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CENTROAFFINE BERNSTEIN PROBLEMS

An-Min LI
Haizhong LI
Udo SIMON

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An-Min Li, Haizhong Li, Udo Simon

Abstract. C.P. Wang [21] studied the Euler-Lagrange equations for the centroaffine area functional of hypersurfaces. We consider classes of examples satisfying these equations together with completeness conditions. We formulate appropriate centroaffine Bernstein problems and give partial solutions.

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1. Introduction

Bernstein's famous uniqueness result on minimal graphs in Euclidean 3-space could be generalized up to dimension $n \leq 7$:

Theorem A (see [16]) *Let $x : M \to R^{n+1}$ be an n-dimensional minimal graph given by*

$$x_{n+1} = f(x_1, \cdots, x_n), \qquad (x_1, \cdots, x_n) \in R^n;$$

if $n \leq 7$ then f is a linear function.

For $n \geq 8$ there also exist other solutions. Uniqueness results similar to Bernstein's theorem were proved in different geometries. This led to the following standard terminology: Consider a hypersurface in some ambient space, subject to at least two conditions:

(a) the hypersurface is a critical point for a given area functional;
(b) the hypersurface satisfies certain completeness condition without being compact.

A classification problem for such hypersurfaces is called a Bernstein problem. Famous examples are the following two versions of the so called affine Bernstein conjecture which are due to S. S. Chern [3] and E. Calabi [1].

Chern's affine Bernstein conjecture: *Consider a locally strongly convex graph $x: R^2 \to A^3$ with vanishing affine mean curvature $H = 0$. Then x is an elliptic paraboloid.*

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Calabi's affine Bernstein conjecture: *Consider a locally strongly convex surface $x: M^2 \to A^3$ with vanishing affine mean curvature $H = 0$ and complete Blaschke metric. Then x is an elliptic paraboloid.*

Recall that, in a certain analogy to the Euclidean case, the vanishing of the affine mean curvature expresses the Euler-Lagrange equation for the affine area functional. In all such Bernstein problems it is the question whether the vanishing of the trace of the shape operator implies the vanishing of the operator itself. While the Euclidean minimal surface equation is a PDE of second order , the corresponding affine Euler-Lagrange equation is a PDE of fourth order. Moreover, the evaluation of the second variation of the affine area functional is based on a very complicated expression which is negative in case the locally strongly convex hypersurface satisfies one of the following two conditions [2]: (i) dimension $M = n = 2$; (ii) $x: R^n \supset M \to A^{n+1}$ is a graph. A more general result in the case of locally strongly convex hypersurfaces is not yet known. Therefore there is no standard terminology for the class of hypersurfaces with vanishing affine mean curvature, but different notions (affine minimal hypersurfaces, affine maximal hypersurfaces, affine extremal hypersurfaces) are used. In recent years, for both versions of the affine Bernstein conjecture there are affirmative solutions: [20] and [5].

In centroaffine differential geometry one studies the properties of hypersurfaces in R^{n+1} which are invariant under the centroaffine transformation group $G = GL(n+1, R)$, where G keeps the origin $0 \in R^{n+1}$ fixed. In this paper, we want to consider centroaffine Bernstein problems. C. P. Wang [21] studied the Euler-Lagrange equation for the area functional of a so called centroaffine hypersurface. The Euler-Lagrange equation is given by a fourth order PDE, namely, trace$\mathcal{T} = 0$, where $\mathcal{T}$ is the so called Tchebychev operator; in contrast to the above mentioned Bernstein problems the operator $\mathcal{T}$ is not related to something like "extrinsic affine curvature". As there are no general results about the sign of the second variation of the centroaffine area integral, we use the terminology *centroaffine extremal hypersurface* in case the equation trace$\mathcal{T} = 0$ is satisfied.

All proper affine spheres satisfy the equation trace$\mathcal{T} = 0$. For proper affine spheres, the Blaschke geometry and the centroaffine geometry coincide, in particular completeness conditions for their metrics. Thus metrically complete proper affine hyperspheres are centroaffine extremal and complete; the ellipsoid is the only compact affine hypersphere. Besides affine spheres there are more examples of centroaffine extremal hypersurfaces [21]. In section 3 we study classes of such examples and give a generalized Calabi-composition to produce a new family of centroaffine extremal hypersurfaces from two given centroaffine extremal hypersurfaces. Moreover, we derive a fourth order equation for an extremal graph. From this it is easy to prove that centroaffine extremal hypersurfaces are invariant under polarization (inversion at a sphere); this in particular yields for the special classes of examples considered. The study of the examples leads to the formulation of different centroaffine Bernstein problems in section 5. In the rest of the paper we formulate and prove our results to give partial solutions of the centroaffine Bernstein problems. Examples of such results are the following:

Theorem 6.1 *Let $x: M \to R^3$ be a complete, non-compact hyperbolic centroaffine extremal surface. If the Gaussian curvature K of the centroaffine metric and the length $|T|$ of the Tchebychev vector field satisfy*

(1) $K \geq 0$,

(2) $|T| < \infty$,
then x *is centroaffinely equivalent to one of the following surfaces*

$$x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} = 1, \quad \alpha_1 > 0, \quad \alpha_2 > 0, \quad \alpha_3 > 0. \tag{1.1}$$

Corollary 6.1 *Let* $x : M \to R^3$ *be a complete hyperbolic affine sphere. If the Gaussian curvature* K *of the centroaffine metric is nonnegative, then* x *is affinely equivalent to the following surface*

$$x_1 x_2 x_3 = 1. \tag{1.2}$$

As an example of one of our centroaffine Bernstein problems we state the following conjecture.

