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Perron Frobenius Theorems for the Numerical Range of Semi-Monic Matrix Polynomials

K.-H. Förster and P. Kallus

Abstract

We present an extension of the Perron-Frobenius theory to the numerical ranges of semi-monic Perron-Frobenius polynomials, namely matrix polynomials of the form

$$Q(\lambda) = \lambda^m - (\lambda^l A_l + \cdots + A_0) = \lambda^m - A(\lambda),$$

where the coefficients are entrywise nonnegative matrices. Our approach relies on the function $\beta \mapsto$ numerical radius $A(\beta)$ and the infinite graph $G_m(A_0, \ldots, A_l)$. Our main result describes the cyclic distribution of the elements of the numerical range of $Q(\cdot)$ on the circles with radius β satisfying $\beta^m =$ numerical radius $A(\beta)$.

MSC. 15B48; 15A60; 15A22
Keywords. Perron-Frobenius theory; numerical range; matrix polynomials

1 Introduction

We consider matrix polynomials of the form

$$\lambda^m - (\lambda^l A_l + \cdots + A_0) \tag{1.1}$$

where $l \in \mathbb{N}_0$ and $m \in \mathbb{N}$. We do not assume that these polynomials are monic, i.e. $m > l$. Matrix polynomials of this type have appeared in different situations (see [2], [3], [5], [7], [14], [15]).

In [5], [6], [7] and [8] the authors considered spectral properties of matrix polynomials of this type, where the coefficients are entrywise nonnegative. Here we consider for such matrix polynomials properties of their numerical range.

For a single matrix A_0 with nonnegative entries the Perron-Frobenius theory is applied to the spectrum (= set of eigenvalues; see [10], [17]) and the numerical range (see [12], [16], [18]). One of the main results of this theory says that the peripheral part of the spectrum and the numerical range of an irreducible entrywise nonnegative matrix have the same cyclic properties, the proof thereof relying on Wielandt's Lemma.

Properties of the numerical range of monic polynomials with entrywise nonnegtive coefficients were considered in [18]. The proofs are based on the linearisation by the companion matrix C_Q and by applying the Perron-Frobenius theory to C_Q. Linearisation does not help in the semi-monic case, instead our analysis is based on the function $\mathbb{R}_+ \to \mathbb{R}_+ : \beta \mapsto \mathrm{nur}(A(\beta)) =$ numerical radius of $A(\beta)$ and an infinite graph $G_m(A_0, \ldots, A_l)$.

The presentation of the paper is arranged as follows. In Section 2 we recall notations, definitions and include some known results for the numerical range of (entrywise nonnegative) matrices. In Section 3 we consider the function

$$\mathrm{nur}_A : \mathbb{R}_+ \to \mathbb{R}_+ : \beta \mapsto \sup_{|\lambda|=\beta} \mathrm{nur}(A(\lambda)) \tag{1.2}$$

and show its relevance for the numerical range of general matrix functions analytical in an annulus; at the ned we show that $\text{nur}_A(\beta) = \text{nur}(A(\beta))$ when the coefficients are entrywise nonnegative matrices. In Section 4 we define the graph $G_m(A_0, \ldots, A_l)$ and its index of phase imprimitivity, where the sum $A_0 + \cdots + A_l$ is an irreducible matrix. In the semi-monic setting this index can be seen as the analogue to the index of imprimitivity of the graph associated with an irreducible matrix in the Perron-Frobenius theory. Finally in Section 5 we proof the main result on the number and cyclic distribution of values in the numerical range of $Q(\cdot)$ on circles with radius β such that $\beta^m = \text{nur}\, A(\beta)$, where the coefficients A_j of $Q(\cdot)$ are entrywise nonnegative matrices and their sum is an irreducible matrix. As a related result we also prove the rotation invariance of the numerical range of $Q(\cdot)$ under certain angles. We close the section with some illustrative examples.

2 Notations, Definitions and Preliminaries

We largely follow the notation used in [13].

- $\langle n \rangle$ the set of integers $1, \ldots, n$ and $\langle n \rangle_0 = \langle n \rangle \cup \{0\}$,
- $(\cdot, \cdot)$ denotes the standard inner product in $\mathbb{C}^n$,
- $\mathbb{T}_\rho = \{\lambda \in \mathbb{C} : |\lambda| = \rho\}$ the circle in the complex plane $\mathbb{C}$ centred at the origin with radius $\rho > 0$,
- $\mathbb{A}_{\rho_1,\rho_2} = \{\lambda \in \mathbb{C} : \rho_1 < |\lambda| < \rho_2\}$ the open annulus in the complex plane $\mathbb{C}$ centred at the origin with radii $0 \leq \rho_1 < \rho_2$,
- $\mathbb{S}^n = \{x \in \mathbb{C}^n : \|x\| = 1\}$,
- $\mathbb{C}^{n,n}$ the set of complex $n \times n$ matrices and $\mathbb{R}_+^{n,n}$ the set of entrywise nonnegative $n \times n$ matrices,
- $\Sigma(B)$ the spectrum (=set of eigenvalues) and $\text{spr}(B)$ the spectral radius of $B \in \mathbb{C}^{n,n}$,
- $\Theta(B)$ the numerical range (=field of values) and $\text{nur}(B)$ the numerical radius of $B \in \mathbb{C}^{n,n}$,
- for $x \in \mathbb{R}_+^n$ write $x \gg 0$ iff all entries of x are strictly positive.

Recall that for a matrix $B \in \mathbb{C}^{n,n}$ the numerical range is defined as

$$\Theta(B) := \{(Bx, x) : x \in \mathbb{S}^n\}$$

and the numerical radius of B is defined as

$$\text{nur}(B) := \sup\{|\lambda| : \lambda \in \Theta(B)\} = \sup_{x \in \mathbb{S}^n} |(Bx, x)|.$$

We briefly collect some well known properties of the numerical range of a matrix $B \in \mathbb{C}^{n,n}$:

- $\Theta(B)$ is compact and convex,
- if $B \in \mathbb{R}_+^{n,n}$ then $\operatorname{nur}(B) \in \Theta(B)$ (see e.g. [16, p.51]),
- for $B \in \mathbb{R}_+^{n,n}$ there holds

$$\operatorname{nur}(B) = \sup_{x \in \mathbb{S}^n} |(Bx, x)| = \sup_{x \in \mathbb{S}^n \cap \mathbb{R}_+^n} |(Bx, x)|.$$

- for irreducible $B \in \mathbb{R}_+^{n,n}$ and $x \in \mathbb{S}^n \cap \mathbb{R}_+^n$ such that $(Bx, x) = \operatorname{nur}(B)$, there holds $x \gg 0$ (see e.g. [16, Theorem 3.1.(i)]).

