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Continuous-Time Mean Field Games with Finite State Space and Common Noise

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Abstract
We formulate and analyze a mathematical framework for continuous-time mean field games with finitely many states and common noise, including a rigorous probabilistic construction of the state process and existence and uniqueness results for the resulting equilibrium system. The key insight is that we can circumvent the master equation and reduce the mean field equilibrium to a system of forward-backward systems of (random) ordinary differential equations by conditioning on common noise events. In the absence of common noise, our setup reduces to that of Gomes, Mohr and Souza (Appl Math Optim 68(1): 99–143, 2013) and Cecchin and Fischer (Appl Math Optim 81(2):253–300, 2020).

Keywords Mean field games · Common noise · Markov chains · Regime shifts

Mathematics Subject Classification 60J27 · 93E20 · 91A15

1 Introduction

Since the seminal contributions of Lasry and Lions [44] and Huang, Malhamé and Caines [39], mean field games have become an active field of mathematical research with a wide range of applications, including economics [13,16,27,33,41,50], sociology

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[35], finance [17,45], epidemiology [23,26,46] and computer science [40]; see also the overview article [29] and the monograph [9].

Mean field games constitute a class of dynamic, multi-player stochastic differential games with identical agents. The key characteristic of the mean field approach is that (i) the payoff and state dynamics of each agent depend on other agents' decisions only through an aggregate statistic (typically, the aggregate distribution of states); and (ii) no individual agent's actions can change the aggregate outcome. Thus, in solving an individual agent's optimization problem, the feedback effect of his own actions on the aggregate outcome can be discarded, breaking the notorious vicious circle ("the optimal strategy depends on the aggregate outcome, which depends on the strategy, which depends …"). This significantly facilitates the identification of rational expectations equlibria. A standard assumption that further simplifies the analysis is that randomness is idiosyncratic (equivalently, there is no common noise), i.e. that the random variables appearing in one agent's optimization are independent of those in any other's. As a result, all randomness is "averaged out" in the aggregation of individual decisions, and the equilibrium dynamics of the aggregate distribution are deterministic.

In the literature, mean field games are most often studied in settings with a continuous state space and deterministic or diffusive dynamics, i.e. stochastic differential equations (SDEs) driven by Brownian motion. The corresponding dynamic programming equations thus become parabolic partial differential equations, and the aggregate dynamics are represented by a flow of Borel probability measures; see, e.g., the monographs [4] and [9] and the references therein. Formally, the mean field game is typically formulated in terms of a controlled McKean-Vlasov SDE, where the coefficients depend on the current state and control and the distribution of the solution; intuitively, these McKean-Vlasov dynamics codify the dynamics that pertain to a representative agent. The mathematical link to N-player games is subsequently made through suitable propagation of chaos results in the mean field limit $N \to \infty$; see, e.g., [14,25,28,42,43]. In this context, the analysis of McKean-Vlasov SDEs has also seen significant progress recently; see, e.g., [6,8,19,48]. In the presence of common noise, i.e. sources of risk that affect all agents and do not average out in the mean field limit, the mathematical analysis becomes even more involved as the dynamics of the aggregate distribution become stochastic, leading to conditional McKean-Vlasov dynamics; see, e.g., [1,12,21,51]. We refer to [10] for background and further references on continuous-state mean field games with common noise.

There is also a strand of literature on mean field games with finite state spaces, including [2,15,18,24,30,31,34,49] as well as [9, §7.2]. In a recent article, [22] provide an extension of [31] to mean field interactions that occur not only through the agents' states, but also through their controls. To the best of our knowledge, however, to date there has been no extension of these results to settings that include common noise. In the context of finite-state mean field games, we are only aware of two contributions that include common stochasticity (both via the master equation and with a different focus/setting than this paper): [5] analyze the master equation for finite-state mean field games with common noise, and [3] include a common continuous-time Gaussian noise in the aggregate distribution dynamics.

In this article, we set up a mathematical framework for finite-state mean field games with common noise.[1] Our setup extends that of [31] and [15] by common noise events at fixed points in time. We provide a rigorous formulation of the underlying stochastic dynamics, and we establish a verification theorem for the optimal strategy and an aggregation theorem to determine the resulting aggregate distribution. This leads to a characterization of the mean field equilibrium in terms of a system of (random) forward-backward differential equations. The key insight is that, after conditioning on common noise configurations, we obtain classical piecewise dynamics subject to jump conditions at common noise times.

The remainder of this article is organized as follows: In Sect. 2 we set up the mathematical model, provide a probabilistic construction of the state dynamics, and formulate the agent's optimization problem. In Sect. 3 we state the dynamic programming equation and establish a verification theorem for the agent's optimization, given an *ex ante* aggregate distribution (Theorem 6). Section 4 provides the dynamics of the *ex post* distribution (Theorem 9) and, on that basis, a system of random forward-backward ODEs for the mean field equilibrium (Definition 10) as well as corresponding existence and uniqueness results (Theorems 13 and 16). In Sect. 5 we showcase our results in two benchmark applications: agricultural production and infection control. The Appendix provides the proofs of Theorems 13 and 16.

2 Mean Field Model

We first provide an *informal* description of the individual agents' state dynamics, optimization problem, and the resulting mean field equilibrium. The agent's state process $X = \{X_t\}$ takes values in the finite set $\mathbb{S}$. Between common noise events, transitions from state i to state j occur with intensity $Q^{ij}(t, W_t, M_t, \nu_t)$, where W_t represents the common noise events that have occurred up to time t; M_t the time-t aggregate distribution of agents; and ν_t the agent's control. In addition, upon the realization of a common noise event W_k at time T_k, the state jumps from X_{T_k-} to $X_{T_k} = J^{X_{T_k-}}(T_k, W_{T_k}, M_{T_k-})$. With this, the agent aims to maximize

$$\mathbb{E}^{\nu}\Big[\int_0^T \psi^{X_t}(t, W_t, M_t, \nu_t)\mathrm{d}t + \Psi^{X_T}(W_T, M_T)\Big]$$

where ψ and Ψ are suitable reward functions and the aggregate distribution process $M = \{M_t\}$ is given by

$$M_t \triangleq \mu(t, W_t) \quad \text{for } t \in [0, T].$$

Here μ represents the aggregate distribution of states as a function of the common noise factors. We obtain a rational expectations equilibrium by determining μ such that

[1] We wish to point out that our focus is *not* on the mean field limit of multi-player games; rather, we directly investigate the mean field equilibrium via the corresponding McKean-Vlasov dynamics (see also Remark 7 and [11] in that context).

the representative agent's *ex ante* expectations equal the *ex post* aggregate distribution resulting from all agents' optimal decisions, i.e.

$$\mathbb{P}^{\widehat{\nu}}(X_t \in \cdot \mid W_t) = \widehat{\mu}(t, W_t) \quad \text{for all } t \in [0, T],$$

where $\widehat{\nu}$ and $\widehat{\mu}$ denote the equilibrium strategy and the equilibrium aggregate distribution. In the remainder of this section, we provide a rigorous mathematical formulation of this model.

2.1 Probabilistic Setting and Common Noise

Throughout, we fix a time horizon $T > 0$ and a finite set $\mathbb{W}$ and work on a probability space $(\Omega, \mathfrak{A}, \mathbb{P})$ that carries a finite sequence $W_1, \ldots, W_n$ of i.i.d. random variables that are uniformly distributed[2] on $\mathbb{W}$. We refer to $W_1, \ldots, W_n$ as *common noise factors* and to $\mathbb{P}$ as the *reference probability*. The common noise factor W_k is revealed at time T_k, where

$$0 \triangleq T_0 < T_1 < T_2 < \cdots < T_n < T_{n+1} \triangleq T.$$

Both n and the common noise times $T_0, T_1, \ldots, T_{n+1}$ are fixed and deterministic. The piecewise constant filtration $\mathfrak{G} = \{\mathfrak{G}_t\}$ generated by common noise events is given by

$$\mathfrak{G}_t \triangleq \sigma\big(W_k \,:\, k \in [1:n],\ T_k \leq t\big) \vee \mathfrak{N} \quad \text{for } t \in [0, T]$$

where $\mathfrak{N}$ denotes the set of $\mathbb{P}$-null sets. For each configuration of common noise factors $w \in \mathbb{W}^n$ we write

$$w_t \triangleq (w_1, \ldots, w_k) \text{ for } t \in [T_k, T_{k+1}\rangle,\ k \in [0:n],$$

where for $0 \leq s \leq t \leq T$ we set $[s, t\rangle \triangleq [s, t)$ if $t < T$ and $[s, T\rangle \triangleq [s, T]$. With this convention, $W = \{W_t\}$ represents a piecewise constant, $\mathfrak{G}$-adapted process.

Definition 1 A function $f : [0, T] \times \mathbb{W}^n \to \mathbb{R}^m$ is *non-anticipative* if for all $t \in [0, T]$

$$f(t, w) = f(t, \bar{w}) \quad \text{whenever } w, \bar{w} \in \mathbb{W}^n \text{ are such that } w_t = \bar{w}_t.$$

Moreover, f is *regular* if $f(\cdot, w)$ is absolutely continuous on $[T_k, T_{k+1}\rangle$ for all $k \in [0:n]$. □

With a slight abuse of notation, if $f : [0, T] \times \mathbb{W}^n \to \mathbb{R}^m$ is non-anticipative, we write

$$f(t, w_t) \triangleq f(t, w) \quad \text{for } w \in \mathbb{W}^n,\ t \in [0, T].$$

[2] While the common noise factors are i.i.d. uniformly distributed under $\mathbb{P}$, the distribution of $W_1, \ldots, W_n$ in the agent's optimization problem can be modeled arbitrarily via the functions $\kappa_1, \ldots, \kappa_n$ introduced below; see also Lemma 2.

Note that for f regular, the one-sided limits $f(T_k-, w) \triangleq \lim_{t\uparrow T_k} f(t, w)$ exist for all $k \in [1:n]$, $w \in \mathbb{W}^n$.

2.2 Optimization Problem

The agent's state and action spaces are given by

$$\mathbb{S} \triangleq [1:d] \quad \text{and} \quad \mathbb{U} \subseteq \mathbb{R}^k, \quad \text{where } d, k \in \mathbb{N} \text{ and } \mathbb{U} \neq \varnothing,$$

and we identify the space of aggregate distributions on $\mathbb{S}$ with the space of probability vectors

$$\mathbb{M} \triangleq \Big\{ m \in [0,\infty)^{1\times d} \ :\ \sum_{i=1}^d m^i = 1 \Big\}.$$

The coefficients in the state dynamics and payoff functional are bounded and Borel measurable functions

$$\begin{aligned} &Q :\ [0,T]\times\mathbb{W}^n\times\mathbb{M}\times\mathbb{U} \to \mathbb{R}^{d\times d} \qquad && J :\ [0,T]\times\mathbb{W}^n\times\mathbb{M} \to \mathbb{S}^d \\ &\psi :\ [0,T]\times\mathbb{W}^n\times\mathbb{M}\times\mathbb{U} \to \mathbb{R}^{d} && \Psi :\ \mathbb{W}^n\times\mathbb{M} \to \mathbb{R}^d \end{aligned}$$

such that $Q(\cdot,\cdot,m,u)$, $\psi(\cdot,\cdot,m,u)$ and $J(\cdot,\cdot,m)$ are non-anticipative for all fixed $m \in \mathbb{M}$ and $u \in \mathbb{U}$; Q satisfies the intensity matrix conditions $Q^{ij}(t,w,m,u) \geq 0, i, j \in \mathbb{S}$, $i \neq j$ and $\sum_{j\in\mathbb{S}} Q^{ij}(t,w,m,u) = 0, i \in \mathbb{S}$, for $(t,w,m,u) \in [0,T]\times\mathbb{W}^n\times\mathbb{M}\times\mathbb{U}$; and for each $k \in [1:n]$ the function

$$\kappa_k :\ \mathbb{W}^k \times \mathbb{M} \to [0,1], \qquad (w_k, w_1, \ldots, w_{k-1}, m) \mapsto\ \kappa_k(w_k|w_1,\ldots,w_{k-1},m),$$

is Borel measurable with $\sum_{\bar{w}_k\in\mathbb{W}} \kappa_k(\bar{w}_k|w_1,\ldots,w_{k-1},m) = 1$ for all $w_1,\ldots,w_{k-1} \in \mathbb{W}$ and $m \in \mathbb{M}$.

We further suppose that $(\Omega, \mathfrak{A}, \mathbb{P})$ supports, for each $i, j \in \mathbb{S}$, $i \neq j$, a standard (i.e., unit intensity) Poisson process $N^{ij} = \{N_t^{ij}\}$ and an $\mathbb{S}$-valued random variable X_0 such that

$$X_0 \quad \text{and} \quad N^{ij},\ i,j\in\mathbb{S},\ i\neq j \quad \text{and} \quad W_1,\ldots,W_n \quad \text{are independent.}$$

The corresponding full filtration $\mathfrak{F} = \{\mathfrak{F}_t\}$ is given by

$$\mathfrak{F}_t \triangleq \sigma\big(X_0,\ W_s,\ N_s^{ij}\ :\ s\in[0,t];\ i,j\in\mathbb{S},\ i\neq j\big) \vee \mathfrak{N} \quad \text{for } t \in [0,T].$$

Note that $\mathfrak{G}_t \subseteq \mathfrak{F}_t$ for all $t \in [0,T]$, that both $\mathfrak{G}$ and $\mathfrak{F}$ satisfy the usual conditions, and that N^{ij} is a standard $(\mathfrak{F},\mathbb{P})$-Poisson process for $i, j \in \mathbb{S}$, $i \neq j$. Given a regular, non-anticipative function μ, the $\mathfrak{G}$-adapted, $\mathbb{M}$-valued *ex ante* aggregate distribution $M = \{M_t\}$ is given by

$$M_t \triangleq \mu(t, W_t) \quad \text{for } t \in [0, T]$$

and the agent's *optimization problem* reads[3]

$$\mathbb{E}^\nu\Big[\int_0^T \psi^{X_t}(t, W_t, M_t, \nu_t)\mathrm{d}t + \Psi^{X_T}(W_T, M_T)\Big] \underset{\nu\in\mathcal{A}}{\longrightarrow} \max! \tag{P$_\mu$}$$

where the class of *admissible strategies* for (P$_\mu$) is given by the set of closed-loop controls

$$\mathcal{A} \triangleq \big\{\nu : [0, T] \times \mathbb{S}^{[0,T]} \times \mathbb{W}^n \to \mathbb{U} \ : \ \nu \text{ is Borel measurable and} \\ \nu(\,\cdot\,, x, \,\cdot\,) \text{ is non-anticipative for all } x \in \mathbb{S}^{[0,T]}\big\}.$$

Note that $\mathcal{A}$ subsumes the class of Markovian feedback controls considered in, e.g., [31] or [34], and that each $\nu \in \mathcal{A}$ canonically induces an $\mathfrak{F}$-adapted $\mathbb{U}$-valued process via

$$\nu_t \triangleq \nu\big(t, X_{(\,\cdot\,\wedge t)-}, W_t\big) \quad \text{for } t \in [0, T].$$

$\mathbb{E}^\nu[\,\cdot\,]$ denotes the expectation operator with respect to the probability measure $\mathbb{P}^\nu$ given by

$$\frac{\mathrm{d}\mathbb{P}^\nu}{\mathrm{d}\mathbb{P}} = \prod_{\substack{i,j\in\mathbb{S},\\ i\neq j}} \left(\exp\left\{\int_0^T \big(1 - Q^{ij}(t, W_t, M_t, \nu_t)\big)\mathrm{d}t\right\} \cdot \prod_{\substack{t\in(0,T],\\ \Delta N_t^{ij}\neq 0}} Q^{ij}(t, W_t, M_t, \nu_t)\right) \\ \times\ |\mathbb{W}|^n \cdot \prod_{k=1}^n \kappa_k\big(W_k | W_1, \ldots, W_{k-1}, M_{T_k-}\big); \tag{1}$$

and the agent's state process X is given by

$$\mathrm{d}X_t = \sum_{\substack{i,j\in\mathbb{S},\\ i\neq j}} \mathbb{1}_{\{X_{t-}=i\}}(j - i)\mathrm{d}N_t^{ij} \quad \text{for } t \in [T_k, T_{k+1}\rangle,\ k \in [0 : n], \tag{2}$$

subject to the jump conditions

$$X_{T_k} = J^{X_{T_k-}}\big(T_k, W_{T_k}, M_{T_k-}\big) \text{ for } k \in [1 : n]. \tag{3}$$

[3] For notational simplicity, we write X_t instead of X_{t-}, M_t instead of M_{t-}, etc., where it does not make a difference.

Here N^{ij} triggers transitions from state i to state j, and $\mathbb{P}^\nu$ is defined in such a way that N^{ij} has $\mathbb{P}^\nu$-intensity $Q^{ij}(t, W_t, M_t, \nu_t)$; see Lemma 2 below.[4] In summary, in order to formulate a mean field model within the above setting, it suffices to specify

- the agent's state space $\mathbb{S}$, action space $\mathbb{U}$ and the common noise space $\mathbb{W}$,
- the transition intensities $Q(t, w, m, u)$, transition kernels $\kappa_k(w_k|w_1, \dots, w_{k-1}, m)$ and common noise jumps $J(t, w, m)$, and finally
- the reward functions $\psi(t, w, m, u)$ and $\Psi(w, m)$.

2.3 State Dynamics

In what follows, we show that the preceding construction implies the dynamics described informally above.

Lemma 2 ($\mathbb{P}^\nu$-dynamics) *For each admissible strategy $\nu \in \mathcal{A}$, $\mathbb{P}^\nu$ is a well-defined probability measure on $(\Omega, \mathfrak{A})$, absolutely continuous with respect to $\mathbb{P}$, and satisfies*

$$\mathbb{P}^\nu = \mathbb{P} \quad on\ \sigma(X_0).$$

Moreover, N^{ij} is a counting process with $(\mathfrak{F}, \mathbb{P}^\nu)$-intensity $\lambda^{ij} = \{\lambda^{ij}_t\}$, where

$$\lambda^{ij}_t \triangleq Q^{ij}(t, W_t, M_t, \nu_t) \quad for\ t \in [0, T]\ and\ i, j \in \mathbb{S},\ i \neq j.$$

Finally, for all $k \in [1 : n]$ we have

$$\mathbb{P}^\nu\big(W_k = w_k | \mathfrak{G}_{T_k-}\big) = \kappa_k(w_k | W_1, \dots, W_{k-1}, M_{T_k-}) \quad for\ all\ w_1, \dots, w_k \in \mathbb{W}$$

where $\mathfrak{G}$ denotes the common noise filtration and, in particular,

$$\mathbb{P}^{\nu_1} = \mathbb{P}^{\nu_2} \quad on\ \mathfrak{G}_T \quad for\ all\ admissible\ strategies\ \nu_1, \nu_2 \in \mathcal{A}.$$

Proof We fix $\nu \in \mathcal{A}$ and split the proof into four steps.

Step 1: $\mathbb{P}^\nu$ is well-defined by (1). Since N^{ij} is a standard Poisson process under $\mathbb{P}$, the compensated process $\bar{N}^{ij}_t \triangleq N^{ij}_t - t$, $t \geq 0$, is an $(\mathfrak{F}, \mathbb{P})$-martingale for all $i, j \in \mathbb{S}$, $i \neq j$. We define $\theta^\nu = \{\theta^\nu_t\}$ via[5]

$$\theta^\nu_t \triangleq \sum_{\substack{i,j \in \mathbb{S},\\ i \neq j}} \int_0^t \big(Q^{ij}\big(s, W_s, \mu(s, W_s), \nu_s\big) - 1\big) \mathrm{d}\bar{N}^{ij}_s, \quad t \in [0, T],$$

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[4] See also [7, Sect. 3.3] for a similar change-of-measure construction of jump processes with stochastic intensities.