Centroaffine Bernstein Conjecture *Let* $x : M \to R^{n+1} (n \geq 2)$ *be a complete, non-compact hyperbolic centroaffine extremal hypersurface. If the Ricci curvature of the centroaffine metric is non-negative, then* x *is centroaffinely equivalent to one of the following hypersurfaces*

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1, \quad \alpha_1 > 0, \cdots, \alpha_{n+1} > 0. \tag{1.3}$$

2. Centroaffine hypersurfaces in R^{n+1}

We summarize basic formulas of centroaffine hypersurface theory in terms of Cartan's moving frames (compare [8], chapters 1-2; for an approach in the invariant calculus see [19], chapters 4-6). We restrict to locally strongly convex hypersurfaces as in this case the so called centroaffine metric is a Riemannian metric; see section 4.3.3 in [19].

Let $x : M \to R^{n+1}$ be a locally strongly convex hypersurface and assume that the position vector x is transversal to the tangent hyperplane $x_*(TM)$ at each point $p \in M$. In particular, this implies that $O \notin x(M)$. In a standard terminology, a hypersurface normalized by its transversal position vector is called a centroaffine hypersurface. According to the type of the hypersurface one uses different orientations for the normalization to get a positive definite centroaffine metric:

1. Hyperbolic type: For any point $x(p) \in R^{n+1}$, the origin of R^{n+1} and the hypersurface are on different sides of the tangent hyperplane $x_*(TM)$; the centroaffine normal vector field is given by $e_{n+1} = x$ (examples are hyperbolic affine hyperspheres in R^{n+1} centered at $0 \in R^{n+1}$).

2. Elliptic type: For any point $x(p) \in R^{n+1}$, the origin of R^{n+1} and the hypersurface are on the same side of the tangent hyperplane $x_*(TM)$; the centroaffine normal vector field is given by $e_{n+1} = -x$ (examples are elliptic affine hyperspheres in R^{n+1} centered at $0 \in R^{n+1}$).

As already stated in the introduction, in centroaffine differential geometry we study the properties of hypersurfaces in R^{n+1} that are invariant under the centroaffine transformation group G. For the hypersurface, we choose a centroaffine frame field $\{e_1, \cdots, e_n, e_{n+1}\}$ with $e_{n+1} = -\epsilon x$ ($\epsilon = 1$ for elliptic type, $\epsilon = -1$ for hyperbolic type) and $e_1, \cdots, e_n \in T_x M$; we denote by $\{\omega^1, \cdots, \omega^n\}$ the dual frame field of the tangential frame field. The structure equations read

$$dx = \sum_i \omega^i e_i, \quad \omega^{n+1} = 0, \tag{2.1}$$

$$de_i = \sum_j \omega_i^j e_j + \omega_i^{n+1} e_{n+1}, \tag{2.2}$$

$$de_{n+1} = \sum_i \omega_{n+1}^i e_i, \quad \omega_{n+1}^{n+1} = 0, \quad \omega_{n+1}^i = -\epsilon\omega^i. \tag{2.3}$$

Differentiation of (2.1) – (2.3) gives the integrability conditions (2.4) – (2.6).

$$d\omega^i = \sum_j \omega^j \wedge \omega_j^i, \quad \sum_i \omega^i \wedge \omega_i^{n+1} = 0, \tag{2.4}$$

$$d\omega_i^j = \sum_k \omega_i^k \wedge \omega_k^j - \epsilon\omega_i^{n+1} \wedge \omega^j, \quad d\omega_i^{n+1} = \sum_j \omega_i^j \wedge \omega_j^{n+1}, \tag{2.5}$$

$$d\omega_{n+1}^i = \sum_j \omega_{n+1}^j \wedge \omega_j^i. \tag{2.6}$$

From the second equation of (2.4), we have

$$\omega_i^{n+1} = \sum_{i,j} h_{ij}\omega^j, \quad h_{ij} = h_{ji}. \tag{2.7}$$

For locally strongly convex hypersurfaces, the quadratic form

$$h = \sum_{i,j} h_{ij}\omega^i\omega^j \tag{2.8}$$

is positive definite by appropriate choice of the orientation; h is called the centroaffine metric of the hypersurface. It is well known that h is independent of the choice of the frame $\{e_1, \cdots, e_n\}$ and that h is invariant under transformations of the group G. The centroaffine metric is the first fundamental invariant of centroaffine hypersurface theory.

