Time Evolution of Quantum Resonance States

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Abstract

Let H_0 be self-adjoint with E_0 a possibly degenerate eigenvalue embedded in the continuous spectrum $\sigma_c(H_0)$ and ψ_0 a normalized eigenfunction. For a certain class of perturbations W and $H = H_0 + W$ we investigate the asymptotics of the (naive) resonance state $e^{-itH}\psi_0$ in the limit $W \to 0$. This amplifies previous results of Merkli and Sigal.

1 Introduction and Results

Let H_0 be a self-adjoint Hamiltonian in a complex Hilbert space $\langle \mathcal{H}, \|\cdot\|\rangle$. Let E_0 be a possibly degenerate eigenvalue of H_0, embedded in its continuous spectrum $\sigma_c(H_0)$, with (normalized) eigenfunction ψ_0: $H_0\psi_0 = E_0\psi_0$. Let Π_0 be the orthogonal projection onto $\mathrm{Ker}(H_0 - E_0)$ and

$$\overline{\Pi}_0 := \mathbb{1} - \Pi_0 \,. \tag{1.1}$$

Now let H_0 be perturbed by an operator W, where

(C0) W is symmetric in $\mathcal{H}$, $H := H_0 + W$ is self-adjoint in $\mathcal{H}$ and $\mathcal{D}(H_0) = \mathcal{D}(H)$.

For a small perturbation W – where "small" is specified by (C2) through (C5) below – E_0 should turn into a resonance; see [AHSk] for results in this direction. More naively, one may directly investigate $e^{-itH}\psi_0$. One expects that $e^{-itH}\psi_0$ shows the typical behavior of a resonance: Up to the order of the expected lifetime (given by the Fermi golden rule) $\|\Pi_0 e^{-itH}\psi_0\|$ decays (roughly) exponentially; see (1.23). For large times $e^{-itH}\psi_0$ may tunnel completely to the spectral complement $\mathrm{Ran}\,\overline{\Pi}_0$, but there it is (in some weak sense) *outgoing*. This can be defined to mean that $e^{-itH}\psi_0$ belongs to a subspace of large spectral values for an operator A conjugate to H (as specified below). Thus the last statement may be rephrased by saying that the weighted norm $\|\langle A\rangle^{-\alpha}\overline{\Pi}_0 e^{-itH}\psi_0\|$ is small uniformly in time.

Such an approach was introduced in [MerSi], following [SoWei]. It is the main purpose of this paper to show that the statements above - although they are not explicitly proved in [MerSi] - actually follow from the estimates of [MerSi] by standard techniques.

To formulate our results more precisely, we shall introduce some notations and briefly recall the central result of [MerSi]. For any bounded interval I let $g_I \in C_0^\infty$ be a smoothed out version of the characteristic function 1_Δ, i.e.

$$g_I(\mu) = \begin{cases} 1 & , \quad \mu \in I \\ 0 & , \quad \mu \text{ outside some neighborhood of } I \end{cases} \,. \tag{1.2}$$

We fix some neighborhood Δ of E_0 (assumed to contain no eigenvalue of H_0 different from E_0) and an interval $\Delta \subset \Delta'$ a little bigger than Δ. $g_\Delta(H)$ is a smoothed out version of the

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spectral projection $E_\Delta(H) = 1_\Delta(H)$. We assume that $\operatorname{supp} g_\Delta \cap \operatorname{supp}(1 - g_{\Delta'}) = \emptyset$ and Δ' also contains no eigenvalues of H_0 different from E_0. We set

$$\overline{g_I} := \mathbb{1} - g_I \tag{1.3}$$

for any interval I and

$$\overline{H} := \overline{\Pi}_0 H \overline{\Pi}_0\,. \tag{1.4}$$

Assume that there exists a self-adjoint operator A in $\mathcal{H}$ and $\alpha > 2$ such that

(C1) $\|\langle A\rangle^\alpha \Pi_0\| < \infty, \quad \langle A\rangle := (|A|^2 + 1)^{1/2}\,.$

Next we state, in addition to (C0), further conditions on W.

(C2) $\kappa := \|\langle A\rangle^\alpha W \Pi_0\| < \infty\,.$

Remark: κ is a measure for the size of the perturbation W. In this work we are interested in κ small.

(C3) The k-fold commutators $ad_A^k(H)$, recursively defined by $ad_A(\cdot) := [A, \cdot]$, are H-bounded for all $k \in \{1, 2, \ldots, n\}$ and some $n > \alpha + 1 > 3$, uniformly in $\kappa < \kappa_0$ for some κ_0 sufficiently small.

(C4) For all $\phi \in \mathcal{D}(\langle A\rangle^\alpha)$ and $t \geq 0$ the following local decay estimate holds:

$$\|\langle A\rangle^{-\alpha} e^{-it\overline{H}} g_{\Delta'}(\overline{H})\overline{\Pi}_0 \phi\| \leq C\,\langle t\rangle^{-\alpha} \|\langle A\rangle^\alpha \overline{\Pi}_0 \phi\|$$

for some $C < \infty$, independent of t and $\kappa < \kappa_0$ for some κ_0 small enough.
$\langle t\rangle := (1 + |t|^2)^{1/2}$ $(t \in \mathbb{R})$; $\overline{\Pi}_0,\ \overline{H},\ \Delta',\ g_{\Delta'}$ are defined in (1.1) - (1.4).

Remark: (C4) is a consequence of the Mourre estimate.

(C5) Non vanishing of the Fermi golden rule holds, i.e.

$$\Gamma := \pi \cdot \Pi_0 W \delta(\overline{H} - E_0)\overline{\Pi}_0 W \Pi_0\,, \quad \Gamma \restriction \operatorname{Ran}\Pi_0 \geq c_0 \kappa^2 \tag{1.5}$$

for some $c_0 > 0$, uniformly in $\kappa < \kappa_0$ for some κ_0 sufficiently small.

Remark: In analogy to the well known formula

$$\lim_{\varepsilon \downarrow 0} (x - i\varepsilon)^{-1} = \text{P.V.}\left(\frac{1}{x}\right) + i \cdot \pi\delta(x)\,,$$

which holds in the space of tempered distributions, we define

$$\langle A\rangle^{-\alpha}\delta(\overline{H} - E_0)\overline{\Pi}_0\langle A\rangle^{-\alpha} := \frac{1}{\pi} \cdot \operatorname{Im}\left(\text{s-}\lim_{\varepsilon\downarrow 0}\langle A\rangle^{-\alpha}(\overline{H} - E_0 - i\varepsilon)^{-1}\overline{\Pi}_0\langle A\rangle^{-\alpha}\right), \tag{1.6}$$

$$\langle A\rangle^{-\alpha}\text{P.V.}(\overline{H} - E_0)^{-1}\overline{\Pi}_0\langle A\rangle^{-\alpha} := \operatorname{Re}\left(\text{s-}\lim_{\varepsilon\downarrow 0}\langle A\rangle^{-\alpha}(\overline{H} - E_0 - i\varepsilon)^{-1}\overline{\Pi}_0\langle A\rangle^{-\alpha}\right), \tag{1.7}$$

whenever the limits on the r.h.s. exist. In fact the existence of the limits follows from (C4) (see A Appendix). Obviously $\Gamma \geq 0$. The actual assumption in (C5) is the positivity of Γ on $\operatorname{Ran}\Pi_0$. Note that $\Gamma = O(\kappa^2)$ by (C2), if the limit in (1.6) exists.