[5] Note that $\int_0^t Q^{ij}(s, W_s, \mu(s, W_s), \nu_s) - 1)\mathrm{d}\bar{N}^{ij}_s = \int_0^t Q^{ij}(s, W_{s-}, \mu(s, W_{s-}), \nu_s) - 1)\mathrm{d}\bar{N}^{ij}_s$ $\mathbb{P}$-a.s.

and observe that the Doléans-Dade exponential $\mathcal{E}[\theta^\nu]$ is a local $(\mathfrak{F}, \mathbb{P})$-martingale with

$$\mathcal{E}[\theta^\nu]_t = \prod_{\substack{i,j\in\mathbb{S},\\ i\neq j}} \left(\exp\left\{ \int_0^t \left(1 - Q^{ij}\big(s, W_s, \mu(s, W_s), \nu_s\big)\right)\mathrm{d}s \right\} \cdot \prod_{\substack{s\in(0,t],\\ \Delta N_s^{ij}\neq 0}} Q^{ij}(s, W_s, \mu(s, W_s), \nu_s) \right) \tag{4}$$

for $t \in [0, T]$. Next, we define $\vartheta = \{\vartheta_t\}$ via

$$\vartheta_t \triangleq \sum_{\substack{k\in[1:n],\\ T_k\leq t}} \Big(|\mathbb{W}| \cdot \kappa_k\big(W_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})\big) - 1 \Big), \quad t \in [0, T],$$

and note that ϑ is an $(\mathfrak{F}, \mathbb{P})$-martingale. Indeed, for each $k \in [0 : n]$ we have $\vartheta_t = \vartheta_{T_k}$ for $t \in [T_k, T_{k+1}\rangle$ and, using that W_k is independent of $\mathfrak{F}_{T_k-}$ and uniformly distributed on $\mathbb{W}$ under $\mathbb{P}$, it follows that

$$\begin{aligned}
\mathbb{E}\big[\vartheta_{T_k} | \mathfrak{F}_{T_k-}\big] &= \vartheta_{T_k-} + \mathbb{E}\big[|\mathbb{W}| \cdot \kappa_k\big(W_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})\big) - 1 \big| \mathfrak{F}_{T_k-}\big] \\
&= \vartheta_{T_k-} - 1 + |\mathbb{W}| \cdot \sum_{w_k\in\mathbb{W}} \mathbb{P}\big(W_k = w_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})\big) \\
&\qquad \times \kappa_k\big(w_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})\big) \\
&= \vartheta_{T_k-} - 1 + |\mathbb{W}| \cdot \sum_{w_k\in\mathbb{W}} \frac{1}{|\mathbb{W}|} \cdot \kappa_k\big(w_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})\big) = \vartheta_{T_k-}.
\end{aligned}$$

Hence the Doléans-Dade exponential $\mathcal{E}[\vartheta]$ is a local $(\mathfrak{F}, \mathbb{P})$-martingale, and we have

$$\mathcal{E}[\vartheta]_t = \prod_{s\in(0,t]} (1 + \Delta\vartheta_s) = \prod_{\substack{k\in[1:n],\\ T_k\leq t}} \Big(|\mathbb{W}| \cdot \kappa_k\big(W_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})\big) \Big) \tag{5}$$

for $t \in [0, T]$. Since $\Delta N_{T_k}^{ij} = 0$ for all $i, j \in \mathbb{S}$, $i \neq j$, and $k \in [1 : n]$ a.s., we have $[\theta^\nu, \vartheta] = 0$, and thus the process $Z^\nu \triangleq \mathcal{E}[\theta^\nu + \vartheta] = \mathcal{E}[\theta^\nu] \cdot \mathcal{E}[\vartheta]$, i.e.

$$\begin{aligned}
Z_t^\nu = &\prod_{\substack{i,j\in\mathbb{S},\\ i\neq j}} \left(\exp\left\{ \int_0^t \left(1 - Q^{ij}\big(s, W_s, \mu(s, W_s), \nu_s\big)\right)\mathrm{d}s \right\} \cdot \prod_{\substack{s\in(0,t],\\ \Delta N_s^{ij}\neq 0}} Q^{ij}(s, W_s, \mu(s, W_s), \nu_s) \right) \\
&\times \prod_{\substack{k\in[1:n],\\ T_k\leq t}} \Big(|\mathbb{W}| \cdot \kappa_k\big(W_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})\big) \Big)
\end{aligned} \tag{6}$$

is a local $(\mathfrak{F}, \mathbb{P})$-martingale. Since

$$\sup_{t\in[0,T]} |\mathcal{E}[\theta^\nu]_t| \leq \mathrm{e}^{d^2 T} \cdot \ell^Y \tag{7}$$

where $\ell \triangleq \max_{i,j\in\mathbb{S},\,i\neq j} \|Q^{ij}\|_\infty$ and $Y \triangleq \sum_{i,j\in\mathbb{S},\,i\neq j} N_T^{ij} \sim_{\mathbb{P}} \mathsf{Poisson}(d(d-1)T)$ and

$$\sup_{t\in[0,T]} |\mathcal{E}[\vartheta]_t| \leq |\mathbb{W}|^n \tag{8}$$

it follows that $\sup_{t\in[0,T]} |Z_t^\nu|$ is $\mathbb{P}$-integrable, so Z^ν is in fact an $(\mathfrak{F},\mathbb{P})$-martingale. Since Z^ν is non-negative with $Z_0^\nu = 1$ by construction, we conclude that $\mathbb{P}^\nu$ is a well-defined probability measure on $\mathfrak{A}$, absolutely continuous with respect to $\mathbb{P}$, with density process

$$\left.\frac{\mathrm{d}\mathbb{P}^\nu}{\mathrm{d}\mathbb{P}}\right|_{\mathfrak{F}_t} = Z_t^\nu, \quad t \in [0,T].$$

Step 2: $\mathbb{P}^\nu$-intensity of N^{ij}. Let $i, j \in \mathbb{S}$ with $i \neq j$. Since $\mathbb{P}^\nu \ll \mathbb{P}$ it is clear that N^{ij} is a $\mathbb{P}^\nu$-counting process, so it suffices to show that the process ${}^\nu\bar{N}^{ij} = \{{}^\nu\bar{N}_t^{ij}\}$ given by

$${}^\nu\bar{N}_t^{ij} \triangleq N_t^{ij} - \int_0^t Q^{ij}(s, W_s, \mu(s, W_s), \nu_s)\mathrm{d}s, \quad t \in [0,T], \tag{9}$$

is a local $(\mathfrak{F},\mathbb{P}^\nu)$-martingale. To show this, by Step 1 it suffices to demonstrate that $Z^\nu \cdot {}^\nu\bar{N}^{ij}$ is a local $(\mathfrak{F},\mathbb{P})$-martingale. Noting that

- $[N^{k\ell}, N^{ij}] = \sum_{s\in(0,\,\cdot\,]} \Delta N_s^{k\ell} \cdot \Delta N_s^{ij} = 0$ whenever $k, \ell \in \mathbb{S}$ and $(k,\ell) \neq (i,j)$,
- $\mathrm{d}Z_t^\nu = Z_{t-}^\nu \mathrm{d}\theta_t^\nu + Z_{t-}^\nu \mathrm{d}\vartheta_t = \sum_{k,\ell\in\mathbb{S},k\neq\ell} Z_{t-}^\nu \big(Q^{k\ell}(t, W_t, \mu(t, W_t), \nu_t) - 1\big)\,\mathrm{d}\bar{N}_t^{k\ell} + Z_{t-}^\nu \mathrm{d}\vartheta_t,$
- $\mathrm{d}[Z^\nu, {}^\nu\bar{N}^{ij}]_t = Z_{t-}^\nu(Q^{ij}(t, W_t, \mu(t, W_t), \nu_t) - 1)\mathrm{d}N_t^{ij},$

and using integration by parts, the local martingale property follows since

$$\begin{aligned}
\mathrm{d}\big(Z_t^\nu \cdot {}^\nu\bar{N}_t^{ij}\big) &= Z_{t-}^\nu \mathrm{d}{}^\nu\bar{N}_t^{ij} + {}^\nu\bar{N}_{t-}^{ij}\mathrm{d}Z_t^\nu + \mathrm{d}[Z^\nu, {}^\nu\bar{N}^{ij}]_t \\
&= Z_{t-}^\nu \mathrm{d}N_t^{ij} - Z_{t-}^\nu Q^{ij}(t, W_t, \mu(t, W_t), \nu_t)\mathrm{d}t + {}^\nu\bar{N}_{t-}^{ij}\mathrm{d}Z_t^\nu \\
&\quad + Z_{t-}^\nu Q^{ij}(t, W_t, \mu(t, W_t), \nu_t)\mathrm{d}N_t^{ij} - Z_{t-}^\nu \mathrm{d}N_t^{ij} \\
&= {}^\nu\bar{N}_{t-}^{ij}\mathrm{d}Z_t^\nu + Z_{t-}^\nu Q^{ij}(t, W_t, \mu(t, W_t), \nu_t)\mathrm{d}\bar{N}_t^{ij}.
\end{aligned}$$

Step 3: $\mathbb{P}^\nu = \mathbb{P}$ on $\sigma(X_0)$. For any function $g : \mathbb{S} \to \mathbb{R}$ we have

$$\mathbb{E}^\nu[g(X_0)] = \mathbb{E}[g(X_0)\cdot Z_T^\nu] = \mathbb{E}\big[g(X_0)\cdot \mathbb{E}[Z_T^\nu|\mathfrak{F}_0]\big] = \mathbb{E}[g(X_0)\cdot Z_0^\nu] = \mathbb{E}[g(X_0)]$$

by the $(\mathfrak{F},\mathbb{P})$-martingale property of Z^ν.

Step 4: Distribution of W_k *under* $\mathbb{P}^\nu$. Let $k \in [1:n]$ and $w_1, \ldots, w_k \in \mathbb{W}$. Since $\mathcal{E}[\theta^\nu]_{T_k} = \mathcal{E}[\theta^\nu]_{T_k-}$ a.s. and W_k is uniformly distributed on $\mathbb{W}$ and independent of $\mathfrak{F}_{T_k-}$ under $\mathbb{P}$, iterated conditioning yields

$$
\begin{aligned}
&\mathbb{P}^\nu\big(W_1 = w_1, \ldots, W_k = w_k\big) = \mathbb{E}\big[Z^\nu_{T_k} \cdot \mathbb{1}_{\{W_k = w_k\}} \cdot \mathbb{1}_{\{W_1 = w_1, \ldots, W_{k-1} = w_{k-1}\}}\big] \\
&= \mathbb{E}\big[Z^\nu_{T_k-} \cdot |\mathbb{W}| \cdot \kappa_k(W_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})) \cdot \mathbb{1}_{\{W_k = w_k\}} \cdot \mathbb{1}_{\{W_1 = w_1, \ldots, W_{k-1} = w_{k-1}\}}\big] \\
&= |\mathbb{W}| \cdot \kappa_k(w_k | w_1, \ldots, w_{k-1}, \mu(T_k-, w_{T_k-})) \cdot \mathbb{E}\big[Z^\nu_{T_k-} \cdot \mathbb{1}_{\{W_1 = w_1, \ldots, W_{k-1} = w_{k-1}\}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \mathbb{P}(W_k = w_k | \mathfrak{F}_{T_k-})\big] \\
&= \kappa_k(w_k | w_1, \ldots, w_{k-1}, \mu(T_k-, w_{T_k-})) \cdot \mathbb{P}^\nu\big(W_1 = w_1, \ldots, W_{k-1} = w_{k-1}\big).
\end{aligned}
$$

Thus we have $\mathbb{P}^\nu(W_k = w_k | \mathfrak{G}_{T_k-}) = \kappa_k(w_k | W_1, \ldots, W_{k-1}, M_{T_k-})$ and the proof is complete. $\square$

Lemma 2 implies in particular that $\mathbb{P}^\nu(\Delta N^{ij}_t \neq 0) = 0$ for every $t \in [0, T]$, so as a consequence we have

$$
\Delta X_t = 0 \quad \mathbb{P}^\nu\text{-a.s. for all } t \in [0, T] \setminus \{T_1, \ldots, T_n\}.
$$

Moreover, since $\mathbb{P}^{\nu_1} = \mathbb{P}^{\nu_2}$ on $\mathfrak{G}_T$ for all admissible controls $\nu_1, \nu_2 \in \mathcal{A}$ and $M_t = \mu(t, W_t)$ for $t \in [0, T]$, the agent's *ex ante* beliefs concerning the common noise factors are the same, irrespective of his control.

3 Solution of the Optimization Problem

In the following, we solve the agent's maximization problem (P_μ) using the associated dynamic programming equation (DPE). This is the same methodology as in [31] and [15]; see [22] for an alternative approach (to extended mean field games, but without common noise) based on backward SDEs.

The DPE for the value function of the agent's optimization problem (P_μ) reads

$$
0 = \sup_{u \in \mathbb{U}} \left\{ \frac{\partial v^i}{\partial t}(t, w) + \psi^i\big(t, w, \mu(t, w), u\big) + Q^{i\cdot}\big(t, w, \mu(t, w), u\big) \cdot v(t, w) \right\}
$$

for $i \in \mathbb{S}$, subject to suitable consistency conditions for $t = T_k$, $k \in [1:n]$, and the terminal condition

$$
v(T, w) = \Psi\big(w, \mu(T, w)\big) \quad \text{for all } w \in \mathbb{W}^n.
$$

Assumption 3 There exists a Borel measurable function $h : [0, T] \times \mathbb{W}^n \times \mathbb{M} \times \mathbb{R}^d \to \mathbb{U}^d$ such that for every $i \in \mathbb{S}$ and all $(t, w, m, v) \in [0, T] \times \mathbb{W}^n \times \mathbb{M} \times \mathbb{R}^d$ we have

$$
h^i(t, w, m, v) \in \arg\max_{u \in \mathbb{U}} \big\{ \psi^i(t, w, m, u) + Q^{i\cdot}(t, w, m, u) \cdot v \big\}.
$$

Assumption 3 is satisfied e.g. if $\mathbb{U}$ is compact and Q and ψ are continuous with respect to $u \in \mathbb{U}$. Note that, since $\psi^i(\cdot,\cdot,m,u)$ and $Q^{i\cdot}(\cdot,\cdot,m,u)$ are non-anticipative for $m \in \mathbb{M}$, $u \in \mathbb{U}$, we can assume without loss of generality that $h(\,\cdot\,,\,\cdot\,,m,v)$ is non-anticipative for $m \in \mathbb{M}$, $v \in \mathbb{R}^d$. With this, we define

$$\begin{aligned}
&\widehat{Q}:\ [0,T]\times\mathbb{W}^n\times\mathbb{M}\times\mathbb{R}^d \to \mathbb{R}^{d\times d}, && \widehat{Q}^{ij}(t,w,m,v) \triangleq Q^{ij}\big(t,w,m,h^i(t,w,m,v)\big),\\
&\widehat{\psi}:\ [0,T]\times\mathbb{W}^n\times\mathbb{M}\times\mathbb{R}^d \to \mathbb{R}^{d}, && \widehat{\psi}^{i}(t,w,m,v) \triangleq \psi^{i}\big(t,w,m,h^i(t,w,m,v)\big)
\end{aligned}$$

and thus obtain the following reduced-form DPE, which we use in the following:

Definition 4 Let $\mu:\ [0,T]\times\mathbb{W}^n \to \mathbb{M}$ be regular and non-anticipative. A function $v:[0,T]\times\mathbb{W}^n \to \mathbb{R}^d$ is called a *solution* of (DP$_\mu$) subject to (CC$_\mu$), (TC$_\mu$) if v is non-anticipative and satisfies the ordinary differential equation (ODE)[6]

$$\dot{v}(t,w) = -\widehat{\psi}\big(t,w,\mu(t,w),v(t,w)\big) - \widehat{Q}\big(t,w,\mu(t,w),v(t,w)\big)\cdot v(t,w) \quad (\mathrm{DP}_\mu)$$

for $t \in [T_k, T_{k+1}\rangle$, $k \in [0:n]$, subject to the consistency and terminal conditions

$$v(T_k-,w) = \Psi_k\big(w,\mu(T_k-,w),v(T_k,\,\cdot\,)\big), \quad (\mathrm{CC}_\mu)$$

$$v(T,w) = \Psi\big(w,\mu(T,w)\big) \quad (\mathrm{TC}_\mu)$$

for $k \in [1:n]$ and all $w \in \mathbb{W}^n$. Here, for $k \in [1:n]$, the jump operator Ψ_k is defined via

$$\Psi_k^i(w,m,\bar{v}) \triangleq \sum_{\bar{w}_k\in\mathbb{W}} \kappa_k\big(\bar{w}_k|w_1,\ldots,w_{k-1},m\big)\cdot \bar{v}^{J^i(T_k,(w_{-k},\bar{w}_k),m)}(w_{-k},\bar{w}_k),\ i\in\mathbb{S}, \tag{10}$$

where $\bar{v}:\mathbb{W}^n \to \mathbb{R}^d$ and $(w_{-k},\bar{w}_k) \triangleq (w_1,\ldots,w_{k-1},\bar{w}_k,w_{k+1},\ldots,w_n)$ for $\bar{w}_k \in \mathbb{W}$, $w \in \mathbb{W}^n$. □

Observe that (DP$_\mu$) represents a system of (random) ODEs, coupled via $w \in \mathbb{W}^n$. The ODEs run backward in time on each segment $[T_k, T_{k+1}\rangle \times \mathbb{W}^n$, $k \in [0:n]$, and their terminal conditions for $t \uparrow T_{k+1}$ are specified by (TC$_\mu$) for $k=n$ and by (CC$_\mu$) for $k<n$. Note that for $t \in [T_k, T_{k+1}\rangle$ the relevant common noise factors $W_1,\ldots,W_k$ are known.

Remark 5 While the significance of the DPE (DP$_\mu$) and the terminal condition (TC$_\mu$) are clear, the consistency conditions (CC$_\mu$) warrant a brief comment: For $i \in \mathbb{S}$, $k \in [1:n]$ and $w \in \mathbb{W}^n$ the state process jumps from state i to state $j \triangleq J^i(T_k,(w_{-k},W_k),\mu(T_k-,w_{T_k-}))$ on $\{X_{T_k-}=i\}\cap\{W_{T_k-}=w_{T_k-}\}$ when the common noise factor W_k is revealed at time T_k. □

[6] All ODEs in this article are taken in the sense of Carathéodory; see [36, §I.5]. Briefly, $x:\ I\to\mathbb{R}$ is a solution of $\dot{x}(t)=f(t,x(t))$, $x(t_0)=x_0$, in the sense of Carathéodory if $t\mapsto f(t,x(t))$ is Lebesgue integrable and satisfies $x(t)=x_0+\int_{t_0}^t f(s,x(s))\mathrm{d}s$ for all $t\in I$.

We next link the solution of the DPE to the underlying stochastic control problem.

Theorem 6 (Verification) *Suppose $\mu : [0,T] \times \mathbb{W}^n \to \mathbb{M}$ is regular and non-anticipative and v is a solution of (DP$_\mu$) subject to (CC$_\mu$) and (TC$_\mu$). Then v is the agent's value function for problem (P$_\mu$), i.e.*

$$\sum_{i\in\mathbb{S}} \mathbb{P}(X_0 = i) v^i(0) = \sup_{\nu\in\mathcal{A}} \mathbb{E}^\nu\Big[\int_0^T \psi^{X_t}(t, W_t, M_t, \nu_t)\mathrm{d}t + \Psi^{X_T}(W_T, M_T)\Big],$$

and an optimal control is given by $\widehat{\nu} \in \mathcal{A}$ with

$$\widehat{\nu}\big(t, X_{(\,\cdot\,\wedge t)-}, W_t\big) = h^{X_{t-}}\big(t, W_t, \mu(t, W_t), v(t, W_t)\big) \quad \textit{for } t \in [0,T].$$

Proof Let $\nu \in \mathcal{A}$ be an admissible strategy. Until further notice we fix $k \in [0:n]$.

