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Regularization of nonlinear equations of motion of multibody systems by index reduction with preseving the solution manifold

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Abstract

Different types of solution behavior for equations of motion of multibody systems with respect to deviate from the solution manifold and numerical instabilities are considered. An algorithm is presented that reduces the index of linear and nonlinear equations of motion of multibody systems in the usually used form by preserving all information about the solution manifold. The reduction is obtained by analyzing only the constraint matrix, the mass matrix and the transformation matrix. This technique allows the construction of a strangeness-free form which is suitable for numerical integration using stiff ODE solvers. The here presented algorithm is the generalization of the already developed algorithm for linear equations of motion.
The obtained results are illustrated by a numerical example.

Keywords. multibody systems, equations of motion, solution manifold, deviation, index reduction, strangeness

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1 Introduction and Preliminaries

In this paper we discuss an approach for the dynamic simulation of *multibody systems*. The dynamic simulation is an important topic in the construction and design of machinery like robots, cars, aircrafts, see, e.g., [1], [21], [32]. The simulation is based on the equations of motion of multibody systems which form a system of differential equations arising from the dynamics of the system combined with algebraic equations which represent different constraints. The numerical integration of such systems of *differential-algebraic equations* is the topic of several publications, e.g., [5], [18], [19], [25], [26]. Numerical solution techniques for the *equations of motion* of multibody systems which form a system of differential-algebraic equations with special structure are considered, e.g., in [8], [27], [28], [34].

The multibody system approach usually yields the standard form of the equations of motion (EoM)

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{Z}(\mathbf{p})\mathbf{v}, & \text{(1a)} \\ \mathbf{M}(\mathbf{p})\dot{\mathbf{v}} &= \mathbf{f}(\mathbf{p},\mathbf{v},t) - \mathbf{Z}^T(\mathbf{p})\mathbf{G}^T(\mathbf{p},t)\boldsymbol{\lambda}, & \text{(1b)} \\ 0 &= \mathbf{g}(\mathbf{p},t) & \text{(1c)} \end{aligned}$$

arising from the *descriptor form*

$$\begin{aligned} \tilde{\mathbf{M}}(\mathbf{p})\ddot{\mathbf{v}} &= \tilde{\mathbf{f}}(\mathbf{p},\dot{\mathbf{p}},t) - \tilde{\mathbf{G}}^T(\mathbf{p},t)\boldsymbol{\lambda}, \\ 0 &= \tilde{\mathbf{g}}(\mathbf{p},t) \end{aligned} \tag{2}$$

by order reduction and by introducing velocity variables $\mathbf{v}$ in an appropriate way. In addition, initial values

$$\mathbf{p}_0 = \mathbf{p}(t_0),\ \mathbf{v}_0 = \mathbf{v}(t_0),\ \boldsymbol{\lambda}_0 = \boldsymbol{\lambda}(t_0)$$

are necessary to uniquely determine a solution [5]. Here and in the following we will use the dot operator as the derivative with respect to t, i.e., $\dot{\mathbf{p}} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}$. The state of the multibody system is given by the position variables $\mathbf{p} \in \mathbb{R}^{n_\mathbf{p}}$ and the (generalized) velocity variables $\mathbf{v} \in \mathbb{R}^{n_\mathbf{p}}$. Depending on the choice of the components of $\mathbf{v}$, the nonsingular transformation matrix $\mathbf{Z}(\mathbf{p}) \in \mathbb{R}^{n_\mathbf{p},n_\mathbf{p}}$ in (1a) may be the identity. In this paper we will not consider a quaternion formulation (for more details we refere to [30]). The inertia of the different bodies of the multibody system is represented by the positive definite mass matrix $\mathbf{M}(\mathbf{p}) \in \mathbb{R}^{n_\mathbf{p},n_\mathbf{p}}$. The applied forces are given by $\mathbf{f}(\mathbf{p},\mathbf{v},t) \in \mathbb{R}^{n_\mathbf{p}}$. Furthermore, there are certain holonomic constraints $\mathbf{g}(\mathbf{p},t) \in \mathbb{R}^{n_{\boldsymbol{\lambda}}}$ depending only on position and time. The constraint forces $-\mathbf{Z}^T(\mathbf{p})\mathbf{G}^T(\mathbf{p},t)\boldsymbol{\lambda}$ are determined by the constraint matrix $\mathbf{G}(\mathbf{p},t) \in \mathbb{R}^{n_{\boldsymbol{\lambda}},n_\mathbf{p}}$ given by the Jacobian

$$\mathbf{G}(\mathbf{p},t) = \frac{\partial \mathbf{g}}{\partial \mathbf{p}}(\mathbf{p},t).$$

The equations of motion (1) form a nonlinear system of differential-algebraic equations. One of the most important quantities for differential-algebraic equations is the *differentiation index* (d-index).

The differentiation index of a system of nonlinear differential-algebraic equations

$$0 = \mathbf{F}(\dot{\mathbf{x}},\mathbf{x},t) \tag{3}$$

has been introduced by Gear (1988) in [14]. A definition is given in [7] as follows, see also [5].

Definition 1 *Suppose that (3) is a solvable differential-algebraic equation on an open set* Ω*. Let the derivative array* $\tilde{\mathbf{F}}_l(\mathbf{z}, \dot{\mathbf{x}}, \mathbf{x}, t)$ *be defined by (see [6])*

$$\tilde{\mathbf{F}}_l(\mathbf{z}, \dot{\mathbf{x}}, \mathbf{x}, t) = \begin{bmatrix} \mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, t) \\ \vdots \\ \frac{\mathrm{d}^l}{\mathrm{d}t^l}\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, t) \end{bmatrix}$$

where

$$\mathbf{z} = [\mathbf{x}^{(l+1)}, ..., \ddot{\mathbf{x}}].$$

If $\dot{\mathbf{x}}$ *is considered locally as an algebraic variable* $\mathbf{y}$*, and that* $\mathbf{y}$ *is uniquely determined by* $\mathbf{x}$*,* t*, and* $\tilde{\mathbf{F}}_l(\mathbf{z}, \mathbf{y}, \mathbf{x}, t) = 0$ *for all consistent values. If* ν_d *is the smallest integer* l *such that this holds true, then we call* ν_d *the differentiation index of the differential-algebraic equation.*

Proposition 2 *The equations of motion (1) form a nonlinear system of differential-algebraic equations with differentiation index 3 if they satisfy the condition:*

$$\mathbf{G}(\mathbf{p}, t)\mathbf{Z}(\mathbf{p})\mathbf{M}^{-1}(\mathbf{p})\mathbf{Z}^T(\mathbf{p})\mathbf{G}^T(\mathbf{p}, t) \in \mathbb{R}^{n_{\boldsymbol{\lambda}}, n_{\boldsymbol{\lambda}}} \text{ is nonsingular for all } \mathbf{p}, t \text{ which satisfy (1c).} \quad (4)$$

Proof in [2], for example.

In this paper we only consider multibody systems that satisfy (4) and thus have differentiation index 3. Condition (4). implies in particular that

$$\mathrm{rank}(\mathbf{G}(\mathbf{p}, t)) = n_{\boldsymbol{\lambda}} \quad \text{for all } \mathbf{p}, t \text{ which satisfy (1c).} \quad (5)$$

Since the position variables $\mathbf{p}$ are restricted by $n_{\boldsymbol{\lambda}}$ holonomic, nonredundant constraints, the *degree of freedom* of the multibody system is

$$n_f = n_{\mathbf{p}} - n_{\boldsymbol{\lambda}}. \quad (6)$$

Furthermore, from (5) it follows $n_{\boldsymbol{\lambda}} \leq n_{\mathbf{p}}$. If $n_{\boldsymbol{\lambda}} = n_{\mathbf{p}}$, then the degree of freedom is zero and therefore, the motion of the multibody system is completely determined by the constraints $\mathbf{g}(\mathbf{p}, t) = 0$.

Differential-algebraic equations with a d-index higher than 1 are called *higher index problems*. Difficulties arising in the numerical solution of higher index problems are discussed in [5], [14], [16], [19], [29]. Furthermore, it was demonstrated in [16] that a differential-algebraic equation in the form (3) is solvable in general, only if the d-index is at most one. An index reduction by differentiation seems to be a way out, but then drift-off phenomenon may occur, see [5],[19]. Another important problem is the possible order reduction in the numerical integration of DAEs of higher index, for example, by using implicit Runge-Kutta-Methods. This phenomenon is considered in [17]. Possibilities to regularize DAEs, especially the equations of motion of multibody systems, are discussed in [4, 9, 15, 20]
Below, we will present an algorithm that regularizes the equations of motion by reducing the index of the equations of motion (1). This algorithm is performed in such a way that the solution manifold is preserved.

The paper is organized as follows. First, we describe the solution manifold for special choices of constraints in Section 3. In Section 4 we briefly consider index reduction for linear equations of motion. Afterwards, we discuss index reduction for nonlinear equations of motion in Section 5. In Section 6 we illustrate the results of the previous sections with a small-dimensional example.

2 Notations

Below, we will frequently use total time derivatives of some of the equations of motion, e.g., of the constraint equations

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g}(\mathbf{p}(t), t) = \frac{\partial}{\partial \mathbf{p}}\mathbf{g}(\mathbf{p}, t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}(t) + \frac{\partial}{\partial t}\mathbf{g}(\mathbf{p}, t).$$

To avoid the large effort in deriving higher derivatives and to keep the paper readable we will use a notation for the partial derivatives introduced in the following definition.

Definition 3 *Let* $\mathbf{A} : (\mathbb{R}^{m_1,1} \times ... \times \mathbb{R}^{m_K,1}) \to \mathbb{R}^{m,n}$*. Then the komma operator defined by*

$$\mathbf{A}_{,\mathbf{z}^{i_1}\dots\mathbf{z}^{i_L}}(\mathbf{z}^1, ..., \mathbf{z}^K) = \frac{\partial^L}{\partial \mathbf{z}^{i_1} \cdots \partial \mathbf{z}^{i_L}}\mathbf{A}(\mathbf{z}^1, ..., \mathbf{z}^K) \quad \textit{with } i_k \in \{1, ..., K\} \quad \textit{for all } k = 1, ..., L,$$

denotes the partial derivatives of $\mathbf{A}(\mathbf{z}^1, ..., \mathbf{z}^K)$ *with respect to* $\mathbf{z}^{i_1}$*, ...,* $\mathbf{z}^{i_L}$

By using this notation the first time derivative of the constraint equations takes the form

$$0 = \dot{\mathbf{g}}(\mathbf{p}, t) = \mathbf{g}_{,\mathbf{p}}(\mathbf{p}, t)\dot{\mathbf{p}}(t) + \mathbf{g}_{,t}(\mathbf{p}, t).$$

Furthermore, the second time derivative of $\mathbf{g}(\mathbf{p}(t), t)$ would require the time derivative

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{g}_{,\mathbf{p}}(\mathbf{p}, t)\dot{\mathbf{p}}\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{g}_{,\mathbf{p}}(\mathbf{p}, t)\Big)\dot{\mathbf{p}}(t) + \mathbf{g}_{,\mathbf{p}}(\mathbf{p}, t)\ddot{\mathbf{p}}(t).$$

The term $\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{g}_{,\mathbf{p}}(\mathbf{p}, t)\Big)$ leads to a product of a tensor of third order $\mathbf{g}_{,\mathbf{pp}}(\mathbf{p}, t)$, and a vector (tensor of first order) $\dot{\mathbf{p}}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{g}_{,\mathbf{p}}(\mathbf{p}, t)\Big) = \mathbf{g}_{,\mathbf{pp}}(\mathbf{p}, t)\dot{\mathbf{p}}(t) + \mathbf{g}_{,\mathbf{p}t}(\mathbf{p}, t)$$

which again has to be multiplied by $\dot{\mathbf{p}}(t)$. For more clearness and readability we introduce a notation which describes the tensor multiplication of one tensor higher order and several vectors arising from higher time derivatives of vector valued functions like $\frac{\mathrm{d}^l}{\mathrm{d}t^l}\mathbf{g}(\mathbf{p}(\mathbf{t}), t)$, $l \in \mathbb{N}$.

Definition 4 *Let* $\mathbf{A}(\mathbf{z}^1, ..., \mathbf{z}^K) : (\mathbb{R}^{m_1,1} \times ... \times \mathbb{R}^{m_K,1}) \to \mathbb{R}^{m,n}$*. Then, with respect to partial derivatives of* $\mathbf{A}(\mathbf{z}^1, ..., \mathbf{z}^K)$*, the operator* $\cdot \odot (\cdot)$ *is defined by*

$$\mathbf{A}_{,\mathbf{z}^{i_1}\dots\mathbf{z}^{i_L}}(\mathbf{z}^1, ..., \mathbf{z}^K) \odot (\mathbf{x}^1, ..., \mathbf{x}^L) = \left[\sum_{r_1=1}^{n_1} \cdots \sum_{r_L=1}^{n_L} \frac{\partial^L A_{ij}(\mathbf{z}^1, ..., \mathbf{z}^K)}{\partial z^{i_1}_{r_1} \cdots \partial z^{1_L}_{r_L}} x^1_{r_1} \cdot ... \cdot x^L_{r_L}\right]_{i=1,...,m,\ j=1,...,n}$$

with $i_k \in \{1, ..., K\}$ *for all* $k = 1, ..., L$*. Here* $\mathbf{z}^j = [z^j_1, ..., z^j_{n_j}]^T$ *and* $\mathbf{x}_k, \mathbf{z}^{i_k} \in \mathbb{R}^{n_k,1}$*. Furthermore,* $\mathbf{A}_{,\mathbf{z}^{i_1}\dots\mathbf{z}^{i_L}}(\mathbf{z}^1, ..., \mathbf{z}^K) \odot (\mathbf{x}^1, ..., \mathbf{x}^L) \in \mathbb{R}^{m,n}$*.*

By using this notation the second time derivative of the constraint equations takes the form

$$0 = \ddot{\mathbf{g}}(\mathbf{p}, t) = \mathbf{g}_{,\mathbf{pp}}(\mathbf{p}, t) \odot (\dot{\mathbf{p}}(t), \dot{\mathbf{p}}(t)) + \mathbf{g}_{,\mathbf{p}t}(\mathbf{p}, t)\dot{\mathbf{p}}(t) + \mathbf{g}_{,\mathbf{p}}(\mathbf{p}, t)\ddot{\mathbf{p}}(t).$$

3 Solution submanifold and deviation

Differential-algebraic equations of d-index higher than one contain *hidden constraints*. These are algebraic constraints which do not appear explicitly in the original form of the system. By differentiating the differential-algebraic equations l times with respect to time t and by applying algebraic transformations one can determine the hidden constraints of level l.
Let us consider the equations of motion (1). Here, the explicit constraints are the so-called *constraints on position level* (1c). Since we have assumed that

$$\frac{\partial \mathbf{g}}{\partial \mathbf{p}}(\mathbf{p}, t) = \mathbf{G}(\mathbf{p}, t)$$

has full row rank $n_{\boldsymbol{\lambda}}$, the constraints on position level are $n_{\boldsymbol{\lambda}}$ nonredundant restrictions. Therefore, the position variables $\mathbf{p}$ are restricted to the $(n_{\mathbf{p}} + n_{\mathbf{p}})$-dimensional time varying *position manifold*

$$\mathcal{M}_{\mathbf{p}}(t) := \{(\mathbf{p}, \mathbf{v}, \boldsymbol{\lambda}) \ : \ 0 = \mathbf{g}(\mathbf{p}, t)\} \subset \mathbb{R}^{n_{\mathbf{p}}+n_{\mathbf{p}}+n_{\boldsymbol{\lambda}}} \tag{7}$$

by the *position constraints*

$$0 = \mathbf{g}(\mathbf{p}, t). \tag{8}$$

Differentiating the constraint equations (1c) with respect to time and substituting $\dot{\mathbf{p}}$ by $\mathbf{Z}(\mathbf{p})\mathbf{v}$ using (1a), we get the hidden constraints on level one, the so-called *constraints on velocity level*

$$\begin{aligned} 0 &= \dot{\mathbf{g}}(\mathbf{p}, t) \\ &= \mathbf{g}_{,\mathbf{p}}(\mathbf{p}, t)\dot{\mathbf{p}} + \mathbf{g}_{,t}(\mathbf{p}, t) \\ &= \mathbf{G}(\mathbf{p}, t)\dot{\mathbf{p}} + \mathbf{g}_{,t}(\mathbf{p}, t) && (9) \\ &= \mathbf{G}(\mathbf{p}, t)\mathbf{Z}(\mathbf{p})\mathbf{v} + \mathbf{g}_{,t}(\mathbf{p}, t) =: \mathbf{g}^I(\mathbf{p}, \mathbf{v}, t). && (10) \end{aligned}$$