So the class of perturbations in question is

$$\mathcal{W}_{\kappa_0} := \{\, W \mid W \text{ satisfies (C0) – (C5) for } \kappa < \kappa_0 \}\,.$$

Assuming (C0) - (C5), results about time evolution of resonance states have been proved in [MerSi, Theorem 2.1]. These results are formulated in terms of the bounded operator [MerSi, p.559/560 and (A.17)]

$$\Lambda := E_0\Pi_0 + \Pi_0 W B \Pi_0 - \Pi_0 W(\overline{H} - E_0 - i0)^{-1} g_{\Delta'}(\overline{H})\overline{\Pi}_0 W \Pi_0 \,.$$

For κ sufficiently small,

$$B \;:=\; \big(\mathbb{1} - \overline{g_{\Delta'}}(\overline{H})\overline{\Pi}_0 g_\Delta(H)\big)^{-1} \;=\; \mathbb{1} + O(\kappa) \tag{1.8}$$

exists by a Neumann series expansion, because $\overline{g_{\Delta'}}(\overline{H})\overline{\Pi}_0 g_\Delta(H) = O(\kappa)$ $(\kappa \to 0)$; see [MerSi, Proposition 3.1]. The main result of [MerSi] is

Theorem 1.1 [MerSi, part of Theorem 2.1]

Assume (C0) - (C5). Let $\psi(t) = e^{-iHt}\psi(0)$ with initial condition $\psi(0) \in \mathrm{Ran}(E_\Delta(H)) \cap \mathcal{D}(\langle A\rangle^\alpha)$. Let $0 \le \beta < \min\{\frac{1}{2}, \alpha - 2\}$. Then there exists a constant κ_0 (depending on $\alpha,\ \beta,\ |\Delta|$) such that for $t \ge 0$ one has the following expansion:

$$\psi(t) \;=\; B\Pi_0\psi(t) + \psi_{disp}(t) \qquad (\kappa \to 0) \quad \text{with}$$

$$\psi_{disp}(t) \;:=\; B g_{\Delta'}(\overline{H})\overline{\Pi}_0\psi(t)\,, \tag{1.9}$$

$$\Pi_0\psi(t) \;=\; e^{-i\Lambda t}\Pi_0\psi(0) + O(\kappa^{1-4\beta}\langle t\rangle^{-\beta})\,, \quad (\kappa \to 0) \tag{1.10}$$

$$\|\langle A\rangle^{-\alpha}\psi_{disp}(t)\| \;\le\; C\big(\|\langle A\rangle^\alpha \overline{\Pi}_0\psi(0)\|\,\langle t\rangle^{-\alpha} + \kappa^{1-2\beta}\langle t\rangle^{-\beta}\big)\,, \tag{1.11}$$

uniformly in $W \in \mathcal{W}_{\kappa_0}$.

Remark: Under the conditions outlined in this section, H has no eigenvalues in Δ (cf. [MerSi, Corollary 2.2]).

To understand the action of $e^{-i\Lambda t}$ in more detail, one needs a suitable expansion of $e^{-i\Lambda t}$. As a preparation, we collect results of [MerSi, Proposition 3.3] and [MerSi, A. Appendix, p.573 ff.]:

Proposition 1.2 [MerSi, cp. Proposition 3.3]

Λ has the representation

$$\Lambda \;=\; E_0\Pi_0 + \Pi_0 W\Pi_0 - \Pi_0 W\left(P.V.(\overline{H} - E_0)^{-1}\right)\overline{\Pi}_0 W\Pi_0 - i\Gamma + K\,, \tag{1.12}$$

where

$$K = O(\kappa^3) \qquad (\kappa \to 0)\,, \tag{1.13}$$

$$\Gamma = O(\kappa^2) \qquad (\kappa \to 0)\,, \tag{1.14}$$

$$\Pi_0 W\big(P.V.(\overline{H} - E_0)^{-1}\big)\overline{\Pi}_0 W\Pi_0 = O(\kappa^2) \qquad (\kappa \to 0)\,,$$

uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small.

Remark: Explicitly

$$K \;:=\; \Pi_0 W\overline{g_{\Delta'}}(\overline{H})\overline{\Pi}_0 \int_{\mathbb{C}} (\overline{H} - z)^{-1} W\Pi_0\,\Pi_0 W\,(H - z)^{-1}\,\frac{1}{2\pi}\,\big(\overline{\partial}\tilde{g}_\Delta\big)(z)\;dx\;dy\,\Pi_0 \;+$$

$$+\;\Pi_0 W \sum_{j=2}^{\infty}\Big(\overline{\Pi}_0\overline{g_{\Delta'}}(\overline{H})\,g_\Delta(H)\Big)^j\,\Pi_0\,, \tag{1.15}$$

where $\tilde{g}_\Delta$ is an almost analytic extension of g_Δ in the sense of Lemma 3.1. Our proof does not need this explicit representation of K.

Setting

$$G := E_0\Pi_0 + \Pi_0 W \Pi_0 - \Pi_0 W \left(P.V.(\overline{H} - E_0)^{-1}\right) \overline{\Pi}_0 W \Pi_0 \,, \qquad Q := G - i\Gamma \,, \tag{1.16}$$

we have by (1.12) $\Lambda = Q + K$.

Now we are ready to describe in more detail the asymptotic behavior of $\Pi_0\psi(t)$ (i.e. the decay of the resonance state), valid up to the expected lifetime, which is $O(\kappa^{-2})$. The following theorems are the main result of our paper.