Step 1: Dynamics on $[T_k, T_{k+1}\rangle$. From Itô's lemma, applicable due to regularity of v, we obtain

$$\begin{aligned} v^{X_{T_k}}\big(T_k, W_{T_k}\big) &= v^{X_{T_{k+1}-}}\big(T_{k+1}-, W_{T_{k+1}-}\big) \\ &\quad - \sum_{\substack{i,j\in\mathbb{S},\\ i\neq j}} \int_{T_k}^{T_{k+1}} \mathbb{1}_{\{X_{s-}=i\}} \Big(v^j(s, W_s) - v^i(s, W_s)\Big)\, \mathrm{d}^\nu \bar{N}_s^{ij} \\ &\quad - \int_{T_k}^{T_{k+1}} \sum_{i=1}^d \mathbb{1}_{\{X_s=i\}} \Big(\dot{v}^i(s, W_s) + Q^{i\cdot}(s, W_s, \mu(s, W_s), \nu_s) \cdot v(s, W_s)\Big)\, \mathrm{d}s. \end{aligned} \tag{11}$$

Step 2: Jump dynamics at T_k. We recall from Lemma 2 that

$$\begin{aligned} \mathbb{P}^\nu(W_k = \bar{w}_k | X_{T_k-}, W_1, \ldots, W_{k-1}) &= \mathbb{P}^\nu(W_k = \bar{w}_k | W_1, \ldots, W_{k-1}) \\ &= \kappa_k\big(\bar{w}_k | W_1, \ldots, W_{k-1}, \mu(T_k-, W_{T_k-})\big). \end{aligned}$$

In view of the jump dynamics (3) and the consistency condition (CC$_\mu$), we thus obtain

$$\begin{aligned} &\mathbb{E}^\nu\big[v^{X_{T_k}}\big(T_k, W_{T_k}\big)\big|\sigma\big(X_{T_k-}, W_{T_k-}\big)\big] \\ &= \mathbb{E}^\nu\big[v^{J^{X_{T_k-}}(T_k, (W_{T_k-}, W_k), \mu(T_k-, W_{T_k-}))}\big(T_k, (W_{T_k-}, W_k)\big)\big| X_{T_k-}, W_{T_k-}\big] \\ &= \sum_{\bar{w}_k\in\mathbb{W}} \kappa_k\big(\bar{w}_k | W_{T_k-}, \mu(T_k-, W_{T_k-})\big) v^{J^{X_{T_k-}}(T_k, (W_{T_k-}, \bar{w}_k), \mu(T_k-, W_{T_k-}))}\big(T_k, (W_{T_k-}, \bar{w}_k)\big) \\ &= \Psi_k^{X_{T_k-}}\big(W_{T_k-}, \mu(T_k-, W_{T_k-}), v(T_k, \,\cdot\,)\big) = v^{X_{T_k-}}(T_k-, W_{T_k-}). \end{aligned} \tag{12}$$

Step 3: Optimality. Combining (11) and (12) for $k = [1:n]$ and using (TC$_\mu$) yields

$$
\begin{aligned}
v^{X_0}(0) &= v^{X_T}(T, W_T) + \sum_{k=1}^{n}\Big(v^{X_{T_k-}}\big(T_k-, W_{T_k-}\big) - v^{X_{T_k}}\big(T_k, W_{T_k}\big)\Big) \\
&\quad - \sum_{k=0}^{n}\Bigg[\sum_{\substack{i,j\in\mathbb{S},\\ i\neq j}} \int_{T_k}^{T_{k+1}} \mathbb{1}_{\{X_{s-}=i\}}\left(v^j(s, W_s) - v^i(s, W_s)\right) \mathrm{d}^{\nu}\bar{N}_s^{ij} \\
&\quad + \int_{T_k}^{T_{k+1}} \sum_{i=1}^{d} \mathbb{1}_{\{X_s=i\}}\left(\dot{v}^i(s, W_s) + Q^{i\cdot}(s, W_s, \mu(s, W_s), \nu_s)\cdot v(s, W_s)\right)\mathrm{d}s\Bigg] \\
&= \Psi^{X_T}(W_T, \mu(T, W_T)) + \sum_{k=1}^{n}\Big(\mathbb{E}^{\nu}\big[v^{X_{T_k}}\big(T_k, W_{T_k}\big)\big|\sigma\big(X_{T_k-}, W_{T_k-}\big)\big] - v^{X_{T_k}}\big(T_k, W_{T_k}\big)\Big) \\
&\quad - \sum_{\substack{i,j\in\mathbb{S},\\ i\neq j}} M_T^{ij} - \int_0^T \sum_{i=1}^{d} \mathbb{1}_{\{X_s=i\}}\left(\dot{v}^i(s, W_s) + Q^{i\cdot}(s, W_s, \mu(s, W_s), \nu_s)\cdot v(s, W_s)\right)\mathrm{d}s,
\end{aligned}
\tag{13}
$$

where for $i, j \in \mathbb{S}, i \neq j$ the local $(\mathfrak{F}, \mathbb{P}^{\nu})$-martingale M^{ij} is given by

$$
M_t^{ij} \triangleq \int_0^t \mathbb{1}_{\{X_{s-}=i\}}\left(v^j(s, W_s) - v^i(s, W_s)\right)\mathrm{d}^{\nu}\bar{N}_s^{ij} \quad \text{for } t \in [0, T].
$$

Since ${}^{\nu}\bar{N}^{ij}$ is a compensated counting process and v and Q are bounded, M^{ij} is in fact an $(\mathfrak{F}, \mathbb{P}^{\nu})$-martingale. Hence taking $\mathbb{P}^{\nu}$-expectations in (13), using the tower property of conditional expectation and the fact that $\mathbb{P}^{\nu}$ and $\mathbb{P}$ coincide on $\sigma(X_0)$ by Lemma 2, and finally that v solves the DPE, we obtain

$$
\begin{aligned}
\sum_{i\in\mathbb{S}} \mathbb{P}(X_0 = i)v^i(0) = \mathbb{E}\big[v^{X_0}(0)\big] &= \mathbb{E}^{\nu}\big[v^{X_0}(0)\big] \\
&= \mathbb{E}^{\nu}\Bigg[\Psi^{X_T}(W_T, \mu(T, W_T)) \\
&\quad - \int_0^T \sum_{i=1}^{d} \mathbb{1}_{\{X_s=i\}}\left(\dot{v}^i(s, W_s) + Q^{i\cdot}(s, W_s, \mu(s, W_s), \nu_s)\cdot v(s, W_s)\right)\mathrm{d}s\Bigg] \\
&\geq \mathbb{E}^{\nu}\left[\Psi^{X_T}(W_T, M_T) + \int_0^T \psi^{X_s}(s, W_s, M_s, \nu_s)\,\mathrm{d}s\right].
\end{aligned}
\tag{14}
$$

If we replace ν with $\widehat{\nu}$, the same argument applies with equality in (14); we thus conclude that v is the value function of (P_{μ}), and that the strategy $\widehat{\nu}$ is optimal. □

The optimal strategy is Markovian in the agent's state; this is unsurprising given the literature, see e.g. [31, Theorem 1] or [22, Proposition 3.9] and [15, Theorem 4]. Note, however, that the time-t optimal strategy may depend on *all* common noise events that have occurred up to time t, as $W_t = (W_1, \ldots, W_k)$ for $t \in [T_k, T_{k+1}\rangle$. In the following, we denote by $\widehat{\mathbb{P}}$ the probability measure

$$
\widehat{\mathbb{P}} \triangleq \mathbb{P}^{\widehat{\nu}}
$$

where $\widehat{\nu}$ is the optimal control specified in Theorem 6. It follows from Lemma 2 that N^{ij} has $\widehat{\mathbb{P}}$-intensity $\widehat{\lambda}^{ij} = \{\widehat{\lambda}_t^{ij}\}$ for $i, j \in \mathbb{S}, i \neq j$, where

$$\widehat{\lambda}_t^{ij} \triangleq Q^{ij}\big(t, W_t, \mu(t, W_t), h^{X_{t-}}(t, W_t, \mu(t, W_t), v(t, W_t))\big) \quad \text{for } t \in [0, T]. \tag{15}$$

4 Equilibrium

Having solved the agent's optimization problem for a given *ex ante* function μ, we now turn to the resulting mean field equilibrium. We first identify the aggregate distribution resulting from the optimal control.

Remark 7 This paper generally adopts a "representative agent" point of view; an alternative justification of mean field equilibrium is via convergence of Nash equilibria of symmetric N-player games in the limit $N \to \infty$; see, among others, [2,14,15,18,20,22,24,28]. In the setting of this article (albeit under additional regularity conditions) a mean field limit justification can be provided along the lines of the proof of Theorem 7 in [31] by conditioning on common noise configurations, similarly as in the proof of Theorem 9 below. □

4.1 Aggregation

Given an *ex ante* aggregate distribution specified in terms of a regular, non-anticipative function μ and a corresponding solution v of (DP_μ) subject to (CC_μ), (TC_μ), Theorem 6 yields an optimal strategy $\widehat{\nu}$ for the agent's optimization problem (P_μ). With $\widehat{\mathbb{P}}$ denoting the probability measure associated with $\widehat{\nu}$, the resulting *ex post* aggregate distribution is given by the $\mathbb{M}$-valued, $\mathfrak{G}$-adapted process $\widehat{M} = \{\widehat{M}_t\}$,

$$\widehat{M}_t \triangleq \widehat{\mathbb{P}}(X_t \in \cdot \mid \mathfrak{G}_t) \quad \text{for } t \in [0, T]$$

where $\mathfrak{G}$ denotes the common noise filtration. We note that $\widehat{M}$ is càdlàg since $\mathfrak{G}$ is piecewise constant and X is càdlàg. Equilibrium obtains if $\widehat{M}_t = \mu(t, W_t)$ for all $t \in [0, T]$. To proceed, we aim for a more explicit description of $\widehat{M}$ and, in particular, its dynamics. Thus we define for $k \in [1 : n]$

$$\Phi_k : \mathbb{W}^n \times \mathbb{M} \times \mathbb{M} \to \mathbb{M}, \quad \Phi_k(w, m, \bar{m}) \triangleq m \cdot P_k(w, \bar{m}), \tag{16}$$

where $P_k : \mathbb{W}^n \times \mathbb{M} \to \{0, 1\}^{d \times d}$ is given by

$$P_k^{ij}(w, \bar{m}) \triangleq \mathbb{1}_{\{J^i(T_k, w_1, \ldots, w_k, \bar{m}) = j\}} \quad \text{for } i, j \in \mathbb{S}$$

and we set

$$m_0 \triangleq \mathbb{P}(X_0 \in \cdot\,) = \widehat{\mathbb{P}}(X_0 \in \cdot\,) \in \mathbb{M}.$$

Lemma 8 *Let $\mu : [0, T] \times \mathbb{W}^n \to \mathbb{M}$ and $v : [0, T] \times \mathbb{W}^n \to \mathbb{R}^d$ be regular and non-anticipative, and suppose that $Y = \{Y_t\}$ is an $\mathbb{M}$-valued stochastic process with dynamics*

$$Y_0 = m_0, \quad Y_t = Y_{T_k} + \int_{T_k}^t Y_s \cdot \widehat{Q}\big(s, W_s, \mu(s, W_s), v(s, W_s)\big)\mathrm{d}s$$
$$\textit{for } t \in [T_k, T_{k+1}\rangle,\ k \in [0:n] \tag{17}$$

that satisfies the consistency conditions

$$Y_{T_k} = \Phi_k\big(W_{T_k}, Y_{T_k-}, \mu(T_k-, W_{T_k-})\big) \ \ \textit{for } k \in [1:n].$$

Then Y is $\mathfrak{G}$-adapted.

Proof *Step 1: Existence and uniqueness of Carathéodory solutions.* For each $k \in [0:n]$ and $w \in \mathbb{W}^n$, since μ and v are regular and Q is bounded, the function

$$f : [T_k, T_{k+1}] \times \mathbb{R}^{1\times d} \to \mathbb{R}^{1\times d}, \quad f(t, y) \triangleq y \cdot \widehat{Q}\big(t, w, \mu(t, w), v(t, w)\big)$$

is measurable in the first and Lipschitz continuous in the second argument. Thus, using that μ, v and $\widehat{Q}$ are non-anticipative, a classical result, see [36, Theorem I.5.3], implies that for each initial condition $y \in \mathbb{R}^{1\times d}$ there exists a unique Carathéodory solution $\varphi_k^{y,w_{T_k}} : [T_k, T_{k+1}\rangle \to \mathbb{R}^{1\times d}$ of

$$\dot{y}(t) = y(t) \cdot \widehat{Q}\big(t, w_{T_k}, \mu(t, w_{T_k}), v(t, w_{T_k})\big) \text{ for } t \in [T_k, T_{k+1}\rangle, \qquad y(T_k) = y.$$

Step 2: Y is $\mathfrak{G}$-adapted. First note that $Y_0 = m_0$ is clearly $\mathfrak{G}_0$-measurable. Next, suppose that Y_{T_k} is $\mathfrak{G}_{T_k}$-measurable, and note that for $t \in [T_k, T_{k+1}\rangle$ we have $W_t = W_{T_k}$, so

$$Y_t = Y_{T_k} + \int_{T_k}^t Y_s \cdot \widehat{Q}\big(s, W_{T_k}, \mu(s, W_{T_k}), v(s, W_{T_k})\big)\mathrm{d}s.$$

Thus from uniqueness in part (a) it follows that we have the representation

$$Y_t = \varphi_k^{Y_{T_k}, W_{T_k}}(t) \quad \text{for } t \in [T_k, T_{k+1}\rangle.$$

Hence Y_t is $\mathfrak{G}_{T_k}$-measurable for all $t \in [T_k, T_{k+1}\rangle$. Finally, for all $k \in [0:(n-1)]$ the consistency condition implies that $Y_{T_{k+1}} = \Phi_{k+1}(W_{T_{k+1}}, Y_{T_{k+1}-}, \mu(T_{k+1}-, W_{T_{k+1}-}))$ is $\mathfrak{G}_{T_{k+1}}$-measurable, so the claim follows by induction on $k \in [0:n]$. □

Theorem 9 (Aggregation) *Let $\mu : [0, T] \times \mathbb{W}^n \to \mathbb{M}$ be regular and non-anticipative with $\mu(0) = m_0$. Suppose v is a solution of (DP_μ) subject to (CC_μ), (TC_μ), and the agent implements his optimal strategy $\widehat{\nu}$ as defined in Theorem 6. Then the aggregate distribution $\widehat{M}$ has the $\widehat{\mathbb{P}}$-dynamics*

$$\mathrm{d}\widehat{M}_t = \widehat{M}_t \cdot \widehat{Q}\big(t, W_t, \mu(t, W_t), v(t, W_t)\big)\mathrm{d}t \ \ \textit{for } t \in [T_k, T_{k+1}\rangle,\ k \in [0:n], \tag{M}$$

and satisfies the initial condition

$$\widehat{M}_0 = m_0 \tag{M$_0$}$$

and the jump conditions

$$\widehat{M}_{T_k} = \Phi_k\big(W_{T_k}, \widehat{M}_{T_k-}, \mu(T_k-, W_{T_k-})\big) \quad \textit{for } k \in [1:n]. \tag{M$_k$}$$

Proof Let $w \in \mathbb{W}^n$ be a common noise configuration. Since X is defined path by path, see (2) and (3), we first note that $X = X^w$ on $\{W_T = w\}$, where X^w satisfies (2) and

$$X^w_{T_k} = J^{X^w_{T_k-}}\big(T_k, w_{T_k}, \mu(T_k-, w_{T_k-})\big) \text{ for } k \in [1:n]. \tag{18}$$

We define $\zeta(w) = \{\zeta(w)_t\}$ via

$$\begin{aligned}\zeta(w)_t \triangleq \prod_{\substack{i,j\in\mathbb{S},\\ i\neq j}} \Bigg(&\exp\Big\{\int_0^t \Big(1 - Q^{ij}\big(s, w_s, \mu(s, w_s), h^{X^w_{s-}}(s, w_s, \mu(s, w_s), v(s, w_s))\big)\Big)\mathrm{d}s\Big\}\\ &\times \prod_{\substack{s\in(0,t],\\ \Delta N^{ij}_s\neq 0}} Q^{ij}\big(s, w_s, \mu(s, w_s), h^{X^w_{s-}}(s, w_s, \mu(s, w_s), v(s, w_s))\big)\Bigg).\end{aligned}$$

Using analogous arguments as in Step 1 of the proof of Lemma 2 (see in particular (4) and (7)), it follows that there exists a probability measure $\widehat{\mathbb{P}}^w$ with density process

$$\left.\frac{\mathrm{d}\widehat{\mathbb{P}}^w}{\mathrm{d}\mathbb{P}}\right|_{\mathfrak{H}_t} \triangleq \zeta(w)_t \quad \text{for } t \in [0, T],$$

where the filtration $\mathfrak{H} = \{\mathfrak{H}_t\}$ is given by

$$\mathfrak{H}_t \triangleq \sigma\big(X_0,\ N^{ij}_s\ :\ s \in [0, t];\ i, j \in \mathbb{S},\ i \neq j\big) \vee \mathfrak{N} \quad \text{for } t \in [0, T].$$

Furthermore, in view of (4) and (15) we have

$$\zeta(w) = \mathcal{E}[\theta^{\widehat{v}}] \quad \text{on } \{W_T = w\}. \tag{19}$$

Step 1: Conditional Kolmogorov dynamics. Throughout Step 1, we fix a common noise configuration $w \in \mathbb{W}^n$. It follows exactly as in the proof of Lemma 2 (with $\widehat{\mathbb{P}}^w$ in place of $\widehat{\mathbb{P}}$) that

$$\widehat{\mathbb{P}}^w \ll \mathbb{P}, \qquad \widehat{\mathbb{P}}^w = \mathbb{P} \quad \text{on } \sigma(X_0),$$

and that for $i, j \in \mathbb{S}$, $i \neq j$, the process N^{ij} is a counting process with $(\mathfrak{H}, \widehat{\mathbb{P}}^w)$-intensity

$$Q^{ij}\big(t, w_t, \mu(t, w_t), h^{X^w_{t-}}(t, w_t, \mu(t, w_t), v(t, w_t))\big) \quad \text{for } t \in [0, T].$$

Boundedness of Q implies that for each $z \in \mathbb{R}^d$ the process $L^w[z] = \{L^w_t[z]\}$,

$$L^w_t[z] \triangleq \sum_{\substack{i,j \in \mathbb{S}, \\ i \neq j}} \int_0^t \mathbb{1}_{\{X^w_{s-}=i\}} \cdot (z^j - z^i) \mathrm{d}\, {}^w\bar{N}^{ij}_s \quad \text{for } t \in [0, T],$$

is an $(\mathfrak{H}, \widehat{\mathbb{P}}^w)$-martingale, where ${}^w\bar{N}^{ij} = \{{}^w\bar{N}^{ij}_t\}$ is given by

$${}^w\bar{N}^{ij}_t \triangleq N^{ij}_t - \int_0^t Q^{ij}\big(s, w_s, \mu(s, w_s), h^{X^w_{s-}}(s, w_s, \mu(s, w_s), v(s, w_s))\big)\mathrm{d}s, \quad t \in [0, T].$$

Using Itô's lemma and the fact that $\widehat{\lambda}^{ij}_t = \widehat{Q}^{ij}(t, W_t, \mu(t, W_t), v(t, W_t))$ on $\{X_{t-} = i\}$, $t \in [0, T]$, by (15), we have for each $z \in \mathbb{R}^d$, $k \in [0 : n]$ and $t \in [T_k, T_{k+1}\rangle$

$$z^{X^w_t} = z^{X^w_{T_k}} + L^w_t[z] - L^w_{T_k}[z] + \sum_{i=1}^d \int_{T_k}^t \mathbb{1}_{\{X^w_s=i\}} \cdot \widehat{Q}^{i\cdot}(s, w_s, \mu(s, w_s), v(s, w_s)) \cdot z \,\mathrm{d}s.$$

Taking expectations with respect to $\widehat{\mathbb{P}}^w$ and using Fubini's theorem yields

$$\widehat{\mathbb{E}}^w\big[z^{X^w_t}\big] = \widehat{\mathbb{E}}^w\big[z^{X^w_{T_k}}\big] + \sum_{i=1}^d \int_{T_k}^t \widehat{\mathbb{P}}^w(X^w_s = i) \cdot \widehat{Q}^{i\cdot}(s, w_s, \mu(s, w_s), v(s, w_s)) \cdot z \,\mathrm{d}s,$$

so with $z = e_i$, $i \in \mathbb{S}$, we get

$$\widehat{\mathbb{P}}^w(X^w_t = i) = \widehat{\mathbb{P}}^w(X^w_{T_k} = i) + \sum_{j=1}^d \int_{T_k}^t \widehat{\mathbb{P}}^w(X^w_s = j) \cdot \widehat{Q}^{ji}(s, w_s, \mu(s, w_s), v(s, w_s))\mathrm{d}s. \tag{20}$$

It follows from (20) that $\eta(w) = \{\eta(w)_t\}$,

$$\eta(w)_t \triangleq \widehat{\mathbb{P}}^w(X^w_t \in \cdot\,), \quad t \in [0, T] \tag{21}$$

satisfies, for all $i \in \mathbb{S}$ and $k \in [0 : n]$,

$$\eta(w)^i_t = \eta(w)^i_{T_k} + \int_{T_k}^t \eta(w)_s \cdot \widehat{Q}^{\cdot i}\,(s, w_s, \mu(s, w_s), v(s, w_s))\,\mathrm{d}s \quad \text{for } t \in [T_k, T_{k+1}\rangle. \tag{22}$$

Moreover, since $\widehat{\mathbb{P}}^w = \mathbb{P}$ on $\sigma(X_0)$ and $X^w_0 = X_0$, $\eta(w)$ satisfies the initial condition

$$\eta(w)_0 = \widehat{\mathbb{P}}^w(X^w_0 \in \cdot\,) = \mathbb{P}(X^w_0 \in \cdot\,) = \mathbb{P}(X_0 \in \cdot\,) = m_0. \tag{23}$$

Finally, consider a common noise time $t = T_k$ and note that for all $i \in \mathbb{S}$ the jump condition (18) implies

$$\begin{aligned}\eta(w)^i_{T_k} &= \widehat{\mathbb{P}}^w\big(X^w_{T_k} = i\big) = \widehat{\mathbb{P}}^w\big(J^{X^w_{T_k-}}(T_k, w_{T_k}, \mu(T_k-, w_{T_k-})) = i\big)\\ &= \sum_{j=1}^d \widehat{\mathbb{P}}^w\big(J^j(T_k, w_{T_k}, \mu(T_k-, w_{T_k-})) = i \,\big|\, X^w_{T_k-} = j\big) \cdot \widehat{\mathbb{P}}^w(X^w_{T_k-} = j)\\ &= \sum_{j=1}^d \mathbb{1}_{\{J^j(T_k, w_{T_k}, \mu(T_k-, w_{T_k-}))=i\}} \cdot \widehat{\mathbb{P}}^w(X^w_{T_k-} = j)\\ &= \sum_{j=1}^d P^{ji}_k(w_{T_k}, \mu(T_k-, w_{T_k-})) \cdot \eta(w)^j_{T_k-} = \Phi^i_k\big(w_{T_k}, \eta(w)_{T_k-}, \mu(T_k-, w_{T_k-})\big).\end{aligned} \tag{24}$$

Since $\eta(W_T) = \sum_{w\in\mathbb{W}^n} \mathbb{1}_{\{W_T=w\}} \cdot \eta(w)$, in view of (22), (23) and (24) it follows from Lemma 8 that the process $\eta(W_T)$ is $\mathfrak{G}$-adapted.