Analogously to the spectrum the numerical range is extended to a matrix function $Q(\cdot)$ via

$$\Theta(Q(\cdot)) := \{\lambda : 0 \in \Theta(Q(\lambda))\}.$$

3 The Numerical Range of Semi-monic Matrix Functions

One of our main tool in analysing the numerical range of semi-monic matrix functions

$$Q(\cdot) : \mathbb{A}_{\rho_1,\rho_2} \to \mathbb{C}^{n,n} : \lambda \mapsto \lambda^m - A(\lambda),$$

where $m \in \mathbb{N}$, will be the real function

$$\operatorname{nur}_A : (\rho_1, \rho_2) \to \mathbb{R}_+ : \rho \mapsto \sup_{|\lambda|=\rho} \operatorname{nur}(A(\lambda)),$$

where

$$A(\cdot) : \mathbb{A}_{\rho_1,\rho_2} \to \mathbb{C}^{n,n} : \lambda \mapsto \sum_{j \in \mathbb{Z}} \lambda^j A_j,$$

$A_j \in \mathbb{C}^{n,n}$, the domain being the open annulus $\mathbb{A}_{\rho_1,\rho_2}$. We state some properties of nur_A.

Proposition 3.1. *For an analytic matrix function $A(\cdot) : \mathbb{A}_{\rho_1,\rho_2} \to \mathbb{C}^{n,n}$ the function nur_A is geometrically convex, that is for all $\sigma_1, \sigma_2 \in (\rho_1, \rho_2)$ and $\theta \in [0, 1]$ the functional inequality*

$$\operatorname{nur}_A(\sigma_1^\theta \sigma_2^{1-\theta}) \leq (\operatorname{nur}_A(\sigma_1))^\theta (\operatorname{nur}_A(\sigma_2))^{1-\theta}$$

holds.

Proof. Note that for any $x \in \mathbb{S}^n$ by [1, Lemma 3.4.6] the function $\log |(A(\cdot)x, x)|$ is subharmonic. By [9, Theorem 2.13] it follows that

$$\beta \mapsto \log \sup_{|\lambda|=\beta} |(A(\lambda)x, x)|$$

is convex in $\log \beta$.
This implies that for $\sigma_1, \sigma_2 \in (\rho_1, \rho_2)$, $\theta \in [0,1]$

$$\begin{aligned}\sup_{|\lambda|=\sigma_1^\theta \sigma_2^{1-\theta}} |(A(\lambda)x,x)| &\leq \left(\sup_{|\lambda|=\sigma_1} |(A(\lambda)x,x)|\right)^\theta \left(\sup_{|\lambda|=\sigma_2} |(A(\lambda)x,x)|\right)^{1-\theta} \\ &\leq \operatorname{nur}_A(\sigma_1)^\theta \operatorname{nur}_A(\sigma_2)^{1-\theta}\end{aligned}$$

holds and by exchanging the suprema we obtain

$$\begin{aligned}\operatorname{nur}_A(\sigma_1^\theta s_2^{1-\theta}) &= \sup_{|\lambda|=\sigma_1^\theta \sigma_2^{1-\theta}} \sup_{x\in\mathbb{S}^n} |(A(\lambda)x,x)| \\ &= \sup_{x\in\mathbb{S}^n} \sup_{|\lambda|=\sigma_1^\theta \sigma_2^{1-\theta}} |(A(\lambda)x,x)| \\ &\leq (\operatorname{nur}_A(\sigma_1))^\theta (\operatorname{nur}_A(\sigma_2))^{1-\theta}.\end{aligned}$$

□

The following proposition characterizes the behaviour of nur_A on intervals.

Proposition 3.2. *Let $A(\cdot) : \mathbb{A}_{\rho_1,\rho_2} \to \mathbb{C}^{n,n}$ be an analytic matrix function and $\sigma_1, \sigma_2 \in (\rho_1, \rho_2)$ with $\sigma_1 \leq \sigma_2$ and $\operatorname{nur}_A(\sigma_j) = \sigma_j^m, j = 1, 2$ for some $m \in \mathbb{N}$. Then exactly one of the following assertions is true.*

(i) $\operatorname{nur}_A(\beta) = \beta^m$ *for all* $\beta \in (\sigma_1, \sigma_2)$.

(ii) $\operatorname{nur}_A(\beta) < \beta^m$ *for all* $\beta \in (\sigma_1, \sigma_2)$.

Proof. The proof follows from well known facts about convex functions since $\tau \mapsto \log \operatorname{nur}_A(e^\tau)$ is convex and $\tau \mapsto \log(e^\tau)^m$ is linear. □

The next two results provide conditions under which the numerical range of semi-monic matrix functions is disjoint to some annulus around the origin.

Proposition 3.3. *Let $A(\cdot) : \mathbb{A}_{\rho_1,\rho_2} \to \mathbb{C}^{n,n}$ be an analytic matrix function and $Q(\cdot)$ its corresponding semi-monic matrix function for a $m \in \mathbb{N}$. Let $\sigma_1, \sigma_2 \in (\rho_1, \rho_2)$, $\sigma_1 < \sigma_2$, such that $\operatorname{nur}_A(\beta) < \beta^m$ for all $\beta \in (\sigma_1, \sigma_2)$. Then*

$$\Theta(Q(\cdot)) \cap \mathbb{A}_{\sigma_1,\sigma_2} = \emptyset.$$

Proof. Assume there exists $\lambda_0 \in \Theta(Q(\cdot)) \cap \mathbb{A}_{\sigma_1,\sigma_2}$. This means that

$$\begin{aligned}& 0 \in \Theta(Q(\lambda_0)) \\ \Leftrightarrow\ & 0 \in \Theta(\lambda_0^m - A(\lambda_0)) \\ \Leftrightarrow\ & \lambda_0^m \in \Theta(A(\lambda_0)).\end{aligned}$$