Since

$$\frac{\partial \mathbf{g}^I}{\partial [\mathbf{p}^T \ \mathbf{v}^T]^T}(\mathbf{p}, \mathbf{v}, t) = [*, \ \mathbf{G}(\mathbf{p}, t)\mathbf{Z}(\mathbf{p})]$$

has full row rank $n_{\boldsymbol{\lambda}}$, the constraints on velocity level are $n_{\boldsymbol{\lambda}}$ nonredundant restrictions which (particularly) restrict the velocity $\mathbf{v}$ on an additional $(n_{\mathbf{p}} + n_{\mathbf{p}})$-dimensional time varying *velocity manifold*

$$\mathcal{M}_{\mathbf{v}}(t) := \{(\mathbf{p}, \mathbf{v}, \boldsymbol{\lambda}) \ : \ 0 = \mathbf{g}^I(\mathbf{p}, \mathbf{v}, t)\} \subset \mathbb{R}^{n_{\mathbf{p}}+n_{\mathbf{p}}+n_{\boldsymbol{\lambda}}}$$

by the *velocity constraints*

$$0 = \mathbf{g}^I(\mathbf{p}, \mathbf{v}, t). \tag{11}$$

Furthermore, if we differentiate the constraint equations (1c) twice, replace $\dot{\mathbf{p}}$ again by $\mathbf{Z}(\mathbf{p})\mathbf{v}$, and replace $\dot{\mathbf{v}}$ by (1b), we get the so-called *constraints on acceleration level*

$$
\begin{aligned}
0 &= \ddot{\mathbf{g}}(\mathbf{p}, t) \\
&= \big((\mathbf{GZ})_{,\mathbf{p}} \odot (\dot{\mathbf{p}}) + (\mathbf{GZ})_{,t}\big)\mathbf{v} + \mathbf{GZ}\dot{\mathbf{v}} + \mathbf{g}_{,t\mathbf{p}}\dot{\mathbf{p}} + \mathbf{g}_{,tt} \\
&= \big((\mathbf{GZ})_{,\mathbf{p}} \odot (\mathbf{Zv}) + (\mathbf{GZ})_{,t}\big)\mathbf{v} + \mathbf{GZ}\dot{\mathbf{v}} + \mathbf{g}_{,t\mathbf{p}}\mathbf{Zv} + \mathbf{g}_{,tt} \\
&= \mathbf{GZ}\dot{\mathbf{v}} + \tilde{\mathbf{g}}^{I\!I}(\mathbf{p}, \mathbf{v}, t) \qquad (12) \\
&= \big((\mathbf{GZ})_{,\mathbf{p}} \odot (\mathbf{Zv}) + (\mathbf{GZ})_{,t}\big)\mathbf{v} + \mathbf{GZM}^{-1}(\mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda}) + \mathbf{g}_{,t\mathbf{p}}\mathbf{Zv} + \mathbf{g}_{,tt} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{GZ})\mathbf{v} + \mathbf{GZM}^{-1}(\mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda}) + \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{g}}^{I}(\mathbf{p}, t) \qquad (13) \\
&=: \mathbf{g}^{I\!I}(\mathbf{p}, \mathbf{v}, \boldsymbol{\lambda}, t). \qquad (14)
\end{aligned}
$$

Here and in the following, we will often omit the dependencies, (e.g., of $\mathbf{G}$) on $\mathbf{p}$, $\mathbf{v}$, $\boldsymbol{\lambda}$, and t unless we want to focus on some of them.
Now $\mathbf{g}^{I\!I}(\mathbf{p}, \mathbf{v}, \boldsymbol{\lambda}, t)$ represents the hidden constraints on second level which allow the computation of the Lagrange-multiplier by given positions $\mathbf{p}$ and velocities $\mathbf{v}$ at a certain time t. Since

$$\frac{\partial \mathbf{g}^{I\!I}}{\partial [\mathbf{p}^T \ \mathbf{v}^T \ \boldsymbol{\lambda}^T]^T}(\mathbf{p}, \mathbf{v}, \boldsymbol{\lambda}, t) = [*, \ *, \ \mathbf{GZM}^{-1}\mathbf{Z}^T\mathbf{G}^T]$$

has full row rank $n_{\boldsymbol{\lambda}}$, the constraints on acceleration level are $n_{\boldsymbol{\lambda}}$ nonredundant restrictions and, again, represent an additional $(n_{\mathbf{p}} + n_{\mathbf{p}})$-dimensional time varying *acceleration manifold*

$$\mathcal{M}_{\mathbf{a}}(t) := \{(\mathbf{p}, \mathbf{v}, \boldsymbol{\lambda}) \ : \ 0 = \mathbf{g}^{I\!I}(\mathbf{p}, \mathbf{v}, \boldsymbol{\lambda}, t)\} \subset \mathbb{R}^{n_{\mathbf{p}} + n_{\mathbf{p}} + n_{\boldsymbol{\lambda}}}.$$

by satisfying the *acceleration constraints*

$$0 = \mathbf{g}^{I\!I}(\mathbf{p}, \mathbf{v}, \boldsymbol{\lambda}, t). \qquad (15)$$

In summary, the solution $(\mathbf{p}, \mathbf{v}, \boldsymbol{\lambda})$ has to satisfy the constraints on position level, on velocity level as well as the constraints on acceleration level, i.e.,

$$(\mathbf{p}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t)) \overset{!}{\in} \mathcal{M}(t)$$

where the *solution manifold* $\mathcal{M}(t)$ is the intersection of all manifolds

$$
\begin{aligned}
\mathcal{M}(t) &:= \mathcal{M}_{\mathbf{p}}(t) \cap \mathcal{M}_{\mathbf{v}}(t) \cap \mathcal{M}_{\mathbf{a}}(t) \qquad (16) \\
&= \left\{ (\mathbf{p}, \mathbf{v}, \boldsymbol{\lambda}) \ : \ 0 = \begin{bmatrix} \mathbf{g}(\mathbf{p}, t) \\ \mathbf{g}^{I}(\mathbf{p}, \mathbf{v}, t) \\ \mathbf{g}^{I\!I}(\mathbf{p}, \mathbf{v}, \boldsymbol{\lambda}, t) \end{bmatrix} \right\}.
\end{aligned}
$$

Since

$$\frac{\partial \left[\mathbf{g}^T \ (\mathbf{g}^{I})^T \ (\mathbf{g}^{I\!I})^T\right]^T}{\partial [\mathbf{p}^T \ \mathbf{v}^T \ \boldsymbol{\lambda}^T]^T} = \begin{bmatrix} \mathbf{G} & 0 & 0 \\ * & \mathbf{GZ} & 0 \\ * & * & \mathbf{GZM}^{-1}\mathbf{Z}^T\mathbf{G}^T \end{bmatrix}$$

has full row rank $3n_{\boldsymbol{\lambda}}$, all constraints together are $3n_{\boldsymbol{\lambda}}$ nonredundant restrictions. Therefore, the solution manifold has the dimension $(n_{\mathbf{p}} + n_{\mathbf{p}} + n_{\boldsymbol{\lambda}}) - 3n_{\boldsymbol{\lambda}} = 2n_f$.

The classical approach (see, e.g., [11]) for modelling and simulating multibody systems is using *minimal coordinates* $\mathbf{q}$. In this approach a minimal number of coordinates is used to describe the motion of the multibody system. Possible kinematic constraints resulting for example from joints which restrict the motion are eliminated. Therefore, in the equations of motion modeled in minimal coordinates, no constraints and no constraint forces appear. The equations of motion arise in the *state space form*

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} \;\;=\;\; \mathbf{f}(\dot{\mathbf{q}}, \mathbf{q}, t). \tag{17}$$

This is an ordinary differential equation (ODE) of second order that can be reduced to a first order ODE. Then it can be treated in the usual way given by the theory and numerical methods for ODEs.
The determination of the state space form plays a key role in the classical approach. In [8] it is described how to determine and to compute the state space form for linear or linearized equations of motion. Furthermore, the determination and computation of the state space form of general equations of motion are considered in [13] and [32]. Here, it becomes clear, that for general nonlinear mechanical systems, especially those with closed loops, the state space form can only be established locally, and that the complete process of reduction to state space form is quite laborious and therefore time consuming.
Therefore, in modern approaches (see e.g. [8], [10], [33], [34], [35]) multibody systems are modeled by using constraints and nonminimal coordinates. The equations of motion then appear in descriptor form (2).

In the following, we consider the different formulations of the equations of motion and analyse when the conditions (8),(11), and (15) are satidfied, which describe the solution manifold (16). These different formulations of the equations of motion are the underlying ordinary differential equation (uODE), the equations of motion using acceleration constraints (15) (EoM_1), the equations of motion using velocity constraints (11) (EoM_2), and the original equations of motion using position constraints (8) (EoM).

3.1 Underlying ordinary differential equation

Differentiating of the constraints (1c) three times yields

$$\begin{aligned}
0 \;&=\; \frac{\mathrm{d}^3}{\mathrm{d}t^3}\mathbf{g}(\mathbf{p}, t) \\
&=\; \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g}^{II}(\mathbf{p}, \mathbf{v}, \boldsymbol{\lambda}, t) \\
&=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big(\big((\mathbf{GZ})_{,\mathbf{p}} \odot (\mathbf{Zv}) + (\mathbf{GZ})_{,t}\big)\mathbf{v} + \mathbf{GZM}^{-1}(\mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda}) + \mathbf{g}_{,t\mathbf{p}}\mathbf{Zv} + \mathbf{g}_{,tt}\Big) \\
&=\; \frac{\mathrm{d}}{\mathrm{d}t}\Big(\big((\mathbf{GZ})_{,\mathbf{p}} \odot (\mathbf{Zv}) + (\mathbf{GZ})_{,t}\big)\mathbf{v} + \mathbf{GZM}^{-1}\mathbf{f} + \mathbf{g}_{,t\mathbf{p}}\mathbf{Zv} + \mathbf{g}_{,tt}\Big) \\
&\quad\; -\frac{\mathrm{d}}{\mathrm{d}t}\Big(\mathbf{GZM}^{-1}\mathbf{Z}^T\mathbf{G}^T\Big)\boldsymbol{\lambda} - \mathbf{GZM}^{-1}\mathbf{Z}^T\mathbf{G}^T\dot{\boldsymbol{\lambda}} \\
&=\; \tilde{\mathbf{g}}^{III}(\mathbf{p}, \mathbf{v}, \boldsymbol{\lambda}, t) - \mathbf{GZM}^{-1}\mathbf{Z}^T\mathbf{G}^T\dot{\boldsymbol{\lambda}} \qquad (18) \\
&=:\; \mathbf{g}^{III}(\mathbf{p}, \mathbf{v}, \boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}, t),
\end{aligned}$$

by using (1a) and (1b). Replacing the constraints (1c) in the equations of motion (1) by their third derivatives yields the *underlying ordinary differential equation* (uODE)

$$
\begin{aligned}
\dot{\mathbf{p}} &= \mathbf{Z}\mathbf{v}, && \text{(19a)}\\
\dot{\mathbf{v}} &= \mathbf{M}^{-1}\big(\mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda}\big), && \text{(19b)}\\
\dot{\boldsymbol{\lambda}} &= \big(\mathbf{G}\mathbf{Z}\mathbf{M}^{-1}\mathbf{Z}^T\mathbf{G}^T\big)^{-1}\tilde{\mathbf{g}}^{\mathit{III}}. && \text{(19c)}
\end{aligned}
$$

By setting $\gamma(t) = \mathbf{g}(\mathbf{p}(t), t)$ (compare with [12]), the last equation of the uODE corresponds to the ODE

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}\gamma(t) = 0$$

which has the solution

$$\gamma(t) = \gamma(t_0) + \dot{\gamma}(t_0)t + \frac{1}{2}\ddot{\gamma}(t_0)t^2$$

$$
\text{with} \quad
\begin{aligned}
&\gamma(t_0) = \mathbf{g}(\mathbf{p}_0, t_0),\\
&\dot{\gamma}(t_0) = \mathbf{g}^{I}(\mathbf{p}_0, \mathbf{v}_0, t_0),\\
&\ddot{\gamma}(t_0) = \mathbf{g}^{II}(\mathbf{p}_0, \mathbf{v}_0, \boldsymbol{\lambda}_0, t_0).
\end{aligned}
$$

Since $\gamma(t)$ represents the residual of the position constraints, the solution behavior when using the underlying ODE is described in the following proposition.

Proposition 5 *The uODE (19) has a solution for every set of initial values $\mathbf{p}_0$, $\mathbf{v}_0$, and $\boldsymbol{\lambda}_0$. If the initial values $\mathbf{p}_0$, $\mathbf{v}_0$, and $\boldsymbol{\lambda}_0$ are consistent with the EoM (1), i.e., if*

$$
\begin{aligned}
\mathbf{g}(\mathbf{p}_0, t_0) &= 0,\\
\mathbf{g}^{I}(\mathbf{p}_0, \mathbf{v}_0, t_0) &= 0,\\
\mathbf{g}^{II}(\mathbf{p}_0, \mathbf{v}_0, \boldsymbol{\lambda}_0, t_0) &= 0,
\end{aligned}
$$

then it follows that

$$
\begin{aligned}
\gamma(t) &= \mathbf{g}(\mathbf{p}(t), t) &&\equiv 0 \;\forall t,\\
\dot{\gamma}(t) &= \mathbf{g}^{I}(\mathbf{p}(t), \mathbf{v}(t), t) &&\equiv 0 \;\forall t,\\
\ddot{\gamma}(t) &= \mathbf{g}^{II}(\mathbf{p}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), t) &&\equiv 0 \;\forall t.
\end{aligned}
$$

The analytic solution satisfies all constraints and, consequently, it lies in the solution manifold.
If the initial values $\mathbf{p}_0$, $\mathbf{v}_0$, or $\boldsymbol{\lambda}_0$ are not consistent with the EoM (1), then it follows that

$$\gamma(t_0) \neq 0, \;\; \dot{\gamma}(t_0) \neq 0, \;\; \ddot{\gamma}(t_0) \neq 0$$

and thus

$$
\begin{aligned}
\gamma(t) &= \gamma(t_0) + \dot{\gamma}(t_0)t + \tfrac{1}{2}\ddot{\gamma}(t_0)t^2 &&= \mathbf{g}(\mathbf{p}(t), t) &&\not\equiv 0,\\
\dot{\gamma}(t) &= \dot{\gamma}(t_0) + \ddot{\gamma}(t_0)t &&= \mathbf{g}^{I}(\mathbf{p}(t), \mathbf{v}(t), t) &&\not\equiv 0,\\
\ddot{\gamma}(t) &= \ddot{\gamma}(t_0) &&= \mathbf{g}^{II}(\mathbf{p}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), t) &&\not\equiv 0.
\end{aligned}
$$

In this case the analytic solution does not satisfy the constraints and consequently, it does not lie in the solution manifold. In particular, the solution is deviating from the position manifold $\mathcal{M}_{\mathbf{p}}(t)$, such that the residual of the constraints on position level has quadratic behavior in time t. Likewise, the solution is deviating from the velocity manifold $\mathcal{M}_{\mathbf{v}}(t)$ with a residual of constraints on velocity level behaving linearly in time t. The residual of the acceleration constraints is constant.

Remark 6 Aside from numerical aspects of the solution of ODEs, no problems arise from initial values or computed solutions at intermediate steps.

3.2 Equations of motion using acceleration constraints

If one substitutes constraints on position level (1c) in the equations of motion (1) by the constraints on acceleration level (14), then one gets the equations of motion in the form

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{Z}(\mathbf{p})\mathbf{v}, && (20a)\\ \mathbf{M}(\mathbf{p})\dot{\mathbf{v}} &= \mathbf{f}(\mathbf{p},\mathbf{v},t) - \mathbf{Z}^T(\mathbf{p})\mathbf{G}^T(\mathbf{p},t)\boldsymbol{\lambda}, && (20b)\\ 0 &= \mathbf{g}^{II}(\mathbf{p},\mathbf{v},\boldsymbol{\lambda},t) && (20c) \end{aligned}$$

which have d-index 1. We will use also the abbreviation EoM_1 for (20). By setting $\gamma(t) = \mathbf{g}(\mathbf{p}(t),t)$, we obtain from (20c) with (14) the ODE

$$\ddot{\gamma}(t) = 0.$$

The solution of this ODE is given by

$$\gamma(t) = \gamma(t_0) + \dot{\gamma}(t_0)t$$

$$\text{with} \qquad \begin{aligned} \gamma(t_0) &= \mathbf{g}(\mathbf{p}_0,t_0),\\ \dot{\gamma}(t_0) &= \mathbf{g}^{I}(\mathbf{p}_0,\mathbf{v}_0,t_0). \end{aligned}$$

Since $\gamma(t)$ still represents the residual of the position constraints, we get the solution behavior for the d-index 1 equations of motion using acceleration constraints as follows.