Step 2: Identification of $\eta(W_T)$. Recall that $\mathfrak{G}_T = \sigma(W_T) \vee \mathfrak{N}$ and let $w \in \mathbb{W}^n$. For $t \in [0, T]$ and $i \in \mathbb{S}$ we have by (6) and (19)

$$\begin{aligned}&\widehat{\mathbb{E}}\big[\mathbb{1}_{\{W_T=w\}} \cdot \mathbb{1}_{\{X_t=i\}}\big] = \mathbb{E}\big[\mathbb{1}_{\{W_T=w\}} \cdot \mathbb{1}_{\{X^w_t=i\}} \cdot Z^{\widehat{\nu}}_T\big] = \mathbb{E}\big[\mathbb{1}_{\{W_T=w\}} \cdot \mathbb{1}_{\{X^w_t=i\}} \cdot \zeta(w)_T \cdot \mathcal{E}[\vartheta]_T\big]\\ &= \prod_{k=1}^n \big(|\mathbb{W}| \cdot \kappa_k(w_k|w_1, \ldots, w_{k-1}, \mu(T_k-, w_{T_k-}))\big) \cdot \mathbb{E}\big[\mathbb{1}_{\{W_T=w\}} \cdot \mathbb{1}_{\{X^w_t=i\}} \cdot \zeta(w)_T\big]\\ &= |\mathbb{W}|^n \cdot \widehat{\mathbb{P}}(W_T = w) \cdot \mathbb{P}(W_T = w) \cdot \widehat{\mathbb{P}}^w(X^w_t = i) = \widehat{\mathbb{E}}\big[\mathbb{1}_{\{W_T=w\}} \cdot \eta(W_T)^i_t\big],\end{aligned}$$

where in the final line the first identity is due to Lemma 2 and $\mathbb{P}$-independence of $(\zeta(w), X^w)$ and $\mathfrak{G}_T$; and the second is due to (21) and the fact that $\mathbb{P}(W_T = w) = 1/|\mathbb{W}|^n$. Thus

$$\widehat{\mathbb{P}}(X_t \in \cdot\, |\mathfrak{G}_T) = \eta(W_T)_t \quad \widehat{\mathbb{P}}\text{-a.s. for } t \in [0, T].$$

Step 3: Dynamics of $\widehat{M}$. By Step 2 and the tower property of conditional expectation, we find that for each $i \in \mathbb{S}$ and $t \in [0, T]$

$$\widehat{M}^i_t = \widehat{\mathbb{P}}(X_t = i|\mathfrak{G}_t) = \widehat{\mathbb{E}}\big[\widehat{\mathbb{E}}[\mathbb{1}_{\{X_t=i\}}|\mathfrak{G}_T]|\mathfrak{G}_t\big] = \widehat{\mathbb{E}}\big[\eta(W_T)^i_t|\mathfrak{G}_t\big] = \eta(W_T)^i_t \quad \widehat{\mathbb{P}}\text{-a.s.},$$

where the final identity is due to the fact that $\eta(W_T)$ is $\mathfrak{G}$-adapted by Step 1 and $\widehat{\mathbb{E}}$ denotes $\widehat{\mathbb{P}}$-expectation. Since both $\widehat{M}$ and $\eta(W_T)$ are càdlàg, it follows that $\widehat{M} = \eta(W_T)$ $\widehat{\mathbb{P}}$-a.s., and (M), (M_0) and (M_k) follow from (22), (23) and (24). □

As a by-product, the preceding proof yields the alternative representation

$$\widehat{M}_t = \widehat{\mathbb{P}}(X_t \in \cdot\, | \mathfrak{G}_T) \quad \text{for } t \in [0, T],\ \widehat{\mathbb{P}}\text{-a.s.}$$

4.2 Mean Field Equilibrium System

As discussed above, equilibrium obtains if the agents' ex ante beliefs coincide with the ex post outcome. This holds if and only if the ex post aggregate distribution process $\widehat{M}$ from (M) satisfies

$$\widehat{\mathbb{P}}(X_t \in \cdot \,|\mathfrak{G}_t) = \widehat{M}_t \overset{!}{=} M_t = \mu(t, W_t) \quad \text{for all } t \in [0, T].$$

Definition 10 (Equilibrium System). A pair (μ, v) of regular and non-anticipative functions

$$\mu : \; [0, T] \times \mathbb{W}^n \to \mathbb{M} \qquad \text{and} \qquad v : \; [0, T] \times \mathbb{W}^n \to \mathbb{R}^d$$

is called a *rational expectations equilibrium*, or briefly an *equilibrium*, if for all $w \in \mathbb{W}^n$

$$\dot{\mu}(t, w) = \mu(t, w) \cdot \widehat{Q}\big(t, w, \mu(t, w), v(t, w)\big) \tag{E1}$$

$$\dot{v}(t, w) = -\widehat{\psi}\big(t, w, \mu(t, w), v(t, w)\big) - \widehat{Q}\big(t, w, \mu(t, w), v(t, w)\big) \cdot v(t, w) \tag{E2}$$

for $t \in [T_k, T_{k+1}\rangle$, $k \in [0 : n]$, subject to the consistency conditions[7]

$$\mu(T_k, w) = \Phi_k\big(w, \mu(T_k-, w)\big) \tag{E3}$$

$$v(T_k-, w) = \Psi_k\big(w, \mu(T_k-, w), v(T_k, \,\cdot\,)\big) \tag{E4}$$

for $k \in [1 : n]$, and the initial/terminal conditions

$$\mu(0, w) = m_0 \tag{E5}$$

$$v(T, w) = \Psi\big(w, \mu(T, w)\big). \tag{E6}$$

We also refer to (E1)-(E6) as the *equilibrium system*. □

In combination, Theorem 6 and Theorem 9 demonstrate that, given a solution (μ, v) of the equilibrium system, v is the value function of the agent's optimization problem (P_μ) with *ex ante* aggregate distribution μ; and the *ex post* distribution resulting from the corresponding optimal strategy is given by μ itself. Thus we can identify a mean field equilibrium with common noise by producing a solution of the equilibrium system (E1)-(E6). We provide some illustrations in Sect. 5. Theorems 13 and 16 below ensure that this is feasible by showing that, under suitable continuity and monotonicity conditions, there exists a unique solution of the equilibrium system. The proofs are

[7] With a slight abuse of notation, here and subsequently we set $\Phi_k(w, m) \triangleq \Phi_k(w, m, m)$ for $k \in [1 : n]$, $w \in \mathbb{W}^n$, $m \in \mathbb{M}$. Recall that Ψ_k and Φ_k are defined in (10) and (16), respectively.

ramifications of classical arguments, based on Schauder's fixed point theorem and monotonicity arguments, respectively.

We set

$$Q_{\max} \triangleq \sup_{\substack{t\in[0,T],\, w\in\mathbb{W}^n\\ m\in\mathbb{M},\, u\in\mathbb{U}}} \left\| Q(t,w,m,u) \right\|, \quad \psi_{\max} \triangleq \sup_{\substack{t\in[0,T],\, w\in\mathbb{W}^n\\ m\in\mathbb{M},\, u\in\mathbb{U}}} \left\| \psi(t,w,m,u) \right\|,$$

$$\Psi_{\max} \triangleq \sup_{\substack{m\in\mathbb{M}\\ w\in\mathbb{W}^n}} \|\Psi(w,m)\|$$

and

$$v_{\max} \triangleq \left(\Psi_{\max} + T\cdot\psi_{\max}\right)\cdot \mathrm{e}^{Q_{\max}\cdot T}. \tag{25}$$

Note that these constants depend only on the underlying model coefficients.

Assumption 11 (i) The reduced-form running reward function $\widehat{\psi}$ satisfies

$$\|\widehat{\psi}(t,w,m_1,v_1) - \widehat{\psi}(t,w,m_2,v_2)\| \le L_{\widehat{\psi}}\cdot\left(\|m_1-m_2\| + \|v_1-v_2\|\right)$$

for all $t\in[0,T]$, $w\in\mathbb{W}^n$, $m_1,m_2\in\mathbb{M}$ and $v_1,v_2\in\mathbb{R}^d$ with $\|v_1\|,\|v_2\|\le v_{\max}$, for some $L_{\widehat{\psi}}>0$.

(ii) The reduced-form intensity matrix function $\widehat{Q}$ satisfies

$$\left\|\widehat{Q}(t,w,m_1,v_1) - \widehat{Q}(t,w,m_2,v_2)\right\| \le L_{\widehat{Q}}\cdot\left(\|m_1-m_2\| + \|v_1-v_2\|\right)$$

for all $t\in[0,T]$, $w\in\mathbb{W}^n$, $m_1,m_2\in\mathbb{M}$ and $v_1,v_2\in\mathbb{R}^d$ with $\|v_1\|,\|v_2\|\le v_{\max}$, for some $L_{\widehat{Q}}>0$.

(iii) The terminal reward function Ψ is continuous with respect to m, i.e. for every $w\in\mathbb{W}^n$ the map $\Psi(w,\,\cdot\,)$ is continuous.

(iv) For each $k\in[1:n]$ and all $i\in\mathbb{S}$, $w\in\mathbb{W}^n$ and $v\in\mathbb{R}^d$ with $\|v\|\le v_{\max}$, the map

$$\mathbb{M}\ni m \mapsto \sum_{\bar{w}_k\in\mathbb{W}} \kappa_k\left(\bar{w}_k|w_1,\ldots,w_{k-1},m\right) v^{J^i(T_k,(w_{-k},\bar{w}_k),m)} \in \mathbb{R} \quad \text{is continuous.}$$

(v) For each $k\in[1:n]$ and $w\in\mathbb{W}^n$ the map $\Phi_k(w,\,\cdot\,)$ is continuous. □

Since all norms on $\mathbb{R}^d$ are equivalent, the concrete specification is immaterial for Assumption 11. For the sake of convenience, in the following we use the maximum norm on $\mathbb{R}^d$ and a compatible matrix norm on $\mathbb{R}^{d\times d}$; moreover, we suppose that (ii) holds for both $\widehat{Q}$ and $\widehat{Q}^{\mathsf{T}}$.

Remark 12 Sufficient conditions for Assumptions 11(i)-(ii) in terms of the model's primitives can be found in, e.g., [31] or [15]. Furthermore, in the special case where the jump map J is independent of $m\in\mathbb{M}$, Assumption 11(v) is trivially satisfied, and continuity of the transition kernels κ_k with respect to m is sufficient for Assumption 11(iv) to hold. □

Theorem 13 (Existence of Equilibria) *If Assumption 11 holds, then there exists a solution of the equilibrium system* (E1)– (E6).

Proof See Appendix A. □

The *reduced-form Hamiltonian* $\widehat{\mathcal{H}}: [0,T] \times \mathbb{W}^n \times \mathbb{M} \times \mathbb{R}^d \to \mathbb{R}^d$ is defined via

$$\begin{aligned} \widehat{\mathcal{H}}^i(t,w,m,v) &\triangleq \sup_{u \in \mathbb{U}} \psi^i(t,w,m,u) + Q^{i\cdot}(t,w,m,u) \cdot v \\ &= \widehat{\psi}^i(t,w,m,v) + \widehat{Q}^{i\cdot}(t,w,m,v) \cdot v. \end{aligned}$$

Assumption 14 Let Assumptions 11(i) and (ii) hold, and suppose that:

(i) The terminal payoff function Ψ is monotone with respect to $m \in \mathbb{M}$, i.e.

$$(m_1 - m_2) \cdot \left[\Psi(w,m_1) - \Psi(w,m_2)\right] \le 0 \quad \text{for all } w \in \mathbb{W}^n,\ m_1, m_2 \in \mathbb{M}.$$

(ii) The reduced-form Hamiltonian $\widehat{\mathcal{H}}$ is convex with respect to v, i.e. for all $i \in \mathbb{S}$, $t \in [0,T]$, $w \in \mathbb{W}^n$, $m \in \mathbb{M}$ and $v_1, v_2 \in \mathbb{R}^d$ satisfying $\|v_1\|, \|v_2\| \le v_{\max}$ we have

$$\widehat{\mathcal{H}}^i(t,w,m,v_2) - \widehat{\mathcal{H}}^i(t,w,m,v_1) - \widehat{Q}^{i\cdot}(t,w,m,v_1) \cdot (v_2 - v_1) \ge 0.$$

(iii) The reduced-form Hamiltonian $\widehat{\mathcal{H}}$ satisfies a uniform monotonicity condition with respect to $m \in \mathbb{M}$, i.e. there exist $\alpha, \gamma > 0$ such that

$$\begin{aligned} m_1 \cdot \left[\widehat{\mathcal{H}}(t,w,m_2,v_2) - \widehat{\mathcal{H}}(t,w,m_1,v_2)\right] + m_2 \cdot \big[\widehat{\mathcal{H}}(t,w,m_1,v_1) \\ -\widehat{\mathcal{H}}(t,w,m_2,v_1)\big] \ge \gamma \cdot \|m_1 - m_2\|^\alpha \end{aligned}$$

for all $t \in [0,T]$, $w \in \mathbb{W}^n$, $m_1, m_2 \in \mathbb{M}$ and $v_1, v_2 \in \mathbb{R}^d$ with $\|v_1\|, \|v_2\| \le v_{\max}$.

(iv) For $k \in [1:n]$ the maps κ_k and J satisfy the following monotonicity conditions in $m \in \mathbb{M}$: For all $w \in \mathbb{W}^n$, $m_1, m_2 \in \mathbb{M}$ and $v_1, v_2 \in \mathbb{R}^d$ satisfying $\|v_1\|, \|v_2\| \le v_{\max}$ as well as

$$\left[\Phi_k(w,m_1) - \Phi_k(w,m_2)\right] \cdot (v_1 - v_2) \le 0$$

we have

$$\begin{aligned} &\left[\kappa_k(w_k|w_1,\dots,w_{k-1},m_1) - \kappa_k(w_k|w_1,\dots,w_{k-1},m_2)\right] \\ &\quad \cdot \Big(m_2 \cdot v_1^{J^\cdot(T_k,w,m_2)} - m_1 \cdot v_1^{J^\cdot(T_k,w,m_1)} \\ &\quad + m_2 \cdot v_2^{J^\cdot(T_k,w,m_2)} - m_1 \cdot v_2^{J^\cdot(T_k,w,m_1)}\Big) \ge 0 \end{aligned} \tag{26}$$

and

$$
\begin{aligned}
&\kappa_k(w_k|w_1,\dots,w_{k-1},m_2)\cdot m_1\cdot\left(v_2^{J^{\cdot}(T_k,w,m_2)}-v_2^{J^{\cdot}(T_k,w,m_1)}\right)\\
&\quad+\kappa_k(w_k|w_1,\dots,w_{k-1},m_1)\cdot m_2\cdot\left(v_1^{J^{\cdot}(T_k,w,m_1)}-v_1^{J^{\cdot}(T_k,w,m_2)}\right)\geq 0. \quad (27)
\end{aligned}
$$

The constant $v_{\max} > 0$ in 14(ii)-(iv) is defined in (25). Conditions 14(i)-(iii) are standard given the literature; see, e.g., Assumptions 1-3 in [31].[8]

Remark 15 Assumption 14 simplifies if some model coefficients do not depend on the mean field parameter $m \in \mathbb{M}$:

(a) If $\widehat{Q}$ is independent of m, 14(iii) reduces to a monotonicity condition for $\widehat{\psi}$.
(b) In 14(iv), (26) is trivially satisfied if the probability weights κ_k do not depend on m.
(c) In 14(iv), (27) is trivially satisfied if the jump map J is independent of m. □

Theorem 16 (Uniqueness of Equilibria) *Under the monoticity conditions stated in Assumption 14, the equilibrium system (E1)– (E6) possesses at most one solution.*

Proof See Appendix B. □

5 Applications

Before we illustrate our results in two showcase examples, we briefly discuss our numerical approach to the equilibrium system (E1)-(E6). (E1)-(E2) is a forward-backward system of $2d$ ODEs with boundary conditions (E3)-(E6), coupled through the parameter $w \in \mathbb{W}^n$ representing common noise configurations. The special case $n = 0$ (no common noise) corresponds to the setting of [31] and [15], with the equilibrium system reducing to a single $2d$-dimensional forward-backward ODE. For $n \geq 1$, the consistency conditions (E3)-(E4) specify initial conditions for μ on $[T_k, T_{k+1}\rangle$ and terminal conditions for v on $[T_{k-1}, T_k\rangle$, $k \in [1 : n]$; since these conditions are interconnected, there is in general *no* segment $[T_k, T_{k+1}\rangle \times \mathbb{W}^n$ where the equilibrium system yields both an explicit initial condition for μ and an explicit terminal condition for v, so we cannot simply split the problem into subintervals. Rather, the equilibrium system can be regarded as a *multi-point boundary value problem* where for each of the $|\mathbb{W}|^k$ conceivable combinations of common noise factors on $[T_k, T_{k+1}\rangle$, $k \in [0 : n]$, we have to solve a coupled forward-backward system of ODEs in $2d$ dimensions, resulting in a tree of such systems of size

$$
\sum_{k=0}^{n}|\mathbb{W}|^k=\frac{|\mathbb{W}|^{n+1}-1}{|\mathbb{W}|-1}\in\mathcal{O}(|\mathbb{W}^n|).
$$

Our approach to solving (E1)-(E6) numerically is to rely on the probabilistic interpretation as a fixed-point system, based on Theorem 13. Thus, starting from an initial

[8] Note, however, that our result does not require uniform convexity in 14(ii).

flow of probability weights $\mu_0(t, w)$, $(t, w) \in [0, T] \times \mathbb{W}^n$ with $\mu_0(0, w) = m_0$ for all $w \in \mathbb{W}^n$, we solve (DP_μ) subject to (TC_μ) and (CC_μ) backward in time for all non-negligible common noise configurations $w \in \mathbb{W}^n$ to obtain the value $v_0(t, w)$, $(t, w) \in [0, T] \times \mathbb{W}^n$, of the agents' optimal response to the given belief μ_0. This, in turn, is used to solve (M) subject to (M_0) and (M_k) forward in time. As a result, we obtain an ex post aggregate distribution $\mu_1(t, w)$, $(t, w) \in [0, T] \times \mathbb{W}^n$; we then iterate this with μ_1 in place of μ_0, etc.[9]

5.1 A Decentralized Agricultural Production Model

As a first (stylized) example we consider a mean field game of agents, each of which owns (an infinitesimal amount of) land of identical size and quality within a given area. If it is farmed, each field has a productivity $f(w_k) > 0$ depending on the common weather condition w_k. We assume that weather is either good, bad or catastrophic, so $w_k \in \mathbb{W} \triangleq \{\uparrow, \downarrow, \lightning\}$, and changes at given common noise times $T_1, \ldots, T_n$.
Each agent is in exactly one state $i \in \mathbb{S} \triangleq \{0, 1\}$ depending on whether he grows crops on his field ($i = 1$, the agent is a farmer) or not ($i = 0$). The selling price p for his harvest depends on *aggregate* production, and thus in particular on the proportion $m^1 \in [0, 1]$ of farmers; the mean field interaction is transmitted through the market price of the crop. We assume that p is a strictly decreasing function of overall production $f(w_k) \cdot m^1$; see Fig. 1 for illustration.