Additionally we have $|\lambda_0| \in (\sigma_1, \sigma_2)$. It follows that

$$\begin{aligned}|\lambda_0|^m &\leq \operatorname{nur}(A(\lambda_0)) \\ &\leq \sup_{|\lambda|=|\lambda_0|} \operatorname{nur}(A(\lambda)) \\ &= \operatorname{nur}_A(|\lambda_0|) \\ &< |\lambda_0|^m,\end{aligned}$$

which is a contradiction. $\square$

Theorem 3.4. *Let $A(\cdot) : \mathbb{A}_{\rho_1,\rho_2} \to \mathbb{C}^{n,n}$ be an analytic matrix function and $Q(\cdot)$ its corresponding semi-monic matrix function for a $m \in \mathbb{N}$. Let $\sigma_1, \sigma_2 \in (\rho_1, \rho_2)$, $\sigma_1 < \sigma_2$, with $\operatorname{nur}_A(\sigma_j) = \sigma_j^m$, $j = 1, 2$. If there exists a $\beta \in (\sigma_1, \sigma_2)$ such that $\operatorname{nur}_A(\beta) < \beta^m$ then*

$$\Theta(Q(\cdot)) \cap \mathbb{A}_{\sigma_1,\sigma_2} = \emptyset.$$

Proof. Proposition 3.2 implies that $\operatorname{nur}_A(\beta) < \beta^m$ for all $\beta \in (\sigma_1, \sigma_2)$. The assertion then follows from Proposition 3.3. $\square$

Up until now we did not pose any requirements on the coefficients of $A(\cdot)$. In the sequel we will require the A_j to be entrywise nonnegative, i.e. $A_j \in \mathbb{R}_+^{n,n}$. In this case we can give a simpler representation of the function nur_A.

Proposition 3.5. *Let $A(\cdot) : \mathbb{A}_{\rho_1,\rho_2} \to \mathbb{C}^{n,n}$ be an analytic matrix function with coefficients $A_j \in \mathbb{R}_+^{n,n}$. Then*

$$\operatorname{nur}_A(\beta) = \operatorname{nur}(A(\beta)), \qquad \beta \in (\rho_1, \rho_2).$$

Proof. Since the A_j are entrywise nonnegative we have for $\lambda \in \mathbb{A}_{\rho_1,\rho_2}$ and $x \in \mathbb{S}^n$

$$|(A(\lambda)x, x)| \leq (A(|\lambda|)|x|, |x|)$$

noting that $|x| \in \mathbb{S}^n$ whenever $x \in \mathbb{S}^n$. It follows for $\beta \in (\rho_1, \rho_2)$

$$\begin{aligned}
\operatorname{nur}_A(\beta) &= \sup_{|\lambda|=\beta} \sup_{x\in\mathbb{S}^n} |(A(\lambda)x, x)| \\
&\leq \sup_{|\lambda|=\beta} \sup_{x\in\mathbb{S}^n} (A(|\lambda|)|x|, |x|) \\
&= \sup_{x\in\mathbb{S}^n} (A(\beta)|x|, |x|) \\
&\leq \sup_{x\in\mathbb{S}^n} |(A(\beta)x, x)| \\
&= \operatorname{nur}(A(\beta)) \leq \operatorname{nur}_A(\beta).
\end{aligned}$$ $\square$

In [8] the author considered the function $\operatorname{spr}_A : \mathbb{A}_{\rho_1,\rho_2} \to \mathbb{R}_+ : \rho \mapsto \sup_{|\lambda|=\rho} \operatorname{spr}(A(\lambda))$, where $\operatorname{spr}(B)$ denotes the spectral radius of a $B \in \mathbb{C}$, and proved corresponding results for the spectrum of $Q(\cdot)$. Additionally different classes of coefficients were treated, e.g. where the A_j are positive semidefinite. It is easy to see that Proposition 3.5 holds when the coefficients are positive semidefinite.

4 The Infinite Graph $G_m(A_0, \ldots, A_l)$

Our second tool is the infinite graph $G_m(A_0, \ldots, A_l)$ for matrices $A_0, \ldots, A_l \in \mathbb{R}_+^{n,n}$ and $m \in \mathbb{N}$; we define the set of vertices of $G_m(A_0, \ldots, A_l)$ by

$$V = \{(r, p) : r \in \langle n \rangle, p \in \mathbb{Z}\}$$

and the set of its edges

$$E = \{[(r,p),(s,q)] : A_{m-p+q}(r,s) > 0\},$$

where $A_{m+p-q}(r,s)$ denotes the entry with coordinates (r,s) in the matrix A_{m+p-q}. For $(r,p) \in V$ we call r the *phase* and p the *level* of the vertex. A sequence of edges

$$[(r_0,p_0),(r_1,p_1)],[(r_1,p_1),(r_2,p_2)],\ldots,[(r_{w-1},p_{w-1}),(r_w,p_w)]$$

is called a *path* of length w connecting (r_0,p_0) with (r_w,p_w) and we might also write

$$(r_0,p_0) \to (r_1,p_1) \to \cdots \to (r_w,p_w).$$

For the above path the number $p_w - p_0$ is called its *level displacement*. Furthermore we call a path $(r_0,p_0) \to \cdots \to (r_w,p_w)$ a *phase cycle* if $r_0 = r_w$. Then the *index of phase imprimitivity* of the graph $G_m(A_0,\ldots,A_l)$ is defined as the greatest common divisor (g.c.d.) of the level displacements of all of its phase cycles. In the case where every phase cycle has level displacement 0 (which can happen, see [6, Example 4.3]) the index of phase imprimitivity is defined as 0. Moreover the index of phase imprimitivity is defined to be nonnegative.

A concept similar to the infinite graph above was considered in [7, p.132]. Note that the graph associated with a single matrix $A_0 \in \mathbb{R}_+^{n,n}$ can be expressed by $G_1(A_0)$. Then the index of phase imprimitivity of $G_1(A_0)$ coincides with the usual index of imprimitivity of A_0 used in the Perron-Frobenius theory. Thus the graph $G_m(A_0,\ldots,A_l)$ can be seen as an extension of the usual graph associated with an entrywise nonnegative matrix.
In the unpublished doctoral thesis of N. Hartanto [8] the infinite graph was used to derive results for the spectrum of semi-monic Perron-Frobenius polynomials similar to our main result for the numerical range in the following section. We collect some properties of these infinite graphs which will be needed in the proof of our main result. The proofs are straight forward and can be found in [8, pp. 66].

Lemma 4.1. *For the graph $G_m(A_0,\ldots,A_l)$ we have that $[(r,p),(s,q)] \in E$ implies that $[(r,p+u),(s,q+u)] \in E$ for all $u \in \mathbb{Z}$.*
Moreover for any $u \in \mathbb{Z}$ the paths $(r_0,p_0) \to (r_1,p_1) \to \cdots \to (r_w,p_w)$ and $(r_0,p_0+u) \to (r_1,p_1+u) \to \cdots \to (r_w,p_w+u)$ have the same level displacement.

Lemma 4.2. *Let $A_0,\ldots,A_l \in \mathbb{R}_+^{n,n}$ and $m \in \mathbb{N}$. Then for $r,s \in \langle n\rangle$ the following assertions are equivalent:*

(i) There exists a path from r to s in the directed graph associated with the matrix $A_0 + \cdots + A_l \in \mathbb{R}_+^{n,n}$.