We sketch how to derive a second fundamental invariant. We choose a centroaffine tangential frame $\{e_1, \cdots, e_n\}$ on M such that $h_{ij} = \delta_{ij}$, i.e.,

$$\omega_i^{n+1} = \omega^i. \tag{2.9}$$

Differentiate (2.9) and use (2.5); this implies

$$d\omega^i = \sum_j \omega_{ij} \wedge \omega^j. \tag{2.10}$$

(2.4) and (2.10) give

$$d\omega^i = \sum_j \omega^j \wedge [\frac{1}{2}(\omega_{ji} - \omega_{ij})]. \tag{2.11}$$

The expression $\frac{1}{2}(\omega_{ji} - \omega_{ij})$ is skew-symmetric and $\{\omega^1, \cdots, \omega^n\}$ is an orthonormal coframe of h. (2.11) and the fundamental theorem of Riemannian geometry imply that the Levi-Civita connection of h satisfies

$$\tilde{\omega}_{ji} = \frac{1}{2}(\omega_{ji} - \omega_{ij}), \quad \tilde{\omega}_{ji} = -\tilde{\omega}_{ij}. \tag{2.12}$$

Define

$$\omega_{ij} - \tilde{\omega}_{ij} = \frac{1}{2}(\omega_{ij} + \omega_{ji}) = \sum_k A_{ijk}\omega^k. \tag{2.13}$$

This gives the symmetry relation

$$A_{ijk} = A_{jik}. \tag{2.14}$$

Combine (2.10) with (2.11) and use (2.13):

$$\sum_{j,k} A_{ijk}\omega_j \wedge \omega_k = 0,$$

this implies the total symmetry of the form

$$A = \sum_{i,j,k} A_{ijk}\omega^i\omega^j\omega^k,$$

namely

$$A_{ijk} = A_{ikj} = A_{jik}. \tag{2.15}$$

The form A is called the centroaffine cubic form of the hypersurface. Again it is well known that this form is independent of the choice of the frame and invariant under transformations of the group G. The vanishing of its traceless part characterizes hyperquadrics (see [19], section 7.1; [6], Lemma 2.1 and Remark 2.2).

The uniqueness part of the fundamental theorem of centroaffine hypersurface theory states that the forms h and A together build a fundamental system of centroaffine invariants of the hypersurface, that means that they completely describe the geometry of x which is invariant under the transformations of G. Considering integrability conditions, one also can state an existence theorem using the forms h and A.

We need the following two important geometric invariants built from h and A:

$$J = \frac{1}{n(n-1)} \sum_{i,j,k} A_{ijk}^2 \tag{2.16}$$

is called the centroaffine Pick invariant. The tangent vector field

$$T = \sum_i T_i e_i, \quad T_i = \frac{1}{n}\sum_{j=1}^{n} A_{jji} \tag{2.17}$$

is called the centroaffine Tchebychev vector field of x. For locally strongly convex hypersurfaces the metric is positive definite, thus the vanishing of J implies that of A and T, and therefore that of the traceless part of A; the hypersurface must be a quadric. In the context of relative geometry and in terms of volume forms, the geometric meaning of T was studied in section 4.4.8, 4.4.9 in [19]. In the centroaffine case, there is an additional well known relation between T, the so called centroaffine Tchebychev function ψ and the support function ρ of the Blaschke geometry. To state this relation, we recall the following definition from section 2 of [9].

Definition 2.1 The positive function ψ, given by

$$\psi = \frac{\det(h_{ij})}{[e_1, \cdots, e_n, x]^2}, \tag{2.18}$$

is independent of the choice of the frame $\{e_1, \cdots, e_n\}$ and is invariant under transformations of G, where $[\cdots]$ is the determinant. We call the function ψ the Tchebychev function of x. Choosing $i = j$ in (2.13) and summing up over i, we get

$$\begin{aligned} \sum_{i,k} A_{iik}\omega^k &= \sum_i \omega_{ii} \\ &= d(\log[e_1, \cdots, e_n, x]) \\ &= -\tfrac{1}{2} d\log\psi. \end{aligned} \tag{2.19}$$

One can compare invariants from different relative geometries of a hypersurface (see section 5 in [19]); from (2.19) (c.f. formula (2) in [9]) it follows that the equiaffine support function ρ (section 4.13 in [19]), the centroaffine Tchebychev function ψ defined above, and the Tchebychev vector field T satisfy the relation

$$T_i = -\frac{1}{2n}(\log\psi)_i = \frac{(n+2)}{2n}(\log\rho)_i, \tag{2.20}$$

The relation

$$\rho = const$$

characterizes proper affine spheres (section 7.2 in [17]); this is equivalent to the centroaffine relation $T = 0$. Our foregoing remarks clarify the geometric meaning of the invariants J and T.

For later applications we list the integrability conditions in terms of the metric and the cubic form. In a standard local notation, by a comma we indicate covariant differentiation in terms of the Levi-Civita connection. The sign of the Riemannian curvature tensor $\Omega = \sum R_{ijkl}\omega^i \otimes \omega^j \otimes \omega^k \otimes \omega^l$ of h is fixed by

$$d\tilde{\omega}_{ij} - \sum_k \tilde{\omega}_{ik} \wedge \tilde{\omega}_{kj} = -\frac{1}{2}\sum_{k,l} R_{ijkl}\omega^k \wedge \omega^l. \tag{2.21}$$

In terms of the frame considered ($h_{ij} = \delta_{ij}$), the Gauss equations read

$$R_{ijkl} = \sum (A_{jkm}A_{mil} - A_{ikm}A_{mjl}) + \epsilon(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}), \tag{2.22}$$

while the cubic form satisfies Codazzi equations, that means the covariant derivative is totally symmetric:

$$A_{ijk,l} = A_{ijl,k}. \tag{2.23}$$

Here, as mentioned above, $A_{ijk,l}$ are the components of the covariant derivative of A with respect to the Levi-Civita connection of h. Contraction of (2.22) gives the following important relations

$$R_{ik} = \sum A_{iml}A_{mlk} - n\sum_m T_m A_{mik} + \epsilon(n-1)\delta_{ik}, \tag{2.24}$$

where R_{ik} denote the components of the Ricci tensor, and the "centroaffine theorema egregium"

$$n(n-1)\kappa = R = n(n-1)(J+\epsilon) - n^2|T|^2, \quad |T|^2 = \sum (T_i)^2, \tag{2.25}$$

where κ denotes the normalized scalar curvature.