Theorem 1.3 Assume (C0) - (C5). Let $\psi(t) = e^{-iHt}\psi(0)$ with $\psi(0) \in \mathrm{Ran}(E_\Delta(H)) \cap \mathcal{D}(\langle A\rangle^\alpha)$. Then there exists $\epsilon \in (0,1]$ and $C > 0$ such that for $0 \le t \le C\kappa^{-2}$

$$\Pi_0\psi(t) = e^{-iGt}e^{-\Gamma t}e^{-iKt}\Pi_0\psi(0) + O(\kappa^2 t) + O(\kappa^\epsilon) \,, \qquad (\kappa \to 0) \tag{1.17}$$

where

$$e^{-\Gamma t}e^{-iKt}\Pi_0\psi(0) = e^{-\Gamma t}\Pi_0\psi(0) + e^{-\Gamma t}O(\kappa)\Pi_0\psi(0) \,. \qquad (\kappa \to 0) \tag{1.18}$$

In particular

$$\|\Pi_0\psi(t)\| = \|e^{-\Gamma t}\Pi_0\psi(0)\| + O(\kappa^2 t) + O(\kappa) + O(\kappa^\epsilon) \,, \qquad (\kappa \to 0) \tag{1.19}$$

where

$$e^{-c\kappa^2 t}\|\Pi_0\psi(0)\| \le \|e^{-\Gamma t}\Pi_0\psi(0)\| \le e^{-c_0\kappa^2 t}\|\Pi_0\psi(0)\| \tag{1.20}$$

for some $c_0 > 0$, $c > 0$, uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small. (See (1.16), (1.15), (1.5) for the definitions of $G,\ K,\ \Gamma$.)

We shall now show that $\psi(0)$ in Theorem 1.3 can be replaced by ψ_0 and that $\psi(t)$ is outgoing (in the sense described above).

Theorem 1.4 Let E_0 be an embedded eigenvalue of H_0, $H_0\psi_0 = E_0\psi_0$. Assume (C0) - (C5). Then for $0 \le t \le C\kappa^{-2}$ and some $\epsilon \in (0,1]$ the results of Theorem 1.3 yield

$$\Pi_0 e^{-itH}\psi_0 = e^{-iGt}e^{-\Gamma t}e^{-iKt}\psi_0 + O(\kappa^2 t) + O(\kappa) + O(\kappa^\epsilon), \ (\kappa \to 0) \tag{1.21}$$

$$e^{-\Gamma t}e^{-iKt}\psi_0 = e^{-\Gamma t}\psi_0 + O(\kappa) \,, \qquad (\kappa \to 0) \tag{1.22}$$

$$\|\Pi_0 e^{-itH}\psi_0\| = \|e^{-\Gamma t}\psi_0\| + O(\kappa^2 t) + O(\kappa) + O(\kappa^\epsilon) \,, \qquad (\kappa \to 0) \tag{1.23}$$

$$e^{-c\kappa^2 t}\|\psi_0\| \le \|e^{-\Gamma t}\psi_0\| \le e^{-c_0\kappa^2 t}\|\psi_0\| \tag{1.24}$$

for some $c_0 > 0$, $c > 0$, uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small. For $t \ge 0$ and some $\epsilon \in (0,1]$

$$\|\langle A\rangle^{-\alpha}\overline{\Pi}_0 e^{-itH}\psi_0\| = O(\kappa^\epsilon) + O(\kappa) \,, \qquad (\kappa \to 0) \tag{1.25}$$

uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small.

We shall prove these Theorems in Section 2 and 3.

2 Proof of Theorem 1.3

In the following $\langle B(\mathcal{H}), \|\cdot\|\rangle$ will denote the Banach space of bounded linear operators on $\mathcal{H}$, $\sigma(\cdot)$ the spectrum of an operator and C a generic positive constant, independent of κ and t.

We have the following decomposition of $e^{-i\Lambda t}$:

Lemma 2.1 For $0 \le t \le C\kappa^{-2}$ with some $C > 0$ the following is true:

$$e^{-i\Lambda t} = e^{-iGt} e^{-\Gamma t} e^{-iKt} + F(t) \tag{2.1}$$

where $B(\mathcal{H}) \ni F(t) = F_1(t) + F_2(t) + F_3(t) = O(\kappa^2 t) \quad (\kappa \to 0)$ and

$$F_1(t) = O(\kappa^2 t) \qquad (\kappa \to 0)\,, \tag{2.2}$$

$$F_2(t) = O(\kappa^3 t) \qquad (\kappa \to 0)\,, \tag{2.3}$$

$$F_3(t) = O(\kappa^5 t^2) \qquad (\kappa \to 0)\,, \tag{2.4}$$

uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small.
$G,\ K,\ \Gamma$ are defined in (1.16), (1.15), (1.5).

Remarks on Γ:

Since $\Gamma = O(\kappa^2)$ is self-adjoint on $\mathcal{H}$, positive on $\operatorname{Ran}\Pi_0$ and (C5) holds, we have $\sup\sigma(\Gamma) = \|\Gamma\| \le c\kappa^2$ for some $c > 0$, and for $\phi \in \mathcal{H}$ and $t \ge 0$ we get via functional calculus

$$e^{-c\kappa^2 t}\|\Pi_0\phi\| \le \|e^{-\Gamma t}\Pi_0\phi\| \le e^{-c_0\kappa^2 t}\|\Pi_0\phi\|\,, \quad e^{-c\kappa^2 t} \le \|e^{-\Gamma t}\| \le 1 \tag{2.5}$$

for some $c_0 > 0\,,\ c > 0$. In particular for any $0 < C < \infty$ and $0 \le t \le C\kappa^{-2}$

$$e^{\Gamma t} = O(1)\,, \qquad (\kappa \to 0) \tag{2.6}$$

uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small.

To prove Lemma 2.1, we will need

Lemma 2.2 Let $K = O(\kappa^3)$ $(\kappa \to 0)$ as in (1.13). Let $\varepsilon > 0$. Then for any $0 < C < \infty$ and $0 \le t \le C\kappa^{-3+\varepsilon}$ we have

$$e^{\pm iKt} = \mathbb{1} + O(\kappa^\varepsilon)\,, \qquad (\kappa \to 0) \tag{2.7}$$

uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small.

Proof: Since $K \in B(\mathcal{H})$ we have $\|e^{\pm iKt} - \mathbb{1}\| \le \sum_{j=1}^{\infty} \frac{1}{j!}\|Kt\|^j$. Then for $0 \le t \le C\kappa^{-3+\varepsilon}$ we have $Kt = O(\kappa^\varepsilon)$ $(\kappa \to 0)$ by (1.13). Thus

$$e^{\pm iKt} = \mathbb{1} + O(\kappa^\varepsilon)\sum_{j=1}^{\infty}\frac{1}{j!} = \mathbb{1} + O(\kappa^\varepsilon)\,, \qquad (\kappa \to 0)$$

uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small.