(ii) For all $p \in \mathbb{Z}$ there exists a $q \in \mathbb{Z}$ such that there is a path from (r,p) to (s,q) in $G_m(A_0,\ldots,A_l)$.

(iii) For all $q \in \mathbb{Z}$ there exists a $p \in \mathbb{Z}$ such that there is a path from (r,p) to (s,q) in $G_m(A_0,\ldots,A_l)$.

Lemma 4.3. *Let $A_0, \dots, A_l \in \mathbb{R}_+^{n,n}$ and $m \in \mathbb{N}$. Then the following assertions are equivalent:*

(i) $A_0 + \cdots + A_l \in \mathbb{R}_+^{n,n}$ *is irreducible.*

(ii) For all $r, s \in \langle n \rangle$ and $p \in \mathbb{Z}$ there exists a $q \in \mathbb{Z}$ such that there is a path from (r, p) to (s, q) in $G_m(A_0, \dots, A_l)$.

(iii) For all $r, s \in \langle n \rangle$ and $q \in \mathbb{Z}$ there exists a $p \in \mathbb{Z}$ such that there is a path from (r, p) to (s, q) in $G_m(A_0, \dots, A_l)$.

We will need one more lemma which is usually attributed to I. Schur. A proof can be found in [4, Lemma 3.4.2].

Lemma 4.4. *Let M be a nonempty set of integers which is closed under addition and let $d \in \mathbb{N}$ be the greatest common divisor of M. Then we have $kn \in M$ for all but finitely many $k \in \mathbb{N}$.*

5 Main Result

In the remainder we will restrict ourselves to matrix polynomials with entrywise non-negative coefficients, i.e. functions of the form

$$A(\cdot) : \mathbb{C} \to \mathbb{C}^{n,n} : \lambda \mapsto \sum_{j=0}^{l} \lambda^j A_j,$$

where $A_j \in \mathbb{R}_+^{n,n}$, $l \in \mathbb{N}$ and the corresponding semi-monic[1] polynomial

$$Q(\cdot) : \mathbb{C} \to \mathbb{C}^{n,n} : \lambda \mapsto \lambda^m - A(\lambda) = \lambda^m - \sum_{j=0}^{l} \lambda^j A_j$$

for some $m \in \mathbb{N}$. Note that $Q(\cdot)$ gives rise to an associated infinite graph $G_m(A_0, \dots, A_l)$. We also give one more definition that extends the concept of irreducibility from matrices to matrix polynomials.

Definition 5.1. Let $A(\cdot)$ be a matrix polynomial with entrywise nonnegtive coefficients. Then we say the polynomial $A(\cdot)$ is *irreducible* if the matrix $A(\beta) \in \mathbb{R}_+^{n,n}$ is irreducible for one (and then for all) $\beta > 0$.

Theorem 5.2. *Let $A(\lambda) = \sum_{j=0}^{l} \lambda^j A_j$ be an irreducible matrix polynomial with entrywise nonnegative coefficients and $Q(\lambda) = \lambda^m - A(\lambda)$ its corresponding semi-monic polynomial for some $m \in \mathbb{N}$. Let further d be the index of phase imprimitivity of the associated graph $G_m(A_0, \dots, A_l)$. Then for all $\beta > 0$ with*

$$\beta^m = \operatorname{nur}(A(\beta))$$

the following statements hold:

[1] Here 'semi-monic' refers to the fact that we do not require for m to be greater than l.

(i) If $d = 0$ then $\mathbb{T}_\beta \subseteq \Theta(Q(\cdot))$.

(ii) If $d \geq 1$ then $\Theta(Q(\cdot)) \cap \mathbb{T}_\beta = \{e^{i\frac{2\pi}{d}k} : k = 0, \dots, d-1\}$.

Proof. (ii) We first proof the second part of the theorem by showing both inclusions.
"$\subseteq$": Take an element of $\Theta(Q(\cdot)) \cap \mathbb{T}_\beta$, i.e. an element that can be written as $\beta\omega$ with $\omega \in \mathbb{T}_1$. Then there exists some $x \in \mathbb{S}^n$ such that

$$(\beta\omega)^m = (A(\beta\omega)x, x). \tag{5.1}$$

Now we define $y = |x|$, and show that

$$\beta^m = (A(\beta)y, y).$$

To see this write

$$\beta^m = |(\beta\omega)^m| = |(A(\beta\omega)x, x)| \leq (A(|\beta\omega|)|x|, |x|) = (A(\beta)y, y) \leq \operatorname{nur}(A(\beta)) = \beta^m$$

Since $A(\cdot)$ is irreducible it then also follows that $|x| = y \gg 0$ (see Section 2).
We proceed by breaking down the above equalities.

$$\begin{aligned}\beta^m &= (A(\beta)y, y) \\ &= \sum_{j=1}^{l} \beta^j (A_j y, y) \\ &= \sum_{j=1}^{l}\sum_{r=1}^{n}\sum_{s=1}^{n} \beta^j \bar{y}_r A_j(r, s) y_s\end{aligned}$$

and dividing by the left-hand side we arrive at

$$1 = \sum_{j=1}^{l}\sum_{r=1}^{n}\sum_{s=1}^{n} \beta^{j-m} \bar{y}_r A_j(r, s) y_s. \tag{5.2}$$

Here the terms of the sum on the right-hand side are all nonnegative. Doing the same for (5.1) we get

$$\begin{aligned}1 &= \sum_{j=1}^{l}\sum_{r=1}^{n}\sum_{s=1}^{n} (\beta\omega)^{j-m} \bar{x}_r A_j(r, s) x_s \\ &= \sum_{j=1}^{l}\sum_{r=1}^{n}\sum_{s=1}^{n} \left[\beta^{j-m} \bar{y}_r A_j(r, s) y_s\right] \left[\omega^{j-m} \frac{\bar{x}_r}{\bar{y}_r} \frac{x_s}{y_s}\right]. \end{aligned} \tag{5.3}$$

Moreover

$$\left|\omega^{j-m} \frac{\bar{x}_r}{\bar{y}_r} \frac{x_s}{y_s}\right| = \left|\omega^{j-m}\right| \left|\frac{\bar{x}_r}{\bar{y}_r}\right| \left|\frac{x_s}{y_s}\right| = 1. \tag{5.4}$$