We assume that $f(\uparrow) \geq f(\downarrow) = f(\lightning) \geq 0$. Moreover, on the catastrophic event $\{W_k = \lightning\}$ all agents are reduced to being non-farmers, and thus

$$J^i(t, w, m) \triangleq \begin{cases} 0 & \text{if } t \in \{T_1, \ldots, T_n\},\ t = T_k,\ w_k = \lightning, \\ i & \text{else} \end{cases}$$

for $(i, t, w, m) \in \mathbb{S} \times [0, T] \times \mathbb{W}^n \times \mathbb{M}$. Each agent can make an effort $u \in \mathbb{U} \triangleq [0, \infty)$ to become being a farmer; the intensity matrix for state transitions is given by

$$Q(t, w, m, u) = \begin{bmatrix} -u \cdot q_{\text{entry}} & u \cdot q_{\text{entry}} \\ q_{\text{exit}} & -q_{\text{exit}} \end{bmatrix} \quad \text{for } (t, w, m, u) \in [0, T] \times \mathbb{W}^n \times \mathbb{M} \times \mathbb{U},$$

where $q_{\text{entry}}, q_{\text{exit}} \geq 0$ are given transition rates. The running rewards capture the fact that both efforts to building up farming capacities and production itself are costly, while revenues from the sales of the crop generate profits; thus

$$\psi^0(t, w, m, u) = -\frac{1}{2} c_{\text{entry}} \cdot u^2 \quad \text{and} \quad \psi^1(t, w, m, u) = p\big(f(w_k) \cdot m^1\big) \cdot f(w_k) - c_{\text{prod}}$$

for $t \in [T_k, T_{k+1}\rangle$, $k \in [0 : n]$, where $w_0 \triangleq \uparrow$ and $c_{\text{entry}}, c_{\text{prod}} \geq 0$. The terminal reward is zero. It follows that the maximizer h^0 in Assumption 3 is unique and given by

[9] Under suitable Lipschitz conditions, one can establish a variant of Theorem 13 based on Banach's (rather than Schauder's) fixed point theorem; see, e.g., [15, Theorem 6] for the case without common noise. This guarantees convergence of the fixed point iteration for sufficiently short time horizons.

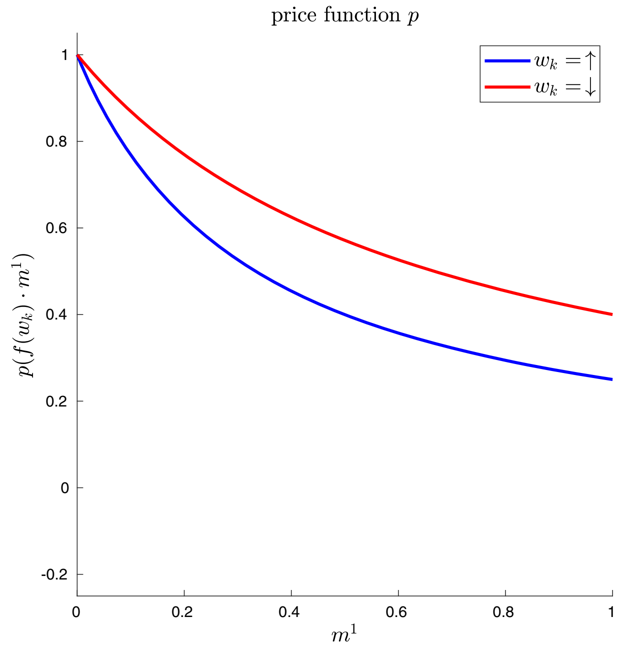

Fig. 1 Price function p (parameters as in Table 1)

Table 1 Coefficients in the agricultural production model

Parameter	T	n	T_k	q_{entry}	q_{exit}	$f(\uparrow)$	$f(\downarrow)$	$p(q)$	c_{prod}	c_{entry}
Value	1	4	$k/5$	1	0.1	1	0.5	$1/(1+3q)$	0.3	0.1

$$h^0(t, w, m, v) = \frac{q_{\text{entry}}}{c_{\text{entry}}}(v^1 - v^0)^+;$$

a specification of h^1 is immaterial. We choose $m_0^1 \triangleq 10\%$ for the initial proportion of farmers, and report the relevant coefficients in Table 1.

Our results for the evolution of the mean field equilibrium are shown in Figs. 2 and 3 for various common noise configurations $w \in \mathbb{W}^n$ and the following two baseline models:

(nC) Catastrophic weather conditions do not occur; we use

$$\kappa_k(\uparrow | w_1, \ldots, w_{k-1}, m) = \kappa_k(\downarrow | w_1, \ldots, w_{k-1}, m) = 0.5$$

for all $w \in \mathbb{W}^n$ and $m \in \mathbb{M}$.

(C) Catastrophic events are likely; we use

$$\kappa_k(\uparrow | w_1, \ldots, w_{k-1}, m) = 0.25, \quad \kappa_k(\downarrow | w_1, \ldots, w_{k-1}, m) = 0.25,$$
$$\kappa_k(\lightning | w_1, \ldots, w_{k-1}, m) = 0.5$$

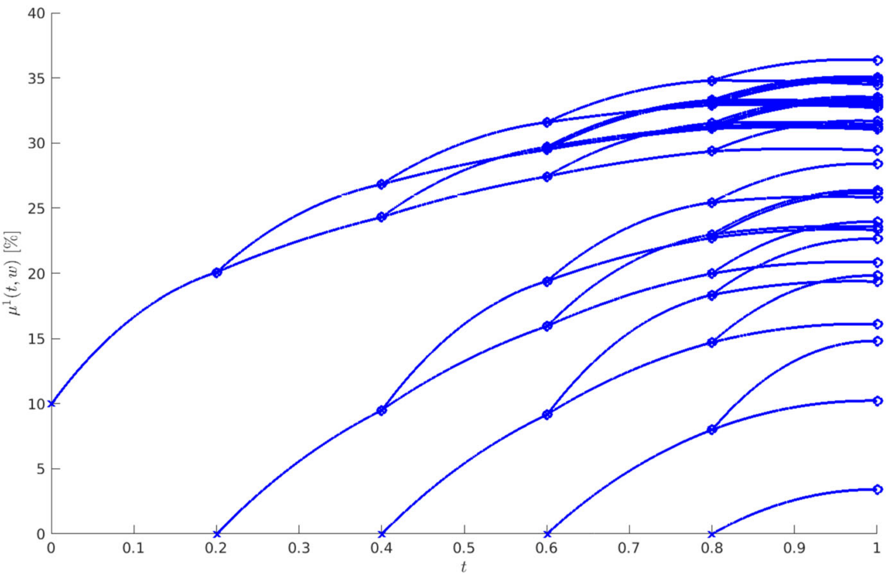

Fig. 2 Proportion of farmers in model (C) for all possible common noise configurations $w \in \mathbb{W}^n$

for all $w \in \mathbb{W}^n$ and $m \in \mathbb{M}$.

The model specified above satisfies both Assumption 11 and Assumption 14, so Theorems 13 and 16 guarantee the existence of a unique mean field equilibrium characterized by (E1)-(E6). Figure 2 illustrates the tree of all possible equilibrium evolutions in model (C). Figs. 3, 4 and 5 illustrate the resulting equilibrium proportions of farmers, optimal actions, and market prices for some fixed common noise configurations. To illustrate the effect of uncertainty about future weather conditions we also show, for each common noise configuration, the theoretical perfect-foresight equilibria that would pertain if future weather conditions were known; these are plotted using dashed lines in Figs. 3, 4 and 5, and the subscript $\circ$ indicates the relevant *deterministic* common noise path. Equilibrium prices are stochastically modulated by the prevailing weather conditions, both directly and indirectly: First, prices jump at common noise times due to weather-related changes in productivity. Second, weather conditions indirectly affect market prices through their effect on the proportion of farming agents. Thus, with consistently good weather conditions, agents are strongly incentivized to become farmers, see Fig. 4; the fraction of farmers increases, see Fig. 3; and hence increased production drives down prices, see Fig. 5. By contrast, under bad weather conditions, incentives are weaker and prices remain higher. Both effects are dampened if a catastrophic event may occur. In addition, efforts tend to decrease between common noise times; this is due to the uncertainty of future weather conditions; this effect is more pronounced in the presence of catastrophic events.

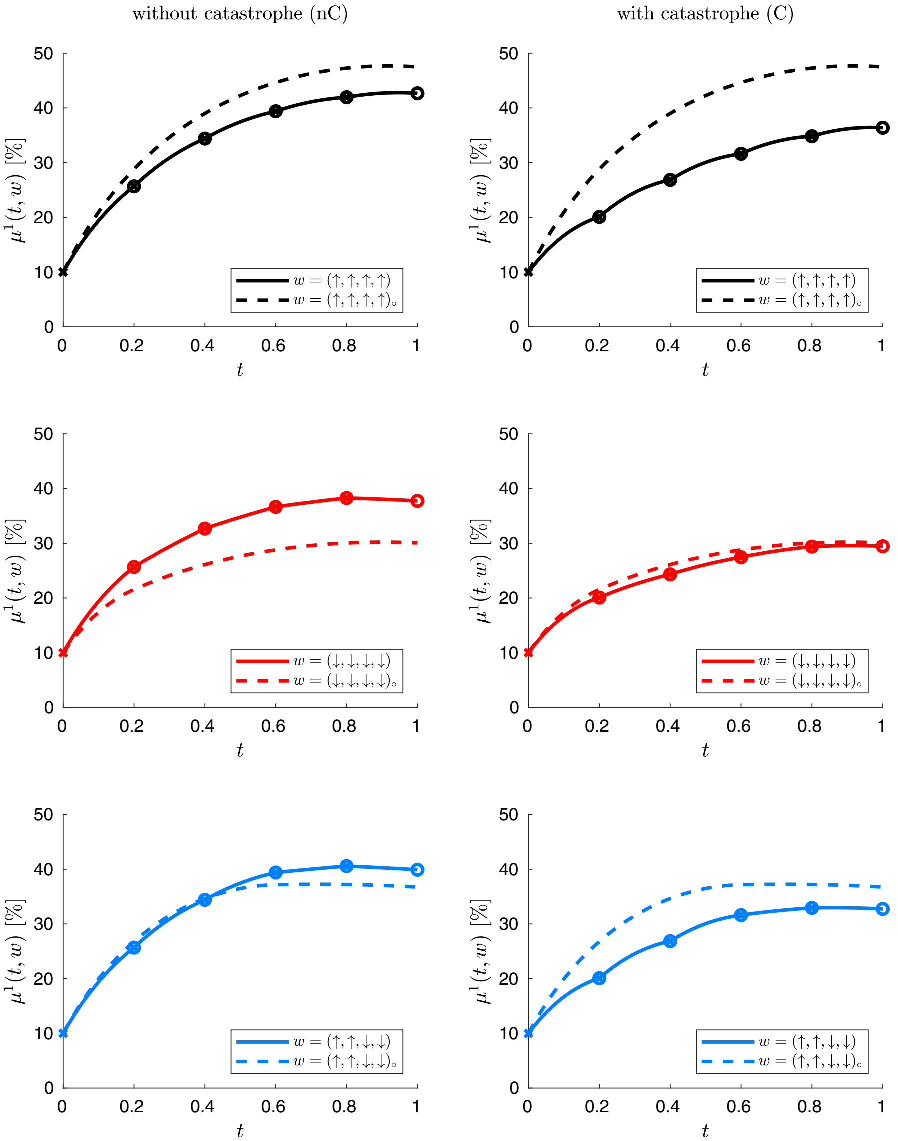

Fig. 3 Proportion of farmers in models (nC) and (C)

5.2 An SIR Model with Random One-Shot Vaccination

Our second application is a mean field game of agents that are confronted with the spread of an infectious disease. Our main focus is to illustrate the qualitative effects of common noise on the equilibrium behavior of the system. We consider a classical SIR model setup with $\mathbb{S} = \{S, I, R\}$: Each agent can be either *susceptible* to infection (S), *infected* and simultaneously infectious for other agents (I), or *recovered* and thus immune to (re-)infection (R); see Fig. 6.

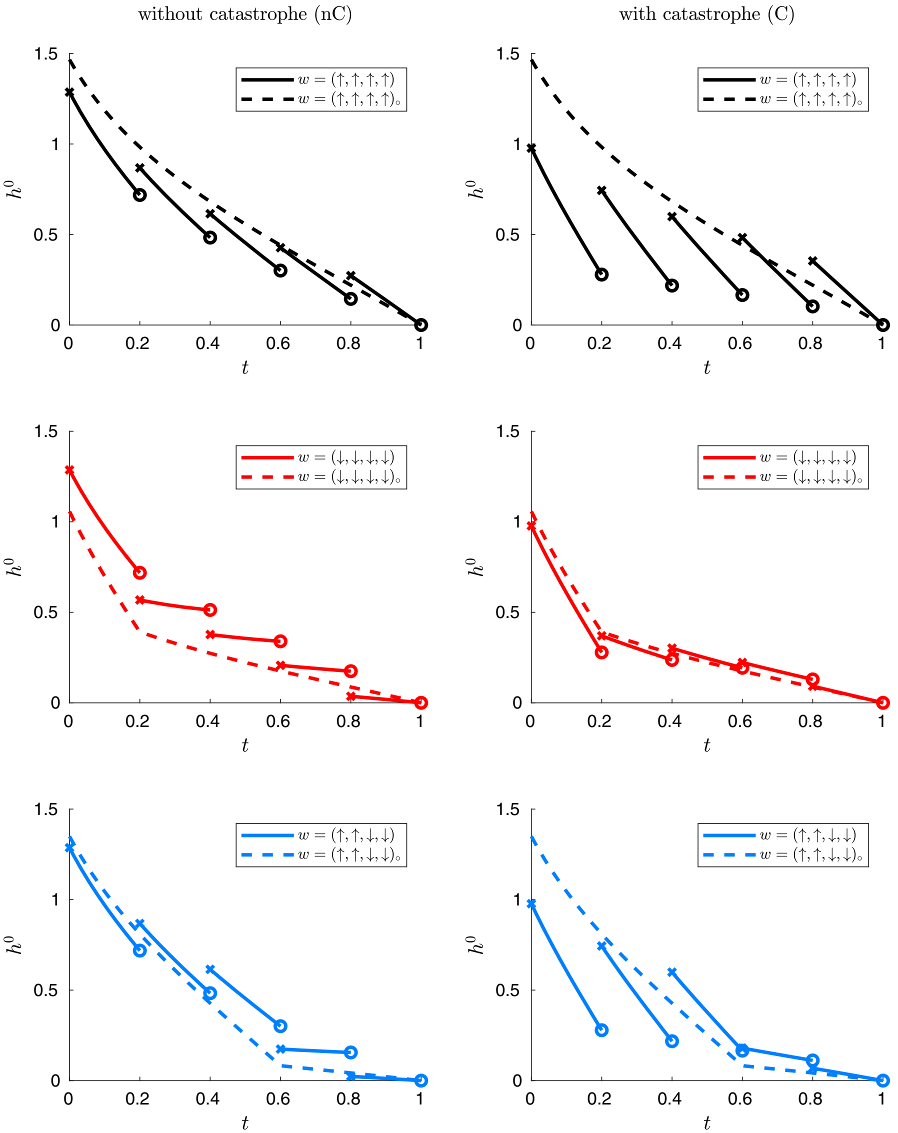

Fig. 4 Optimal action h^0 of non-farmers in models (nC) and (C)

The infection rate is proportional to the prevalence of the disease, i.e. the percentage of currently infected agents. Susceptible agents can make individual efforts of size $u \in \mathbb{U} \triangleq [0, 1]$ to protect themselves against infection and thus reduce intensity of infection. The transition intensities are given by

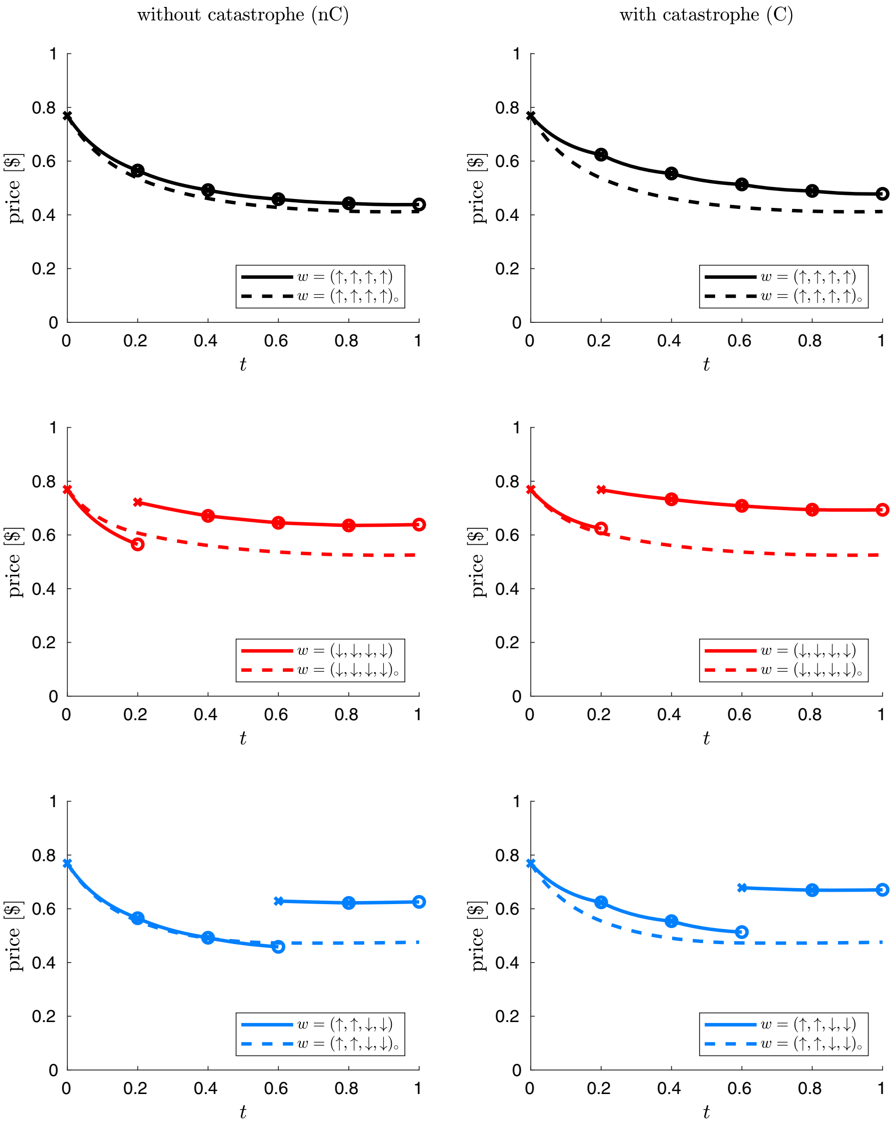

Fig. 5 Equilibrium market prices in models (nC) and (C)

$$Q(t, w, m, u) \triangleq \begin{bmatrix} -q_{\text{inf}}(t, w, m, u) & q_{\text{inf}}(t, w, m, u) & 0 \\ 0 & -q_{\text{IR}} & q_{\text{IR}} \\ 0 & 0 & 0 \end{bmatrix}$$

for $(t, w, m, u) \in [0, T] \times \mathbb{W}^n \times \mathbb{M} \times \mathbb{U}$, where $q_{\text{IR}} \geq 0$ denotes the recovery rate of infected agents and the infection rate is given by