$\square$

Proof of Lemma 2.1: In general $Q,\ G,\ K$ and Γ do not commute. But

$$e^{-i\Lambda t} \quad = \quad e^{-iQt}e^{-iKt}\,R(t)\,, \tag{2.8}$$

where the operator-valued remainder $R(t) := e^{iKt}e^{iQt}e^{-i\Lambda t}$ solves the initial value problem $\frac{d}{dt}\,R(t) = i\,e^{iKt}[K\,,\,e^{iQt}]\,e^{-i\Lambda t}\,,\quad R(0) = \mathbb{1}$. Thus

$$R(t) \quad = \quad \mathbb{1} \; + \; \int_0^t i\,e^{iKs}[K\,,\,e^{iQs}]\,e^{-i\Lambda s}\;ds\,. \tag{2.9}$$

Analogously

$$e^{-iQt} \quad =: \quad e^{-iGt}e^{-\Gamma t}\,\tilde{R}(t)\,, \tag{2.10}$$

where

$$\tilde{R}(t) \quad = \quad \mathbb{1} \; + \; \int_0^t e^{\Gamma s}\,[\Gamma\,,\,e^{iGs}]\,e^{-iQs}\;ds\,. \tag{2.11}$$

Using (2.8) and (2.10) we obtain

$$e^{-i\Lambda t} \quad = \quad \Big(e^{-iGt}e^{-\Gamma t}\,\tilde{R}(t)\Big)e^{-iKt}\,R(t) \;\; = \;\; e^{-iGt}e^{-\Gamma t}e^{-iKt} + F(t)$$

where, using (2.11) and (2.9), $\quad F(t) = F_1(t) + F_2(t) + F_3(t) \quad$ with

$$F_1(t) \quad := \quad e^{-iGt}e^{-\Gamma t}\int_0^t e^{\Gamma s}\,[\Gamma\,,\,e^{iGs}]\,e^{-iQs}\;ds\;e^{-iKt}\,,$$

$$F_2(t) \quad := \quad e^{-iGt}e^{-\Gamma t}e^{-iKt}\int_0^t i\,e^{iKs}[K\,,\,e^{iQs}]\,e^{-i\Lambda s}\;ds\,, \tag{2.12}$$

$$F_3(t) \quad := \quad F_1(t)\cdot e^{iKt}e^{\Gamma t}e^{iGt}F_2(t)\,. \tag{2.13}$$

We shall now estimate $F_1(t),\ F_2(t),\ F_3(t)$.

Upper Bounds on $F_1(t)$: To estimate $F_1(t)$, we observe that G is self-adjoint. Since $Q \in B(\mathcal{H})$, we have the representation $e^{-iQt} = \lim\limits_{n\to\infty}(e^{-i\mathrm{Re}Q\,t/n}e^{\mathrm{Im}Q\,t/n})^n$, due to the Lie product formula [RS, Theorem VIII.29], which shows

$$\|e^{-iQs}\| \leq \|e^{\mathrm{Im}Q\,s}\| \quad \text{ with } \quad \mathrm{Im}Q \overset{(1.16)}{=} -\Gamma\,. \qquad (s \geq 0) \tag{2.14}$$

Thus

$$\|F_1(t)\| \quad \leq \quad \|e^{-\Gamma t}\|\;\|e^{-iKt}\|\int_0^t \|e^{\Gamma s}\|\;\Big\|[\Gamma\,,\,e^{iGs}]\Big\|\;\|e^{-\Gamma s}\|\;ds\,.$$

Using (1.14), we have $[\Gamma\,,\,e^{iGt}] = O(\kappa^2)$ $(\kappa \to 0)$. Using in addition Lemma 2.2 and (2.6), which holds for $0 \leq t \leq C\kappa^{-2}$, we obtain (2.2).

□

Upper Bounds on $F_2(t)$: Equation (2.12) yields

$$\|F_2(t)\| \quad \leq \quad \|e^{-iKt}\|\int_0^t \|e^{iKs}\|\;\Big(2\;\|K\|\;\|e^{-\mathrm{Im}Q\,s}\|\Big)\|e^{\mathrm{Im}\Lambda s}\|\;ds\,. \tag{2.15}$$

We have $\mathrm{Im}\Lambda \overset{(1.12)}{=} -\Gamma + \mathrm{Im}K \overset{(1.13)}{\underset{(1.14)}{=}} O(\kappa^2) + O(\kappa^3) = O(\kappa^2) \quad (\kappa \to 0)$, uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small. Hence for any $0 < C < \infty$ and $0 \le s \le C\kappa^{-2}$

$$\|e^{-i\Lambda s}\| \le \|e^{\mathrm{Im}\Lambda s}\| = O(1)\,, \qquad (\kappa \to 0) \tag{2.16}$$

uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small. Using (1.13), (2.7), (2.14) and (2.16) in (2.15), we obtain (2.3).

□

Upper Bounds on $F_3(t)$: By use of (2.13) it suffices to show that $e^{iKt}e^{\Gamma t}e^{iGt} = O(1)$ $(\kappa \to 0)$ for $0 \le t \le C\kappa^{-2}$, uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small. This follows from (2.7), (2.6) and the fact that G is self-adjoint. Thus by (2.2) and (2.3) we obtain (2.4).

□

Finally by (2.2), (2.3) and (2.4) we arrive at $F(t) = O(\kappa^2 t)$ $(\kappa \to 0)$ for $0 \le t \le C\kappa^{-2}$ with some $C > 0$. This completes the proof of Lemma 2.1.

■

Proof of Theorem 1.3: By Theorem 1.1 (1.10) we have for $t \ge 0$

$$\Pi_0\psi(t) = e^{-i\Lambda t}\Pi_0\psi(0) + f(t) \tag{2.17}$$

with

$$\mathcal{H} \ni f(t) = O(\kappa^{1-4\beta}\langle t\rangle^{-\beta}) \qquad (\kappa \to 0)$$

for $\beta \in \big[0, \min\{\frac{1}{2}, \alpha - 2\}\big)$, uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small. Then, possibly decreasing β to $\beta \in \big[0, \min\{\frac{1}{4}, \alpha - 2\}\big)$, there exists an $\epsilon \in (0, 1]$ such that $f(t) = O(\kappa^\epsilon)$ $(\kappa \to 0)$ uniformly in $t \ge 0$ and $\kappa < \kappa_0$.
Substitution of (2.1) into (2.17) yields

$$\Pi_0\psi(t) \quad = \quad e^{-iGt}e^{-\Gamma t}e^{-iKt}\Pi_0\psi(0) + F(t)\Pi_0\psi(0) + f(t)\,,$$

which shows (1.17) by use of Lemma 2.1. (1.18) is given by Lemma 2.2 with $\varepsilon = 1$. Substitution of (1.18) into (1.17) yields

$$\|\Pi_0\psi(t)\| = \|e^{-iGt}e^{-\Gamma t}\Pi_0\psi(0)\| + O(\kappa^2 t) + O(\kappa) + O(\kappa^\epsilon)\,, \qquad (\kappa \to 0)$$

which proves (1.19), since G is self-adjoint. (1.20) follows from (2.5). This completes the proof of Theorem 1.3.