Due to (5.2) we now see that (5.3) is a convex combination of numbers with absolute value 1 whose sum is equal to 1. This is only possible if for all j, r, s where $A_j(r, s) > 0$ we have that

$$1 = \omega^{j-m} \frac{\bar{x}_r}{\bar{y}_r} \frac{x_s}{y_s}$$

or equivalently

$$\omega^{j-m} \frac{x_s}{y_s} = \frac{\bar{y}_r}{\bar{x}_r}. \tag{5.5}$$

Now take any phase cycle $(r_0, p_0) \to \ldots \to (r_w, p_w)$ in $G_m(A_0, \ldots, A_l)$ and denote its level displacement by $\tilde{d}$. Then $r_0 = r_w$ and $A_{m-p_{h-1}+p_h}(r_{h-1}, r_h) > 0$ for $h \in \langle w \rangle$. Setting $j_h = m - p_{h-1} + p_h$ we can then write $\tilde{d} = \sum_{h=1}^{w} j_h - m$ and with (5.5) it follows that

$$\begin{aligned}
\omega^{\tilde{d}} &= \prod_{h=1}^{w} \omega^{j_h - m} \frac{x_{r_h}}{y_{r_h}} \prod_{h=1}^{w} \frac{y_{r_h}}{x_{r_h}} \\
&= \prod_{h=1}^{w} \frac{\bar{y}_{r_{h-1}}}{\bar{x}_{r_{h-1}}} \prod_{h=1}^{w} \frac{y_{r_h}}{x_h} \\
&= \prod_{h=1}^{w} \frac{\bar{y}_{r_{h-1}} y_{r_{h-1}}}{\bar{x}_{r_{h-1}} x_{r_{h-1}}} \\
&= \prod_{h=1}^{w} \frac{|y_{r_{h-1}}|^2}{|x_{r_{h-1}}|^2} = 1
\end{aligned}$$

where the third equality follows from $r_0 = r_w$. Thus $\omega^{\tilde{d}} = 1$. In order to see that then also $\omega^d = 1$ consider the set

$$M = \left\{ \begin{array}{c} \text{the set of all level displacements of phase cycles} \\ \text{in } G_m(A_0, \ldots, A_l) \text{ and their sums} \end{array} \right\}$$

which is closed unter addition. Obviously for an element $\hat{d} \in M$ there still holds $\omega^{\hat{d}} = 1$. Moreover the index of phase imprimitivity d is the greatest common divisor of M. Now by Lemma 4.4 there exists a $k \in \mathbb{N}$ such that kd and $(k+1)d$ are both in M, i.e. $\omega^{kd} = \omega^{(k+1)d} = 1$. Thus the increment d must also fulfil $\omega^d = 1$. It follows that

$$\beta\omega \in \{e^{i\frac{2\pi}{d}k} : k = 0, \ldots, d-1\}.$$

"$\supseteq$": For the inclusion choose an $\omega \in \mathbb{T}$ with $\omega^d = 1$. Since $A(\cdot)$ is irreducible we can find a strictly positive $y \in \mathbb{S}^n$ satisfying $\beta^m = (A(\beta)y, y)$. Our goal is to construct an $x \in \mathbb{S}^n$ satisfying $(\beta\omega)^m = (A(\beta\omega)x, x)$. Set $x_1 = y_1$. For $s \in \langle 2, n \rangle$ take a path $(r_0, p_0) \to \ldots \to (r_w, p_w)$ in $G_m(A_0, \ldots, A_l)$ such that $r_0 = 1$ and $r_w = s$ (which is possible by Lemma 4.3). Further by Lemma 4.1 we can assume w.l.o.g. that $p_w = 0$. Thus the path will have level displacement $-p_0$.
Now define x_s recursively via

$$x_{r_h} = y_{r_h} \frac{\bar{y}_{r_{h-1}}}{\bar{x}_{r_{h-1}}} \omega^{p_{h-1} - p_h}, \qquad h \in \langle w \rangle.$$

Claim: The above construction is well defined, i.e. it is independent of the specific path.
To see the claim note that it can be easily shown via induction that $|x_{r_h}| = |y_{r_h}|$, $h \in \langle w \rangle_0$ and we can thus write

$$x_{r_h} = y_{r_h} \frac{\bar{y}_{r_{h-1}}}{\bar{x}_{r_{h-1}}} \omega^{p_{h-1}-p_h} = y_{r_h} \frac{x_{r_{h-1}}}{y_{r_{h-1}}} \frac{|y_{r_{h-1}}|^2}{|x_{r_{h-1}}|^2} \omega^{p_{h-1}-p_h} = y_{r_h} \frac{x_{r_{h-1}}}{y_{r_{h-1}}} \omega^{p_{h-1}-p_h}.$$

Then

$$\begin{aligned} x_s &= y_s \frac{x_{r_{w-1}}}{y_{r_{w-1}}} \omega^{p_{w-1}-p_w} \\ &= y_s \frac{y_{r_{w-1}}}{y_{r_{w-1}}} \frac{x_{r_{w-2}}}{y_{r_{w-2}}} \omega^{p_{w-2}-p_{w-1}} \omega^{p_{w-1}-p_w} \\ &\vdots \\ &= y_s \omega^{p_0 - p_w} = y_s \omega^{p_0} \end{aligned}$$

Now take another path $(\tilde{r}_0, \tilde{p}_0) \to \ldots \to (\tilde{r}_{\tilde{w}}, \tilde{p}_{\tilde{w}})$ satisfying $\tilde{r}_0 = 1$ and $\tilde{r}_{\tilde{w}} = s$ and again w.l.o.g. $\tilde{p}_{\tilde{w}} = 0$ (so that this path has level displacement $-\tilde{p}_0$). This path gives rise to a $\tilde{x}_s$ and by the same calculation as above we get

$$\tilde{x}_s = y_s \omega^{\tilde{p}_0}.$$

In the last step consider a third path from $(s, 0)$ to $(1, \hat{d})$ with level displacement $\hat{d}$ (any such path will do). Attaching this path to any of the previous two paths gives phase cycles with level displacements $\hat{d} - p_0$ and $\hat{d} - \tilde{p}_0$ respectively. We can now divide x_s by $\tilde{x}_s$ to get

$$\frac{x_s}{\tilde{x}_s} = \omega^{p_0 - \tilde{p}_0} = \omega^{-(\hat{d} - p_0)} \omega^{\hat{d} - \tilde{p}_0} = 1,$$

where the last equality holds because both exponents are level displacements of phase cycles and thus divisible by d. The claim is proved.
The vector x was constructed in a way that whenever $A_j(r, s) > 0$ relation (5.5) is satisfied. We can thus write

$$\begin{aligned} (A(\beta\omega)x, x) &= \sum_{j=0}^{l} (\beta\omega)^j (A_j x, x) \\ &= \sum_{j=0}^{l} \sum_{r=1}^{n} \sum_{s=1}^{n} (\beta\omega)^j \bar{x}_r A_j(r, s) x_s \\ &= \sum_{j=0}^{l} \sum_{r=1}^{n} \sum_{s=1}^{n} \left[\beta^j \bar{y}_r A_j(r, s) y_s\right] \left[\omega^{j-m} \frac{\bar{x}_r}{\bar{y}_r} \frac{x_s}{y_s}\right] \omega^m \\ &= (A(\beta)y, y)\omega^m = (\beta\omega)^m \end{aligned}$$

which shows $\beta\omega \in \Theta(Q(\cdot)) \cap \mathbb{T}_\beta$.
(i) Note that $d = 0$ implies that in the proof of the reverse inclusion above we can choose any $\omega \in \mathbb{T}_1$. It then follows that $\mathbb{T}_\beta \subseteq \Theta(Q(\cdot))$. $\square$