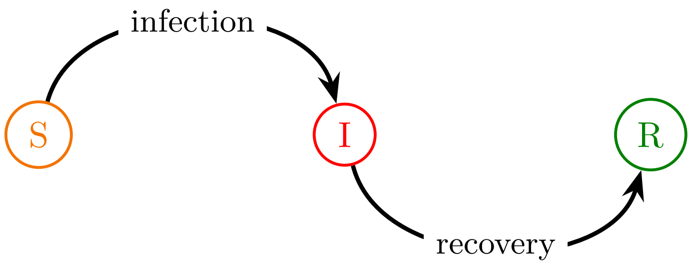

Fig. 6 State space and transitions in the SIR model

$$q_{\inf}(t, w, m, u) \triangleq q_{\mathrm{SI}} \cdot m^{\mathrm{I}} \cdot (1-u) \cdot \mathbb{1}_{\{t<\tau^\star\}}(w)$$

with a given maximum rate $q_{\mathrm{SI}} \geq 0$. The running reward penalizes both protection efforts and time spent in the infected state; with $c_{\mathrm{P}}, \psi_{\mathrm{I}} \geq 0$ we set

$$\psi^{\mathrm{S}}(t, w, m, u) \triangleq -c_{\mathrm{P}} \frac{u}{1-u}, \qquad \psi^{\mathrm{I}}(t, w, m, u) \triangleq -\psi_{\mathrm{I}}, \qquad \psi^{\mathrm{R}}(t, w, m, u) \triangleq 0.$$

In addition, we include the possibility of a one-shot vaccination that becomes available, simultaneously to all agents, at a random point of time $\tau^\star \in \{T_1, \ldots, T_n\} \subset (0, T)$. We set $\mathbb{W} \triangleq \{0, 1\}$ and identify the k^{th} unit vector $e_k = (\delta_{kj})_{j\in[1:n]} \in \mathbb{W}^n$, $k \in [1:n]$ with the indicator of the event $\{\tau^\star = T_k\}$. The event that no vaccine is available until T is represented by $0 \in \mathbb{W}^n$; we set $\tau^\star \triangleq +\infty$ in this case.[10] If and when it is available, all susceptible agents are vaccinated instantaneously, rendering them immune to infection; thus

$$J^{\mathrm{S}}(t, w, m) \triangleq \begin{cases} \mathrm{R} & \text{if } t \in \{T_1, \ldots, T_n\},\ t = T_k = \tau^\star, \\ \mathrm{S} & \text{otherwise} \end{cases} \quad \text{and} \quad J^i(t, w, m) \triangleq i \quad \text{for } i \in \{\mathrm{I}, \mathrm{R}\}.$$

The probability of vaccination becoming available is proportional to the percentage of agents that have already recovered from the disease. Thus for $k \in [1:n]$, $w_1, \ldots, w_k \in \mathbb{W}$ and $m \in \mathbb{M}$ we set

$$\kappa_k(1\,|w_1, \ldots, w_{k-1}, m) \triangleq \begin{cases} \alpha \cdot m^{\mathrm{R}} & \text{if } w_1, \ldots, w_{k-1} = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and $\kappa_k(0\,|w_1, \ldots, w_{k-1}, m) \triangleq 1 - \kappa_k(1\,|w_1, \ldots, w_{k-1}, m)$ where $\alpha \in (0, 1]$. As a consequence, for all $(i, t, w, m, v) \in \mathbb{S} \times [0, T] \times \mathbb{W} \times \mathbb{M} \times \mathbb{R}^3$, a maximizer as

[10] The specification of κ_k, $k \in [1:n]$, below implies that $\tau^\star = +\infty$ is equivalent to $w = 0 \in \mathbb{W}^n$ $\mathbb{P}$-a.s., i.e., the configurations $\mathbb{W}^n \setminus \big(\{0\} \cup \{e_k : k \in [1:n]\}\big)$ are $\mathbb{P}$-negligible.

Table 2 Coefficients in the SIR model

Parameter	T	n	T_k	α	q_{SI}	q_{IR}	c_{P}	ψ_{I}
Value	20	1999	$k \cdot 0.01$	0.1	7.5	0.5	25	100

required in Assumption 3 is given by[11]

$$h^{\mathrm{S}}(t,w,m,v) \triangleq \begin{cases} \left[1-\sqrt{\frac{c_{\mathrm{P}}}{q_{\mathrm{SI}}\cdot m^{\mathrm{I}}\cdot(v^{\mathrm{S}}-v^{\mathrm{I}})}}\right]^{+} & \text{if } v^{\mathrm{S}} > v^{\mathrm{I}},\ m^{\mathrm{I}} > 0 \text{ and } t < \tau^{\star}, \\ 0 & \text{otherwise,} \end{cases}$$

and $h^{i}(t,w,m,v) \triangleq 0$ for $i \in \{\mathrm{I},\mathrm{R}\}$.

Remark 17 (SIR Models in the Literature). Note that, given the above specification of the transition matrix Q, the forward dynamics (E1) within the equilibrium system (E1)-(E6) read as follows:

$$\begin{cases} \dot{\mu}^{\mathrm{S}}(t,w) = -q_{\mathrm{SI}}\cdot\mu^{\mathrm{I}}(t,w)\cdot\big(1-h^{\mathrm{S}}(t,w,\mu(t,w),v(t,w))\big)\cdot\mathbb{1}_{\{t<\tau^{\star}\}}(w)\cdot\mu^{\mathrm{S}}(t,w) \\ \dot{\mu}^{\mathrm{I}}(t,w) = \quad q_{\mathrm{SI}}\cdot\mu^{\mathrm{I}}(t,w)\cdot\big(1-h^{\mathrm{S}}(t,w,\mu(t,w),v(t,w))\big)\cdot\mathbb{1}_{\{t<\tau^{\star}\}}(w)\cdot\mu^{\mathrm{S}}(t,w) - q_{\mathrm{IR}}\cdot\mu^{\mathrm{I}}(t,w) \\ \dot{\mu}^{\mathrm{R}}(t,w) = \qquad q_{\mathrm{IR}}\cdot\mu^{\mathrm{I}}(t,w). \end{cases}$$

Disregarding common noise, these constitute a ramification of the classical SIR dynamics, which are a basic building block of numerous compartmental epidemic models in the literature; see, among others, [32,37,38,47] and the references therein. The SIR mean field game with controlled infection rates, albeit without common noise, has recently been studied in the independent article [26]; we also refer to [46] and [23] for mean field models with controlled vaccination rates. Mathematically similar contagion mechanisms also appear in, e.g., [40,41], §7.2.3 in [9], §7.1.10 in [10], or §4.4 in [52]. □

While Theorem 13 guarantees existence of a mean field equilibrium for (a variant[12] of) the SIR model, the monotonicity conditions of Theorem 16 do not hold in this setup.[13] Nevertheless, our numerical results reliably yield consistent equilibria. For our illustrations, the initial distribution of agents is given by $m_0 \triangleq (0.995, 0.005, 0.00)$, and the model coefficients are reported in Table 2. Note that there are $n = 1999$ common noise times $T_k = k \cdot 0.01$, $k = 1, \ldots, 1999$, at which a vaccine can be administered. The specifications of q_{SI} and q_{IR} imply a basic reproduction number $R_0 \triangleq q_{\mathrm{SI}}/q_{\mathrm{IR}} = 15$ in the absence of vaccination and protection efforts.

Our results for the mean field equilibrium distributions of agents μ and the corresponding optimal protection efforts of susceptible agents h^{S} are displayed in Figs. 7, 8, and 9 for different common noises configurations, i.e. vaccination times $\tau^{\star}$. As in

[11] Note that for given $w \in \mathbb{W}^n$ the stated maximizer h^{S} is unique for times $t < \tau^{\star}$; otherwise its specification is immaterial. The latter applies likewise to h^{I} and h^{R}.

[12] With action space $\mathbb{U} \triangleq [0, u_{\max}]$ for an arbitrary, but fixed $u_{\max} < 1$.

[13] In fact, Assumption 14(iii) is not even satisfied in the baseline SIR model without protection or vaccination.

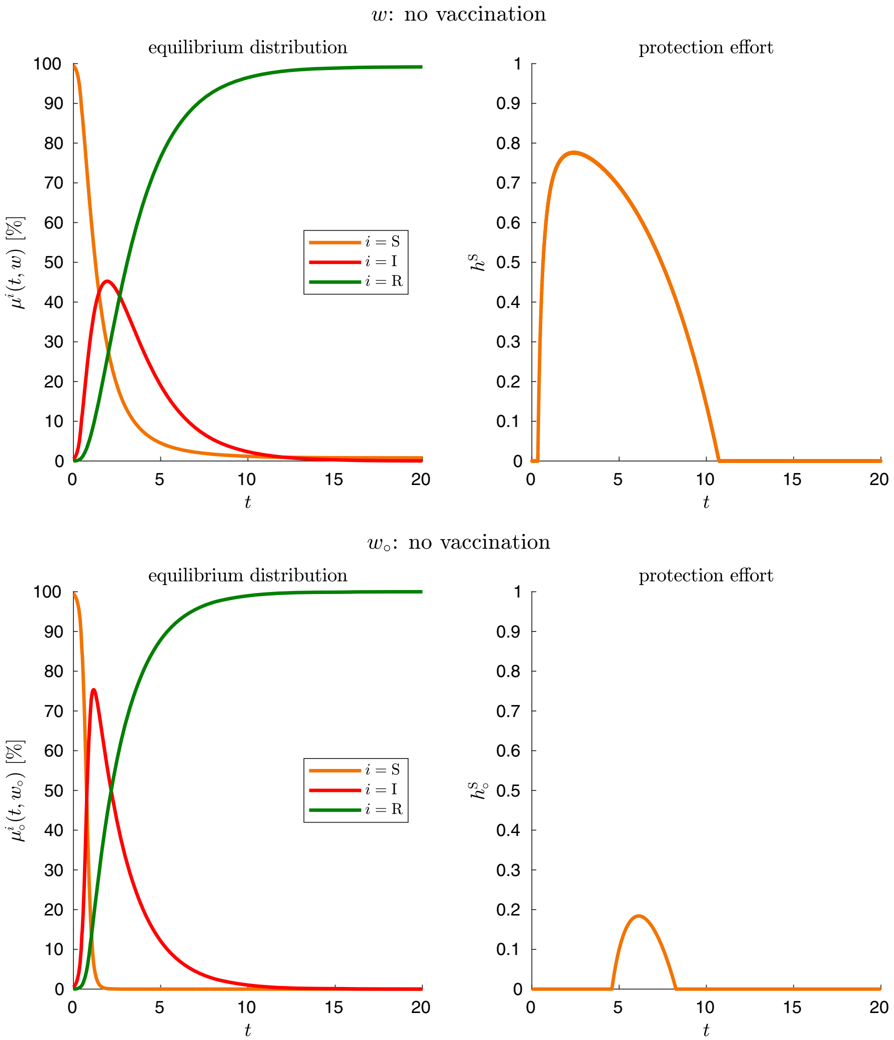

Fig. 7 Equilibrium distribution and protection effort for $\tau^\star = +\infty$: Mean field game with common noise (top) and corresponding perfect-foresight equilibrium (bottom)

Sect. 5.1, we also display the corresponding (theoretical) perfect-foresight equilibria, marked by the subscript $\circ$.

Note that an agent's running reward is the same in state S with zero protection effort and in state R; agents are penalized relative to these in state I and hence aim to avoid that state. Susceptible agents can reach the state R of immunity by two ways: First, they can become infected and overcome the disease; second, they can be vaccinated and jump instantly from state S to state R. While the first alternative is painful, the second comes at no cost and is hence clearly preferable. However, as the availability of

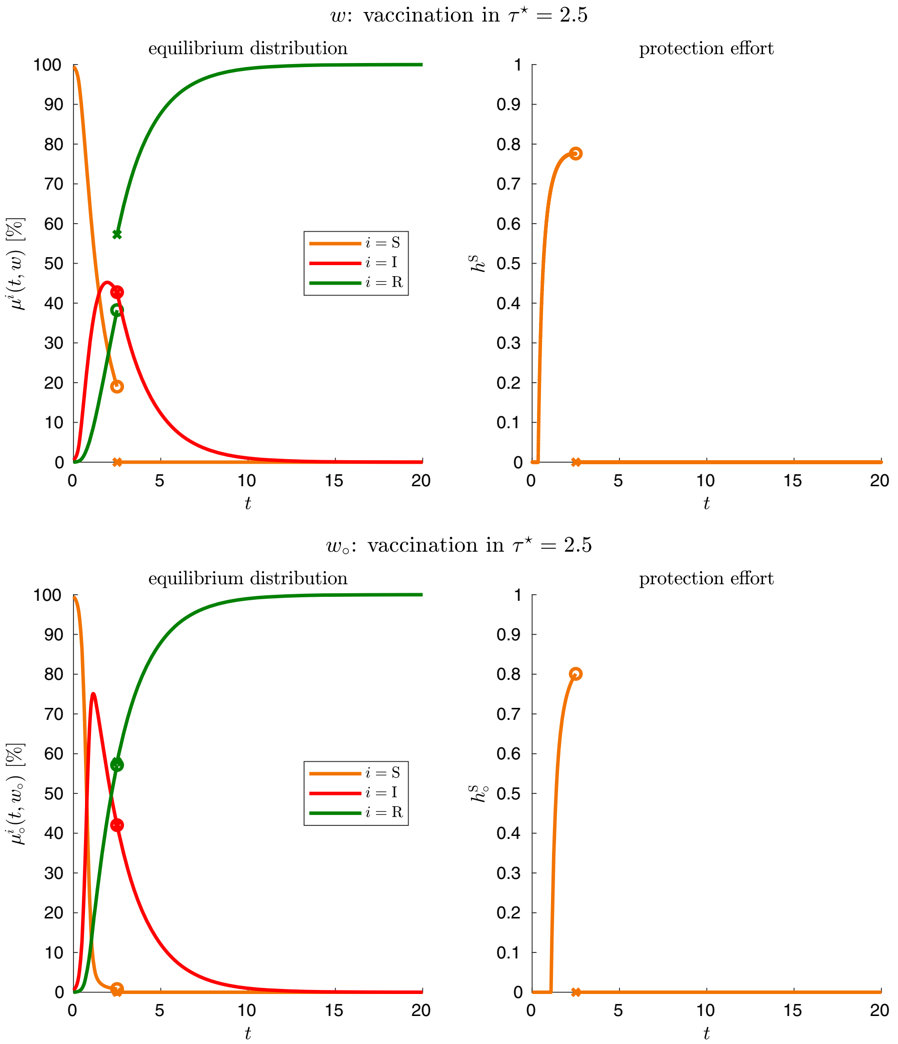

Fig. 8 Equilibrium distribution and protection effort for $\tau^\star = 2.5$: Mean field game with common noise (top) and corresponding perfect-foresight equilibrium (bottom)

a vaccine cannot be directly controlled by the agents, they can only protect themselves against infection at a certain running cost until the vaccine becomes available.

Figures 7, 8, and 9 demonstrate that the possibility of vaccination as a common noise event can dampen the spread of the disease and lower the peak infection rate. This is due to an increase in agents' protection efforts *during the time period when the proportion of infected agents is high*. By contrast, in the perfect-foresight equilibria where the vaccination date is known, agents do not make substantial protection efforts until the vaccination date is imminent, see Figs. 8 and 9; in the scenario without vaccination, see Fig. 7, protection efforts are only ever made by a very small fraction

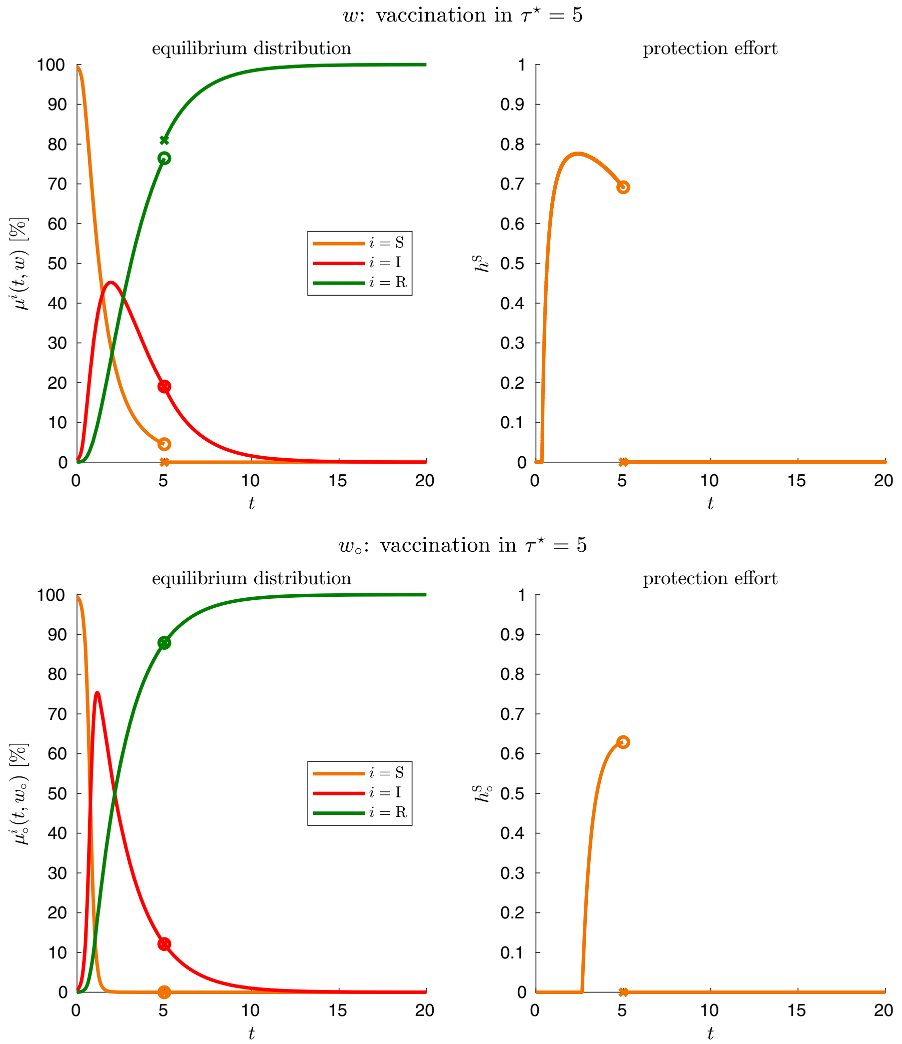

Fig. 9 Equilibrium distribution and protection effort for $\tau^\star = 5$: Mean field game with common noise (top) and corresponding perfect-foresight equilibrium (bottom)

of the population. With perfect foresight, the agents' main rationale is to avoid being in state I when the vaccine becomes available. This highlights the importance of being able to model the vaccination date as a (random) common noise event. Finally, observe that our numerical results indicate convergence to the stationary distribution $\bar{\mu} = (0, 0, 1) \in \mathbb{M}$, showing that the model is able to capture the entire evolution of an epidemic.