Later we will need the following Ricci identities

$$A_{ijk,lm} - A_{ijk,ml} = \sum A_{rjk}R_{rilm} + \sum A_{irk}R_{rjlm} + \sum A_{ijr}R_{rklm}. \tag{2.26}$$

The Codazzi equations for A (or the relations between T and the Tchebychev function) imply

$$T_{i,j} = T_{j,i}. \tag{2.27}$$

If $T_{i,j} = 0$, we say that the Tchebychev vector field T is parallel.

As stated above, for a centroaffine hypersurface the position vector is used for a normalization; from this a Weingarten type equation is trivial, and there is no shape operator describing "exterior curvature" in the standard way. But studies of Wang [21] and other authors ([11], [6]) show that there is another important operator in centroaffine geometry. Wang called this operator originally shape operator, but for the reasons just stated, later the notion was changed to Tchebychev operator. This operator $\mathcal{T} : TM \to TM$ of x is defined by

$$\mathcal{T}(v) := \nabla_v T, \quad v \in TM. \tag{2.28}$$

The foregoing relation $T_{i,j} = T_{j,i}$ implies that $\mathcal{T}$ is a self-adjoint operator with respect to the centroaffine metric h. Moreover, $\mathcal{T} \equiv 0$ if and only if T is parallel. For locally strongly convex hypersurfaces, Wang [21] proved

Theorem 2.1 The relation trace$\mathcal{T} = 0$ is the Euler-Lagrange equation for the centroaffine area functional.

As there is no general statement about the sign of the second variation, we call the critical points of the area functional "extremal centroaffine hypersurfaces" (other authors call them minimal centroaffine hypersurfaces). From calculations of the second variational formulas for the area integral, Wang [21] gave some examples of stable and unstable extremal hypersurfaces.

By (2.20), we obtain

Theorem 2.2 *Let $x : M \to R^{n+1} (n \geq 2)$ be a centroaffine hypersurface with Tchebychev function ψ. Then x is an extremal centroaffine hypersurface if and only if*

$$\Delta(\log\psi) = 0, \tag{2.29}$$

where Δ is the Laplacian of the centroaffine metric h of x.

3. Examples of extremal and complete centroaffine hypersurfaces

In this section, we recall examples of locally strongly convex, extremal centroaffine hypersurfaces; some already were listed in [21]. The convexity condition implies that the centroaffine metric is positive definite for an appropriate orientation of the normalization. It is well known that the hyperellipsoids are the only closed (compact without boundary), centroaffine extremal hypersurfaces; this result is due to C. P. Wang.

Proposition 3.1 (Theorem 1 of [21]) *Let $x : M \to R^{n+1} (n \geq 2)$ be a compact centroaffine hypersurface with constant trace of the Tchebychev operator. Then $x(M)$ is centroaffinely equivalent to a hyperellipsoid centered at $0 \in R^{n+1}$.*

In this section we consider non-compact examples which satisfy at least one of the following completeness conditions:

(i) the centroaffine metric is complete;

(ii) the hypersurface can be represented as graph over a hyperplane.

We will come back to the completeness conditions in section 4 below.

Example 3.1 Proper affine spheres.

According to C.P. Wang [21], any locally strongly convex, proper affine hypersphere is centroaffine extremal. This is a trivial consequence of the fact that the vanishing of the Tchebychev field characterizes proper affine spheres in centroaffine geometry. In the Blaschke geometry, it is well known that hyperbolic affine hyperspheres can be described in terms of solutions of some Monge-Ampère equations; therefore there are many proper affine hyperspheres, and thus this gives a very large class of centroaffine extremal hypersurfaces. For proper affine hyperspheres the unimodular (equiaffine) theory (sometimes called Blaschke theory) and the centroaffine theory coincide modulo a nonzero constant factor. In particular this implies that the notions of completeness with respect to the metrics coincide in both theories. The classification of the locally strongly convex affine hyperspheres, which are complete with respect to the affine metric, was finished about a decade ago; see e.g. [8], chapter 2. Considering proper affine hyperspheres, there are two subclasses, namely the elliptic ones and the hyperbolic ones. While there is only one type of complete elliptic affine hyperspheres, namely the hyperellipsoid, the class of complete hyperbolic affine hyperspheres is described by what Calabi originally stated as a conjecture (see [8], section 2.7); all examples in this latter class are non compact, but they satisfy both completeness conditions (i) and (ii) (in fact, in this case the two completeness conditions are equivalent). From this, any hyperbolic affine hypersphere is an example of a noncompact, centroaffine extremal hypersurface satisfying the two different completeness conditions (i) and (ii). Moreover, their Ricci tensor is bounded below : $\mathrm{Ric} \geq -(n-1)h$.