Proposition 7 *If the initial values $\boldsymbol{\lambda}_0$ are not consistent, then the d-index 1 equations of motion (20) has no solution.*
Furthermore, if the initial values $\mathbf{p}_0$ and $\mathbf{v}_0$ are consistent with the EoM (1), i.e., if

$$\begin{aligned} \mathbf{g}(\mathbf{p}_0,t_0) &= 0,\\ \mathbf{g}^{I}(\mathbf{p}_0,\mathbf{v}_0,t_0) &= 0, \end{aligned}$$

then it follows that

$$\begin{aligned} \gamma(t) &= \mathbf{g}(\mathbf{p}(t),t) &&\equiv 0\ \forall t,\\ \dot{\gamma}(t) &= \mathbf{g}^{I}(\mathbf{p}(t),\mathbf{v}(t),t) &&\equiv 0\ \forall t. \end{aligned}$$

The acceleration constraints (14)

$$\mathbf{g}^{II}(\mathbf{p}(t),\mathbf{v}(t),\boldsymbol{\lambda}(t),t) \equiv 0\ \forall t$$

are enforced by using these constraints explicitly in (20). If the initial values are consistent, then the analytic solution satisfies all constraints (using (20)) and, consequently, lies in the solution manifold.
If the initial values $\boldsymbol{\lambda}_0$ are consistent and $\mathbf{p}_0$ or $\mathbf{v}_0$ are not consistent with the EoM (1), then

$$\gamma(t_0) \neq 0,\ \dot{\gamma}(t_0) \neq 0$$

and thus

$$\begin{array}{rclcll} \gamma(t) & = & \gamma(t_0) + \dot{\gamma}(t_0)t & = & \mathbf{g}(\mathbf{p}(t), t) & \not\equiv \ 0, \\ \dot{\gamma}(t) & = & \dot{\gamma}(t_0) & = & \mathbf{g}^I(\mathbf{p}(t), \mathbf{v}(t), t) & \not\equiv \ 0. \end{array}$$

The acceleration constraints (20c), i.e.,

$$\mathbf{g}^{II}(\mathbf{p}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), t) \quad \equiv \quad 0 \ \forall t,$$

are enforced by using these constraints explicitly in (20). Provided with inconsistent initial values $\mathbf{p}_0$, $\mathbf{v}_0$, the analytic solution deviates from the position manifold $\mathcal{M}_{\mathbf{p}}(t)$ with a linear behavior with respect to the residuals and lies with a constant residual near the velocity manifold $\mathcal{M}_{\mathbf{v}}(t)$. The solution itself lies in the acceleration manifold $\mathcal{M}_{\mathbf{a}}(t)$.

Remark 8 Since the d-index 1 formulation (20) of the EoM has no solution if the initial values $\boldsymbol{\lambda}_0$ are not consistent, one would expect numerical problems. Actually, if these initial values are inconsistent then no solution exits but it is easy to compute consistent initial values $\boldsymbol{\lambda}_0$ by solving the constraints (20c). Furthermore, in all subsequent integration steps the solutions $\mathbf{p}_i$, $\mathbf{v}_i$, and $\boldsymbol{\lambda}_i$ at time t_i satisfies the constraints (20c). Consequently, the intermediate solution $\boldsymbol{\lambda}_i$ is consistent in each intermediate step. The consistency of $\mathbf{p}_i$ and $\mathbf{v}_i$ is not important for numerical aspects. Therefore, neglecting numerical aspects of the solution of ODEs, no problems arise from initial values or computed solutions at intermediate steps.

3.3 Equations of motion using velocity constraints

If we use the constraints on velocity level (10) instead of the constraints on position level (1c) in the equations of motion (1) then we get the form

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{Z}(\mathbf{p})\mathbf{v}, && (21a) \\ \mathbf{M}(\mathbf{p})\dot{\mathbf{v}} &= \mathbf{f}(\mathbf{p}, \mathbf{v}, t) - \mathbf{Z}^T(\mathbf{p})\mathbf{G}^T(\mathbf{p}, t)\boldsymbol{\lambda}, && (21b) \\ 0 &= \mathbf{g}^I(\mathbf{p}, \mathbf{v}, t) && (21c) \end{aligned}$$

which has d-index 2. We will use also the abbreviation EoM_2 for (21). Again, by setting $\gamma(t) = \mathbf{g}(\mathbf{p}(t), t)$ it follows from (21c) with (10) that

$$\dot{\gamma}(t) = 0.$$

The solution of this ODE is given by

$$\gamma(t) = \gamma(t_0) = \mathbf{g}(\mathbf{p}_0, t_0).$$

The following proposition reflects the solution behavior for the d-index 2 equations of motion using velocity constraints.

Proposition 9 *If the initial values $\mathbf{v}_0$ or $\boldsymbol{\lambda}_0$ are inconsistent then the EoM$_2$ (21) has no solution.*
If the initial values $\mathbf{p}_0$ are consistent with the EoM (1), i.e., if

$$\mathbf{g}(\mathbf{p}_0, t_0) = 0,$$

then it follows that

$$\gamma(t) = \mathbf{g}(\mathbf{p}(t), t) \equiv 0 \ \forall t.$$

The velocity constraints (21c)

$$\mathbf{g}^{I}(\mathbf{p}(t), \mathbf{v}(t), t) \equiv 0 \ \forall t,$$

and also their first time derivative, i.e., the acceleration constraints (14)

$$\mathbf{g}^{II}(\mathbf{p}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), t) \equiv 0 \ \forall t$$

are enforced by using the velocity constraints explicitly in (21). With consistent initial values, the analytic solution satisfies all constraints even in the form (21) and, consequently, it lies in the solution manifold.
If the initial values $\mathbf{v}_0$ and $\boldsymbol{\lambda}_0$ are consistent and $\mathbf{p}_0$ are not consistent with the EoM (1), then it follows that

$$\gamma(t_0) \neq 0$$

and thus

$$\gamma(t) = \gamma(t_0) = \mathbf{g}(\mathbf{p}(t), t) \not\equiv 0.$$

The velocity constraints (21c)

$$\mathbf{g}^{I}(\mathbf{p}(t), \mathbf{v}(t), t) \equiv 0 \ \forall t,$$

and also their first time derivative, i.e., the acceleration constraints (14)

$$\mathbf{g}^{II}(\mathbf{p}(t), \mathbf{v}(t), \boldsymbol{\lambda}(t), t) \equiv 0 \ \forall t$$

are enforced by using the velocity constraints explicitly in (21). By using inconsistent initial values $\mathbf{p}_0$ the analytic solution lies near the position manifold $\mathcal{M}_{\mathbf{p}}(t)$ with a constant residual, but the solution lies in the velocity manifold $\mathcal{M}_{\mathbf{v}}(t)$ and in the acceleration manifold $\mathcal{M}_{\mathbf{a}}(t)$.

Remark 10 Since the d-index 2 formulation (21) of the EoM has no solution if the initial values $\mathbf{v}_0$ and $\boldsymbol{\lambda}_0$ are not consistent, one would expect numerical problems. Actually, if these initial values are inconsistent then no solution exits. But it is only possible to compute initial values $\mathbf{v}_0$ by solving the constraints (21c). Furthermore, if one provides consistent initial values, after every numerical integration step the consistency of the solution components $\mathbf{v}_i$ is guaranteed by the explicit appearance of the constraints on velocity level (21c). But, in general, the solution components $\boldsymbol{\lambda}_i$ are not consistent because of rounding errors. If one uses a method which is not L-stable (see [19]), like the implicit trapezoidal rule or the implicit midpoint rule, one can observe oscillations in the Lagrange-multipliers $\boldsymbol{\lambda}_i$. See Figure 3. Therefore, in addition to the numerical aspects of the solution of ODEs, it is not possible to guarantee a convergent behavior for Lagrange-multipliers in general. The consistency of $\mathbf{p}_i$ is not important for the numerical solvability.

3.4 Original equations of motion (using position constraints)

Consider the original equations of motion (1). Since the position constraints are satisfied by their explicit appearance in the original EoM, also their first and second time derivative, i.e., the constraints on velocity level and the constraints on acceleration level are satisfied.

$$
\begin{array}{rcccll}
0 & \equiv & \mathbf{g}(\mathbf{p}(t),t) & \forall t & & \\
\Rightarrow 0 & \equiv & \dot{\mathbf{g}}(\mathbf{p}(t),t) & = & \mathbf{g}^{I}(\mathbf{p}(t),\mathbf{v}(t),t) & \forall t \\
\Rightarrow 0 & \equiv & \ddot{\mathbf{g}}(\mathbf{p}(t),t) & = & \mathbf{g}^{II}(\mathbf{p}(t),\mathbf{v}(t),\boldsymbol{\lambda}(t),t) & \forall t
\end{array}
$$

Proposition 11 *If the initial values $\mathbf{p}_0$, $\mathbf{v}_0$, and $\boldsymbol{\lambda}_0$ are consistent, then the analytic solution lies within the solution manifold.*
If the initial values $\mathbf{p}_0$, $\mathbf{v}_0$, or $\boldsymbol{\lambda}_0$ are inconsistent, then the original equations of motion (1) with d-index 3 have no solution.

Remark 12 Since the original EoM (1) have no solution if the initial values $\mathbf{p}_0$, $\mathbf{v}_0$, and $\boldsymbol{\lambda}_0$ are not consistent, numerical problems are to be expected. Actually, if these initial values are inconsistent, then no solution can be found but it is only possible to compute initial values $\mathbf{p}_0$ by solving the constraints (1c). Therefore, if one provides consistent initial values after each numerical integration step, the consistency of solution components $\mathbf{p}_i$ is enforced by the explicit appearance of the constraints on position level (1c). On the other hand, the solution components $\mathbf{v}_i$ and $\boldsymbol{\lambda}_i$ are not consistent in general because of rounding errors. If one use methods which are not L-stable (see [19]), like the implicit trapezoidal rule or the implicit midpoint rule, one can observe oscillations in the velocity $\mathbf{v}_i$ and in the Lagrange-multipliers $\boldsymbol{\lambda}_i$, see Figure 4. Therefore, in addition to the numerical aspects of the solution of ODEs, it is not possible to guarantee a convergent behavior for the velocity components and the Lagrange-multipliers in general.

Table 1 summarizes all results obtained above. Here, it becomes clear that none of the formulations above is ideally suited as a base of numerical integrations. Either the analytical solution deviates from the solution manifold with quadratic behavior but without possible numerical oscillations by using uODE, or by using the original EoM the solution does not deviate from the solution manifold but the numerical integration may yields numerical oscillations with respect to $\boldsymbol{\lambda}$ and $\mathbf{v}$ or a trade-off something between oscillations and deviation by using EoM_1 or EoM_2.
Different ways out of this dilemma are regularization techniques. For example the approach of Baumgarte [4] manipulates the constraint equations and introduces a control term to the equations of motion which controls or steers a perturbed nonconsistent solution back to the solution manifold $\mathcal{M}$. The dimension of the obtained system is preserved. The index of the Baumgarte-stabilized equations of motion is reduced to d-index 1 but the manifold defined there has a larger dimension as the original solution manifold $\mathcal{M}$ and contains the solution manifold $\mathcal{M}$. The main drawback is that the choice of the introduced control term to make the system robust has been unclear in practice, see [3].
An other important approach is the Gear-Gupta-Leimkuhler-formulation [15] which add the constraints on velocity level (11) to the equations of motion and introduce new Lagrange-multipliers whose purpose is to insure that the constraints on velocity level are satisfied. The dimension of the system is increased by $n_{\boldsymbol{\lambda}}$. The advantage of the Gear-Gupta-Leimkuhler-formulation is the reduced index from d-index 3 to d-index 2 and that the manifold defined

used form	**uODE (19)**	**EoM$_1$ (20)**	**EoM$_2$ (21)**	**original EoM (1)**
last equations in used form	$0 = \mathbf{g}^{III}(\mathbf{p}, \mathbf{v}, \boldsymbol{\lambda}, \dot{\boldsymbol{\lambda}}, t)$	$0 = \mathbf{g}^{II}(\mathbf{p}, \mathbf{v}, \boldsymbol{\lambda}, t)$	$0 = \mathbf{g}^{I}(\mathbf{p}, \mathbf{v}, t)$	$0 = \mathbf{g}(\mathbf{p}, t)$
d-index	0	1	2	3
Deviation caused by inconsistent initial values	deviate from ... $\mathcal{M}_\mathbf{p}$ - quadratic in time $\mathcal{M}_\mathbf{v}$ - linear in time $\mathcal{M}_\mathbf{a}$ - constant in time	deviate from ... $\mathcal{M}_\mathbf{p}$ - linear in time $\mathcal{M}_\mathbf{v}$ - constant in time	deviate from ... $\mathcal{M}_\mathbf{p}$ - constant in time	no deviation
Existence of solution	always	$\boldsymbol{\lambda}_0$ must be consistent	$\mathbf{v}_0$, $\boldsymbol{\lambda}_0$ must be consistent	$\mathbf{p}_0$, $\mathbf{v}_0$, $\boldsymbol{\lambda}_0$ must be consistent
Numerical behavior	no oscillations	no oscillations	oscillations w.r.t. $\boldsymbol{\lambda}$	oscillations w.r.t. $\boldsymbol{\lambda}$, $\mathbf{v}$
Dimension of solution manifold	$n_f + n_f + n_{\boldsymbol{\lambda}} + n_{\boldsymbol{\lambda}} + n_{\boldsymbol{\lambda}}$	$n_f + n_f + n_{\boldsymbol{\lambda}} + n_{\boldsymbol{\lambda}}$	$n_f + n_f + n_{\boldsymbol{\lambda}}$	$n_f + n_f$

Table 1: Deviation and numerical behavior of different forms of equations of motion

there is identical to the solution manifold $\mathcal{M}$, and in addition to the constraints on position level the constraints on velocity level occur explicitely. The disadvantages are the higher index, i.e., larger than 1, and the hidden constraints on acceleration level.
A third approch was introduced by Führer [12],[13] and just adds the constraints on velocity level (11) and the constraints on acceleration level (15) and yields an overdetermined system with $2n_{\boldsymbol{\lambda}}$ equations which are redundant to the others. The d-index is not defined for overdetermined systems and the manifold defined there is identical to the solution manifold $\mathcal{M}$ and all necessary information is contained in the system in an explicit way.
The index of a differential-algebraic system and the way how all information of the solution manifold is given have an essential influence on the quality and the success of numerical integration. Here, a small index, i.e., if possible d-index 0 or d-index 1 (see [16]), is preferred. Furthermore, there should be no hidden constraints, i.e., all information of the solution manifold should be given in an explicit way inside the system which has to be integrated.
Since a d-index 0 formulation would be an ODE of dimension $2n_f$, it corresponds to the state space form (17), which we will not focus on in this paper. We follow the ideas introduced in [22] and [23]. There, the differential-algebraic system is transformed such that the regularized system is of d-index 1 and contains all information of the solution manifold in an explicit way. In [22] and [23] the described strategy is developed for general differential-algebraic equations without a certain structure and of arbitrary index. But the equations of motion of multibody systems have a certain structure (see (1)) and a fixed d-index 3 (see Proposition 2).
The exploitation of this special knowledge about the equations of motion makes it possible to adapt the strategy introduced in [22] and [23] such that the regularization of the equations of motion becomes more efficient. This approach will be considered in the following sections.

4 Index Reduction for Linear Equations of Motion

Usually, multibody systems are described by nonlinear equations of motion as in (1). But in some cases or under special assumptions, e.g., linear behavior of springs and dampers, the equations of motion may appear in linear form

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{Z}(t)\mathbf{v}, && \text{(22a)} \\ \mathbf{M}(t)\dot{\mathbf{v}} &= \mathbf{C}(t)\mathbf{p} + \mathbf{D}(t)\mathbf{v} - \mathbf{Z}(t)^T\mathbf{G}^T(t)\boldsymbol{\lambda}, && \text{(22b)} \\ 0 &= \mathbf{G}(t)\mathbf{p}. && \text{(22c)} \end{aligned}$$

In other cases the motion of the bodies may be assumed to be very small such that the linearized equations of motion (22) reflect a similar but sufficiently accurate behavior of the multibody system as it would be described by the nonlinear equations of motion (1). For the linearization itself we refer to [8] or [2]. For the sake of simplicity we will omit the dependencies on t.

Since the linear equations of motion also have d-index 3, we have a higher index problem and it is clear that the standard formulation (22) is not well suited for the numerical integration. An equivalent formulation of d-index 1 or 0 which explicitly contains all information about the solution manifold would be ideal for the numerical integration since all constraints including the hidden constraints can be numerically satisfied. Such a form of a differential-algebraic equation is called *strangeness-free* [22]. Here, "equivalent" means that both formulations have the same solution set.

In [22] Kunkel and Mehrmann presented an algorithm that reduces the index of general time varying linear differential-algebraic equations of the form

$$\mathbf{E}(t)\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t) \tag{23}$$

of arbitrary index with $\mathbf{E}(t), \mathbf{A}(t) \in \mathbb{R}^{n,n}$, $\mathbf{x}(t), \mathbf{f}(t) \in \mathbb{R}^n$ sufficiently smooth. This algorithm is based on the *linear derivative array*

$$\mathcal{E}_l(t)\dot{\mathbf{z}}_l(t) = \mathcal{A}_l(t)\mathbf{z}_l(t) + \mathcal{F}_l(t) \tag{24}$$

for a particular l, where

$$\begin{aligned}
(\mathcal{E}_l(t))_{ij} &= \binom{i}{j}\mathbf{E}_{i-j}(t) - \binom{i}{j+1}\mathbf{A}_{i-j-1}(t), && i,j = 0,...,l,\\
(\mathcal{A}_l(t))_{ij} &= \begin{cases} \mathbf{A}_i(t) & \text{for } i = 0,...,l,\ j = 0,\\ 0 & \text{else,}\end{cases} && i,j = 0,...,l,\\
(\mathcal{F}_l(t))_i &= \mathbf{f}^{(i)}(t), && i = 0,...,l,\\
(\mathbf{z}_l(t))_i &= \mathbf{x}^{(i)}(t), && i = 0,...,l.
\end{aligned}$$

We use the convention that $\binom{i}{j} = 0$ for $i < 0,\ j < 0$ or $j > i$.
The derivative array of a linear differential-algebraic equation (23) was originally introduced by Campbell [6] and contains all time derivatives of the linear differential-algebraic equation (23) up to order l.