Remark 5.3. The angles of rotation invariance in Theorem 5.2 only depend on the index of phase imprimitivity d of the graph $G_m(A_0, \dots, A_l)$. This implies that if there are several $\beta > 0$ satisfying $\beta^m = \text{nur}(A(\beta))$, then the values of the numerical range lying on $\mathbb{T}_\beta$ will be distributed along the same angles.

Remark 5.4. An equivalent result to Theorem 5.2 holds for the spectrum of $Q(\cdot)$ (see [8, Theorem 4.23]). The eigenvalues are distributed along the d-th roots on the circle $\mathbb{T}_\rho$ where $\rho^m = \text{spr}(A(\rho))$. Clearly in general ρ does not coincide with β (satisfying $\beta^m = \text{nur}(A(\beta)))$ and thus the eigenvalues and values of the numerical range of a semi-monic matrix polynomial $Q(\cdot)$ can be rotation invariant on different circles $\mathbb{T}_\rho$ and $\mathbb{T}_\beta$ but with the same number of values d.

Remark 5.5. Theorem 5.2 can be seen as an extension of [18, Corollary 5.6] for monic Perron-Frobenius polynomials. We have shown that the number of the maximal elements of $\Theta(Q(\cdot))$ is given by the index of phase imprimitiviy of $G_m(A_0, \dots, A_l)$.

Theorem 5.6. *Let $A(\lambda) = \sum_{j=0}^{l} \lambda^j A_j$ be an irreducible matrix polynomial with entrywise nonnegative coefficients and $Q(\lambda) = \lambda^m - A(\lambda)$ its corresponding semi-monic polynomial for some $m \in \mathbb{N}$. Let further d be the index of phase imprimitivity of the associated graph $G_m(A_0, \dots, A_l)$ and assume $d \geq 1$. Then $\Theta(Q(\cdot))$ is invariant under rotation with the angle $\theta = \frac{2\pi}{d}$, i.e.*

$$\Theta(Q(\cdot)) = e^{i\frac{2\pi}{d}} \Theta(Q(\cdot)).$$

Moreover if there exists a $\beta > 0$ such that $\text{nur}(A(\beta)) = \beta^m$ *then θ is the smallest such angle.*

Proof. The proof is conceptually very similar to the second inclusion in part (ii) of the proof of Theorem 5.2. Let $\lambda \in \Theta(Q(\cdot))$ and $\omega \in \mathbb{T}_1$ such that $\omega^d = 1$. Then there exists an $y \in \mathbb{S}^n$ such that $\lambda^m = (A(\lambda)y, y)$. We will construct an $x \in \mathbb{S}^n$ such that $(\lambda\omega)^m = (A(\lambda\omega)x, x)$. Set $x_1 = y_1$. For $s \in \langle 2, n \rangle$ take a path $(r_0, p_0) \to \dots \to (r_w, p_w)$ in $G_m(A_0, \dots, A_l)$ such that $r_0 = 1$ and $r_w = s$ (which is possible by Lemma 4.3). Further by Lemma 4.1 we can assume w.l.o.g. that $p_w = 0$. Thus the path will have level displacement $-p_0$.
Now define x_s recursively via

$$x_{r_h} = \begin{cases} y_{r_h} \frac{\bar{y}_{r_{h-1}}}{\bar{x}_{r_{h-1}}} \omega^{p_{h-1} - p_h}, & \bar{x}_{r_{h-1}} \neq 0 \\ y_{r_h} \omega^{p_0 - p_h}, & \bar{x}_{r_{h-1}} = 0 \end{cases} \qquad , h \in \langle w \rangle.$$

Claim: The above construction is well defined, i.e. it is independent of the specific path.
To see this note that it can be easily shown via induction that $|x_{r_h}| = |y_{r_h}|$, $h \in \langle w \rangle_0$ and we can thus write

$$x_{r_h} = y_{r_h} \frac{\bar{y}_{r_{h-1}}}{\bar{x}_{r_{h-1}}} \omega^{p_{h-1} - p_h} = y_{r_h} \frac{x_{r_{h-1}}}{y_{r_{h-1}}} \frac{|y_{r_{h-1}}|^2}{|x_{r_{h-1}}|^2} \omega^{p_{h-1} - p_h} = y_{r_h} \frac{x_{r_{h-1}}}{y_{r_{h-1}}} \omega^{p_{h-1} - p_h}$$

if $\bar{x}_{r_{h-1}} \neq 0$. Then

$$\begin{aligned} x_s &= y_s \frac{x_{r_{w-1}}}{y_{r_{w-1}}} \omega^{p_{w-1}-p_w} \\ &= y_s \frac{y_{r_{w-1}}}{y_{r_{w-1}}} \frac{x_{r_{w-2}}}{y_{r_{w-2}}} \omega^{p_{w-2}-p_{w-1}} \omega^{p_{w-1}-p_w} \\ &\vdots \\ &= y_s \omega^{p_0 - p_w} = y_s \omega^{p_0} \end{aligned}$$