■

3 Proof of Theorem 1.4

A convenient functional calculus for C_0^∞-functions of self-adjoint operators in Hilbert spaces is due to B. Helffer and J. Sjöstrand [HeSj], using the concept of almost analytic extensions. This calculus can be generalized to smooth functions with non-compact support, but satisfying certain growth conditions. Here we follow [DeGé, Chapter C.2 and C.3]. We use the notations $\overline{\partial} := \partial_x + i\partial_y$, $\mathbb{C} \ni z = x + iy$.

Lemma 3.1 [DeGé, Proposition C.2.2]

Let $\rho \in \mathbb{R}$. Define the following class of smooth functions:

$$\mathbf{S}^\rho := \big\{ f \in C^\infty(\mathbb{R}) \,\big|\, |\partial_\lambda^k f(\lambda)| \le C_k \langle\lambda\rangle^{\rho-k},\ k \ge 0 \big\} \tag{3.1}$$

Then for $f \in \mathbf{S}^\rho$, there exists an almost analytic extension $\tilde f \in C^\infty(\mathbb{C})$ of f in the sense, that $\tilde f\big|_{\mathbb{R}} = f$,

$$|\big(\overline{\partial}\tilde f\big)(z)| \le C_k \langle x\rangle^{\rho-1-k}|y|^k\,, \qquad (k\in\mathbb{N}) \tag{3.2}$$

$$\operatorname{supp}\tilde f \subset \big\{x+iy\,\big|\,|y|\le C\langle x\rangle\big\} \quad \text{and}$$

$$f(\lambda) \;=\; \int\limits_{\mathbb{C}} (\lambda - z)^{-1}\,\frac{1}{2\pi}\,\big(\overline{\partial}\tilde f\big)(z)\;dx\;dy\,. \qquad (\lambda\in\mathbb{R})$$

In particular for any self-adjoint operator T and $f \in \mathbf{S}^\rho$

$$f(T) = \int\limits_{\mathbb{C}} (T-z)^{-1}\,\frac{1}{2\pi}\,\big(\overline{\partial}\tilde f\big)(z)\;dx\;dy\,. \tag{3.3}$$

Remark: If $f \in \mathbf{S}^\rho$ with compact support, we can choose $\tilde f$ with compact support, i.e. $\tilde f \in C_0^\infty(\mathbb{C})$.

We shall use the following result from [DeGé, Lemma C.3.2]:

Lemma 3.2 Let T, S be self-adjoint operators with $\big\|[T,S]\big\| < \infty$. If $f \in \mathbf{S}^\rho$ with $\rho < 1$, then

$$\big\|[f(T),S]\big\| \le C\,\big\|[T,S]\big\|$$

for some $C < \infty$.

Our proof of Theorem 1.4 (respectively of Proposition 3.3 and Proposition 3.4) uses the following expansions and estimates:

For linear operators T and S we formally have

$$T^m S \;=\; ST^m + \sum_{\substack{j+l=m\\ j,l\ge 1}} c_{jl}\; ad_T^j(S)T^l + ad_T^m(S) \tag{3.4}$$

for all $m\in\mathbb{N}$ and some $c_{jl}\in\mathbb{R}$. Furthermore for T self-adjoint and any Borel-function f

$$ad_T^m([f(T),S]) = f(T)ad_T^m(S) - ad_T^m(S)f(T)\,. \qquad (m\in\mathbb{N}) \tag{3.5}$$

The proofs of (3.4) and (3.5) are by induction.

Assume (C0) - (C3). Let $g \in C_0^\infty(\mathbb{R})$, let $\tilde g \in C_0^\infty(\mathbb{C})$ be an almost analytic extension of g in the sense of Lemma 3.1. By functional calculus and (C3)

$$\|[H,A](H-z)^{-1}\| \le c\,(1+|z|)\;|\mathrm{Im}\,z|^{-1} \tag{3.6}$$

for some $c<\infty$. By induction

$$\|ad_A^k\big((H-z)^{-1}\big)\| \le C\sum_{j=2}^{k+1}|\mathrm{Im}\,z|^{-j} \qquad (k\in\{1,\dots,n\}) \tag{3.7}$$

for some $C<\infty$, locally uniformly in z. Consequently for $k\in\{1,\dots,n\}$ and some $C<\infty$

$$\begin{aligned}\|ad_A^k\big(g(H)\big)\| \;&\overset{(3.3)}{\le}\; \int\limits_{\mathbb{C}} \|ad_A^k\big((H-z)^{-1}\big)\|\,\frac{1}{2\pi}\,\big|\big(\overline{\partial}\tilde g\big)(z)\big|\;dx\;dy\\ &\overset{(3.7)}{\le}\; C\int\limits_{\mathbb{C}}\sum_{j=2}^{k+1}|y|^{-j}\,\frac{1}{2\pi}\,\big|\big(\overline{\partial}\tilde g\big)(z)\big|\;dx\;dy \;\overset{(3.2)}{<}\; \infty\,.\end{aligned} \tag{3.8}$$

To prove Theorem 1.4, we will need

Proposition 3.3 Let E_0 be an embedded eigenvalue of H_0, $H_0\psi_0 = E_0\psi_0$. Assume (C0) - (C3). Let Ω be an interval around E_0 such that $\operatorname{supp} g_\Omega \subset \Delta$. Let $\psi(0) := g_\Omega(H)\psi_0$. Then $\psi(0)$ fulfils the requirements of Theorem 1.1 and Theorem 1.3, i.e. $\psi(0) \in \operatorname{Ran}(E_\Delta(H)) \cap \mathcal{D}(\langle A\rangle^\alpha)$. Furthermore

$$\psi(0) = \psi_0 + O(\kappa), \qquad (\kappa \to 0) \tag{3.9}$$

uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small.

Proof: $\psi(0) \in \operatorname{Ran}(E_\Delta(H))$ is obvious, since $\psi(0) := g_\Omega(H)\psi_0$ and $\operatorname{supp} g_\Omega \subset \Delta$. To prove $\psi(0) \in \mathcal{D}(\langle A\rangle^\alpha)$, it suffices to show $\langle A\rangle^\alpha g_\Omega(H)\Pi_0 \in B(\mathcal{H})$.