A particular example in this class is one sheet of a two-sheeted hyperboloid $H(c, n)$:

$$(x_{n+1})^2 = c^2 + (x_1)^2 + \cdots + (x_n)^2, \quad (x_1, \cdots, x_n) \in R^n, \quad c > 0. \tag{3.1}$$

We have (see [8])

$$A_{ijk} = 0, \qquad 1 \leq i, j, k \leq n.$$

Thus it is a centroaffine extremal hypersurface satisfying two different completeness conditions; for a hyperboloid the Pick invariant vanishes: $J \equiv 0$. The Riemannian curvature tensor of the centroaffine metric and its Ricci curvature tensor satisfy

$$R_{ijkl} = -c^{-\frac{2n+2}{n+2}} (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{3.2}$$

$$R_{ik} = -(n-1)c^{-\frac{2n+2}{n+2}} h_{ik}. \tag{3.3}$$

Obviously the sectional curvature, the Ricci curvature and the scalar curvature of the metric of $H(c, n)$ are negative constants.

Example 3.2 Centroaffine graphs with constant trace of the Tchebychev operator.

Let $x : M \to R^{n+1}$ be a locally strongly convex hypersurface with transversal position vector x at each point M. Then we have a local representation of x as graph:

$$x_{n+1} = f(x_1, x_2, \cdots, x_n). \tag{3.4}$$

We have the centroaffine frame

$$e_i = (0, \cdots, 1, \cdots, 0, f_{x_i}), \quad 1 \le i \le n, \quad e_{n+1} = (x_1, x_2, \cdots, x_n, f), \tag{3.5}$$

where $f_{x_i} = \frac{\partial f}{\partial x_i}$. The structure equations read

$$dx = \sum_i \omega^i e_i, \tag{3.6}$$

$$de_i = \sum_j \omega_i^j e_i + \sum_j h_{ij}\omega^j e_{n+1}, \tag{3.7}$$

thus we have

$$[e_1, \cdots, e_n, x] = \begin{vmatrix} 1 & 0 & \cdots & 0 & f_{x_1} \\ 1 & 1 & \cdots & 0 & f_{x_2} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & f_{x_n} \\ x_1 & x_2 & \cdots & x_n & f \end{vmatrix} = f - \sum_i x_i f_{x_i},$$

$$(h_{ij}) = (\frac{f_{x_i x_j}}{f - x_1 f_{x_1} - \cdots x_n f_{x_n}}) \tag{3.8}$$

and

$$\det(h_{ij}) = \frac{1}{(f - \sum_i x_i f_{x_i})^n} \cdot \det(f_{x_i x_j}). \tag{3.9}$$

The Tchebychev function ψ is given by

$$\psi = \frac{\det(h_{ij})}{[e_1, \cdots, e_n, x]^2} = \frac{1}{(f - \sum_i x_i f_{x_i})^{n+2}} \cdot \det(f_{x_i x_j}). \tag{3.10}$$

Therefore x is a centroaffine local graph with constant value a for the trace of the Tchebychev operator if and only if the Tchebychev function ψ satisfies the following nonlinear PDE of fourth order:

$$\Delta\{\log(\frac{\det(h_{ij})}{[e_1, \cdots, e_n, x]^2})\} = \Delta\{\log(\frac{\det(f_{x_i x_j})}{(f - \sum_i x_i f_{x_i})^{n+2}})\} = a. \tag{3.11}$$

As above, Δ is the Laplacian of the centroaffine metric h of x. In particular, we get a nonlinear PDE of fourth order for centroaffine extremal hypersurfaces. This allows us to consider a centroaffine Bernstein problem using this PDE.

Proposition 3.2 *Let x be a locally strongly convex graph given by the functon f in (3.4). Then x is centroaffine extremal if and only if f satisfies the PDE*

$$\Delta\{\log(\frac{\det(f_{x_ix_j})}{(f-\sum_i x_i f_{x_i})^{n+2}})\} = 0. \tag{3.12}$$

Remark 3.1 (i) We can rewrite the PDE (3.12) in a simpler form using the Legendre function. It follows from the convexity of f that the Hessian $(f_{x_ix_j})$ is positive definite. The Legendre transformation relative to f is defined by (see chapter 2 of [8])

$$F: D \to R^n, \quad (x_1, \cdots, x_n) \to (\xi_1, \cdots, \xi_n),$$

where $D \subset R^n$ is the Legendre transform domain, and

$$\xi_i = f_{x_i} = \frac{\partial f}{\partial x_i}, \quad i = 1, \cdots, n.$$

The Legendre function u is defined by

$$u(\xi_1, \cdots, \xi_n) = \sum_i x_i f_{x_i}(x_1, \cdots, x_n) - f(x_1, \cdots, x_n). \tag{3.13}$$

We know that $(\frac{\partial^2 u}{\partial\xi_i\partial\xi_j})$ is the inverse matrix of the Hessian $(f_{x_ix_j})$ (see [8]). Thus the PDE (3.12) of the centroaffine extremal graph can be rewritten as

$$\Delta\{\log((-u)^{n+2} \cdot \det(\frac{\partial^2 u}{\partial\xi_i\partial\xi_j}))\} = 0. \tag{3.14}$$

Equations (3.12) and (3.14) show the following: in terms of a graph function, the Euler-Lagrange equation for the centroaffine extremal hypersurfaces is a highly complicated non-linear fourth order PDE. From the global classification of locally strongly convex hyperbolic affine spheres we know about earlier difficulties to solve the much simpler equation (3.15).

(ii) We recall that the PDE of a hyperbolic hypersphere with constant affine mean curvature H, in terms of the Legendre function, is (see [8], p. 132)

$$(-u)^{n+2} \cdot \det(\frac{\partial^2 u}{\partial\xi_i\partial\xi_j}) = (-H)^{-n-2}. \tag{3.15}$$

Example 3.3 Wang's class of centroaffine extremal hypersurfaces.