Note that the above recursive expansion of x_s might stop early if the vector y has a zero entry, but the result will remain the same. Now take another path $(\tilde{r}_0, \tilde{p}_0) \to \ldots \to (\tilde{r}_{\tilde{w}}, \tilde{p}_{\tilde{w}})$ satisfying $\tilde{r}_0 = 1$ and $\tilde{r}_{\tilde{w}} = s$ and again w.l.o.g. $\tilde{p}_{\tilde{w}} = 0$ (so that this path has level displacement $-\tilde{p}_0$). This path gives rise to a $\tilde{x}_s$ and by the same calculation as above we get

$$\tilde{x}_s = y_s \omega^{\tilde{p}_0}.$$

Consider a third path from $(s, 0)$ to $(1, \hat{d})$ with level displacement $\hat{d}$ (any such path will do). Attaching this path to any of the previous two paths gives phase cycles with level displacements $\hat{d} - p_0$ and $\hat{d} - \tilde{p}_0$ respectively. We can now divide x_s by $\tilde{x}_s$ to get

$$\frac{x_s}{\tilde{x}_s} = \omega^{p_0 - \tilde{p}_0} = \omega^{-(\hat{d} - p_0)} \omega^{\hat{d} - \tilde{p}_0} = 1,$$

where the last equality holds because both exponents are level displacements of phase cycles and thus divisible by d. This ends the proof of the claim.
The vector x was constructed in a way that whenever $A_j(r, s) > 0$ the following relation (which is the equivalent to relation (5.5)) is satisfied:

$$\omega^{j-m} \bar{x}_r x_s = \bar{y}_r y_s. \tag{5.6}$$

In the following equation we will use $\sum'$ to denote that we leave out all elements of the sum that are equal to zero. We can then write

$$\begin{aligned} (A(\lambda\omega)x, x) &= \sum_{j=0}^{l} (\lambda\omega)^j (A_j x, x) \\ &= \sum_{j=0}^{l}{}' \sum_{r=1}^{n}{}' \sum_{s=1}^{n}{}' (\lambda\omega)^j \bar{x}_r A_j(r, s) x_s \\ &= \sum_{j=0}^{l}{}' \sum_{r=1}^{n}{}' \sum_{s=1}^{n}{}' \left[\lambda^j \bar{y}_r A_j(r, s) y_s\right] \left[\omega^{j-m} \frac{\bar{x}_r}{\bar{y}_r} \frac{x_s}{y_s}\right] \omega^m \\ &= (A(\lambda)y, y)\omega^m = (\lambda\omega)^m \end{aligned}$$

which shows $\lambda\omega \in \Theta(Q(\cdot))$.
The second assertion follows immediately from Theorem 5.2 since a smaller angle would imply that we would get additional elements of the numerical range on the circle $\mathbb{T}_\beta$. $\square$

Example 5.7. We consider the polynomial $A(\lambda) = \lambda^2 A_2 + A_0$ where

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A_0 = \begin{bmatrix} 0 & 0 \\ 4 & 0 \end{bmatrix}.$$

It is obvious that $A(\cdot)$ is irreducible. Moreover there exists exactly one phase cycle in the infinite graph $G_m(A_0, A_2)$ for $m \in \mathbb{N}$ starting in phase 1 and level 0.

$$(1, 0) \to (2, 2 - m) \to (1, 2 - 2m)$$

It follows that the level displacement of all phase cycles in $G_m(A_0, A_2)$ is equal to $2-2m$ and thus the index of phase imprimitivity is

$$d = |2 - 2m|. \tag{5.7}$$

We proceed by calculating the values $\rho, \beta \in (0, \infty)$ for which $\operatorname{spr}(A(\rho)) = \rho^m$ and $\operatorname{nur}(A(\beta)) = \beta^m$. The eigenvalues of $A(\rho) \in \mathbb{R}_+^{n,n}$ are located at

$$\lambda_\pm = \pm 2\rho$$

and thus $\operatorname{spr}(A(\rho)) = 2\rho$. Assuming $m \geq 2$ we then have

$$\operatorname{spr}(A(\rho)) \stackrel{!}{=} \rho^m$$

for $\rho \in \{0, 2^{\frac{1}{m-1}}\}$. Since we require $\rho > 0$ we conclude that the only solution is $\rho = 2^{\frac{1}{m-1}}$. We can now calculate the eigenvalues of the semi-monic polynomial $Q_m(\lambda) = \lambda^m - A(\lambda)$ on the circle $\mathbb{T}_\rho$. Take a complex ω with $|\omega| = 1$. Then the eigenvalues of $A(\rho\omega) \in \mathbb{C}^{n,n}$ are located at $\lambda_\pm = \pm 2\rho\omega$. In order for $\rho\omega \in \Sigma(Q_m(\cdot)) \cap \mathbb{T}_\rho$ to hold we need ω to satisfy

$$\begin{aligned} (\rho\omega)^m &\stackrel{!}{=} \pm 2\rho\omega \\ \Leftrightarrow \qquad \omega^m &= \pm\omega \\ \Leftrightarrow \qquad \omega^{m-1} &= \pm 1. \end{aligned} \tag{5.8}$$

We will now treat the numerical range of $Q_m(\cdot)$. For $\beta \in (0, \infty)$ by [11, Lemma 1.3.3] the numerical radius of $A(\beta) \in \mathbb{R}_+^{n,n}$ is equal to $2 + \frac{1}{2}\beta^2$ and the elements of $\Theta(A(\beta))$ with maximal modulus are located at $\pm(2 + \frac{1}{2}\beta^2)$. In order for $\operatorname{nur}(A(\beta)) = \beta^m$ to hold we thus need β to satisfy

$$\operatorname{nur}(A(\beta)) = 2 + \frac{1}{2}\beta^2 \stackrel{!}{=} \beta^m. \tag{5.9}$$

Elements in $\Theta(Q_m(\cdot)) \cap \mathbb{T}_\beta$ will be of the form $\beta\omega$ with $|\omega| = 1$. We claim that $\Theta(A(\beta\omega)) = \omega\Theta(A(\beta))$. To see this write

$$\begin{aligned} \Theta(A(\beta\omega)) &= \{(A(\beta\omega)x, x) : x \in \mathbb{S}^n\} = \{\omega^2\beta^2\bar{x}_1 x_2 + 4x_1\bar{x}_2 : x \in Sn\} \\ &= \left\{\omega\beta^2\overline{x_1\overline{\omega^{1/2}}}x_2\omega^{1/2} + \omega 4x_1\overline{\omega^{1/2}x_2\omega^{1/2}} : x \in \mathbb{S}^n\right\} \\ &= \omega\left\{\left(A(\beta)\begin{pmatrix} x_1\overline{\omega^{1/2}} \\ x_2\omega^{1/2}\end{pmatrix}, \begin{pmatrix} x_1\overline{\omega^{1/2}} \\ x_2\omega^{1/2}\end{pmatrix}\right) : x \in \mathbb{S}^n\right\} \\ &= \omega\Theta(A(\beta)), \end{aligned}$$

where the last equality follows because

$$\mathbb{S}^n = \left\{ \begin{pmatrix} x_1\overline{\omega^{1/2}} \\ x_2\omega^{1/2} \end{pmatrix} : x \in \mathbb{S}^n \right\}.$$