Let $N := \lfloor\alpha\rfloor \in \mathbb{N}$ be the floor of α, i.e. $\alpha = N + \varepsilon$ for some $\varepsilon \in [0, 1)$. Let

$$f(A) := (A + i)^{-N}\langle A\rangle^\alpha . \tag{3.10}$$

Then $\langle A\rangle^\alpha = (A + i)^N f(A)$ and $f \in \mathbf{S}^\varepsilon$; for the definition of $\mathbf{S}^\varepsilon$ see (3.1). Using (3.10) we get

$$\langle A\rangle^\alpha g_\Omega(H)\Pi_0 = (A + i)^N g_\Omega(H)f(A)\Pi_0 + (A + i)^N[f(A), g_\Omega(H)]\Pi_0 . \tag{3.11}$$

To estimate (3.11), we shall use the following spectral argument: Since for any $k \geq 0$ there exists $c \geq 0$ such that for all $\lambda \in \mathbb{R}$

$$(\lambda^2 + 1)^{k/2} \leq c(|\lambda|^k + 1) \qquad (k \geq 0),$$

functional calculus yields

$$\|(A + i)^N\phi\| = \|\langle A\rangle^N\phi\| \leq c\left(\|A^N\phi\| + \|\phi\|\right). \qquad \left(\phi \in \mathcal{D}(|A|^N)\right) \tag{3.12}$$

By use of (3.12) in (3.11), we obtain

$$\|\langle A\rangle^\alpha g_\Omega(H)\Pi_0\| \leq c\Big(\|A_1\| + \|A_2\| + \|A_3\| + \|A_4\|\Big)$$

for some $c < \infty$, where

$$A_1 := A^N g_\Omega(H)f(A)\Pi_0, \quad A_2 := A^N[f(A), g_\Omega(H)]\Pi_0, \quad A_3 := g_\Omega(H)f(A)\Pi_0,$$
$$A_4 := [f(A), g_\Omega(H)]\Pi_0 .$$

To finish the proof of $\langle A\rangle^\alpha g_\Omega(H)\Pi_0 \in B(\mathcal{H})$, we shall now prove the boundedness of A_j ($j \in \{1, 2, 3, 4\}$): By (C1) and functional calculus

$$A^k f(A)\Pi_0 \text{ is bounded for } k \in \{0, 1, \ldots, N\}. \tag{3.13}$$

This proves $A_3 \in B(\mathcal{H})$. By Lemma 3.2 we have for some $C < \infty$

$$\big\|[f(A), g_\Omega(H)]\big\| \leq C\big\|[A, g_\Omega(H)]\big\| \overset{(3.8)}{<} \infty . \tag{3.14}$$

Thus (3.14) yields $A_4 \in B(\mathcal{H})$. Applying (3.4) to $A^N g_\Omega(H)$ yields

$$\begin{aligned} A_1 &= g_\Omega(H)A^N f(A)\Pi_0 + \sum_{\substack{j+l=N\\ j,l\geq 1}} c_{lj}\, ad_A^j(g_\Omega(H))A^l f(A)\Pi_0 + \\ &\quad + ad_A^N(g_\Omega(H))f(A)\Pi_0 . \end{aligned}$$

Combining (3.13) with (3.8) and using $N < n$ (see (C3)), we get $A_1 \in B(\mathcal{H})$. First applying (3.4) to $A^N[f(A), g_\Omega(H)]$ and then using (3.5) gives

$$\begin{aligned} A_2 &= [f(A), g_\Omega(H)]A^N\Pi_0 + \\ &\quad + \sum_{\substack{j+l=N\\ j,l\geq 1}} c_{lj}\Big(f(A)ad_A^j(g_\Omega(H)) - ad_A^j(g_\Omega(H))f(A)\Big)A^l\Pi_0 + \\ &\quad + f(A)ad_A^N(g_\Omega(H))\Pi_0 - ad_A^N(g_\Omega(H))f(A)\Pi_0 . \end{aligned} \tag{3.15}$$

By (C1) and functional calculus

$$A^k\Pi_0 \text{ is bounded for } 0 \le k \le N \le \alpha\,. \tag{3.16}$$

Using (3.10) and (3.12) for $N=1$, we obtain

$$\|f(A)\phi\| = \|\langle A\rangle^\varepsilon\phi\| \le \|\langle A\rangle\phi\| \le c\left(\|A\phi\|+\|\phi\|\right). \qquad (\phi\in\mathcal{D}(|A|)) \tag{3.17}$$

Thus (3.17) gives

$$\begin{aligned}\|f(A)ad_A^j(g_\Omega(H))A^l\Pi_0\| &\le c\left(\|A\,ad_A^j(g_\Omega(H))A^l\Pi_0\|+\|ad_A^j(g_\Omega(H))A^l\Pi_0\|\right)\\ &\le c\big(\|ad_A^j(g_\Omega(H))A^{l+1}\Pi_0\|+\|ad_A^{j+1}(g_\Omega(H))A^l\Pi_0\|+\|ad_A^j(g_\Omega(H))A^l\Pi_0\|\big)\end{aligned} \tag{3.18}$$

and a very similar estimate for $f(A)ad_A^N(g_\Omega(H))\Pi_0$. Finally $A_2\in B(\mathcal{H})$ follows from using (3.18) in (3.15) and then taking into account (3.8), (3.13), (3.14) and (3.16).

□

To prove (3.9), we observe that

$$\psi(0) := g_\Omega(H)\psi_0 = \psi_0 + \big(g_\Omega(H)-g_\Omega(H_0)\big)\Pi_0\psi_0\,, \tag{3.19}$$

which follows from $\Pi_0\psi_0=\psi_0$ and $g_\Omega(H_0)\psi_0=\psi_0$. By use of (3.3) and the second resolvent equation we get

$$\big(g_\Omega(H)-g_\Omega(H_0)\big)\Pi_0 = -\int_{\mathbb{C}}(H-z)^{-1}W\Pi_0(H_0-z)^{-1}\,\frac{1}{2\pi}\,\big(\overline{\partial}\tilde g_\Omega\big)(z)\,dx\,dy\,, \tag{3.20}$$

where $\tilde g_\Omega\in C_0^\infty(\mathbb{C})$ is an almost analytic extension of g_Ω in the sense of Lemma 3.1. Then using $\|W\Pi_0\|\le\kappa$ and (3.2) we obtain

$$\begin{aligned}&\|\big(g_\Omega(H)-g_\Omega(H_0)\big)\Pi_0\|\\ &\le \kappa\cdot\int_{\mathbb{C}}|\mathrm{Im}\,z|^{-2}\,\frac{1}{2\pi}\,\big|\big(\overline{\partial}\tilde g_\Omega\big)(z)\big|\,dx\,dy \;=\; O(\kappa)\,. \qquad (\kappa\to 0)\end{aligned} \tag{3.21}$$

Thus (3.9) follows from (3.19) and (3.21). This completes the proof of Proposition 3.3.