The algorithm starts with the linear derivative array (24), where $l = \nu_d - 1$ (see Definition 1), and creates two essential projectors, namely $\mathbf{P}_1(t) \in \mathbb{R}^{n_d,n}$ which extracts the differential part of n_d differential equations from (23) and $\mathbf{P}_2(t) \in \mathbb{R}^{n_a,n(l+1)}$ which extracts the algebraic part of n_a (hidden) algebraic equations from (24). Finally, the equivalent strangeness-free differential-algebraic equation has the form

$$\left[\begin{array}{c} \mathbf{P}_1^T(t)\mathbf{E}(t) \\ \hline 0 \end{array}\right]\dot{\mathbf{x}}(t) = \left[\begin{array}{c} \mathbf{P}_1^T(t)\mathbf{A}(t) \\ \hline \mathbf{P}_2^T(t)\mathcal{A}(t)\begin{bmatrix}\mathbf{I}_n \\ 0 \\ \vdots \\ 0\end{bmatrix} \end{array}\right]\mathbf{x}(t) + \left[\begin{array}{c} \mathbf{P}_1^T(t)\mathbf{f}(t) \\ \hline \mathbf{P}_2^T(t)\mathcal{F}(t) \end{array}\right].$$

This algorithm is implemented in the code `GELDA`, see [25], which is applicable for the numerical integration of general time varying linear differential-algebraic equations of arbitrary index. Furthermore, it is also the basic idea of the code `GENDA`, see [26], for the numerical integration of general nonlinear differential-algebraic equations with arbitrary index. In principle, this ideas also allow the consideration of over- and underdetermined systems, too, see [24].
Its generality provides no exploitation of structures which are possibly known for many problems.

In [2] it is shown how to create the equivalent strangeness-free formulation for linear equations of motion exploiting their special structure. There the matrices $\mathbf{S_p} \in \mathbb{R}^{n_f, n_\mathbf{p}}$ and $\mathbf{S_v} \in \mathbb{R}^{n_f, n_\mathbf{p}}$ are used which we will use in the following algorithm as well. These matrices select a certain linear combination of the kinematic and dynamic equations of motion. Therefore, we will call the matrices $\mathbf{S_p}$ and $\mathbf{S_v}$ as *selectors* or, more precisely, *kinematic selector* and *dynamic selector*, respectively.

Algorithm 13 (Solution manifold conserving strangeness deletion of linear EoM)

The linear equations of motion are given in form (22). The index reduction is done by choosing two selectors $\mathbf{S_p}$ and $\mathbf{S_v}$ in the following way.

1. **Determination of the kinematic selector $\mathbf{S_p}$**

   (a) Determine $\mathbf{G}^- \in \mathbb{R}^{n_\mathbf{p}, n_f}$ such that the columns of $\mathbf{G}^-$ span $\ker(\mathbf{G})$.

   (b) Determine the kinematic selector $\mathbf{S_p} \in \mathbb{R}^{n_f, n_\mathbf{p}}$ such that $\mathbf{S_p}\mathbf{G}^-$ is nonsingular.

2. **Determination of the dynamic selector $\mathbf{S_v}$**

   (a) Determine $\mathbf{G}^-_\mathbf{Z} \in \mathbb{R}^{n_\mathbf{p}, n_f}$ such that the columns of $\mathbf{G}^-_\mathbf{Z}$ span $\ker(\mathbf{GZ})$.

   (b) Determine the dynamic selector $\mathbf{S_v} \in \mathbb{R}^{n_f, n_\mathbf{p}}$ such that $\mathbf{S_v}\mathbf{M}\mathbf{G}^-_\mathbf{Z}$ is nonsingular.

   Both selectors $\mathbf{S_p}$ and $\mathbf{S_v}$ depend on time t, i.e., $\mathbf{S_p} = \mathbf{S_p}(t)$ and $\mathbf{S_v} = \mathbf{S_v}(t)$.

3. **Strangeness-free form of the equations of motion**
   By appending the constraints on velocity level (10) and the constraints on acceleration level (14) in the linearized form, the strangeness-free form of the equations of motion is

$$
\begin{aligned}
\mathbf{S_p}\dot{\mathbf{p}} &= \mathbf{S_p}\mathbf{Z}\mathbf{v}, && \text{(25a)}\\
\mathbf{S_v}\mathbf{M}\dot{\mathbf{v}} &= \mathbf{S_v}\mathbf{C}\mathbf{p} + \mathbf{S_v}\mathbf{D}\mathbf{v} - \mathbf{S_v}\mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda}, && \text{(25b)}\\
0 &= \mathbf{G}\mathbf{p}, && \text{(25c)}\\
0 &= \mathbf{G}\mathbf{Z}\mathbf{v} + \dot{\mathbf{G}}\mathbf{p}, && \text{(25d)}\\
0 &= \left(\ddot{\mathbf{G}} + \mathbf{G}\mathbf{Z}\mathbf{M}^{-1}\mathbf{C}\right)\mathbf{p} + \left(2\dot{\mathbf{G}}\mathbf{Z} + \mathbf{G}\dot{\mathbf{Z}} + \mathbf{G}\mathbf{Z}\mathbf{M}^{-1}\mathbf{D}\right)\mathbf{v} && \text{(25e)}\\
&\quad -\left(\mathbf{G}\mathbf{Z}\mathbf{M}^{-1}\mathbf{Z}^T\mathbf{G}^T\right)\boldsymbol{\lambda}.
\end{aligned}
$$

Remark 14 1) The algebraic equations (25d) and (25e) correspond to the first and second time derivative of the constraints on position level (25c). Therefore, roughly speaking, one can say that the index reduction algorithm is performed by appending the first and second time derivative of the constraints contained in the original EoM and removing some of the kinematic equations of motion (22a) and some of the dynamic equations of motion (22b) in an appropriate way, i.e., by determining and applying the selectors $\mathbf{S_p}$ and $\mathbf{S_v}$ as described in (25).
2) Since the matrix $\mathbf{G}\mathbf{Z}\mathbf{M}^{-1}\mathbf{Z}^T\mathbf{G}^T$ is assumed to be nonsingular by (4), the Lagrange-multipliers $\boldsymbol{\lambda}$ are completely determined by the last equation (25e) and by the position and velocity variables. Therefore, there is no freedom in the choice of the Lagrange-multipliers.

5 Index Reduction for Nonlinear Equations of Motion

In this section we will consider the regularization of the nonlinear equations of motion (1) via index reduction. As shown in Section 3, it is important to preserve and to determine all information about the solution manifold, i.e., in the regularized form, in addition to the current explicit given constraints (1c), all hidden constraints (11) and (15) should occur in an explicit way.
Before we are able to investigate the process of index reduction, we need some technical preparations. An ,important task is the possibility to determine a smooth frame of the tangent space of the position manifold. This is provided by Theorem 2 in [31] which we will recall in Lemma 16 and adapt to our situation in Lemma 18.

Definition 15 *Let $F \in \mathcal{C}^r(\mathbb{S}, \mathbb{R}^m)$, let $\mathbb{S}$ open in $\mathbb{R}^n$, and let $n > m$. A point $x_0 \in \mathbb{S}$ is called regular if the first derivative $F_{,x}(x_0)$ of $F(x)$ with respect to x evaluated at the point x_0 has full rank m.*

Lemma 16 *Let $F \in \mathcal{C}^r(\mathbb{S}, \mathbb{R}^m)$, let $\mathbb{S}$ open in $\mathbb{R}^n$, and let $p = n - m > 0$. Furthermore, let $\mathcal{N}_0$ be an open subset of the manifold $\mathcal{N}$ defined by*

$$\mathcal{N} = \{x \in \mathbb{S} \ : \ F(x) = 0, \ x \ \textit{regular}\}.$$

For any $x \in \mathcal{N}_0$, let $U(x)$ be an orthonormal basis matrix of the tangent space $T_x\mathcal{N} := \ker(F_{,x}(x))$. Then it is possible to compute an orthogonal matrix $Q = Q(x)$ (e.g. by using Algorithm 3.3 in [31]) such that the map $x \mapsto U(x)Q(x) \in L(\mathbb{R}^{p,n})$, $x \in \mathcal{N}_0$ is of class $\mathcal{C}^{r-1}$ on $\mathcal{N}_0$ and defines an orthonormal moving frame on $\mathcal{N}_0$.

Proof: See [31]. □

Definition 17 *Let $F : \mathbb{S}_1 \times \mathbb{S}_2 \to \mathbb{R}^m$, $\mathbb{S}_1$ open in $\mathbb{R}^{n_1}$, $\mathbb{S}_2$ open in $\mathbb{R}^{n_2}$ and $n_1 > m$, then $F \in \mathcal{C}^{r_1,r_2}(\mathbb{S}_1 \times \mathbb{S}_2, \mathbb{R}^m)$ if $F(\cdot, y) \in \mathcal{C}^{r_1}(\mathbb{S}_1, \mathbb{R}^m)$ w.r.t. the first component and $F(x, \cdot) \in \mathcal{C}^{r_2}(\mathbb{S}_2, \mathbb{R}^m)$ w.r.t. the second component.*

Lemma 18 *Let $F \in \mathcal{C}^{r_1,r_2}(\mathbb{S}_1 \times \mathbb{S}_2, \mathbb{R}^m)$, $r_1, r_2 \in \mathbb{N}_0$. Furthermore, let $\mathcal{N}_0(y) \subset \mathcal{N}_0$ be an open subset of the y-dependent manifold $\mathcal{N}(y)$ defined by*

$$\mathcal{N}(y) = \{x \in \mathbb{S}_1 \ : \ F(x, y) = 0, \ x \ \textit{regular}\} \subset \mathcal{N}.$$

For any $x \in \mathcal{N}_0(y)$, let $U(x, y)$ be an orthonormal basis matrix of the tangent space $T_x\mathcal{N}(y) := \ker(F_{,x}(x, y))$. Then it is possible to compute an orthogonal matrix $Q = Q(x, y)$ (e.g. by using Algorithm 3.3 in [31]) such that the map $x \mapsto U(x, y)Q(x, y) \in L(\mathbb{R}^{p,n})$, $x \in \mathcal{N}_0(y)$ is of class $\mathcal{C}^{r_1-1,r_2}$ on $\mathcal{N}_0$ and defines an orthonormal moving frame on $\mathcal{N}_0$.

Proof: The proof is analogous the proof of Lemma 16 in [31] generalized to parameterized manifolds. □

Lemma 19 *Let $B \in \mathcal{C}^{r_1,r_2}(\mathbb{S}_1 \times \mathbb{S}_2, \mathbb{R}^{m,n})$, $r_1, r_2 \in \mathbb{N}_0$, $m \geq n$ and $\operatorname{rank}(B(x, y)) = n$ for all $(x, y) \in \mathbb{S}_1 \times \mathbb{S}_2$. Then there exists $V \in \mathcal{C}^{r_1,r_2}(\mathbb{S}_1 \times \mathbb{S}_2, \mathbb{R}^{n,m})$ such that for every $(x, y) \in \mathbb{S}_1 \times \mathbb{S}_2$*

$$V(x, y)B(x, y) \ \textit{is nonsingular for every} \ (x, y) \in \mathbb{S}_1 \times \mathbb{S}_2. \tag{26}$$

Proof: Setting $V(x,y) = B^T(x,y) \in \mathcal{C}^{r_1,r_2}(\mathbb{S}_1 \times \mathbb{S}_2, \mathbb{R}^{n,m})$ we obtain that $V(x,y)B(x,y) = B^T(x,y)B(x,y)$ is nonsingular for every (x,y) since $B(x,y)$ has full rank for every $(x,y) \in \mathbb{S}_1 \times \mathbb{S}_2$. $\square$

In the following let $\mathcal{N}$ be a manifold in $\mathbb{R}^n$, then $\mathcal{N}^\epsilon := \{x \in \mathbb{R}^n \ : \ \exists y \in \mathcal{N} \text{ with } \|x-y\| < \epsilon\}$.

Lemma 20 *For all $t \in \mathbb{I} \subset \mathbb{R}$ let every $\mathbf{p} \in \mathcal{M}^\epsilon_{\mathbf{p}}(t)$ be regular, i.e., $\mathbf{G}(\mathbf{p},t)$ has full rank, and let the constraints (1c) be continously differentiable with respect to $\mathbf{p}$ and continuous in t. Then $\mathbf{g} \in \mathcal{C}^{1,0}(\mathcal{M}^\epsilon_{\mathbf{p}} \times \mathbb{I}, \mathbb{R}^{n_{\boldsymbol{\lambda}}})$ and furthermore, there exists a nonsingular possibly orthogonal $\mathbf{G}^- \in \mathcal{C}^{0,0}(\mathcal{M}^\epsilon_{\mathbf{p}} \times \mathbb{I}, \mathbb{R}^{n_{\mathbf{p}}, n_{\mathbf{p}}-n_{\boldsymbol{\lambda}}})$ such that the columns of $\mathbf{G}^-(\mathbf{p},t)$ define an orthonormal moving frame on $\mathcal{M}^\epsilon_0$, continuous in $\mathbf{p}$ and t, i.e.,*

$$\mathbf{G}(\mathbf{p},t)\mathbf{G}^-(\mathbf{p},t) = \mathbf{0} \textit{ for every } (\mathbf{p},t) \in \mathcal{M}^\epsilon_{\mathbf{p}} \times [t_0, t_N]. \tag{27}$$

Proof: The proof follows directly from Lemma 18. $\square$

Lemma 21 *Let $\mathbf{G}^- \in \mathcal{C}^{0,0}(\mathcal{M}^\epsilon_{\mathbf{p}} \times \mathbb{I}, \mathbb{R}^{n_{\mathbf{p}}, n_{\mathbf{p}}-n_{\boldsymbol{\lambda}}})$ have full column rank for every $(\mathbf{p},t) \in \mathcal{M}^\epsilon_{\mathbf{p}} \times \mathbb{I}$. Then a map $\mathbf{S}_{\mathbf{p}} \in \mathcal{C}^{0,0}(\mathcal{M}^\epsilon_{\mathbf{p}} \times \mathbb{I}, \mathbb{R}^{n_{\mathbf{p}}, n_{\mathbf{p}}-n_{\boldsymbol{\lambda}}})$ can be chosen such that*

$$\mathbf{S}_{\mathbf{p}}(\mathbf{p},t)\mathbf{G}^-(\mathbf{p},t) \quad \textit{is nonsingular for every } (\mathbf{p},t) \in \mathcal{M}^\epsilon_{\mathbf{p}} \times \mathbb{I}.$$

Proof: The proof follows directly from Lemma 19. $\square$

Lemma 22 *For all $t \in \mathbb{I} \subset \mathbb{R}$ let every $\mathbf{p} \in \mathcal{M}^\epsilon_{\mathbf{p}}(t)$ be regular, i.e., $\mathbf{G}(\mathbf{p},t)$ has full rank, let the constraints (1c) be continously differentiable with respect to $\mathbf{p}$ and continuous in time t, and let $\mathbf{Z}(\mathbf{p})$ be nonsingular and continuous with respect to $\mathbf{p}$. Then there exists $\mathbf{G}^-_{\mathbf{Z}} \in \mathcal{C}^{0,0}(\mathcal{M}^\epsilon_{\mathbf{p}} \times \mathbb{I}, \mathbb{R}^{n_{\mathbf{p}}, n_{\mathbf{p}}-n_{\boldsymbol{\lambda}}})$, such that the columns of $\mathbf{G}^-_{\mathbf{Z}}(\mathbf{p},t)$ span the kernel of $\mathbf{G}(\mathbf{p},t)\mathbf{Z}(\mathbf{p})$, i.e.,*

$$\mathbf{G}(\mathbf{p},t)\mathbf{Z}(\mathbf{p})\mathbf{G}^-_{\mathbf{Z}}(\mathbf{p},t) = \mathbf{0} \textit{ for every } (\mathbf{p},t) \in \mathcal{M}^\epsilon_{\mathbf{p}} \times [t_0, t_N]. \tag{28}$$

Proof: Lemma 20 provides the existence of $\mathbf{G}^- \in \mathcal{C}^{0,0}(\mathcal{M}^\epsilon_{\mathbf{p}} \times \mathbb{I}, \mathbb{R}^{n_{\mathbf{p}}, n_{\mathbf{p}}-n_{\boldsymbol{\lambda}}})$ such that equation (27) holds. From this we obtain

$$\mathbf{G}(\mathbf{p},t)\mathbf{Z}(\mathbf{p})\mathbf{Z}^{-1}(\mathbf{p})\mathbf{G}^-(\mathbf{p},t) = \mathbf{0} \text{ for every } (\mathbf{p},t) \in \mathcal{M}^\epsilon_{\mathbf{p}} \times [t_0, t_N]$$

and we get $\mathbf{G}^-_{\mathbf{Z}}(\mathbf{p},t) = \mathbf{Z}^{-1}(\mathbf{p})\mathbf{G}^-(\mathbf{p},t) \in \mathcal{C}^{0,0}(\mathcal{M}^\epsilon_{\mathbf{p}} \times \mathbb{I}, \mathbb{R}^{n_{\mathbf{p}}, n_{\mathbf{p}}-n_{\boldsymbol{\lambda}}})$. $\square$

Lemma 23 *Let $\mathbf{G}^-_{\mathbf{Z}} \in \mathcal{C}^{0,0}(\mathcal{M}^\epsilon_{\mathbf{p}} \times \mathbb{I}, \mathbb{R}^{n_{\mathbf{p}}, n_{\mathbf{p}}-n_{\boldsymbol{\lambda}}})$ have full column rank for every $(\mathbf{p},t) \in \mathcal{M}^\epsilon_{\mathbf{p}} \times \mathbb{I}$. Furthermore, let $\mathbf{M}(\mathbf{p})$ be nonsingular and continuous with respect to $\mathbf{p}$. Then a map $\mathbf{S}_{\mathbf{v}} \in \mathcal{C}^{0,0}(\mathcal{M}^\epsilon_{\mathbf{p}} \times \mathbb{I}, \mathbb{R}^{n_{\mathbf{p}}, n_{\mathbf{p}}-n_{\boldsymbol{\lambda}}})$ can be chosen such that*

$$\mathbf{S}_{\mathbf{v}}(\mathbf{p},t)\mathbf{M}(\mathbf{p})\mathbf{G}^-_{\mathbf{Z}}(\mathbf{p},t) \quad \textit{is nonsingular for every } (\mathbf{p},t) \in \mathcal{M}^\epsilon_{\mathbf{p}} \times \mathbb{I}.$$

Proof: Since $\mathbf{M}\mathbf{G}^-_{\mathbf{Z}} \in \mathcal{C}^{0,0}(\mathcal{M}^\epsilon_{\mathbf{p}} \times \mathbb{I}, \mathbb{R}^{n_{\mathbf{p}}, n_{\mathbf{p}}-n_{\boldsymbol{\lambda}}})$ and has full rank, the result follows directly from Lemma 19. $\square$

In preparation for the index reduction of the nonlinear equations of motion (1) we need in addition the following two lemmas. Here we use $n_f = n_{\mathbf{p}} - n_{\boldsymbol{\lambda}}$.