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A Appendix: Proof of Theorem 13

Let $E \subseteq \mathbb{R}^d$ and define the space

$$\mathsf{D}(E) \triangleq \big\{ f : [0,T] \times \mathbb{W}^n \to E \; : \; f \text{ is càdlàg and non-anticipative} \big\}$$

together with the norm $\|f\|_{\sup} \triangleq \sup_{t\in[0,T],\, w\in\mathbb{W}^n} \|f(t,w)\|$ for $f \in \mathsf{D}(E)$. It is clear that $\mathsf{D}(E)$ is a Banach space provided $E \subseteq \mathbb{R}^d$ is closed; the linear subspace of regular non-anticipative functions is denoted by

$$\mathsf{Reg}(E) \triangleq \{ f \in \mathsf{D}(E) \; : \; f \text{ is regular} \}.$$

Lemma A.1 (Backward Gronwall estimate) *Let $f \in D([0,\infty))$ and $\alpha, \beta, \vartheta, \rho, \eta \geq 0$. Suppose that $f(T,w) \leq \rho \cdot \eta$ for all $w \in \mathbb{W}^n$,*

$$\begin{aligned} & f(t,w) \leq f(T_{k+1}-, w) + \alpha (T_{k+1} - t) \cdot \eta + \beta \cdot \int_t^{T_{k+1}} f(s,w) \mathrm{d}s, \\ & t \in [T_k, T_{k+1}\rangle, \; w \in \mathbb{W}^n, \end{aligned} \tag{28}$$

for $k \in [0:n]$, and

$$f(T_k-, w) \leq \sum_{\bar{w}_k \in \mathbb{W}} \gamma_k(w_{-k}, \bar{w}_k) \cdot f(T_k, (w_{-k}, \bar{w}_k)) + \vartheta \cdot \eta, \quad w \in \mathbb{W}^n, \tag{29}$$

for $k \in [1:n]$, where for all $w_1, \ldots, w_{k-1} \in \mathbb{W}$ the family $\{\gamma_k(w_{-k}, \bar{w}_k)\}_{\bar{w}_k \in \mathbb{W}}$ consists of probability weights on $\mathbb{W}$. Then we have

$$\begin{aligned} f(t,w) \leq C \cdot \eta \quad & \textit{for all } (t,w) \in [0,T] \times \mathbb{W}^n, \\ \textit{where} \quad & C \triangleq \big(\rho + \alpha T + (n+1)\vartheta \big) \cdot \mathrm{e}^{\beta T}. \end{aligned}$$

Proof We recursively define $C_{n+1} \triangleq 1$ and

$$C_k \triangleq \big(C_{k+1} + \alpha (T_{k+1} - T_k) + \vartheta \big) \cdot \mathrm{e}^{\beta (T_{k+1} - T_k)} \quad \text{for } k \in [0:n]$$

and note that $C_{n+1} \leq C_n \leq \cdots \leq C_1 \leq C_0 \leq C$. Hence it suffices to show that

$$f(t, w) \leq C_k \cdot \eta \quad \text{for all } (t, w) \in [T_k, T_{k+1}\rangle \times \mathbb{W}^n,\ k \in [0:n]. \tag{30}$$

By assumption, $f(T_{n+1}-, w) = f(T, w) \leq \rho \cdot \eta$ for all $w \in \mathbb{W}^n$. Next let $k \in [0:n]$ and assume that

$$f(T_{k+1}-, w) \leq C_{k+1} \cdot \eta \quad \text{for all } w \in \mathbb{W}^n.$$

It follows from (28) and Gronwall's inequality on $[T_k, T_{k+1}\rangle$ that for all $w \in \mathbb{W}^n$

$$\begin{aligned} f(t, w) &\leq \big(f(T_{k+1}-, w) + \alpha(T_{k+1} - t) \cdot \eta\big) \cdot \mathrm{e}^{\beta(T_{k+1}-t)} \\ &\leq \big(C_{k+1} + \alpha(T_{k+1} - T_k)\big) \cdot \mathrm{e}^{\beta(T_{k+1}-T_k)} \cdot \eta. \end{aligned}$$

In particular, we have $f(t, w) \leq C_k \cdot \eta$ for $t \in [T_k, T_{k+1}\rangle$, and by (29)

$$\begin{aligned} f(T_k-, w) &\leq \sum_{\bar{w}_k \in \mathbb{W}} \gamma_k(w_1, \ldots, w_{k-1}, \bar{w}_k) \cdot f(T_k, (w_{-k}, \bar{w}_k)) + \vartheta \cdot \eta \\ &\leq \big(C_{k+1} + \alpha(T_{k+1} - T_k) + \vartheta\big) \cdot \mathrm{e}^{\beta(T_{k+1}-T_k)} \cdot \eta = C_k \cdot \eta. \end{aligned}$$

Hence (30) follows by backward induction on $k = n, n-1, \ldots, 0$. □

In the following, we first consider the backward system (E2), (E4), (E6) and subsequently the forward system (E1), (E3), (E5).

Lemma A.2 *Suppose that Assumption 11 holds and let $\mu \in D(\mathbb{M})$. Then there exists a unique solution $\bar{v}$ of (E2) subject to (E4) and (E6). Moreover, we have $\bar{v} \in \mathit{Reg}(\mathbb{R}^d)$ and $\|\bar{v}(t, w)\| \leq v_{\max}$ for all $(t, w) \in [0, T] \times \mathbb{W}^n$.*

Proof *Step 1: Construction of $\bar{v}$.* We construct $\bar{v}$ by backward induction on $k \in [0:n]$ on each segment $[T_k, T_{k+1}\rangle \times \mathbb{W}^n$. First, we set $\bar{v}(T, w) \triangleq \Psi(w, \mu(T, w))$ for $w \in \mathbb{W}^n$. Suppose that $k \in [0:n]$, fix $w \in \mathbb{W}^n$, and let $\tilde{v}(T_{k+1}, w_{T_k}) \in \mathbb{R}^d$ be given and independent of $w_{k+1}, \ldots, w_n$. Using Assumptions 11(i)-(ii) it follows that the Carathéodory conditions are satisfied, so [36, Theorem I.5.3] yields the unique Carathéodory solution $\tilde{v}(\,\cdot\,, w_{T_k}) : [T_k, T_{k+1}] \to \mathbb{R}^d$ of

$$\begin{aligned} \tilde{v}(t, w_{T_k}) = \tilde{v}(T_{k+1}, w_{T_k}) &+ \int_t^{T_{k+1}} \Big(\widehat{\psi}\big(s, w_{T_k}, \mu(s, w_{T_k}), \tilde{v}(s, w_{T_k})\big) \\ &\qquad + \widehat{Q}\big(s, w_{T_k}, \mu(s, w_{T_k}), \tilde{v}(s, w_{T_k})\big) \cdot \tilde{v}(s, w_{T_k})\Big)\mathrm{d}s \\ = \tilde{v}(T_{k+1}, w_{T_k}) &+ \int_t^{T_{k+1}} \Big(\widehat{\psi}\big(s, w, \mu(s, w), \tilde{v}(s, w_{T_k})\big) \\ &\qquad + \widehat{Q}\big(s, w, \mu(s, w), \tilde{v}(s, w_{T_k})\big) \cdot \tilde{v}(s, w_{T_k})\Big)\mathrm{d}s, \quad t \in [T_k, T_{k+1}], \end{aligned}$$

where the final identity is due to the fact that $\widehat{\psi}(\,\cdot\,,\,\cdot\,,\bar{m},\bar{v})$ and $\widehat{Q}(\,\cdot\,,\,\cdot\,,\bar{m},\bar{v})$ are non-anticipative. Define

$$\bar{v}(t,w) \triangleq \tilde{v}(t,w_{T_k}) \quad \text{for } t \in [T_k,T_{k+1}\rangle \quad \text{and each } w \in \mathbb{W}^n.$$

By construction, $\bar{v}(\,\cdot\,,w)$ solves (E2) on $[T_k,T_{k+1}\rangle$ and does not depend on $w_{k+1},\dots,w_n$. Having constructed $\bar{v}$ on $[T_k,T_{k+1}\rangle \times \mathbb{W}^n$, we use (E4) and define

$$\bar{v}(T_k,w_{T_{k-1}}) \triangleq \Psi_k(w,\mu(T_k-,w),\bar{v}(T_k,\,\cdot\,))$$

for $w \in \mathbb{W}^n$. By (10) and the fact that μ and J are non-anticipative, it follows that this definition does not depend on $w_k,\dots,w_n$. Consequently, the above construction can be iterated, and hence we obtain $\bar{v}$ as the unique solution of (E2) subject to (E4) and (E6). By definition, $\bar{v}$ is non-anticipative and regular, i.e. $\bar{v} \in \mathsf{Reg}(\mathbb{R}^d)$.

Step 2: A priori bound. For $k \in [0:n]$ and $(t,w) \in [T_k,T_{k+1}\rangle \times \mathbb{W}^n$ we have

$$\begin{aligned}
\|\bar{v}(t,w)\| &\le \|\bar{v}(T_{k+1}-,w)\| + \int_t^{T_{k+1}} \left\|\widehat{\psi}(s,w,\mu(s,w),\bar{v}(s,w))\right\| \\
&\qquad + \left\|\widehat{Q}(s,w,\mu(s,w),\bar{v}(s,w))\right\| \cdot \|\bar{v}(s,w)\| \mathrm{d}s \\
&\le \|\bar{v}(T_{k+1}-,w)\| + \psi_{\max} \cdot (T_{k+1}-t) + Q_{\max} \cdot \int_t^{T_{k+1}} \|\bar{v}(s,w)\| \mathrm{d}s.
\end{aligned} \tag{31}$$

On the other hand, for $k \in [1:n]$, $w \in \mathbb{W}^n$ and $i \in \mathbb{S}$ we observe from (10) that

$$\|\bar{v}(T_k-,w)\| \le \sum_{\bar{w}_k \in \mathbb{W}} \kappa_k(\bar{w}_k|w_1,\dots,w_{k-1},\mu(T_k-,w_{T_k-})) \cdot \|\bar{v}(T_k,(w_{-k},\bar{w}_k))\|. \tag{32}$$

Since $\|\bar{v}(T,w)\| = \|\Psi(w,\mu(T,w))\| \le \Psi_{\max}$ it follows from (31), (32) and Lemma A.1 with $\eta \triangleq \psi_{\max}$, $\vartheta \triangleq 0$ and $\rho \triangleq \Psi_{\max}/\psi_{\max}$ that

$$\|\bar{v}(t,w)\| \le C \cdot \eta \le \left(\Psi_{\max} + T \cdot \psi_{\max}\right) \cdot \mathrm{e}^{Q_{\max} \cdot T} = v_{\max} \quad \text{for all } (t,w) \in [0,T] \times \mathbb{W}^n.$$

□

Lemma A.3 *Suppose that Assumption 11 is satisfied and let $v \in \mathsf{D}(\mathbb{R}^d)$. Then there is a unique solution $\bar{\mu}$ of (E1) subject to (E3) and (E5), and we have $\bar{\mu} \in \mathsf{Reg}(\mathbb{M})$.*

Proof The proof is analogous to (but somewhat simpler than) that of Lemma A.2. □

Proof of Theorem 13 We divide the proof into four steps:

Step 1: Solution operators. We define

$$\overleftarrow{\chi}: \mathsf{D}(\mathbb{M}) \to \mathsf{Reg}(\mathbb{R}^d), \quad \overleftarrow{\chi}[\mu] \triangleq \bar{v},$$

where $\bar{v} \in \mathsf{Reg}(\mathbb{R}^d)$ is the unique solution of (E2) subject to (E4) and (E6) given $\mu \in \mathsf{D}(\mathbb{M})$; $\overleftarrow{\chi}$ is well-defined by Lemma A.2. Moreover, let

$$\overrightarrow{\chi}: \mathsf{D}(\mathbb{R}^d) \to \mathsf{Reg}(\mathbb{M}), \quad \overrightarrow{\chi}[v] \triangleq \bar{\mu},$$

where $\bar{\mu} \in \mathsf{Reg}(\mathbb{M})$ is the unique solution of (E1) subject to (E3) and (E5) given $v \in \mathsf{D}(\mathbb{R}^d)$; $\overrightarrow{\chi}$ is well-defined by Lemma A.3.

Step 2: Continuity of $\overleftarrow{\chi}$. Let $\mu_0 \in \mathsf{D}(\mathbb{M})$, set $\bar{v}_0 \triangleq \overleftarrow{\chi}[\mu_0]$ and fix some $\varepsilon > 0$. We set

$$\alpha \triangleq L_{\widehat{\psi}} + L_{\widehat{Q}} \cdot v_{\max}, \qquad \beta \triangleq L_{\widehat{\psi}} + Q_{\max} + L_{\widehat{Q}} \cdot v_{\max},$$
$$C \triangleq (\alpha T + n + 2) \cdot \mathrm{e}^{\beta T} \qquad \text{and} \qquad \eta \triangleq \frac{\varepsilon}{C}.$$

By Assumptions 11(iii)-(iv), for each $k \in [1:n]$ we can pick $\delta_k > 0$ such that

$$\begin{aligned} &\Bigg| \sum_{\bar{w}_k \in \mathbb{W}} \Big(\kappa_k\big(\bar{w}_k | w_1, \ldots, w_{k-1}, m\big) \bar{v}_0^{J^i(T_k,(w_{-k},\bar{w}_k),m)}(T_k, (w_{-k}, \bar{w}_k)) \\ &\quad - \kappa_k\big(\bar{w}_k | w_1, \ldots, w_{k-1}, \mu_0(T_k-, w)\big) \bar{v}_0^{J^i(T_k,(w_{-k},\bar{w}_k),\mu_0(T_k-,w))}(T_k, (w_{-k}, \bar{w}_k))\Big)\Bigg| \le \eta \\ &\text{for all } i \in \mathbb{S} \text{ and } (m, w) \in \mathbb{M} \times \mathbb{W}^n \text{ with } \|m - \mu_0(T_k-, w)\| \le \delta_k, \end{aligned} \tag{33}$$

and $\delta_{n+1} > 0$ such that

$$\begin{aligned} &\|\Psi(w, m) - \Psi(w, \mu_0(T, w))\| \le \eta \\ &\quad \text{for all } (m, w) \in \mathbb{M} \times \mathbb{W}^n \text{ with } \|m - \mu_0(T, w)\| \le \delta_{n+1}. \end{aligned} \tag{34}$$

We define $\delta > 0$ via

$$\delta \triangleq \eta \wedge \delta_1 \wedge \cdots \wedge \delta_{n+1} \tag{35}$$

and let $\mu \in \mathsf{D}(\mathbb{M})$ such that $\|\mu - \mu_0\|_{\sup} \le \delta$; set $\bar{v} \triangleq \overleftarrow{\chi}[\mu]$. For each $w \in \mathbb{W}^n$, it follows from Assumptions 11(i)-(ii) and (35) that for all $t \in [T_k, T_{k+1}\rangle$, $k \in [0:n]$, we have

$$\begin{aligned} &\big\|\bar{v}(t, w) - \bar{v}_0(t, w)\big\| \le \big\|\bar{v}(T_{k+1}-, w) - \bar{v}_0(T_{k+1}-, w)\big\| \\ &\quad + \int_t^{T_{k+1}} \big\|\widehat{\psi}\big(s, w, \mu(s, w), \bar{v}(s, w)\big) - \widehat{\psi}\big(s, w, \mu_0(s, w), \bar{v}_0(s, w)\big)\big\| \mathrm{d}s \\ &\quad + \int_t^{T_{k+1}} \big\|\widehat{Q}\big(s, w, \mu(s, w), \bar{v}(s, w)\big) \cdot \bar{v}(s, w) - \widehat{Q}\big(s, w, \mu_0(s, w), \bar{v}_0(s, w)\big) \cdot \bar{v}_0(s, w)\big\| \mathrm{d}s \\ &\le \big\|\bar{v}(T_{k+1}-, w) - \bar{v}_0(T_{k+1}-, w)\big\| + \big(L_{\widehat{\psi}} + L_{\widehat{Q}} \cdot v_{\max}\big) \cdot (T_{k+1} - t) \cdot \eta \\ &\quad + \big(L_{\widehat{\psi}} + Q_{\max} + L_{\widehat{Q}} \cdot v_{\max}\big) \cdot \int_t^{T_{k+1}} \big\|\bar{v}(s, w) - \bar{v}_0(s, w)\big\| \mathrm{d}s \end{aligned}$$

$$= \left\|\bar{v}(T_{k+1}-, w) - \bar{v}_0(T_{k+1}-, w)\right\| + \alpha(T_{k+1} - t) \cdot \eta + \beta \cdot \int_t^{T_{k+1}} \left\|\bar{v}(s, w) - \bar{v}_0(s, w)\right\| \mathrm{d}s. \tag{36}$$

Moreover, for $k \in [1:n]$ we obtain from (33) that

$$\begin{aligned} &\left\|\bar{v}(T_k-, w) - \bar{v}_0(T_k-, w)\right\| \\ &\leq \sum_{\bar{w}_k \in \mathbb{W}} \kappa_k\big(\bar{w}_k | w_1, \dots, w_{k-1}, \mu(T_k-, w_{T_k-})\big) \\ &\quad \cdot \left\|\bar{v}(T_k, (w_{-k}, \bar{w}_k)) - \bar{v}_0(T_k, (w_{-k}, \bar{w}_k))\right\| + \eta \end{aligned} \tag{37}$$

and from (34) that

$$\|\bar{v}(T, w) - \bar{v}_0(T, w)\| = \|\Psi(w, \mu(T, w)) - \Psi(w, \mu_0(T, w))\| \leq \eta. \tag{38}$$

In view of (36), (37) and (38), it follows from Lemma A.1 that

$$\|\bar{v}(t, w) - \bar{v}_0(t, w)\| \leq C \cdot \eta = \varepsilon \quad \text{for all } (t, w) \in [0, T] \times \mathbb{W}^n,$$

i.e. $\|\overleftarrow{\chi}[\mu] - \overleftarrow{\chi}[\mu_0]\|_{\sup} = \|\bar{v} - \bar{v}_0\|_{\sup} \leq \varepsilon$. Thus $\overleftarrow{\chi}$ is continuous with respect to $\|\cdot\|_{\sup}$.

Step 3: Continuity of $\overrightarrow{\chi}$. Let $v_0 \in \mathsf{D}(\mathbb{R}^d)$, set $\bar{\mu}_0 \triangleq \overrightarrow{\chi}[v_0]$ and fix some $\varepsilon > 0$. We set $\delta_{n+1} \triangleq \varepsilon$ and $c \triangleq Q_{\max} + L_{\widehat{Q}}$ and recursively determine $\delta_1, \dots, \delta_n \in (0, \varepsilon)$ using Assumption 11(v) such that

$$\begin{aligned} &\|\Phi_k(w, m) - \Phi_k(w, \bar{\mu}_0(T_k-, w))\| \leq \tfrac{\delta_{k+1}}{2} \cdot \mathrm{e}^{-c(T_{k+1}-T_k)} \\ &\qquad \text{for all } (m, w) \in \mathbb{M} \times \mathbb{W}^n \text{ with } \|m - \bar{\mu}_0(T_k-, w)\| \leq \delta_k. \end{aligned} \tag{39}$$

We define $\delta > 0$ by

$$\delta \triangleq \tfrac{\mathrm{e}^{-cT}}{2cT} \cdot \delta_1 \wedge \cdots \wedge \delta_n \tag{40}$$

and let $v \in \mathsf{D}(\mathbb{R}^d)$ such that $\|v - v_0\|_{\sup} \leq \delta$; put $\bar{\mu} \triangleq \overrightarrow{\chi}[v]$. We fix $k \in [0:n]$ and suppose that

$$\|\bar{\mu}(T_k, w) - \bar{\mu}_0(T_k, w)\| \leq \tfrac{\delta_{k+1}}{2} \cdot \mathrm{e}^{-c(T_{k+1}-T_k)}, \quad w \in \mathbb{W}^n. \tag{41}$$

For each $w \in \mathbb{W}^n$, we have from Assumption 11(ii)

$$\begin{aligned} \left\|\bar{\mu}(t, w) - \bar{\mu}_0(t, w)\right\| &\leq \left\|\bar{\mu}(T_k, w) - \bar{\mu}_0(T_k, w)\right\| \\ &\quad + \int_{T_k}^t \left\|\bar{\mu}(s, w) \cdot \widehat{Q}(s, w, \bar{\mu}(s, w), v(s, w)) - \bar{\mu}_0(s, w) \cdot \widehat{Q}(s, w, \bar{\mu}_0(s, w), v_0(s, w))\right\| \mathrm{d}s \\ &\leq \|\bar{\mu}(T_k, w) - \bar{\mu}_0(T_k, w)\| + c(t - T_k) \cdot \|v - v_0\|_{\sup} + c \cdot \int_{T_k}^t \left\|\bar{\mu}(s, w) - \bar{\mu}_0(s, w)\right\| \mathrm{d}s \end{aligned}$$

on $[T_k, T_{k+1}\rangle$, so using Gronwall's inequality it follows that

$$\left\|\bar{\mu}(t,w)-\bar{\mu}_0(t,w)\right\| \leq \left(\left\|\bar{\mu}(T_k,w)-\bar{\mu}_0(T_k,w)\right\| + c(t-T_k)\cdot\|v-v_0\|_{\sup}\right)\cdot \mathrm{e}^{c(t-T_k)}.$$

Since by (40) we have $c(t-T_k)\cdot\delta\cdot\mathrm{e}^{c(t-T_k)} \leq \frac{\delta_{k+1}}{2}$, we obtain

$$\left\|\bar{\mu}(t,w)-\bar{\mu}_0(t,w)\right\| \leq \delta_{k+1} \quad \text{for all } (t,w)\in[T_k,T_{k+1}\rangle\times\mathbb{W}^n. \tag{42}$$

In particular, using (39) we conclude that

$$\begin{aligned}\|\bar{\mu}(T_{k+1},w)-\bar{\mu}_0(T_{k+1},w)\| &= \|\Phi_{k+1}(w,\bar{\mu}(T_{k+1}-,w))-\Phi_{k+1}(w,\bar{\mu}_0(T_{k+1}-,w))\|\\ &\leq \frac{\delta_{k+2}}{2}\cdot\mathrm{e}^{-c(T_{k+2}-T_{k+1})}, \quad w\in\mathbb{W}^n.\end{aligned}$$

Since $\bar{\mu}(0,w)=\bar{\mu}_0(0,w)=m_0$ for all $w\in\mathbb{W}^n$, it follows by induction that (41) holds for all $k\in[0:n]$, and thus (42) implies

$$\|\bar{\mu}(t,w)-\bar{\mu}_0(t,w)\| \leq \varepsilon \quad \text{for all } (t,w)\in[0,T]\times\mathbb{W}^n.$$

Hence $\|\vec{\chi}[v]-\vec{\chi}[v_0]\|_{\sup} = \|\bar{\mu}-\bar{\mu}_0\|_{\sup} \leq \varepsilon$, so $\vec{\chi}$ is continuous with respect to $\|\cdot\|_{\sup}$.