■

We shall now show that the contribution of the dispersive part (see (1.9)) is small, both for $\psi(0)$ and ψ_0. More precisely:

Proposition 3.4 Let E_0 be an embedded eigenvalue of H_0, $H_0\psi_0=E_0\psi_0$. Assume (C0) - (C5). Let Ω be an interval around E_0 such that $\mathrm{supp}\,g_\Omega\subset\Delta$. Let $\psi(0):=g_\Omega(H)\psi_0$, $\psi(t)=e^{-iHt}\psi(0)$. Then:

(1) For $t\ge 0$ and some $\epsilon\in(0,1]$ we have

$$\|\langle A\rangle^{-\alpha}\psi_{disp}(t)\| \;\le\; C\,\|\langle A\rangle^\alpha\overline{\Pi}_0\psi(0)\|+O(\kappa^\epsilon) \qquad (\kappa\to 0) \tag{3.22}$$

for some $C\ge 0$, uniformly in $W\in\mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small. $\psi_{disp}(t)$ is defined as in (1.9). Furthermore

$$\langle A\rangle^\alpha\overline{\Pi}_0\psi(0) \;=\; O(\kappa)\,, \qquad (\kappa\to 0) \tag{3.23}$$

uniformly in $W\in\mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small.

(2) For $t\ge 0$

$$\begin{aligned}\langle A\rangle^{-\alpha}\psi_{disp,0}(t) &:= \langle A\rangle^{-\alpha}Bg_{\Delta'}(\overline{H})\overline{\Pi}_0e^{-iHt}\psi_0\\ &= \langle A\rangle^{-\alpha}\psi_{disp}(t)+O(\kappa)\,, \qquad (\kappa\to 0)\end{aligned} \tag{3.24}$$

uniformly in $W\in\mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small.

Proof of Proposition 3.4, (3.22): (3.22) directly follows from (1.11).

Proof of Proposition 3.4, (3.23): From (3.19) and $\overline{\Pi}_0\psi_0 = 0$ we get

$$\langle A\rangle^\alpha \overline{\Pi}_0\psi(0) = \langle A\rangle^\alpha \overline{\Pi}_0\big(g_\Omega(H) - g_\Omega(H_0)\big)\Pi_0\psi_0\,. \tag{3.25}$$

Using (1.1) we obtain

$$\langle A\rangle^\alpha \overline{\Pi}_0\big(g_\Omega(H) - g_\Omega(H_0)\big)\Pi_0 \;=\; \langle A\rangle^\alpha\big(g_\Omega(H) - g_\Omega(H_0)\big)\Pi_0 + O(\kappa) \tag{3.26}$$

$(\kappa \to 0)$, since by use of (C1) and (3.21)

$$\langle A\rangle^\alpha \Pi_0\big(g_\Omega(H) - g_\Omega(H_0)\big)\Pi_0 = O(\kappa)\,, \qquad (\kappa \to 0) \tag{3.27}$$

uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small. In analogy to (3.20) we get the following estimate for the first term on the r.h.s. of (3.26):

$$\begin{aligned} &\|\langle A\rangle^\alpha\big(g_\Omega(H) - g_\Omega(H_0)\big)\Pi_0\| \;\le \\ &\le \int_{\mathbb{C}} \|\langle A\rangle^\alpha (H-z)^{-1} W\Pi_0\| \cdot \|(H_0 - z)^{-1}\| \,\frac{1}{2\pi}\,\big|\big(\overline{\partial}\tilde{g}_\Omega\big)(z)\big|\,dx\,dy \end{aligned} \tag{3.28}$$

Thus we have to estimate $\|\langle A\rangle^\alpha (H-z)^{-1} W\Pi_0\|$. Let f be defined as in (3.10). Then

$$\begin{aligned} &\langle A\rangle^\alpha (H-z)^{-1} W\Pi_0 \\ &= (A+i)^N (H-z)^{-1} f(A) W\Pi_0 + (A+i)^N [f(A), (H-z)^{-1}] W\Pi_0\,. \end{aligned} \tag{3.29}$$

Using (3.12) for an estimate of (3.29), we obtain

$$\|\langle A\rangle^\alpha (H-z)^{-1} W\Pi_0\| \;\le\; c\,\big(\|B_1\| + \|B_2\| + \|B_3\| + \|B_4\|\big)$$

for some $c < \infty$, where

$$\begin{aligned} &B_1 := A^N (H-z)^{-1} f(A) W\Pi_0\,, \quad &&B_2 := A^N [f(A), (H-z)^{-1}] W\Pi_0\,, \\ &B_3 := (H-z)^{-1} f(A) W\Pi_0\,, &&B_4 := [f(A), (H-z)^{-1}] W\Pi_0\,. \end{aligned}$$

We shall now prove $B_j = O(\kappa)$ $(j \in \{1,2,3,4\})$, locally uniformly in z. Inserting $\langle A\rangle^\alpha (H-z)^{-1} W\Pi_0 = O(\kappa)$, locally uniformly in z, into (3.28) and using (3.2) and (3.25) - (3.27) will then finish the proof of (3.23). By (C2) and functional calculus

$$\|A^k f(A) W\Pi_0\| \le \kappa \quad (0 \le k \le N)\,, \qquad \|A^k W\Pi_0\| \le \kappa \quad (0 \le k \le \alpha)\,. \tag{3.30}$$

Thus we have $\|B_3\| \le \kappa |\mathrm{Im}\, z|^{-1}$. Splitting $(H-z)^{-1}$ into its real and imaginary parts, Lemma 3.2 together with (3.6) yields

$$\big\|[f(A), (H-z)^{-1}]\big\| \le C\,(1 + |z|)\,|\mathrm{Im}\, z|^{-2}\,. \tag{3.31}$$

So $\|B_4\| \le C\,\kappa |\mathrm{Im}\, z|^{-2}$ for some $C \in \mathbb{R}$, locally uniformly in z. Applying (3.4) to $A^N (H-z)^{-1}$ yields

$$\begin{aligned} B_1 \;=\; &(H-z)^{-1} A^N f(A) W\Pi_0 + \sum_{\substack{j+l=N\\ j,l\ge 1}} c_{jl}\, ad_A^j((H-z)^{-1}) A^l f(A) W\Pi_0 + \\ &+ ad_A^N((H-z)^{-1} f(A) W\Pi_0\,. \end{aligned}$$

Combining (3.7) and (3.30) leads to

$$\|B_1\| \;\le\; C\,\kappa \sum_{k=1}^{N+1} |\mathrm{Im}\, z|^{-k}$$

for some $C < \infty$, locally uniformly in z. By first applying (3.4) to $A^N[f(A),(H-z)^{-1}]$ and then using (3.5), we obtain

$$\begin{aligned} B_2 &= [f(A),(H-z)^{-1}]A^N W\Pi_0 + \\ &+ \sum_{\substack{j+l=N \\ j,l\geq 1}} c_{jl} \, \Big(f(A) ad_A^j\big((H-z)^{-1}\big) - ad_A^j\big((H-z)^{-1}\big) f(A) \Big) \, A^l W \Pi_0 + \\ &+ f(A) ad_A^N\big((H-z)^{-1}\big) W\Pi_0 - ad_A^N\big((H-z)^{-1}\big) f(A) W \Pi_0 \, . \end{aligned} \tag{3.32}$$