Lemma 24 *For all $t \in \mathbb{I} \subset \mathbb{R}$ let every $\mathbf{p} \in \mathcal{M}^{\epsilon}_{\mathbf{p}}(t)$ be regular and let the columns of $\mathbf{G}^{-}(\mathbf{p},t) \in \mathbb{R}^{n_{\mathbf{p}},n_f}$ span* $\ker(\mathbf{G}(\mathbf{p},t))$ *with $\mathbf{G}(\mathbf{p},t) \in \mathbb{R}^{n_{\boldsymbol{\lambda}},n_{\mathbf{p}}}$. Furthermore, let the kinematic selector $\mathbf{S}_{\mathbf{p}}(\mathbf{p},t) \in \mathbb{R}^{n_f,n_{\mathbf{p}}}$ be chosen such that $\mathbf{S}_{\mathbf{p}}(\mathbf{p},t)\mathbf{G}^{-}(\mathbf{p},t)$ is nonsingular. Then*

$$\begin{bmatrix} \mathbf{S}_{\mathbf{p}}(\mathbf{p},t) \\ \mathbf{G}(\mathbf{p},t) \end{bmatrix} \textit{ is nonsingular.}$$

Proof: For every $(\mathbf{p},t) \in \mathcal{M}^{\epsilon}_{\mathbf{p}} \times \mathbb{I}$ the columns of $\mathbf{G}^{-}(\mathbf{p},t)$ span $\ker(\mathbf{G}(\mathbf{p},t))$. Furthermore, let $\mathbf{G}^{\sim}(\mathbf{p},t)$ be such that the columns of $\mathbf{G}^{\sim}(\mathbf{p},t)$ span $\operatorname{coker}(\mathbf{G}(\mathbf{p},t))$.

$$\begin{aligned}
& \begin{bmatrix} \mathbf{S}_{\mathbf{p}}(\mathbf{p},t) \\ \mathbf{G}(\mathbf{p},t) \end{bmatrix} \text{ is nonsingular} \\
\Leftrightarrow \quad & \begin{bmatrix} \mathbf{S}_{\mathbf{p}}(\mathbf{p},t) \\ \mathbf{G}(\mathbf{p},t) \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{G}^{-}(\mathbf{p},t) & \mathbf{G}^{\sim}(\mathbf{p},t) \end{bmatrix}}_{\text{nonsingular}} = \begin{bmatrix} \mathbf{S}_{\mathbf{p}}\mathbf{G}^{-}(\mathbf{p},t) & \mathbf{S}_{\mathbf{p}}\mathbf{G}^{\sim}(\mathbf{p},t) \\ \mathbf{G}\mathbf{G}^{-}(\mathbf{p},t) & \mathbf{G}\mathbf{G}^{\sim}(\mathbf{p},t) \end{bmatrix} \\
& \qquad = \begin{bmatrix} \mathbf{S}_{\mathbf{p}}\mathbf{G}^{-}(\mathbf{p},t) & \mathbf{S}_{\mathbf{p}}\mathbf{G}^{\sim}(\mathbf{p},t) \\ 0 & \mathbf{G}\mathbf{G}^{\sim}(\mathbf{p},t) \end{bmatrix} \text{ is nonsingular} \\
\Leftrightarrow \quad & \begin{cases} \mathbf{S}_{\mathbf{p}}(\mathbf{p},t)\mathbf{G}^{-}(\mathbf{p},t) \in \mathbb{R}^{n_f,n_f} \text{ is nonsingular and} \\ \mathbf{G}(\mathbf{p},t)\mathbf{G}^{\sim}(\mathbf{p},t) \in \mathbb{R}^{n_{\boldsymbol{\lambda}},n_{\boldsymbol{\lambda}}} \text{ is nonsingular} \end{cases}
\end{aligned}$$

$\mathbf{S}_{\mathbf{p}}(\mathbf{p},t)\mathbf{G}^{-}(\mathbf{p},t) \in \mathbb{R}^{n_f,n_f}$ was assumed to be nonsingular. $\mathbf{G}(\mathbf{p},t)\mathbf{G}(\mathbf{p},t)^{\sim} \in \mathbb{R}^{n_{\boldsymbol{\lambda}},n_{\boldsymbol{\lambda}}}$ is nonsingular, since

$$\begin{aligned}
\operatorname{rank}(\mathbf{G}(\mathbf{p},t)\mathbf{G}^{\sim}(\mathbf{p},t)) &= \operatorname{rank}([0\ \mathbf{G}(\mathbf{p},t)\mathbf{G}^{\sim}(\mathbf{p},t)]) = \operatorname{rank}(\mathbf{G}(\mathbf{p},t)\underbrace{[\mathbf{G}^{-}(\mathbf{p},t)\ \mathbf{G}^{\sim}(\mathbf{p},t)]}_{\text{non singular}}) \\
&= \operatorname{rank}(\mathbf{G}(\mathbf{p},t)) = n_{\boldsymbol{\lambda}}.
\end{aligned}$$

□

Lemma 25 *For all $t \in \mathbb{I} \subset \mathbb{R}$ let every $\mathbf{p} \in \mathcal{M}^{\epsilon}_{\mathbf{p}}(t)$ be regular and let the columns of $\mathbf{G}^{-}_{Z}(\mathbf{p},t) \in \mathbb{R}^{n_{\mathbf{p}},n_f}$ span* $\ker(\mathbf{G}(\mathbf{p},t)\mathbf{Z}(\mathbf{p}))$ *with $\mathbf{G}(\mathbf{p},t) \in \mathbb{R}^{n_{\boldsymbol{\lambda}},n_{\mathbf{p}}}$ and $\mathbf{Z}(\mathbf{p}) \in \mathbb{R}^{n_{\mathbf{p}},n_{\mathbf{p}}}$, and suppose that the dynamic selector $\mathbf{S}_{\mathbf{v}}(\mathbf{p},t) \in \mathbb{R}^{n_f,n_{\mathbf{p}}}$ is chosen such that $\mathbf{S}_{\mathbf{v}}(\mathbf{p},t)\mathbf{M}(\mathbf{p})\mathbf{G}^{-}_{Z}(\mathbf{p},t)$ is nonsingular. Then*

$$\begin{bmatrix} \mathbf{S}_{\mathbf{v}}(\mathbf{p},t)\mathbf{M}(\mathbf{p},t) \\ \mathbf{G}(\mathbf{p},t)\mathbf{Z}(\mathbf{p},t) \end{bmatrix} \textit{ is nonsingular.}$$

Proof: For every $(\mathbf{p},t) \in \mathcal{M}_{\mathbf{p}}^{\epsilon} \times \mathbb{I}$ the columns of $\mathbf{G}_{\mathbf{Z}}^{-}(\mathbf{p},t)$ span $\ker(\mathbf{G}(\mathbf{p},t)\mathbf{Z}(\mathbf{p},t))$. Furthermore, let $\mathbf{G}_{\mathbf{Z}}^{\sim}(\mathbf{p},t)$ be such that the columns of $\mathbf{G}_{\mathbf{Z}}^{\sim}(\mathbf{p},t)$ span $\operatorname{coker}(\mathbf{G}(\mathbf{p},t)\mathbf{Z}(\mathbf{p},t))$.

$$
\begin{array}{cl}
& \begin{bmatrix} \mathbf{S}_{\mathbf{v}}(\mathbf{p},t)\mathbf{M}(\mathbf{p},t) \\ \mathbf{G}(\mathbf{p},t)\mathbf{Z}(\mathbf{p},t) \end{bmatrix} \text{ is nonsingular.} \\
\Leftrightarrow & \begin{bmatrix} \mathbf{S}_{\mathbf{v}}(\mathbf{p},t)\mathbf{M}(\mathbf{p},t) \\ \mathbf{G}(\mathbf{p},t)\mathbf{Z}(\mathbf{p},t) \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{G}_{\mathbf{Z}}^{-}(\mathbf{p},t) & \mathbf{G}_{\mathbf{Z}}^{\sim}(\mathbf{p},t) \end{bmatrix}}_{\text{non singular}} \\
& = \begin{bmatrix} \mathbf{S}_{\mathbf{v}}(\mathbf{p},t)\mathbf{M}(\mathbf{p},t)\mathbf{G}_{\mathbf{Z}}^{-} & \mathbf{S}_{\mathbf{v}}(\mathbf{p},t)\mathbf{M}(\mathbf{p},t)\mathbf{G}_{\mathbf{Z}}^{\sim}(\mathbf{p},t) \\ \mathbf{G}(\mathbf{p},t)\mathbf{Z}(\mathbf{p},t)\mathbf{G}_{\mathbf{Z}}^{-}(\mathbf{p},t) & \mathbf{G}(\mathbf{p},t)\mathbf{Z}(\mathbf{p},t)\mathbf{G}_{\mathbf{Z}}^{\sim}(\mathbf{p},t) \end{bmatrix} \\
& = \begin{bmatrix} \mathbf{S}_{\mathbf{v}}(\mathbf{p},t)\mathbf{M}(\mathbf{p},t)\mathbf{G}_{\mathbf{Z}}^{-}(\mathbf{p},t) & \mathbf{S}_{\mathbf{v}}(\mathbf{p},t)\mathbf{M}(\mathbf{p},t)\mathbf{G}_{\mathbf{Z}}^{\sim}(\mathbf{p},t) \\ 0 & \mathbf{G}(\mathbf{p},t)\mathbf{Z}(\mathbf{p},t)\mathbf{G}_{\mathbf{Z}}^{\sim}(\mathbf{p},t) \end{bmatrix} \text{ is nonsingular} \\
\Leftrightarrow & \left\{ \begin{array}{l} \mathbf{S}_{\mathbf{v}}(\mathbf{p},t)\mathbf{M}(\mathbf{p},t)\mathbf{G}_{\mathbf{Z}}^{-}(\mathbf{p},t) \in \mathbb{R}^{n_f,n_f} \text{ is nonsingular and} \\ \mathbf{G}(\mathbf{p},t)\mathbf{Z}(\mathbf{p},t)\mathbf{G}_{\mathbf{Z}}^{\sim}(\mathbf{p},t) \in \mathbb{R}^{n_{\boldsymbol{\lambda}},n_{\boldsymbol{\lambda}}} \text{is nonsingular.} \end{array} \right.
\end{array}
$$

$\mathbf{S}_{\mathbf{v}}(\mathbf{p},t)\mathbf{M}(\mathbf{p},t)\mathbf{G}_{\mathbf{Z}}^{-}(\mathbf{p},t) \in \mathbb{R}^{n_f,n_f}$ was assumed to be nonsingular. $\mathbf{G}(\mathbf{p},t)\mathbf{Z}(\mathbf{p},t)\mathbf{G}_{\mathbf{Z}}^{\sim}(\mathbf{p},t) \in \mathbb{R}^{n_{\boldsymbol{\lambda}},n_{\boldsymbol{\lambda}}}$ is nonsingular since

$$
\begin{aligned}
\operatorname{rank}(\mathbf{G}(\mathbf{p},t)\mathbf{Z}(\mathbf{p},t)\mathbf{G}_{\mathbf{Z}}^{\sim}(\mathbf{p},t)) &= \operatorname{rank}([0\ \mathbf{G}(\mathbf{p},t)\mathbf{Z}(\mathbf{p},t)\mathbf{G}_{\mathbf{Z}}^{\sim}(\mathbf{p},t)]) \\
&= \operatorname{rank}(\mathbf{G}(\mathbf{p},t)\underbrace{[\mathbf{Z}(\mathbf{p},t)][\mathbf{G}_{\mathbf{Z}}^{-}(\mathbf{p},t)\ \mathbf{G}_{\mathbf{Z}}^{\sim}(\mathbf{p},t)]}_{\text{non singular}}) \\
&= \operatorname{rank}(\mathbf{G}(\mathbf{p},t)) \\
&= n_{\boldsymbol{\lambda}}.
\end{aligned}
$$

□

Now we have presented all the tools to perform the solution manifold preserving index reduction of the nonlinear equations of motion as shown in the following theorem.

Theorem 26 *For all $t \in \mathbb{I} \subset \mathbb{R}$ let every $\mathbf{p} \in \mathcal{M}_{\mathbf{p}}^{\epsilon}(t)$ be regular and let the constraints (1c) be continously differentiable with respect to $\mathbf{p}$ and continuous in t, i.e., $\mathbf{g} \in \mathcal{C}^{1,0}(\mathcal{M}_{\mathbf{p}}^{\epsilon} \times \mathbb{I}, \mathbb{R}^{n_{\boldsymbol{\lambda}}})$. Furthermore, let $\mathbf{Z}(\mathbf{p})$ and $\mathbf{M}(\mathbf{p})$ be nonsingular and continuous with respect to $\mathbf{p}$, i.e., $\mathbf{Z}, \mathbf{M} \in \mathcal{C}^{0}(\mathcal{M}_{\mathbf{p}}^{\epsilon}, \mathbb{R}^{n_{\mathbf{p}},n_{\mathbf{p}}})$. Then there exists a kinematic selector $\mathbf{S}_{\mathbf{p}} \in \mathcal{C}^{0,0}(\mathcal{M}_{\mathbf{p}}^{\epsilon} \times \mathbb{I}, \mathbb{R}^{n_{\mathbf{p}},n_{\mathbf{p}}-n_{\boldsymbol{\lambda}}})$ and a dynamic selector $\mathbf{S}_{\mathbf{v}} \in \mathcal{C}^{0,0}(\mathcal{M}_{\mathbf{p}}^{\epsilon} \times \mathbb{I}, \mathbb{R}^{n_{\mathbf{p}},n_{\mathbf{p}}-n_{\boldsymbol{\lambda}}})$ such that the differential algebraic system*

$$
\begin{aligned}
\mathbf{S}_{\mathbf{p}}(\mathbf{p},t)\dot{\mathbf{p}} &= \mathbf{S}_{\mathbf{p}}(\mathbf{p},t)\mathbf{Z}(\mathbf{p})\mathbf{v}, && \text{(29a)} \\
\mathbf{S}_{\mathbf{v}}(\mathbf{p},t)\mathbf{M}(\mathbf{p})\dot{\mathbf{v}} &= \mathbf{S}_{\mathbf{v}}(\mathbf{p},t)\mathbf{f}(\mathbf{p},\mathbf{v},t) - \mathbf{S}_{\mathbf{v}}(\mathbf{p},t)\mathbf{Z}^T(\mathbf{p})\mathbf{G}^T(\mathbf{p},t)\boldsymbol{\lambda}, && \text{(29b)} \\
0 &= \mathbf{g}(\mathbf{p},t), && \text{(29c)} \\
0 &= \mathbf{g}^{I}(\mathbf{p},\mathbf{v},t), && \text{(29d)} \\
0 &= \mathbf{g}^{II}(\mathbf{p},\mathbf{v},\boldsymbol{\lambda},t) && \text{(29e)}
\end{aligned}
$$

has the same solution set as the equations of motion (1) and is of differentiation index one, i.e., strangeness-free.