Li-Wang [10] and Wang [21] also listed the following type of hypersurfaces, and Wang proved that they are centroaffine extremal:

$$(x_1)^{\beta_1}(x_2)^{\beta_2}\cdots(x_{n+1})^{\beta_{n+1}} = c, \quad c > 0, \quad \beta_i > 0, \quad 1 \le i \le n+1.$$

It is easy to see that the above hypersurfaces also can be represented by

$$Q(c; \alpha_1, \cdots, \alpha_n; n): \; x_{n+1} = c x_1^{-\alpha_1} x_2^{-\alpha_2} \cdots x_n^{-\alpha_n}, \quad c > 0, \quad 1 \le i \le n, \tag{3.16}$$

where $\alpha_i = \beta_i/\beta_{n+1} > 0$.

Consider the connected component

$$x_{n+1} = \frac{c}{(x_1)^{\alpha_1}(x_2)^{\alpha_2}\cdots(x_n)^{\alpha_n}}, \qquad \text{for } x_1 > 0, \cdots, x_n > 0.$$

This representation of the hypersurface in terms of a graph function

$$f(x_1, \cdots, x_n) = cx_1^{-\alpha_1} \cdots x_n^{-\alpha_n}$$

admits us to apply the calculations from Example 3.2:

$$h_{ii} = \frac{\alpha_i(1+\alpha_i)}{1+\alpha_1+\cdots+\alpha_n} \cdot x_i^{-2}, \quad 1 \le i \le n,$$

$$h_{ij} = \frac{\alpha_i\alpha_j}{1+\alpha_1+\cdots+\alpha_n} \cdot x_i^{-1}x_j^{-1}, \quad 1 \le i \ne j \le n,$$

$$\det(h_{ij}) = \frac{\alpha_1\cdots\alpha_n}{(1+\alpha_1+\cdots+\alpha_n)^{n-1}} x_1^{-2}\cdots x_n^{-2},$$

$$[e_1, e_2, \cdots, e_n, x] = c(1+\alpha_1+\cdots+\alpha_n)x_1^{-\alpha_1}\cdots x_n^{-\alpha_n}.$$

We calculate the Tchebychev function:

$$\begin{aligned} \psi &= \frac{\det(h_{ij})}{[e_1,\cdots,e_n,x]^2} = \frac{1}{(f-\sum_i x_i f_{x_i})^{n+2}} \cdot \det(f_{x_ix_j}) \\ &= \frac{1}{c^2} \cdot \frac{\alpha_1\cdots\alpha_n}{(1+\alpha_1+\cdots+\alpha_n)^{n+1}} x_1^{-2+2\alpha_1}\cdots x_n^{-2+2\alpha_n}. \end{aligned} \tag{3.17}$$

We easily see that the Tchebychev field has constant norm for any hypersurface of this class and that it satisfies $|T| = 0$ if and only if

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1.$$

Thus there is exactly one affine hypersphere in $Q(c; \alpha_1, \cdots, \alpha_n; n)$. As mentioned, it is well known that proper affine spheres, in terms of centroaffine invariants, can be characterized by the vanishing of the Tchebychev field. Thus Wang's large class of centroaffine extremal hypersurfaces contains exactly one proper affine sphere, and within the examples 3.3 the nonvanishing of the Tchebychev field characterizes the hypersurfaces not belonging to the class 3.1. Again, all hypersurfaces of the class 3.3 satisfy both completeness conditions (i) and (ii), stated in the beginning of this section.

To calculate the curvature tensor easily, we introduce new parameters $u_1, u_2, \cdots, u_n$:

$$x_i = e^{u_i}, \quad 1 \le i \le n.$$

Then $Q(c; \alpha_1, \cdots, \alpha_n; n)$ can be represented as graph in terms of $u_1, \cdots, u_n$ by

$$(x_1, \cdots, x_n, x_{n+1}) = (e^{u_1}, e^{u_2}, \cdots, e^{u_n}, ce^{-\alpha_1u_1-\alpha_2u_2-\cdots-\alpha_nu_n}).$$

The coefficients of the centroaffine metric

$$\begin{aligned} h &= \sum_{i,j} h_{ij}dx_idx_j \\ &= \sum_{i,j} \tilde{h}_{ij}du_idu_j, \end{aligned}$$

satisfy

$$(\tilde{h}_{ij}) = \frac{1}{1+\alpha_1+\cdots+\alpha_n}\begin{pmatrix} \alpha_1(1+\alpha_1) & \alpha_1\alpha_2 & \cdots & \alpha_1\alpha_n \\ \alpha_2\alpha_1 & \alpha_2(1+\alpha_2) & \cdots & \alpha_2\alpha_n \\ \cdots & \vdots & \ddots & \vdots \\ \alpha_n\alpha_1 & \alpha_n\alpha_2 & \cdots & \alpha_n(1+\alpha_n) \end{pmatrix}.$$

Since $(\tilde{h}_{ij})$ is a constant matrix, we immediately get that the metric is flat. From [10] we also know

$$A_{ijk,l} = 0, \quad \text{but} \quad J = \text{constant} \neq 0.$$

The properties just stated characterize the class $Q(c; \alpha_1, \cdots, \alpha_n; n)$. A.-M. Li and C. P. Wang proved