Using (3.17) yields

$$\begin{aligned} \|f(A)ad_A^j\big((H-z)^{-1}\big)A^l W\Pi_0\| &\leq c\,\big(\|ad_A^j\big((H-z)^{-1}\big)A^{l+1}W\Pi_0\| + \\ &+ \|ad_A^{j+1}\big((H-z)^{-1}\big)A^l W\Pi_0\| + \|ad_A^j\big((H-z)^{-1}\big)A^l W\Pi_0\| \,\big) \end{aligned} \tag{3.33}$$

and a similar estimate for $f(A)ad_A^N\big((H-z)^{-1}\big)W\Pi_0$. Thus, combining (3.7), (3.30) and (3.31) with (3.32) and (3.33) and using $N+1<n$ (see (C3)), we obtain

$$\|B_2\| \quad \leq \quad C_1\,\kappa \sum_{k=1}^{N+1} \|ad_A^k\big((H-z)\big)^{-1}\| \leq C_2\,\kappa \sum_{k=2}^{N+2} |\mathrm{Im}\, z|^{-k}$$

for some $C_1<\infty$, $C_2<\infty$, locally uniformly in z.

□

Proof of Proposition 3.4, (3.24): Since $Bg_{\Delta'}(\overline{H})\overline{\Pi}_0 e^{-itH} \in B(\mathcal{H})$ (cp. (1.1), (1.2), (1.8)), substitution of (3.9) into (1.9) yields

$$\psi_{disp}(t) = Bg_{\Delta'}(\overline{H})\overline{\Pi}_0 e^{-itH}\psi_0 + O(\kappa)\,, \qquad (\kappa \to 0) \tag{3.34}$$

uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small. Thus (3.24) follows from (3.34). This completes the proof of the proposition.

■

Now we are prepared to give the

Proof of Theorem 1.4: (1.21) – (1.23) follow from using (3.9) in (1.17) – (1.19). The estimate (1.24) directly follows from (2.5). For the proof of (1.25) we need Proposition 3.4: By use of (1.8) and (1.9) we obtain

$$\begin{aligned} \psi_{disp}(t) &= g_{\Delta'}(\overline{H})\overline{\Pi}_0\psi(t) + O(\kappa) \\ &\overset{(1.3)}{=} \overline{\Pi}_0\psi(t) - \overline{g_{\Delta'}}(\overline{H})\overline{\Pi}_0\psi(t) + O(\kappa)\,, \qquad (\kappa \to 0) \end{aligned}$$

uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small. Since $\psi(0) \in \mathrm{Ran}\, E_\Delta(H)$, we have $g_\Delta(H)\psi(t) = \psi(t)$ $(t \geq 0)$ and therefore

$$\psi_{disp}(t) \quad = \quad \overline{\Pi}_0\psi(t) - \overline{g_{\Delta'}}(\overline{H})\overline{\Pi}_0 g_\Delta(H)\psi(t) + O(\kappa)\,, \qquad (\kappa \to 0)$$

for $t \geq 0$, uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small. [MerSi, Proposition 3.1] gives $\overline{g_{\Delta'}}(\overline{H})\overline{\Pi}_0 g_\Delta(H) = O(\kappa)$ $(\kappa \to 0)$. Thus for $t \geq 0$

$$\psi_{disp}(t) = \overline{\Pi}_0\psi(t) + O(\kappa)\,, \qquad (\kappa \to 0) \tag{3.35}$$

uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small. Combining (3.23) and (3.22), we get for $t \geq 0$ and some $\epsilon \in (0,1]$

$$\|\langle A\rangle^{-\alpha}\psi_{disp}(t)\| \quad = \quad O(\kappa^\epsilon) + O(\kappa)\,, \qquad (\kappa \to 0) \tag{3.36}$$

uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small. Substitution of (3.35) into (3.36) gives

$$\|\langle A\rangle^{-\alpha}\overline{\Pi}_0\psi(t)\| \quad = \quad O(\kappa^\epsilon) + O(\kappa) \qquad (\kappa \to 0) \tag{3.37}$$

for $t \geq 0$, uniformly in $W \in \mathcal{W}_{\kappa_0}$ for some κ_0 sufficiently small. Finally inserting (3.9) into (3.37) yields (1.25). This completes the proof of Theorem 1.4.

■

A Appendix

Lemma A.1 Assume (C0) - (C4). Then $\delta(\overline{H}-E_0)$ and P.V.$(\overline{H}-E_0)^{-1}$ exist in the sense of equation (1.6) and (1.7).

Proof: Since $E_0 \notin \operatorname{supp} \overline{g_{\Delta'}}$, we have

$$(\overline{H}-E_0-i0)^{-1}\overline{g_{\Delta'}}(\overline{H}) = (\overline{H}-E_0)^{-1}\overline{g_{\Delta'}}(\overline{H}) \in B(\mathcal{H})\,. \tag{A.1}$$

By [MerSi, Proposition 3.2 (i)] with $t=0$,

$$\text{s-}\lim_{\varepsilon\downarrow 0}\langle A\rangle^{-\alpha}(\overline{H}-E_0-i\varepsilon)^{-1}g_{\Delta'}(\overline{H})\overline{\Pi}_0\langle A\rangle^{-\alpha} \text{ exists.} \tag{A.2}$$

Thus (A.1) and (A.2) imply the existence of the limits in (1.6) and (1.7), since we have

$$\begin{aligned}\langle A\rangle^{-\alpha}(\overline{H}-E_0-i0)^{-1}\overline{\Pi}_0\langle A\rangle^{-\alpha} \overset{(1.3)}{=} & \langle A\rangle^{-\alpha}(\overline{H}-E_0-i0)^{-1}g_{\Delta'}(\overline{H})\overline{\Pi}_0\langle A\rangle^{-\alpha} + \\ & \langle A\rangle^{-\alpha}(\overline{H}-E_0-i0)^{-1}\overline{g_{\Delta'}}(\overline{H})\overline{\Pi}_0\langle A\rangle^{-\alpha}\,.\end{aligned}$$

■

The proof of [MerSi, Proposition 3.2 (i)] uses (C4); see [MerSi, p.573]. And (C4) is in fact a consequence of Mourre estimates. We refer the reader to [CyFKS, Chapter 4] for the definition and important results of Mourre estimates.

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