Proof: In the following, we will omit the dependencies on $\mathbf{p}$, $\mathbf{v}$, $\boldsymbol{\lambda}$, and t. The existence of the kinematic selector $\mathbf{S}_{\mathbf{p}} \in \mathcal{C}^{0,0}(\mathcal{M}_{\mathbf{p}}^{\epsilon} \times \mathbb{I}, \mathbb{R}^{n_{\mathbf{p}},n_{\mathbf{p}}-n_{\boldsymbol{\lambda}}})$ is proved in Lemma 21 and the existence of the dynamic selector $\mathbf{S}_{\mathbf{v}} \in \mathcal{C}^{0,0}(\mathcal{M}_{\mathbf{p}}^{\epsilon} \times \mathbb{I}, \mathbb{R}^{n_{\mathbf{p}},n_{\mathbf{p}}-n_{\boldsymbol{\lambda}}})$ is proved in Lemma 23. The rest of the proof is partitioned into two parts. In the first part we will show that the solution set of (29) is identical to the solution set of (1). In the second part the index of the system (29) will be examined.

a) It is obvious that a solution of (1) is also a solution of (29). The other direction will be discussed in the following.

With the trivial equation $\mathbf{S_p Z v} = \mathbf{S_p Z v}$ and (29d) in form of (10) we get

$$\begin{bmatrix} \mathbf{S_p Z v} \\ \mathbf{G Z v} \end{bmatrix} = \begin{bmatrix} \mathbf{S_p Z v} \\ -\mathbf{g}_{,t} \end{bmatrix}$$

which is equivalent to

$$\mathbf{Z v} = \begin{bmatrix} \mathbf{S_p} \\ \mathbf{G} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{S_p Z v} \\ -\mathbf{g}_{,t} \end{bmatrix}. \tag{30}$$

On the other hand, it follows from (29a) and the from time derivative of (29c) in form (9) that

$$\begin{bmatrix} \mathbf{S_p \dot{p}} \\ \mathbf{G \dot{p}} \end{bmatrix} = \begin{bmatrix} \mathbf{S_p Z v} \\ -\mathbf{g}_{,t} \end{bmatrix}$$

which is equivalent to

$$\dot{\mathbf{p}} = \begin{bmatrix} \mathbf{S_p} \\ \mathbf{G} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{S_p Z v} \\ -\mathbf{g}_{,t} \end{bmatrix}. \tag{31}$$

Therefore, from (30) and (31) we get the kinematic equations of motion (1a)

$$\dot{\mathbf{p}} = \mathbf{Z v}.$$

Furthermore, with the trivial equation $\mathbf{S_v}(\mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda}) = \mathbf{S_v}(\mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda})$ and (29e) in form of (13) we get

$$\begin{bmatrix} \mathbf{S_v}(\mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda}) \\ \mathbf{G Z M}^{-1}(\mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda}) \end{bmatrix} = \begin{bmatrix} \mathbf{S_v}(\mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda}) \\ -\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{G Z})\mathbf{v} - \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{g}}^I \end{bmatrix}$$

which is equivalent to

$$\mathbf{M}^{-1}(\mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda}) = \begin{bmatrix} \mathbf{S_v M} \\ \mathbf{G Z} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{S_v}(\mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda}) \\ -\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{G Z})\mathbf{v} - \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{g}}^I \end{bmatrix}. \tag{32}$$

On the other hand, it follows from (29b) and from the time derivative of (29d) in form (12) that

$$\begin{bmatrix} \mathbf{S_v M \dot{v}} \\ \mathbf{G Z \dot{v}} \end{bmatrix} = \begin{bmatrix} \mathbf{S_v}(\mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda}) \\ -\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{G Z})\mathbf{v} - \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{g}}^I \end{bmatrix}$$

which is equivalent to

$$\dot{\mathbf{v}} = \begin{bmatrix} \mathbf{S_v M} \\ \mathbf{G Z} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{S_v}(\mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda}) \\ -\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{G Z})\mathbf{v} - \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{g}}^I \end{bmatrix}. \tag{33}$$

Therefore, from (32) and (33) we get the dynamic equations of motion (1b)

$$\mathbf{M}\dot{\mathbf{v}} \quad = \quad \mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda}.$$

In addition, the constraint equations (29c) are identical with the constraint equations (1c). Hence, a solution of (29) is also a solution of (1).

b) It remains to show that the system (29) has differentiation index one. Let us compute the underlying ODE of (29). Replacing the equations (29c), (29d), and (29e) by their first time derivatives using (9), (12), and (18),

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g}(\mathbf{p},t) &= \mathbf{G}(\mathbf{p},t)\dot{\mathbf{p}} + \mathbf{g}_{,t}(\mathbf{p},t),\\
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g}^{I}(\mathbf{p},\mathbf{v},t) &= \mathbf{G}\mathbf{Z}\dot{\mathbf{v}} + \tilde{\mathbf{g}}^{II}(\mathbf{p},\mathbf{v},t),\\
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g}^{II}(\mathbf{p},\mathbf{v},\boldsymbol{\lambda},t) &= \tilde{\mathbf{g}}^{III}(\mathbf{p},\mathbf{v},\boldsymbol{\lambda},t) - \mathbf{G}\mathbf{Z}\mathbf{M}^{-1}\mathbf{Z}^T\mathbf{G}^T\dot{\boldsymbol{\lambda}},
\end{aligned}$$

we obtain

$$\begin{aligned}
\mathbf{S}_{\mathbf{p}}\dot{\mathbf{p}} &= \mathbf{S}_{\mathbf{p}}\mathbf{Z}\mathbf{v},\\
\mathbf{S}_{\mathbf{v}}\mathbf{M}\dot{\mathbf{v}} &= \mathbf{S}_{\mathbf{v}}\mathbf{f} - \mathbf{S}_{\mathbf{v}}\mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda},\\
\mathbf{G}\dot{\mathbf{p}} &= -\mathbf{g}_{,t},\\
\mathbf{G}\mathbf{Z}\dot{\mathbf{v}} &= -\tilde{\mathbf{g}}^{II}(\mathbf{p},\mathbf{v},t),\\
\mathbf{G}\mathbf{Z}\mathbf{M}^{-1}\mathbf{Z}^T\mathbf{G}^T\dot{\boldsymbol{\lambda}} &= \tilde{\mathbf{g}}^{III}(\mathbf{p},\mathbf{v},\boldsymbol{\lambda},t).
\end{aligned}$$

From Lemmas 24 and 25 we obtain that

$$\begin{bmatrix} \mathbf{S}_{\mathbf{p}} \\ \mathbf{G} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{S}_{\mathbf{v}}\mathbf{M} \\ \mathbf{G}\mathbf{Z} \end{bmatrix} \quad \text{are nonsingular.}$$

Furthermore, it is assumed that condition (4) holds. Therefore, we get the underlying ODE

$$\begin{aligned}
\dot{\mathbf{p}} &= \begin{bmatrix} \mathbf{S}_{\mathbf{p}} \\ \mathbf{G} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{S}_{\mathbf{p}}\mathbf{Z}\mathbf{v} \\ -\mathbf{g}_{,t} \end{bmatrix},\\
\dot{\mathbf{v}} &= \begin{bmatrix} \mathbf{S}_{\mathbf{v}}\mathbf{M} \\ \mathbf{G}\mathbf{Z} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{S}_{\mathbf{v}}\mathbf{f} - \mathbf{S}_{\mathbf{v}}\mathbf{Z}^T\mathbf{G}^T\boldsymbol{\lambda} \\ -\tilde{\mathbf{g}}^{II}(\mathbf{p},\mathbf{v},t) \end{bmatrix},\\
\dot{\boldsymbol{\lambda}} &= \left[\mathbf{G}\mathbf{Z}\mathbf{M}^{-1}\mathbf{Z}^T\mathbf{G}^T\right]^{-1}\tilde{\mathbf{g}}^{III}(\mathbf{p},\mathbf{v},\boldsymbol{\lambda},t)
\end{aligned}$$

after only one differentiation of the constraint equations (29c), (29d), and (29e). Hence, the system (29) has differentiation index one. □

Algorithm 27 (Solution manifold conserving strangeness deletion of EoM)
The equations of motion are given in form (1) and it is assumed that for all $t \in \mathbb{I} \subset \mathbb{R}$ every $\mathbf{p} \in \mathcal{M}^{\epsilon}_{\mathbf{p}}(t)$ is regular, i.e., $\mathbf{G}(\mathbf{p},t)$ has full rank, and $\mathbf{g} \in \mathcal{C}^{1,0}(\mathcal{M}^{\epsilon}_{\mathbf{p}} \times \mathbb{I}, \mathbb{R}^{n_{\boldsymbol{\lambda}}})$. Furthermore, let $\mathbf{M}(\mathbf{p})$ and $\mathbf{Z}(\mathbf{p})$ are nonsingular for all $(\mathbf{p},t) \in \mathcal{M}^{\epsilon}_{\mathbf{p}} \times \mathbb{I}$, and let $\mathbf{M} \in \mathcal{C}^{0}(\mathcal{M}^{\epsilon}_{\mathbf{p}}, \mathbb{R}^{n_{\mathbf{p}},n_{\mathbf{p}}})$ and $\mathbf{Z} \in \mathcal{C}^{0}(\mathcal{M}^{\epsilon}_{\mathbf{p}}, \mathbb{R}^{n_{\mathbf{p}},n_{\mathbf{p}}})$.
Then the regularization by index reduction is done by choosing a kinematic selector $\mathbf{S}_{\mathbf{p}} \in \mathcal{C}^{0,0}(\mathcal{M}^{\epsilon}_{\mathbf{p}} \times \mathbb{I}, \mathbb{R}^{n_{\mathbf{p}},n_{\mathbf{p}}-n_{\boldsymbol{\lambda}}})$ and a dynamic selector $\mathbf{S}_{\mathbf{v}} \in \mathcal{C}^{0,0}(\mathcal{M}^{\epsilon}_{\mathbf{p}} \times \mathbb{I}, \mathbb{R}^{n_{\mathbf{p}},n_{\mathbf{p}}-n_{\boldsymbol{\lambda}}})$ in the following way.

1. **Determination of a kinematic selector $\mathbf{S_p}$**

   (a) Determine $\mathbf{G}^-(\mathbf{p}, t) \in \mathbb{R}^{n_\mathbf{p}, n_f}$ such that the columns of $\mathbf{G}^-(\mathbf{p}, t)$ span $\ker(\mathbf{G}(\mathbf{p}, t))$ for every $(\mathbf{p}, t) \in \mathcal{M}^\epsilon_\mathbf{p} \times \mathbb{I}$.

   (b) Determine a kinematic selector $\mathbf{S_p}(\mathbf{p}, t) \in \mathbb{R}^{n_f, n_\mathbf{p}}$ such that $\mathbf{S_p}(\mathbf{p}, t)\mathbf{G}^-(\mathbf{p}, t)$ is nonsingular for every $(\mathbf{p}, t) \in \mathcal{M}^\epsilon_\mathbf{p} \times \mathbb{I}$.

2. **Determination of a dynamic selector $\mathbf{S_v}$**

   (a) Determine $\mathbf{G}^-_\mathbf{Z}(\mathbf{p}, t) \in \mathbb{R}^{n_\mathbf{p}, n_f}$ such that the columns of $\mathbf{G}^-_\mathbf{Z}(\mathbf{p}, t)$ span $\ker(\mathbf{G}(\mathbf{p}, t)\mathbf{Z}(\mathbf{p}, t))$ for every $(\mathbf{p}, t) \in \mathcal{M}^\epsilon_\mathbf{p} \times \mathbb{I}$.

   (b) Determine a dynamic selector $\mathbf{S_v}(\mathbf{p}, t) \in \mathbb{R}^{n_f, n_\mathbf{p}}$ such that $\mathbf{S_v}(\mathbf{p}, t)\mathbf{M}(\mathbf{p}, t)\mathbf{G}^-_\mathbf{Z}(\mathbf{p}, t)$ is nonsingular for every $(\mathbf{p}, t) \in \mathcal{M}^\epsilon_\mathbf{p} \times \mathbb{I}$.

3. **Strangeness-free form of the equations of motion** (sfEoM)
   By appending the constraints on velocity level (10) and the constraints on acceleration level (14), the strangeness-free form of the equations of motion is (29).

With this algorithm we are able to determine an equivalent strangeness-free form of the equations of motion (29) which contains all information of the solution manifold (16). The strangeness-free form created in this way is analytically equivalent to the original equations of motion in the sense that both have the same solution set. Furthermore, this form is suitable for numerical integration using stiff ODE solvers like implicit Runge-Kutta-Methods or BDF-Methods.

6 Examples

In this section we want to consider a small-dimensional example consisting of just one body of mass 1 in two dimensional space is moving on the unit circle. The equations of motion are defined as

$$\underbrace{\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix}}_{\dot{\mathbf{p}}} = \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{\mathbf{v}}, \tag{34a}$$

$$\underbrace{\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix}}_{\dot{\mathbf{v}}} = \underbrace{\begin{bmatrix} -p_1 - 2p_1v_1v_2 \\ -v_1 + 2p_1p_2^2 \end{bmatrix}}_{\mathbf{f}(\mathbf{p},\mathbf{v})} - \underbrace{\begin{bmatrix} 2p_1 \\ 2p_2 \end{bmatrix}}_{\mathbf{G}^T(\mathbf{p})} \underbrace{\begin{bmatrix} \lambda_1 \end{bmatrix}}_{\boldsymbol{\lambda}}, \tag{34b}$$

$$0 = \underbrace{\begin{bmatrix} p_1^2 + p_2^2 - 1 \end{bmatrix}}_{\mathbf{g}(\mathbf{p})}, \tag{34c}$$

or in matrix form

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{v}, \\ \mathbf{M}\dot{\mathbf{v}} &= \mathbf{f}(\mathbf{p}, \mathbf{v}) - \mathbf{G}^T(\mathbf{p})\boldsymbol{\lambda}, \\ 0 &= \mathbf{g}(\mathbf{p}), \end{aligned}$$

with initial values

$$\mathbf{p}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{v}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \boldsymbol{\lambda}(0) = 0. \tag{35}$$

The solution is given by

$$\begin{gathered} \mathbf{p}(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}, \ \mathbf{v}(t) = \dot{\mathbf{p}}(t) = \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix}, \ \dot{\mathbf{v}}(t) = \ddot{\mathbf{p}}(t) = \begin{bmatrix} -\sin(t) \\ -\cos(t) \end{bmatrix}, \\ \boldsymbol{\lambda}(t) = [\sin(t) \cdot \cos(t)]. \end{gathered} \tag{36}$$

In Section 3 we have worked out the behavior of the analytic solution depending on the consistency of the initial values, in particular, the deviation from the solution manifold, and we discussed numerical aspects. Furthermore, in Section 5 we presented Algorithm 27 to determine an equivalent strangeness-free form of the EoM preserving the information of the solution manifold. In the following we will support the obtained results by numerical experiments based on the different formulations of the EoM: the uODE (19), EoM_1 (20), EoM_2 (21), EoM (1) and finally the sfEoM (29).

The numerical integration of the different forms is done by using the implicit trapezoidal rule for the differential equations and solving the algebraic equations at discretization point $i+1$. We use the implicite trapezoidal rule since it is not L-stable. Therefore, one can observe oscillations because of the higher index of certain forms of the equations of motion.

To obtain the numerical solutions $\bar{\mathbf{p}}$, $\bar{\mathbf{v}}$, and $\bar{\boldsymbol{\lambda}}$, the considered time interval $t \in [0, 5]$ is discretized in 1000 equidistant steps of length $h = 5/1000$. To compute the next iterates $\bar{\mathbf{p}}_{i+1}$, $\bar{\mathbf{v}}_{i+1}$, and $\bar{\boldsymbol{\lambda}}_{i+1}$, the discretized forms of the equations of motion are solved using the Newton-Method with a tolerance 10^{-8} with respect to the residual of the discretized equations of motion. We start the integrations with nonconsistent initial values

$$\bar{\mathbf{p}}_0 = \begin{bmatrix} 0.0001 \\ 1.0001 \end{bmatrix}, \ \bar{\mathbf{v}}_0 = \begin{bmatrix} 0.999 \\ 0.001 \end{bmatrix}, \ \bar{\boldsymbol{\lambda}}_0 = [0.0001]. \tag{37}$$

which yields initial residuals

$$\begin{array}{rclcl} \gamma(t_0) & = & \mathbf{g}(\bar{\mathbf{p}}_0, t_0) & = & 0.00020002, \\ \dot{\gamma}(t_0) & = & \mathbf{g}^{I}(\bar{\mathbf{p}}_0, \bar{\mathbf{v}}_0, t_0) & = & 0.0022, \\ \ddot{\gamma}(t_0) & = & \mathbf{g}^{II}(\bar{\mathbf{p}}_0, \bar{\mathbf{v}}_0, \bar{\boldsymbol{\lambda}}_0, t_0) & = & -0.0013956. \end{array}$$

In the following figures, the numerical results using the nonconsistent initial values (37) are compared to the analytical solution (36) using the consistent initial values (35).