Proposition 3.3 (see Theorem 1.3 in [10]) *Let $x : M \to R^{n+1}$ be an n-dimensional ($n \geq 2$) centroaffine hypersurface. If its centroaffine metric is flat and its centroaffine Pick form is parallel with respect to its centroaffine metric, then $x(M)$ is centroaffinely equivalent to one of the following hypersurfaces in R^{n+1}*

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1, \quad \alpha_1 > 0, \cdots, \alpha_{n+1} > 0.$$

In particular , any hypersurfaces of type $Q(c; \alpha_1, \cdots, \alpha_n; n)$ is an extremal centroaffine hypersurface with flat centroaffine metric and parallel centroaffine cubic form; contraction gives that the Tchebychev operator vanishes and thus the square of the norm of T is constant (and non-zero for all such hypersurfaces which are not affine spheres). Moreover, the two completeness conditions (i) and (ii) are satisfied.

Example 3.4. Generalized Calabi-composition

We extend the well-known Calabi-composition for hyperbolic affine hypersurfaces to centroaffine extremal hypersurfaces.

Proposition 3.4 *Given two centroaffine hyperbolic extremal hypersurfaces $x : M_1 \to R^{p+1}$ and $y : M_2 \to R^{q+1}$, the generalized Calabi composition $z : R \times M_1 \times M_2 \to R^{p+q+2}$:*

$$z = (C_1 e^u x, C_2 e^{-\lambda u} y), \quad u \in R, \tag{3.18}$$

defines a centroaffine extremal hypersurface, where λ, C_1, C_2 are arbitrary positive real numbers.

When x and y are two hyperbolic affine sphere, choosing $\lambda = \frac{p+1}{q+1}$ in Proposition 3.4, we recover the Calabi-composition of two hyperbolic affine spheres:

Corollary 3.1 (see [8]) *Given two hyperbolic affine spheres $x : M_1 \to R^{p+1}$ and $y : M_2 \to R^{q+1}$, the Calabi-composition $x : R \times M_1 \times M_2 \to R^{p+q+2}$:*

$$z = (C_1 e^u x, C_2 e^{-\frac{p+1}{q+1} u} y), \quad u \in R, \tag{3.19}$$

defines a hyperbolic affine sphere, where C_1, C_2 are any positive real numbers.

Proof of Proposition 3.4 Consider the given centroaffine extremal hypersurfaces x and y in Proposition 3.4. We construct the generalized Calabi composition z defined by (3.18).

Let $\{u_1,\cdots,u_p\}$ and $\{u_{p+1},\cdots,u_{p+q}\}$ be local coordinates for M_1 and M_2, respectively. We denote $u_0=u$ and use the following range of indices:

$$1\le i,j,k\le p;\quad p+1\le \alpha,\beta,\gamma\le p+q;\quad 0\le A,B,C\le p+q.$$

We mark quantities of the hypersurface z by a tilde. Then $e_i=\frac{\partial x}{\partial u_i}$ form a basis for $x_*(TM_1)$, $e_\alpha=\frac{\partial y}{\partial u_\alpha}$ form a basis for $y_*(TM_2)$. Let $\tilde e_A=\frac{\partial z}{\partial u_A}$, i.e.,

$$\tilde e_0=(C_1e^ux,-C_2\lambda e^{-\lambda u}y),\quad \tilde e_i=(C_1e^ue_i,0),\quad \tilde e_\alpha=(0,C_2e^{-\lambda u}e_\alpha). \tag{3.20}$$

Then $\{\tilde e_A\}$ form a basis for $z_*(TR\oplus TM_1\oplus TM_2)$. We have

$$\begin{aligned}&[\tilde e_0,\tilde e_1,\cdots,\tilde e_{p+q},z]\\ =\;&(-1)^pC_1^{p+1}C_2^{q+1}(\lambda+1)e^{[(p+1)-(q+1)\lambda]u}[e_1,\cdots,e_p,x]\cdot[e_{p+1},\cdots,e_{p+q},y]\neq 0.\end{aligned}$$

x and y are centroaffine hypersurfaces, thus z is also a centroaffine hypersurface.

We denote by h_x, h_y, h_z the centroaffine metrics and ∇_x, ∇_y, ∇_z the Levi-Civita connections for x,y,z, respectively. Then, by a direct calculation, we have

$$\begin{aligned}&\frac{\partial^2 z}{\partial^2 u_0}=(1-\lambda)\tilde e_0+\lambda z;\ \frac{\partial^2 z}{\partial u_0\partial u_i}=\tilde e_i;\ \frac{\partial^2 z}{\partial u_0\partial u_\alpha}=-\lambda\tilde e_\alpha,\\ &\frac{\partial^2 z}{\partial u_i\partial u_j}=\frac{1}{\lambda+1}(h_x)_{ij}\tilde e_0+\sum_{k=1}^{p}(\nabla_x)_{ij}^k\tilde e_k+\frac{\lambda}{\lambda+1}(h_x)_{ij}\cdot z,\\ &\frac{\partial^2 z}{\partial u_i\partial u_\alpha}=\frac{\partial^2 z}{\partial u_\alpha\partial u_i}=0,\\ &\frac{\partial^2 z}{\partial u_\alpha\partial u_\beta}=-\frac{1}{\lambda+1}(h_y)_{\alpha\beta}\tilde e_0+\sum_{\gamma=p+1}^{p+q}(\nabla_y)_{\alpha\beta}^\gamma\tilde e_\gamma+\frac{1}{\lambda+1}(h_y)_{\alpha\beta}\cdot z.\end{aligned}\tag{3.21}$$