Therefore the elements of maximal modulus of $\Theta(A(\beta\omega))$ are located at $\pm\omega(2+\frac{1}{2}\beta^2)$. It follows that $\beta\omega \in \Theta(Q_m(\cdot)) \cap \mathbb{T}_\beta$ if

$$\begin{aligned} (\beta\omega)^m &\overset{!}{=} \pm\omega(2+\frac{1}{2}\beta^2) \\ \Leftrightarrow \qquad \omega^m &= \pm\omega \\ \Leftrightarrow \qquad \omega^{m-1} &= \pm 1 \end{aligned} \tag{5.10}$$

(note that this is the same condition as for the eigenvalues).
We will now look at some specific values for m:

- $m = 2$: First note that by the above calculations $\rho^m = \mathrm{spr}(A(\rho))$ and $\beta^m = \mathrm{nur}(A(\beta))$ are both satisfied for $\rho = \beta = 2$. By equations (5.8) and (5.10) it then follows that
$$\Sigma(Q_2(\cdot)) \cap \mathbb{T}_2 = \Theta(Q_2(\cdot)) \cap \mathbb{T}_2 = \{\pm 1\}.$$
This is consistent with Theorem 5.2 as the index of phase imprimitivity of $G_2(A_0, A_2)$ is $d = 2$ (by equation (5.7)).

- $m = 4$: Here we have $\rho^m = \mathrm{spr}(A(\rho))$ for $\rho = 2^{1/3}$ and by numerical approximation $\beta^m = \mathrm{nur}(A(\beta))$ for $\beta \approx 1.298$. In particular $\rho < \beta$.
Nonetheless we get the same cyclic distribution for the spectrum and the numerical range, i.e.
$$\begin{aligned} \Sigma(Q_4(\cdot)) \cap \mathbb{T}_\rho &= \{\rho \exp(\frac{2\pi i k}{6}) : k = 0, \dots, 5\}, \\ \Theta(Q_4(\cdot)) \cap \mathbb{T}_\beta &= \{\beta \exp(\frac{2\pi i k}{6}) : k = 0, \dots, 5\}. \end{aligned}$$
This is again consistent with Theorem 5.2 since by equation (5.7) the index of phase imprimitivity is $d = 6$.

Example 5.8. Continuing from the previous example we want to choose $m = 1$ (implying $d = 0$). However for $m = 1$ equation (5.9) has no real solutions. This can be remedied by altering A_0 to be

$$\tilde{A}_0 = \begin{bmatrix} 0 & 0 \\ 1/2 & 0 \end{bmatrix}.$$

Then $\beta^m = \mathrm{nur}(A(\beta))$ is satisfied for $\beta_1 \approx 0.293$ and $\beta_2 \approx 1.707$. We also see that equation (5.10) is satisfied for all ω and thus

$$\Theta(\tilde{Q}_1(\cdot)) \cap \mathbb{T}_{\beta_1} = \mathbb{T}_{\beta_1} \quad \text{and} \quad \Theta(\tilde{Q}_1(\cdot)) \cap \mathbb{T}_{\beta_2} = \mathbb{T}_{\beta_2}$$

which is again consistent with Theorem 5.2 for $d = 0$. Moreover, while the circles $\mathbb{T}_{\beta_1}$ and $\mathbb{T}_{\beta_2}$ belong to $\Theta(\tilde{Q}_1(\cdot))$, by Theorem 3.4 the domain between them, i.e. the open annulus $\mathbb{A}_{\beta_1,\beta_2}$, is disjoint to $\Theta(\tilde{Q}_1(\cdot))$.
This is in contrast to the set of eigenvalues since $\mathbb{T}_\rho \subseteq \Sigma(Q(\cdot))$ for a single $\rho > 0$ already implies $\Sigma(Q(\cdot)) = \mathbb{C}$.

References

[1] B. Aupetit. *A Primer on Spectral Theory.* Springer, New York, 1991.

[2] T. Y. Azizov, V. Hardt, N. D. Kopachevsky, and R. Mennicken. On the problem of small motions and normal oscillations of a viscous fluid in a partially filled container. *Linear Algebra and its Applications*, 248/249:3–39, 2003.

[3] D. A. Bini, G. Latouche, and B. Meini. Solving Matrix Polynomial Equations Arising in Queueing Problems. *Linear Algebra Appl.*, 340:225–244, 2002.

[4] R. A. Brualdi and H. J. Ryser. *Combinatorial Matrix Theory.* Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 1991.

[5] G. J. Butler, C. R. Johnson, and H. Wolkowicz. Nonnegative Solutions of a Quadratic Matrix Equation Arising from Comparison Theorems in Ordinary Differential Equations. *SIAM J. Alg. Disc. Meth.*, 6:47–53, 1985.

[6] K.-H. Förster and B. Nagy. On Nonmonic Quadratic Matrix Polynomials with Nonnegative Coefficients. *Operator Theory: Advances and Applications*, 162:145–163, 2005.

[7] H. R. Gail, S. L. Hantler, and B. A. Taylor. Spectral Analysis of $M/G/1$ and $G/M/1$ Type Markov Chains. *Adv. Appl. Prob.*, 28:114–165, 1996.

[8] N. V. Hartanto. Spectral Properties and Companion Forms of Operator- and Matrix Polynomials. PhD thesis, TU Berlin, 2011. unpublished.

[9] W. K. Hayman and P. B. Kennedy. *Subharmonic Functions Volume 1.* London Mathematical Society. Academic Press, London, 1976.

[10] R. A. Horn and C. R. Johnson. *Matrix Analysis.* Cambridge University Press, Cambridge, 1985.

[11] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis.* Sigma Series in Applied Mathematics. Cambridge University Press, Cambridge, 1991.

[12] J. N. Issos. The Field of Values of non-Negative Irreducible Marices. PhD thesis, Auburn University, 1966. unpublished.

[13] T. Kato. *Perturbation theory for linear operators*, volume 132 of *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen.* Springer, Berlin, 1966.

[14] N. D. Kopachevsky and S. G. Krein. *Operator Approach to Linear Problems of Hydrodynamics Vol. 1*, volume 128 of *Operator Theory: Advances and Applications.* Bergheuser Verlag, Basel, 2001.

[15] A. S. Markus. *Introduction to the Spectral Theory of Polynomial Operator Pencils*, volume 71 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, Rhode Island, 1988.

[16] J. Maroulas, P. J. Psarrakos, and M. J. Tsatsomeros. Perron–Frobenius type results on the numerical range. *Linear Algebra and its Applications*, 348:49–62, 2002.

[17] H. Minc. *Nonnegative Matrices.* Wiley-Intersection Series in Discrete Mathematics and Optimization. Wiley, New York, 1988.

[18] P. J. Psarrakos and M. J. Tsatsomeros. A Primer of Perron-Frobenius theory for matrix polynomials. *Linear Algebra and its Applications*, 393:333–351, 2004.

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