Step 4: Construction of the fixed point. Let $\chi : \mathsf{D}(\mathbb{M}) \to \mathsf{Reg}(\mathbb{M})$, $\chi \triangleq \vec{\chi}\circ\overleftarrow{\chi}$ and note that χ is continuous with respect to $\|\cdot\|_{\sup}$ by Steps 2 and 3. We define

$$\begin{aligned}\mathsf{Lip}(\mathbb{M}) \triangleq \big\{\mu\in\mathsf{D}(\mathbb{M}) : \mu(\,\cdot\,,w) \text{ is } Q_{\max} - \text{ Lipschitz}\\ \text{on } [T_k,T_{k+1}\rangle,\ k\in[0:n],\ \text{for all } w\in\mathbb{W}^n\big\}\end{aligned}$$

and note from (E1) that $\chi : \mathsf{D}(\mathbb{M}) \to \mathsf{Lip}(\mathbb{M})$, i.e. $\chi[\mu]\in\mathsf{Lip}(\mathbb{M})$ for every $\mu\in\mathsf{D}(\mathbb{M})$. It is clear that $\mathsf{Lip}(\mathbb{M})$ is a non-empty, convex subset of $\mathsf{D}(\mathbb{M})$; we now argue that $\mathsf{Lip}(\mathbb{M})$ is compact. Given a sequence $\{\mu_\ell\}_{\ell\in\mathbb{N}}\subseteq\mathsf{Lip}(\mathbb{M})$, for each $k\in[0:n]$ and $w\in\mathbb{W}^n$ we define

$$\mu_\ell^{(k,w)} : [T_k,T_{k+1}]\to\mathbb{M}, \quad \mu_\ell^{(k,w)}(t)\triangleq\begin{cases}\mu_\ell(t,w) & \text{if } t\in[T_k,T_{k+1}\rangle,\\ \mu_\ell(T_{k+1}-,w) & \text{if } t=T_{k+1},\end{cases}$$

and note that by the Arzelà-Ascoli theorem, the sequence $\{\mu_\ell^{(k,w)}\}_{\ell\in\mathbb{N}}\subseteq\mathsf{C}([T_k,T_{k+1}];\mathbb{M})$ contains a uniformly convergent subsequence. Taking sub-subsequences for $k\in[0:n]$ and $w\in\mathbb{W}^n$, we obtain a subsequence $\{\ell_\nu\}_{\nu\in\mathbb{N}}$ such that $\|\mu_{\ell_\nu}-\mu\|_{\sup}\to 0$ as $\nu\to\infty$ for some $\mu\in\mathsf{D}(\mathbb{M})$. It is easy to see that $\mu\in\mathsf{Lip}(\mathbb{M})$, and thus $\mathsf{Lip}(\mathbb{M})$ is indeed compact. Now Schauder's fixed point theorem implies that the continuous map $\chi : \mathsf{Lip}(\mathbb{M})\to\mathsf{Lip}(\mathbb{M})$ has a fixed point $\mu\in\mathsf{Lip}(\mathbb{M})$; upon setting $v\triangleq\overleftarrow{\chi}[\mu]\in\mathsf{Reg}(\mathbb{R}^d)$ it follows that $\mu=\vec{\chi}[v]\in\mathsf{Reg}(\mathbb{M})$ and that (μ,v) is a solution of (E1)-(E6). $\square$

B Appendix: Proof of Theorem 16

Proof of Theorem 16 Suppose that (μ_1, v_1) and (μ_2, v_2) are solutions of (E1)-(E6). In the following, we omit arguments to simplify notation when there is no ambiguity; e.g., $\widehat{\mathcal{H}}(\mu_1, v_2) = \widehat{\mathcal{H}}(t, w, \mu_1(t, w), v_2(t, w))$.

Step 1: Dynamics between Common Noise Times. Let $k \in [0 : n]$. The product rule and (E1)-(E2) yield

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big[(\mu_1(t,w) - \mu_2(t,w)) \cdot (v_1(t,w) - v_2(t,w))\big] = \frac{\mathrm{d}}{\mathrm{d}t}\big[(\mu_1 - \mu_2) \cdot (v_1 - v_2)\big] \\
&= \big[\mu_1 \cdot \widehat{Q}(\mu_1, v_1) - \mu_2 \cdot \widehat{Q}(\mu_2, v_2)\big] \cdot (v_1 - v_2) - (\mu_1 - \mu_2) \cdot \big[\widehat{\mathcal{H}}(\mu_1, v_1) - \widehat{\mathcal{H}}(\mu_2, v_2)\big] \\
&= \mu_2 \cdot \big[\widehat{\mathcal{H}}(\mu_2, v_1) - \widehat{\mathcal{H}}(\mu_2, v_2) - \widehat{Q}(\mu_2, v_2) \cdot (v_1 - v_2)\big] \\
&\quad + \mu_1 \cdot \big[\widehat{\mathcal{H}}(\mu_1, v_2) - \widehat{\mathcal{H}}(\mu_1, v_1) - \widehat{Q}(\mu_1, v_1) \cdot (v_2 - v_1)\big] \\
&\quad + \mu_1 \cdot \big[\widehat{\mathcal{H}}(\mu_2, v_2) - \widehat{\mathcal{H}}(\mu_1, v_2)\big] + \mu_2 \cdot \big[\widehat{\mathcal{H}}(\mu_1, v_1) - \widehat{\mathcal{H}}(\mu_2, v_1)\big]
\end{aligned}
$$

and thus by Assumptions 14(ii)-(iii) we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\big[(\mu_1(t,w) - \mu_2(t,w)) \cdot (v_1(t,w) - v_2(t,w))\big] \\
&\quad \geq \gamma \cdot \|\mu_1(t,w) - \mu_2(t,w)\|^{\alpha}, \quad (t,w) \in [T_k, T_{k+1}\rangle \times \mathbb{W}^n.
\end{aligned} \tag{43}
$$

Step 2: Boundary Conditions at Common Noise Times. Let $k \in [1 : n]$ and $w \in \mathbb{W}^n$. For $j = 1, 2$ and $\bar{w}_k \in \mathbb{W}$ we briefly write

$$
\begin{aligned}
&\mu_j \triangleq \mu_j\big(T_k, (w_{-k}, \bar{w}_k)\big) \text{ and } \quad \mu_j^- \triangleq \mu_j(T_k-, w) = \mu_j\big(T_k-, (w_{-k}, \bar{w}_k)\big), \\
&v_j \triangleq v_j\big(T_k, (w_{-k}, \bar{w}_k)\big), J(\mu_j^-) \quad \triangleq J\big(T_k, (w_{-k}, \bar{w}_k), \mu_j^-\big) \\
&\quad \text{and} \quad \kappa_k(\mu_j^-) \triangleq \kappa_k\big(\bar{w}_k | w_1, \ldots, w_{k-1}, \mu_j^-\big).
\end{aligned}
$$

By (E3) we have for $j, \ell = 1, 2$

$$
\mu_j \cdot v_\ell = \mu_j^- \cdot v_\ell^{J^\cdot(\mu_j^-)} \quad \text{for all } \bar{w}_k \in \mathbb{W} \tag{44}
$$

where it is understood that $\mu_j, v_\ell, J(\mu_j^-), \kappa_k(\mu_j^-)$ depend on $\bar{w}_k$. Using (E4) in the first identity and (44) in the second and fourth, we find by elementary algebraic manipulations

$$
\begin{aligned}
&(\mu_1(T_k-, w) - \mu_2(T_k-, w)) \cdot (v_1(T_k-, w) - v_2(T_k-, w)) \\
&= \sum_{\bar{w}_k \in \mathbb{W}} \kappa_k(\mu_1^-) \cdot (\mu_1^- - \mu_2^-) \cdot v_1^{J^\cdot(\mu_1^-)} - \sum_{\bar{w}_k \in \mathbb{W}} \kappa_k(\mu_2^-) \cdot (\mu_1^- - \mu_2^-) \cdot v_2^{J^\cdot(\mu_2^-)} \\
&= \frac{1}{2} \sum_{\bar{w}_k \in \mathbb{W}} \kappa_k(\mu_1^-) \cdot (\mu_1^- - \mu_2^-) \cdot v_1^{J^\cdot(\mu_1^-)} \\
&\quad + \frac{1}{2} \sum_{\bar{w}_k \in \mathbb{W}} \kappa_k(\mu_1^-) \cdot \Big\{(\mu_1 - \mu_2) \cdot v_1 - \mu_2^- \cdot \big[v_1^{J^\cdot(\mu_1^-)} - v_1^{J^\cdot(\mu_2^-)}\big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\sum_{\bar{w}_k\in\mathbb{W}}\kappa_k(\mu_2^-)\cdot(\mu_1^- - \mu_2^-)\cdot v_2^{J^\cdot(\mu_2^-)}\\
&-\frac{1}{2}\sum_{\bar{w}_k\in\mathbb{W}}\kappa_k(\mu_2^-)\cdot\Big\{(\mu_1-\mu_2)\cdot v_2+\mu_1^-\cdot\big[v_2^{J^\cdot(\mu_2^-)}-v_2^{J^\cdot(\mu_1^-)}\big]\Big\}\\
&=\frac{1}{2}\sum_{\bar{w}_k\in\mathbb{W}}\big[\kappa_k(\mu_1^-)+\kappa_k(\mu_2^-)\big]\cdot(\mu_1-\mu_2)\cdot(v_1-v_2)\\
&-\frac{1}{2}\sum_{\bar{w}_k\in\mathbb{W}}\big[\kappa_k(\mu_2^-)\cdot v_1-\kappa_k(\mu_1^-)\cdot v_2\big]\cdot(\mu_1-\mu_2)\\
&+\frac{1}{2}\sum_{\bar{w}_k\in\mathbb{W}}\kappa_k(\mu_1^-)\cdot\Big\{(\mu_1^- - \mu_2^-)\cdot v_1^{J^\cdot(\mu_1^-)}-\mu_2^-\cdot\big[v_1^{J^\cdot(\mu_1^-)}-v_1^{J^\cdot(\mu_2^-)}\big]\Big\}\\
&-\frac{1}{2}\sum_{\bar{w}_k\in\mathbb{W}}\kappa_k(\mu_2^-)\cdot\Big\{(\mu_1^- - \mu_2^-)\cdot v_2^{J^\cdot(\mu_2^-)}+\mu_1^-\cdot\big[v_2^{J^\cdot(\mu_2^-)}-v_2^{J^\cdot(\mu_1^-)}\big]\Big\}\\
&=\frac{1}{2}\sum_{\bar{w}_k\in\mathbb{W}}\big[\kappa_k(\mu_1^-)+\kappa_k(\mu_2^-)\big]\cdot(\mu_1-\mu_2)\cdot(v_1-v_2)\\
&-\frac{1}{2}\sum_{\bar{w}_k\in\mathbb{W}}\Big\{\kappa_k(\mu_1^-)\cdot\big[\mu_2^-\cdot v_2^{J^\cdot(\mu_2^-)}-\mu_1^-\cdot v_2^{J^\cdot(\mu_1^-)}\big]+\kappa_k(\mu_2^-)\cdot\big[\mu_1^-\cdot v_1^{J^\cdot(\mu_1^-)}-\mu_2^-\cdot v_1^{J^\cdot(\mu_2^-)}\big]\Big\}\\
&+\frac{1}{2}\sum_{\bar{w}_k\in\mathbb{W}}\kappa_k(\mu_1^-)\cdot\Big\{\mu_1^-\cdot v_1^{J^\cdot(\mu_1^-)}-\mu_2^-\cdot v_1^{J^\cdot(\mu_2^-)}-2\mu_2^-\cdot\big[v_1^{J^\cdot(\mu_1^-)}-v_1^{J^\cdot(\mu_2^-)}\big]\Big\}\\
&-\frac{1}{2}\sum_{\bar{w}_k\in\mathbb{W}}\kappa_k(\mu_2^-)\cdot\Big\{2\mu_1^-\cdot\big[v_2^{J^\cdot(\mu_2^-)}-v_2^{J^\cdot(\mu_1^-)}\big]+\mu_1^-\cdot v_2^{J^\cdot(\mu_1^-)}-\mu_2^-\cdot v_2^{J^\cdot(\mu_2^-)}\Big\}\\
&=\frac{1}{2}\sum_{\bar{w}_k\in\mathbb{W}}\big[\kappa_k(\mu_1^-)+\kappa_k(\mu_2^-)\big]\cdot(\mu_1-\mu_2)\cdot(v_1-v_2)\\
&-\frac{1}{2}\sum_{\bar{w}_k\in\mathbb{W}}\big[\kappa_k(\mu_1^-)-\kappa_k(\mu_2^-)\big]\cdot\big[\mu_2^-\cdot v_1^{J^\cdot(\mu_2^-)}-\mu_1^-\cdot v_1^{J^\cdot(\mu_1^-)}+\mu_2^-\cdot v_2^{J^\cdot(\mu_2^-)}-\mu_1^-\cdot v_2^{J^\cdot(\mu_1^-)}\big]\\
&-\sum_{\bar{w}_k\in\mathbb{W}}\Big\{\kappa_k(\mu_2^-)\cdot\mu_1^-\cdot\big[v_2^{J^\cdot(\mu_2^-)}-v_2^{J^\cdot(\mu_1^-)}\big]+\kappa_k(\mu_1^-)\cdot\mu_2^-\cdot\big[v_1^{J^\cdot(\mu_1^-)}-v_1^{J^\cdot(\mu_2^-)}\big]\Big\}.
\end{aligned}
$$

Hence by Assumption 14(iv) we conclude that

$$
(\mu_1^- - \mu_2^-)\cdot(v_1^- - v_2^-)\leq\frac{1}{2}\sum_{\bar{w}_k\in\mathbb{W}}\big[\kappa_k(\mu_1^-)+\kappa_k(\mu_2^-)\big]\cdot(\mu_1-\mu_2)\cdot(v_1-v_2)
$$
$$
\text{provided that } (\mu_1-\mu_2)\cdot(v_1-v_2)=\big[\Phi_k(w,\mu_1^-)-\Phi_k(w,\mu_2^-)\big]\cdot(v_1-v_2)\leq 0. \tag{45}
$$

Step 3: Backward Propagation of Monotonicity. Suppose that for some $k\in[1:n]$ we have

$$
(\mu_1(T_{k+1}-,w)-\mu_2(T_{k+1}-,w))\cdot(v_1(T_{k+1}-,w)-v_2(T_{k+1}-,w))\leq 0\quad\text{for all } w\in\mathbb{W}^n. \tag{46}
$$

By regularity, the fundamental theorem of calculus applies and using (43) we obtain

$$
\begin{aligned}
&(\mu_1(T_k,w)-\mu_2(T_k,w))\cdot(v_1(T_k,w)-v_2(T_k,w))\\
&=(\mu_1(T_{k+1}-,w)-\mu_2(T_{k+1}-,w))\cdot(v_1(T_{k+1}-,w)-v_2(T_{k+1}-,w))
\end{aligned}
$$

$$- \int_{T_k}^{T_{k+1}} \frac{\mathrm{d}}{\mathrm{d}t}\big[(\mu_1(s,w)-\mu_2(s,w))\cdot(v_1(s,w)-v_2(s,w))\big]\mathrm{d}s \le 0 \quad \text{for all } w \in \mathbb{W}^n.$$

Thus (45) implies that (46) also holds at time T_k-, i.e.

$$(\mu_1(T_k-,w)-\mu_2(T_k-,w))\cdot(v_1(T_k-,w)-v_2(T_k-,w)) \le 0 \quad \text{for all } w \in \mathbb{W}^n.$$

Since (46) is satisfied for $k=n$ by Assumption 14(i), it follows by induction that

$$\begin{aligned}&\big(\mu_1(T_{k+1}-,w)-\mu_2(T_{k+1}-,w)\big)\cdot\big(v_1(T_{k+1}-,w)\\ &\quad - v_2(T_{k+1}-,w)\big) \le 0 \quad \text{for all } k \in [0:n],\ w \in \mathbb{W}^n.\end{aligned} \tag{47}$$

Step 4: Forward Propagation of Uniqueness. Suppose that for some $k \in [0:n]$ we have

$$\mu_1(T_k,w)=\mu_2(T_k,w) \quad \text{for all } w \in \mathbb{W}^n. \tag{48}$$

Using the fundamental theorem of calculus, (43) and (47) we find that

$$\begin{aligned}&\gamma \cdot \int_{T_k}^{T_{k+1}} \|\mu_1(s,w)-\mu_2(s,w)\|^\alpha \mathrm{d}s\\ &\le \int_{T_k}^{T_{k+1}} \Big(\frac{\mathrm{d}}{\mathrm{d}t}\big[(\mu_1(s,w)-\mu_2(s,w))\cdot(v_1(s,w)-v_2(s,w))\big]\Big)\mathrm{d}s\\ &=(\mu_1(T_{k+1}-,w)-\mu_2(T_{k+1}-,w))\cdot(v_1(T_{k+1}-,w)-v_2(T_{k+1}-,w))\\ &\quad-(\mu_1(T_k,w)-\mu_2(T_k,w))\cdot(v_1(T_{k+1}-,w)-v_2(T_{k+1}-,w)) \le 0 \quad \text{for all } w \in \mathbb{W}^n.\end{aligned}$$

As a consequence, we have

$$\mu_1(t,w)=\mu_2(t,w) \quad \text{for all } (t,w) \in [T_k,T_{k+1}\rangle \times \mathbb{W}^n$$

and, in particular, $\mu_1(T_{k+1}-,w) = \mu_2(T_{k+1}-,w)$, implying $\mu_1(T_{k+1},w) = \mu_2(T_{k+1},w)$ for all $w \in \mathbb{W}^n$. Since $\mu_1(0)=m_0=\mu_2(0)$ by (E5), (48) is satisfied for $k=0$, and we conclude that $\mu_1=\mu_2$. Finally, since Assumption 14 subsumes Assumptions 11(i)-(ii), the arguments in the proof of Lemma A.2 yield $v_1=v_2$, and the proof is complete. □

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