Underlying ordinary differential equation:

First we consider the uODE (d-index 0) of the form (19). The discretization using the implicit trapezoidal rule yields

$$\begin{array}{rcl} 0 & = & \bar{\mathbf{p}}_{i+1} - \bar{\mathbf{p}}_i - \dfrac{h}{2}(\bar{\mathbf{v}}_{i+1} + \bar{\mathbf{v}}_i), \\ 0 & = & \mathbf{M}(\bar{\mathbf{v}}_{i+1} - \bar{\mathbf{v}}_i) - \dfrac{h}{2}\big((\mathbf{f}_{i+1} - \mathbf{G}_{i+1}^T \bar{\boldsymbol{\lambda}}_{i+1}) + (\mathbf{f}_i - \mathbf{G}_i^T \bar{\boldsymbol{\lambda}}_i)\big), \\ 0 & = & \bar{\boldsymbol{\lambda}}_{i+1} - \bar{\boldsymbol{\lambda}}_i - \dfrac{h}{2}\big((\mathbf{G}_{i+1}\mathbf{M}^{-1}\mathbf{G}_{i+1}^T)^{-1}\tilde{\mathbf{g}}_{i+1}^{III} + (\mathbf{G}_i\mathbf{M}^{-1}\mathbf{G}_i^T)^{-1}\tilde{\mathbf{g}}_i^{III}\big) \end{array}$$

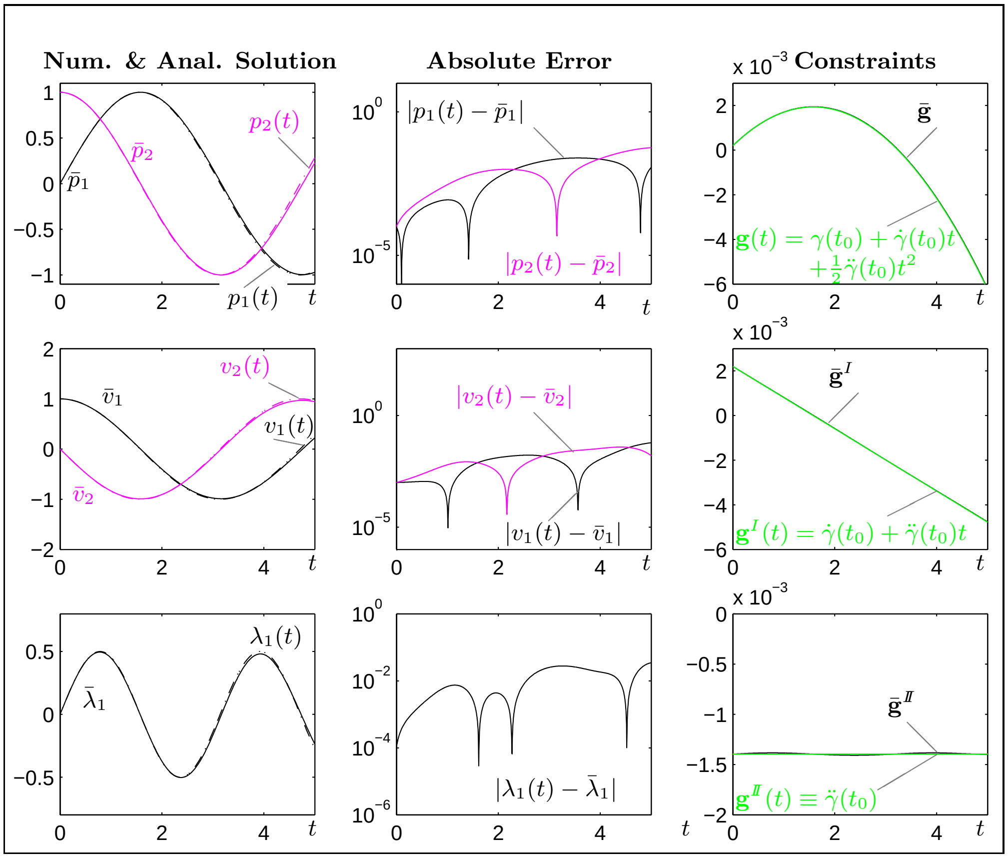

Figure 1: Solution behavior using uODE (d-index 0)

where

$$\mathbf{f}_j = \mathbf{f}(\bar{\mathbf{p}}_j, \bar{\mathbf{v}}_j, \bar{\boldsymbol{\lambda}}_j, t_j), \qquad \mathbf{G}_j = \mathbf{G}(\bar{\mathbf{p}}_j, t_j), \qquad \tilde{\mathbf{g}}_j^{III} = \tilde{\mathbf{g}}^{III}(\bar{\mathbf{p}}_j, \bar{\mathbf{v}}_j, \bar{\boldsymbol{\lambda}}_j, t_j).$$

Since the uODE is an ODE, no numerical problems are expected but deviation. In fact, in the last column of Figure 1, the computed (dark line) residual $\bar{\mathbf{g}}$ of the constraints on position level reflects the predicted (bright line) deviation $\mathbf{g}(t)$ from the position manifold with a quadratic behavior. Furthermore, also the predicted linear deviation of the residual $\mathbf{g}^{I}(t)$ is seen in the numerical solution $\bar{\mathbf{g}}^{I}$. Finally the residual of the constraint on acceleration $\bar{\mathbf{g}}^{II}$ level is (almost) constant as predicted. As anticipated, one cannot detect any oscillating behavior.

d-index 1 equations of motion using acceleration constraints:

The second considered form is EoM_1 (20). Here the constraints on acceleration level (14) are used instead of the constraints on position level (1c). Discretization of (20) using the

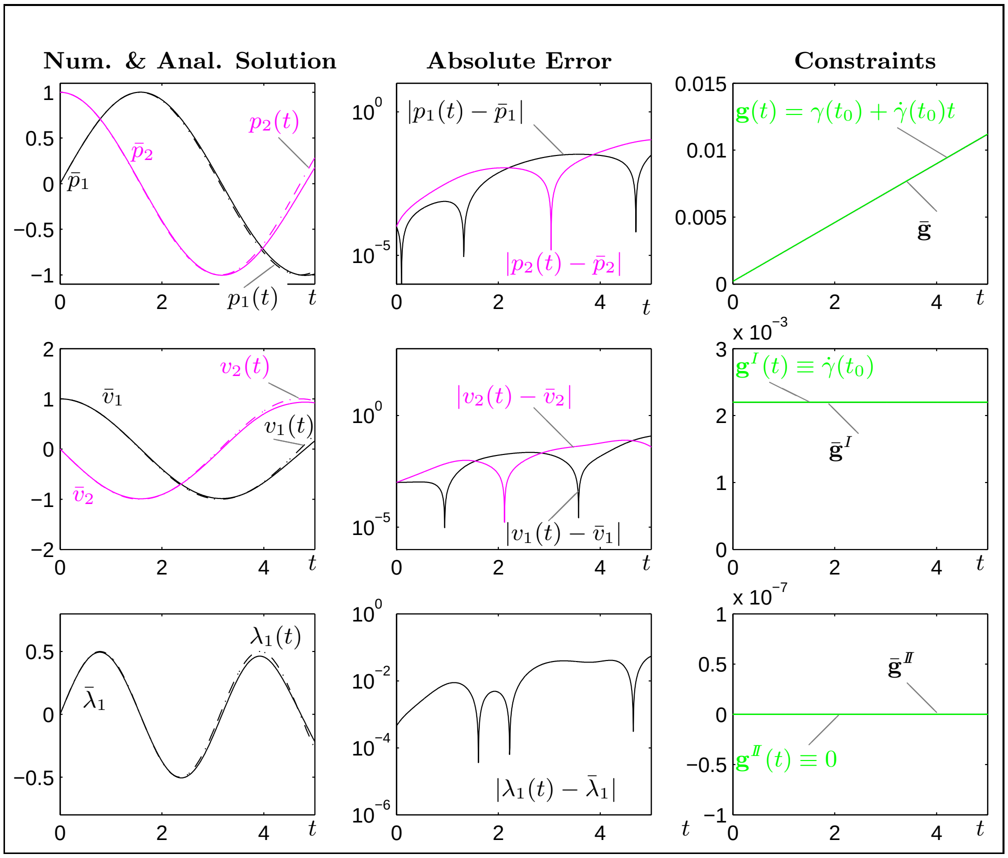

Figure 2: Solution behavior using EoM_1 (d-index 1)

implicit trapezoidal rule yields

$$
\begin{aligned}
0 &= \bar{\mathbf{p}}_{i+1} - \bar{\mathbf{p}}_i - \frac{h}{2}(\bar{\mathbf{v}}_{i+1} + \bar{\mathbf{v}}_i), \\
0 &= \mathbf{M}(\bar{\mathbf{v}}_{i+1} - \bar{\mathbf{v}}_i) - \frac{h}{2}\big((\mathbf{f}_{i+1} - \mathbf{G}_{i+1}^T\bar{\boldsymbol{\lambda}}_{i+1}) + (\mathbf{f}_i - \mathbf{G}_i^T\bar{\boldsymbol{\lambda}}_i)\big), \\
0 &= \mathbf{g}^{II}(\bar{\mathbf{p}}_{i+1}, \bar{\mathbf{v}}_{i+1}, \bar{\boldsymbol{\lambda}}_{i+1}, t_{i+1}),
\end{aligned}
\tag{38}
$$

where

$$
\mathbf{f}_j = \mathbf{f}(\bar{\mathbf{p}}_j, \bar{\mathbf{v}}_j, \bar{\boldsymbol{\lambda}}_j, t_j), \qquad \mathbf{G}_j = \mathbf{G}(\bar{\mathbf{p}}_j, t_j).
$$

Here again, we start the numerical integration by using the nonconsistent initial values (37). The results are depicted in Figure 2. The considerations in Section 3 show that we cannot expect a solution, since the consistency of the Lagrange-multiplier is necessary. But with respect to consistency the Lagrange-multiplier $\boldsymbol{\lambda}_i$ plays no role in the i-th integration step (38). Therefore, the approximated Lagrange-multiplier $\boldsymbol{\lambda}_{i+1}$ is consistent, independent of the consistency of the previous Lagrange-multiplier $\boldsymbol{\lambda}_i$. This fact results in a jump from $\mathbf{g}^{II}(\bar{\mathbf{p}}_0, \bar{\mathbf{v}}_0, \bar{\boldsymbol{\lambda}}_0, t_0) = -0.0013956$ to $\mathbf{g}^{II}(\bar{\mathbf{p}}_1, \bar{\mathbf{v}}_1, \bar{\boldsymbol{\lambda}}_1, t_1) = 0$ which is not visible in Figure 2. Since all iterates of the Lagrange-multiplier are consistent, in the following steps no oscillations occurs.

But again, we can recognize the predicted deviation $\mathbf{g}(t)$ from the position manifold $\mathcal{M}_{\mathbf{p}}$ with a linear behavior of the residual of the constraint on position level reflected by $\bar{\mathbf{g}}$ and also the constant residual $\bar{\mathbf{g}}^I$ with respect to the velocity manifold. The constraints on acceleration level are exactly satisfied with respect to the precision used by the Newton-Method.

d-index 2 equations of motion using velocity constraints:

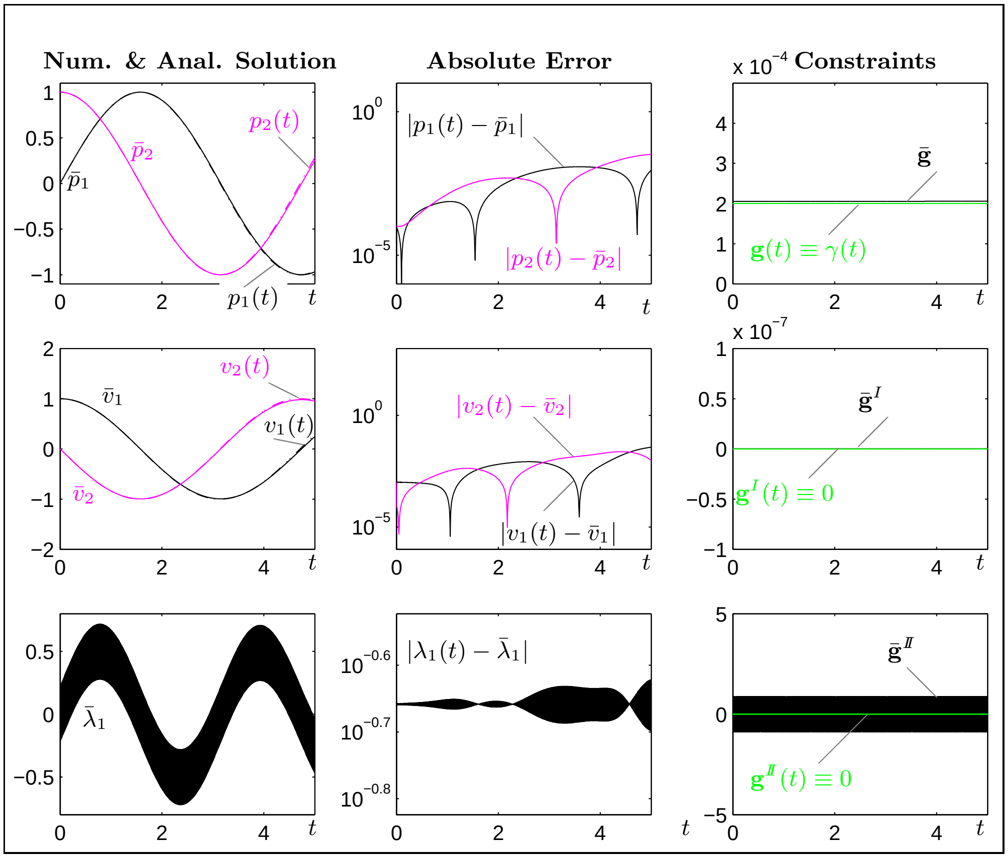

Figure 3: Solution behavior using EoM$_2$ (d-index 2)

The next considered form is obtained from (1) by replacing the constraints on position level (1c) by the constraints on velocity level (10). The corresponding discretized form using the implicit trapezoidal rule is

$$
\begin{aligned}
0 &= \bar{\mathbf{p}}_{i+1} - \bar{\mathbf{p}}_i - \frac{h}{2}(\bar{\mathbf{v}}_{i+1} + \bar{\mathbf{v}}_i), \\
0 &= \mathbf{M}(\bar{\mathbf{v}}_{i+1} - \bar{\mathbf{v}}_i) - \frac{h}{2}\big((\mathbf{f}_{i+1} - \mathbf{G}_{i+1}^T \bar{\boldsymbol{\lambda}}_{i+1}) + (\mathbf{f}_i - \mathbf{G}_i^T \bar{\boldsymbol{\lambda}}_i)\big), \\
0 &= \mathbf{g}^I(\bar{\mathbf{p}}_{i+1}, \bar{\mathbf{v}}_{i+1}, t_{i+1}),
\end{aligned}
$$

where

$$
\mathbf{f}_j = \mathbf{f}(\bar{\mathbf{p}}_j, \bar{\mathbf{v}}_j, \bar{\boldsymbol{\lambda}}_j, t_j), \qquad \mathbf{G}_j = \mathbf{G}(\bar{\mathbf{p}}_j, t_j).
$$

Again, we use the nonconsistent initial values (37) for the numerical integration. Since the constraints on velocity level appear explicitly, the approximated solutions for the velocities $\bar{\mathbf{v}}_1$ and all following $\bar{\mathbf{v}}_i$, $i \geq 1$, are consistent after the first step. But the constraints on acceleration level are only contained as hidden constraints. Therefore, they cannot be satisfied after one integration step. We have to expect numerical problems with respect to the Lagrange-multiplier as can be observed in Figure 3.
On the other hand, we can observe the deviation behavior of the solution. The constraints on velocity level are enforced because they appear explicitly in the current form of the equations of motion. The constraints on acceleration level are not satisfied, because of the oscillating behavior of the Lagrange-multiplier. The predicted behavior of the residual of the constraints on position level $\mathbf{g}(t)$ is reflected by the numerical solution $\bar{\mathbf{g}}$.

d-index 3 original equations of motion using position constraints:

