Günther and U. Feldmann. CAD-based electric-circuit modeling in industry I. Mathematical structure and index of network equations. *Surv. Math. Ind.*, 8:97–129, 1999.

[18] M. Günther and U. Feldmann. CAD-based electric-circuit modeling in industry II. Impact of circuit configurations and parameters. *Surv. Math. Ind.*, 8:131–157, 1999.

[19] E. Hairer, C. Lubich, and M. Roche. *The Numerical Solution of Differential-Algebraic Systems by Runge-Kutta Methods.* Springer-Verlag, Berlin, Germany, 1989.

[20] E. Hairer and G. Wanner. *Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems.* Springer-Verlag, Berlin, Germany, 1st edition, 1991.

[21] M. L. J. Hautus. The formal laplace transform for smooth linear systems. In Lecture Notes in Econ. and Math. Syst., editors, *Proceedings of the International Symposium on the Mathematical System Theory 1975, Udine*, volume 131, pages 29–46, Heidelberg, Germany, 1975. Springer-Verlag.

[22] P. Kunkel and V. Mehrmann. Canonical forms for linear differential-algebraic equations with variable coefficients. *J. Comput. Appl. Math.*, 56:225–259, 1994.

[23] P. Kunkel and V. Mehrmann. A new look at pencils of matrix valued functions. *Lin. Alg. Appl.*, 212/213:215–248, 1994.

[24] P. Kunkel and V. Mehrmann. Generalized inverses of differential-algebraic operators. *SIAM J. Matr. Anal. Appl.*, 17:426–442, 1996.

[25] P. Kunkel and V. Mehrmann. Regular solutions of nonlinear differential-algebraic equations and their numerical determination. *Numer. Math.*, 79:581–600, 1998.

[26] P. Kunkel and V. Mehrmann. Analysis of over- and underdetermined nonlinear differential-algebraic systems with application to nonlinear control problems. *Math. Control, Signals, Sys.*, 14:233–256, 2001.

[27] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations. Analysis and Numerical Solution.* EMS Publishing House, Zürich, Switzerland, 2006.

[28] P. Kunkel, V. Mehrmann, and W. Rath. Analysis and numerical solution of control problems in descriptor form. *Math. Control, Signals, Sys.*, 14:29–61, 2001.

[29] R. März. The index of linear differential algebraic equations with properly stated leading terms. *Res. in Math.*, 42:308–338, 2002.

[30] R. März. Solvability of linear differential algebraic equations with properly stated leading terms. *Res. in Math.*, 45:88–105, 2004.

[31] R. März. Characterizing differential algebraic equations without the use of derivative arrays. *Computers Math. Appl.*, 50:1141–1156, 2005.

[32] The MathWorks, Inc., Natick, MA. *Simulink Version 6.4.1*, 2005.

[33] V. Mehrmann and T. Stykel. Descriptor systems: A general mathematical framework for modelling, simulation and control. *Automatisierungstechnik*, to appear, 2006.

[34] P. J. Rabier and W. C. Rheinboldt. Classical and generalized solutions of time-dependent linear differential-algebraic equations. *Lin. Alg. Appl.*, 245:259–293, 1996.

[35] P. J. Rabier and W. C. Rheinboldt. Time-dependent linear DAEs with discontinuous inputs. *Lin. Alg. Appl.*, 247:1–29, 1996.

[36] P. J. Rabier and W. C. Rheinboldt. *Nonholonomic Motion of Rigid Mechanical Systems from a DAE Viewpoint.* SIAM Publications, Philadelphia, PA, 2000.

[37] P. J. Rabier and W. C. Rheinboldt. *Theoretical and Numerical Analysis of Differential-Algebraic Equations*, volume VIII of *Handbook of Numerical Analysis.* Elsevier Publications, Amsterdam, The Netherlands, 2002.

[38] W. C. Rheinboldt. On the computation of multi-dimensional solution manifolds of parameterized equations. *Numer. Math.*, 53:165–181, 1988.

[39] J. Sand. On implicit euler for high-order high-index daes. *Appl. Numer. Math.*, 42:411–424, 2002.

[40] I. Schumilina. *Charakterisierung der Algebro-Differentialgleichungen mit Traktabilitätsindex 3.* Dissertation, Institut für Mathematik, Humboldt-Universität zu Berlin, Berlin, Germany, 2004.

[41] L. Schwartz. *Theorie des Distributions.* Hermann, Paris, France, 1978.

[42] W. M. Seiler. Indices and solvability for general systems of differential equations. Technical report, Lehrstuhl für Mathematik, Univ. Mannheim, 68131 Mannheim, Germany, 1999.

[43] W.M. Seiler. *Involution-The formal theory of differential equations and its applications in computer algebra and numerical analysis.* PhD thesis, Fakultät für Mathematik, Universität Mannheim, Mannheim, Germany, 2002. Habilitationsschrift.

[44] A. Steinbrecher. *Numerical Solution of Quasi-Linear Differential-Algebraic Equations and Industrial Simulation of Multibody Systems.* Dissertation, Institut für Mathematik, TU Berlin, Berlin, Germany, 2006.

[45] C. Tischendorf. Topological index calculation of differential-algebraic equations in circuit simulation. *Surv. Math. Ind.*, 8:187–199, 1999.

[46] J. Tuomela. On singular points of quasilinear differential and differential-algebraic equations. Research Report A369, Institute of Mathematics, Helsinki University of Technology, Espoo, Finland, 1996.

[47] L. Wunderlich. *Numerical Solution of Second Order Differential-Algebraic Equations.* Diplomarbeit, TU Berlin. Inst. f. Mathematik, Berlin, Germany, 2004.