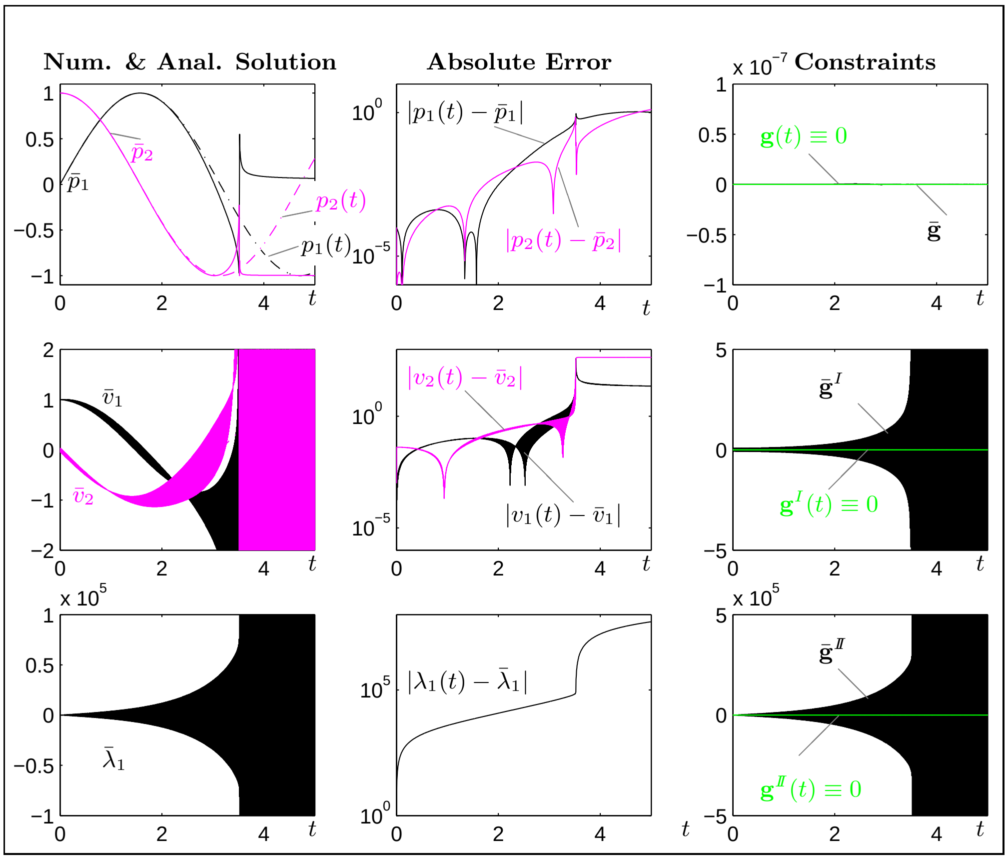

Figure 4: Solution behavior using the original EoM (d-index 3)

The original equations of motion (1) discretized via implicit trapezoidal rule yield

$$\begin{aligned}
0 &= \bar{\mathbf{p}}_{i+1} - \bar{\mathbf{p}}_i - \frac{h}{2}(\bar{\mathbf{v}}_{i+1} + \bar{\mathbf{v}}_i), \\
0 &= \mathbf{M}(\bar{\mathbf{v}}_{i+1} - \bar{\mathbf{v}}_i) - \frac{h}{2}\big((\mathbf{f}_{i+1} - \mathbf{G}_{i+1}^T\bar{\boldsymbol{\lambda}}_{i+1}) + (\mathbf{f}_i - \mathbf{G}_i^T\bar{\boldsymbol{\lambda}}_i)\big), \\
0 &= \mathbf{g}(\bar{\mathbf{p}}_{i+1}, t_{i+1}),
\end{aligned}$$

where

$$\mathbf{f}_j = \mathbf{f}(\bar{\mathbf{p}}_j, \bar{\mathbf{v}}_j, \bar{\boldsymbol{\lambda}}_j, t_j), \qquad \mathbf{G}_j = \mathbf{G}(\bar{\mathbf{p}}_j, t_j).$$

The use of the nonconsistent initial values (37) leads to an extremely oscillating behavior which appears mainly in the Lagrange-multiplier and which is (also) not negligible in the numerical solution of the velocities as is visible in Figure 4.
Because of the oscillating behavior of the numerical solution of the velocities and the Lagrange-multipliers, the satisfaction of the constraints on velocity level and on acceleration level is violated. On the other hand the constraints on position level are satisfied because of their explicit appearance in the equations of motion. Aside from the numerical oscillations, no deviation can be observed.

The considered forms of the equations of motion up to now do not show an acceptable behavior. We get more or less pronounced deviation up to quadratic behavior and we get more or less extreme oscillations. Sections 4 and 5 offered a possibility to transform the EoM into an equivalent strangeness-free form of the equations of motion as demonstrated in the following.

Strangeness-free form of equations of motion (d-index 1): The procedure to transform the EoM of d-index 3 into the equivalent strangeness-free form is given in Algorithm 27 for nonlinear systems and in Algorithm 13 for the linear case. Both are in principle the same. Following Algorithm 27 we first have to determine the matrix $\mathbf{G}^{-}(\mathbf{p})$ such that its columns span the kernel of the constraint matrix

$$\mathbf{G}(\mathbf{p}) = \begin{bmatrix} 2p_1 & 2p_2 \end{bmatrix}.$$

E.g., we get

$$\mathbf{G}^{-}(\mathbf{p}) = \begin{bmatrix} -p_2 \\ p_1 \end{bmatrix}.$$

Afterwards, the kinematic selector $\mathbf{S}_{\mathbf{p}}(\mathbf{p})$ has to be determined such that $\mathbf{S}_{\mathbf{p}}(\mathbf{p})\mathbf{G}^{-}(\mathbf{p})$ is nonsingular. We can choose the kinematic selector in the following way.

$$\mathbf{S}_{\mathbf{p}}(\mathbf{p}) = \begin{cases} \begin{bmatrix} 1 & 0 \end{bmatrix}, & \text{if } p_2 \neq 0 \\ \\ \begin{bmatrix} 0 & 1 \end{bmatrix}, & \text{if } p_1 \neq 0 \end{cases} \tag{39}$$

In general, one can choose $\mathbf{S}_{\mathbf{p}}(\mathbf{p})$ to be

$$\mathbf{S}_{\mathbf{p}}(\mathbf{p}) = \begin{bmatrix} \alpha & \beta \end{bmatrix}, \quad \text{if } (\beta p_1 - \alpha p_2) \neq 0. \tag{40}$$

Since for all α and β there exist p_1 and p_2 such that p_1 and p_2 satisfy the constraint $p_1^2+p_2^2 = L^2$ (34c) but violate the condition $(\beta p_1 - \alpha p_2) \neq 0$ in (40), the choice of the kinematic selector $\mathbf{S}_{\mathbf{p}}(\mathbf{p})$ is not independent of the state $\mathbf{p}$.

Since in our example the transformation matrix $\mathbf{Z}(\mathbf{p})$ is the identity and the mass matrix is a nonsingular diagonal matrix, the kinematic selector $\mathbf{S}_{\mathbf{v}}(\mathbf{p})$ can be chosen in the same way as the dynamic $\mathbf{S_p}(\mathbf{p})$.

$$\mathbf{S_v}(\mathbf{p}) = \begin{cases} \begin{bmatrix} 1 & 0 \end{bmatrix}, & \text{if } p_2 \neq 0 \\ \\ \begin{bmatrix} 0 & 1 \end{bmatrix}, & \text{if } p_1 \neq 0 \end{cases} \tag{41}$$

The strangeness-free form of the equations of motion is given by (29). In particular, by choosing the strategy typified by (39) and (41) we get

$$\begin{aligned}
\underbrace{\begin{bmatrix} \dot{p}_1 \end{bmatrix}}_{[1\ 0]\dot{\mathbf{p}}} &= \underbrace{\begin{bmatrix} v_1 \end{bmatrix}}_{[1\ 0]\mathbf{v}} \\
\underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{[1\ 0]\mathbf{M}} \underbrace{\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix}}_{\dot{\mathbf{v}}} &= \underbrace{\begin{bmatrix} -p_1 - 2p_1v_1v_2 \end{bmatrix}}_{[1\ 0]\mathbf{f}(\mathbf{p},\mathbf{v})} - \underbrace{\begin{bmatrix} 2p_1 \end{bmatrix}}_{[1\ 0]\mathbf{G}^T(\mathbf{p})} \underbrace{\begin{bmatrix} \lambda_1 \end{bmatrix}}_{\boldsymbol{\lambda}} \\
0 &= \underbrace{\begin{bmatrix} p_1^2 + p_2^2 - 1 \end{bmatrix}}_{\mathbf{g}(\mathbf{p})} \\
0 &= \underbrace{\begin{bmatrix} 2p_1v_1 + 2p_2v_2 \end{bmatrix}}_{\mathbf{g}^{I}(\mathbf{p},\mathbf{v})} \\
0 &= \underbrace{\begin{bmatrix} 2v_1^2 + 2p_1(-p_1 - 2p_1v_1v_2 - 2p_1\lambda_1) + 2v_2^2 + 2p_2(-v_1 + 2p_1p_2^2 - 2p_2\lambda_1) \end{bmatrix}}_{\mathbf{g}^{II}(\mathbf{p},\mathbf{v},\boldsymbol{\lambda})}
\end{aligned} \tag{42}$$

if $p_2 \neq 0$, or

$$\begin{aligned}
\underbrace{\begin{bmatrix} \dot{p}_2 \end{bmatrix}}_{[0\ 1]\dot{\mathbf{p}}} &= \underbrace{\begin{bmatrix} v_2 \end{bmatrix}}_{[0\ 1]\mathbf{v}} \\
\underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{[0\ 1]\mathbf{M}} \underbrace{\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix}}_{\dot{\mathbf{v}}} &= \underbrace{\begin{bmatrix} -v_1 + 2p_1p_2^2 \end{bmatrix}}_{[0\ 1]\mathbf{f}(\mathbf{p},\mathbf{v})} - \underbrace{\begin{bmatrix} 2p_2 \end{bmatrix}}_{[0\ 1]\mathbf{G}^T(\mathbf{p})} \underbrace{\begin{bmatrix} \lambda_1 \end{bmatrix}}_{\boldsymbol{\lambda}} \\
0 &= \underbrace{\begin{bmatrix} p_1^2 + p_2^2 - 1 \end{bmatrix}}_{\mathbf{g}(\mathbf{p})} \\
0 &= \underbrace{\begin{bmatrix} 2p_1v_1 + 2p_2v_2 \end{bmatrix}}_{\mathbf{g}^{I}(\mathbf{p},\mathbf{v})} \\
0 &= \underbrace{\begin{bmatrix} 2v_1^2 + 2p_1(-p_1 - 2p_1v_1v_2 - 2p_1\lambda_1) + 2v_2^2 + 2p_2(-v_1 + 2p_1p_2^2 - 2p_2\lambda_1) \end{bmatrix}}_{\mathbf{g}^{II}(\mathbf{p},\mathbf{v},\boldsymbol{\lambda})}
\end{aligned} \tag{43}$$

if $p_1 \neq 0$.

The discretization of the strangeness-free form of the equations of motion (29) using the

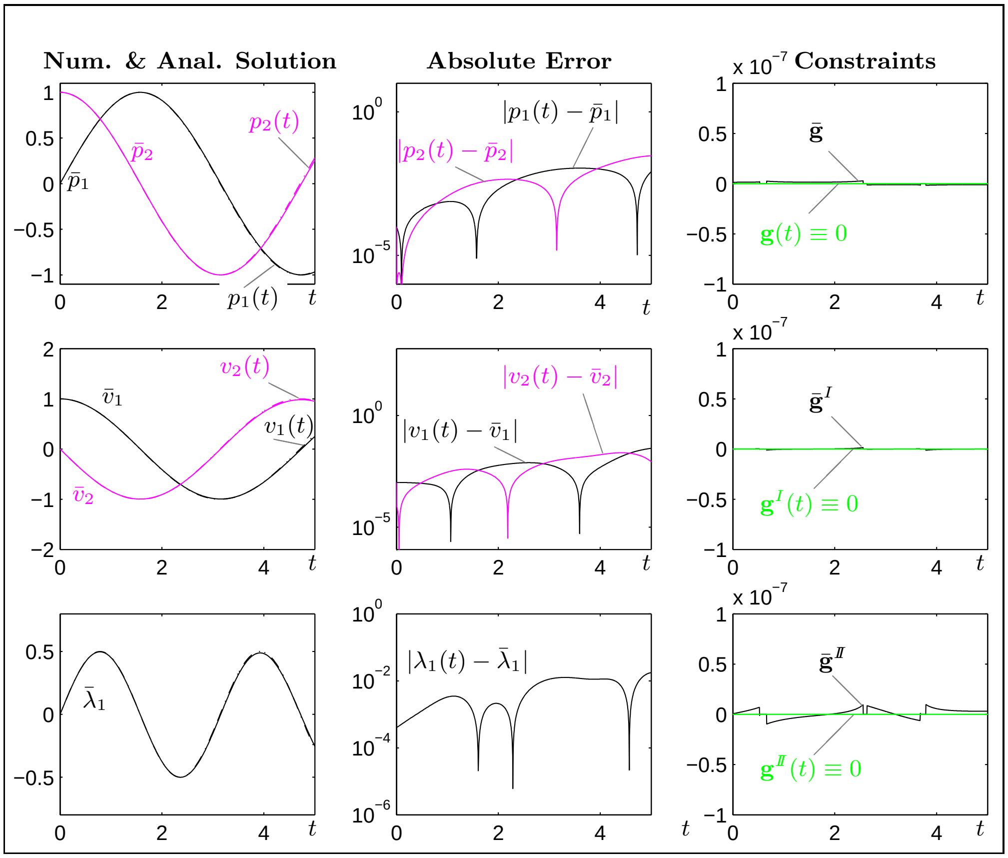

Figure 5: Solution behavior using the strangeness-free form (d-index 0)

implicit trapezoidal rule yields

$$
\begin{aligned}
0 &= \frac{1}{2}(\mathbf{S}_\mathbf{p}^{i+1} + \mathbf{S}_\mathbf{p}^{i})(\bar{\mathbf{p}}_{i+1} - \bar{\mathbf{p}}_i) - \frac{h}{2}(\mathbf{S}_\mathbf{p}^{i+1}\bar{\mathbf{v}}_{i+1} + \mathbf{S}_\mathbf{p}^{i}\bar{\mathbf{v}}_i),\\
0 &= \frac{1}{2}(\mathbf{S}_\mathbf{v}^{i+1} + \mathbf{S}_\mathbf{v}^{i})\mathbf{M}(\bar{\mathbf{v}}_{i+1} - \bar{\mathbf{v}}_i) - \frac{h}{2}\big((\mathbf{S}_\mathbf{v}^{i+1}\mathbf{f}_{i+1} - \mathbf{S}_\mathbf{v}^{i+1}\mathbf{G}_{i+1}^T\bar{\boldsymbol{\lambda}}_{i+1}) + (\mathbf{S}_\mathbf{v}^{i}\mathbf{f}_i - \mathbf{S}_\mathbf{v}^{i}\mathbf{G}_i^T\bar{\boldsymbol{\lambda}}_i)\big),\\
0 &= \mathbf{g}(\bar{\mathbf{p}}_{i+1}, t_{i+1}), \qquad (44)\\
0 &= \mathbf{g}^I(\bar{\mathbf{p}}_{i+1}, \bar{\mathbf{v}}_{i+1}, t_{i+1}),\\
0 &= \mathbf{g}^{II}(\bar{\mathbf{p}}_{i+1}, \bar{\mathbf{v}}_{i+1}, \bar{\boldsymbol{\lambda}}_{i+1}, t_{i+1})
\end{aligned}
$$

where

$$\mathbf{f}_j = \mathbf{f}(\bar{\mathbf{p}}_j, \bar{\mathbf{v}}_j, \bar{\boldsymbol{\lambda}}_j, t_j),\ \mathbf{G}_j = \mathbf{G}(\bar{\mathbf{p}}_j, t_j), \quad \mathbf{S}_\mathbf{p}^j = \mathbf{S}_\mathbf{p}(\mathbf{p}_j),\ \mathbf{S}_\mathbf{v}^j = \mathbf{S}_\mathbf{v}(\mathbf{p}_j).$$

Using the selectors

$$
\begin{aligned}
\mathbf{S}_\mathbf{p}(\mathbf{p}) = \mathbf{S}_\mathbf{v}(\mathbf{p}) = [1\ 0] \ \text{ if } p_1 \le 1/2 \ \ \text{(corresponding to (42))},\\
\mathbf{S}_\mathbf{p}(\mathbf{p}) = \mathbf{S}_\mathbf{v}(\mathbf{p}) = [0\ 1] \ \text{ if } p_1 > 1/2 \ \ \text{(corresponding to (43))},
\end{aligned}
$$

we get the numerical results depicted in Figure 5.

Since all information about the whole solution manifold is contained explicitly in the strangeness-free form of the equations of motion, the formulation does not contain any hidden constraints and the solution has no deviation from the solution manifold. Because of this, the nonconsistent initial values are corrected to consistency and we do not get any numerical problems. Figure 5 exactly shows this behavior - no deviation and no oscillations.

7 Summary

In this paper we have studied the solution behavior of different forms of the equations of motion by using nonconsistent initial values. We have illustrated the arising deviation as well as the arising numerical instabilities depending on the choice of the form of the equations of motion. Alltogether, the properties of the different formulations of EoM cover the range from quadratic deviation in time and no possible oscillations by using the underlying ODE with d-index 0 to no deviation and extreme oscillations by using the originally equations of motion with d-index 3.

Furthermore, a possibility to avoid these disadvantages by transforming the equations of motion into an equivalent strangeness-free form is presented. We have constructed an algorithm to determine an equivalent strangeness-free form of the equations of motion which is suitable for numerical integration using stiff ODE solvers like implicit Runge-Kutta-Methods or BDF-Methods.

The index reduction algorithm is based on the determination of two selectors by mainly analyzing the constraint matrix and also the transformation matrix and the mass matrix. The strangeness-free form created in this way is analytically equivalent to the original equations of motion in the sense that both have the same solution set.

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