By definition, the centroaffine metric of z is

$$\begin{aligned}h_z&=\lambda(du_0)^2+\tfrac{\lambda}{\lambda+1}h_x+\tfrac{1}{\lambda+1}h_y\\ &=:\sum_{A,B=0}^{p+q}\tilde h_{AB}du_Adu_B.\end{aligned}\tag{3.22}$$

If h_x, h_y are complete metrics, h_z is a complete metric. The Tchebychev function $\tilde\psi$ of z is

$$\begin{aligned}\tilde\psi&=\frac{\det(\tilde h_{AB})}{[\tilde e_0,\tilde e_1,\cdots,\tilde e_{p+q},z]^2}\\ &=\frac{\lambda(\frac{\lambda}{\lambda+1})^p(\frac{1}{\lambda+1})^q\det(h^x_{ij})\cdot\det(h^y_{\alpha\beta})}{C_1^{2(p+1)}C_2^{2(q+1)}(\lambda+1)^2e^{2[(p+1)-(q+1)\lambda]u}[e_1,\cdots,e_p,x]^2\cdot[e_{p+1},\cdots,e_{p+q},y]^2}\\ &=C_1^{-2(p+1)}C_2^{-2(q+1)}\lambda^{p+1}(\lambda+1)^{-2-p-q}\cdot e^{-2[(p+1)-(q+1)\lambda]u}\psi_x\cdot\psi_y,\end{aligned}\tag{3.23}$$

where ψ_x and ψ_y are the Tchebychev functions of x and y, respectively. Thus

$$\begin{aligned}\log\tilde{\psi} &= [(p+1)\log\lambda - (2+p+q)\log(\lambda+1) - 2(p+1)\log C_1 - (q+1)\log C_2] \\ &\quad -2[(p+1)-(q+1)\lambda]u + \log\psi_x + \log\psi_y.\end{aligned} \tag{3.24}$$

The Laplacian $\tilde{\Delta}$ of h_z is given by

$$\tilde{\Delta} = \frac{1}{\lambda}\frac{\partial^2}{\partial^2 u} \oplus \frac{\lambda+1}{\lambda}\Delta_x \oplus (1+\lambda)\Delta_y, \tag{3.25}$$

thus we have

$$\tilde{\Delta}(\log\tilde{\psi}) = 0, \tag{3.26}$$

where Δ_x (resp. Δ_y) is the Laplacian of h_x (rep. h_y). From Theorem 2.2 and (3.18), $z : R \times M_1 \times M_2 \to R^{p+q+2}$ is a $(p+q+1)$-dimensional centroaffine extremal hypersurface. In particular, if the Tchebychev operators of x and y vanish, then $\tilde{\mathcal{T}} \equiv 0$.

Proof of Corollary 3.1 If $x : M_1 \to R^{p+1}$ and $y : M_2 \to R^{q+1}$ are two hyperbolic affine spheres, then

$$(\log\psi)_x = \text{constant}, \quad (\log\psi)_y = \text{constant}. \tag{3.27}$$

Choosing

$$\lambda = \frac{p+1}{q+1}, \tag{3.28}$$

from (3.24) we have

$$\log\tilde{\psi} = \text{constant}.$$

Thus $x : R \times M_1 \times M_2 \to R^{p+q+2}$:

$$z = (C_1 e^u x, C_2 e^{-\frac{p+1}{q+1}u} y), \quad u \in R,$$

is a hyperbolic affine sphere.

Example 3.4-A Taking $x(M_1) = H(1,p)$, $y(M_2) = H(1,q)$ and $C_1 = C_2 = 1$ in Proposition 3.4, we obtain a family of centroaffine extremal hypersurfaces $z : R \times M_1 \times M_2 \to R^{p+q+2}$

$$[z_{p+1}^2 - (z_1^2 + \cdots + z_p^2)] \cdot [z_{p+q+2}^2 - (z_{p+2}^2 + \cdots + z_{p+q+1}^2)]^{\frac{1}{\lambda}} = 1, \quad \lambda > 0.$$

We note that z is a hyperbolic affine sphere if and only if $\lambda = \frac{p+1}{q+1}$.

Example 3.4-B Taking $x(M_1) = H(1,p)$, $y(M_2) = Q(1; \alpha_1, \cdots, \alpha_q; q)$ and $C_1 = C_2 = 1$ in Proposition 3.4, we obtain a family of centroaffine extremal hypersurfaces $z : R \times M_1 \times M_2 \to R^{p+q+2}$

$$[z_{p+1}^2 - (z_1^2 + \cdots + z_p^2)]^{\frac{(1+\alpha_1+\cdots+\alpha_q)\lambda}{2}} z_{p+2}^{\alpha_1} \cdots z_{p+q+1}^{\alpha_q} \cdot z_{p+q+2} = 1,$$

where $\alpha_1 > 0, \cdots, \alpha_q > 0$. We note that z is a hyperbolic affine sphere if and only if $\lambda = \frac{p+1}{q+1}$ and $\alpha_1 = \cdots = \alpha_q = 1$.

Example 3.5 Polar Hypersurfaces

We recall the following construction, called inversion at the unit sphere, which is well known from Euclidean hypersurface theory. For this, we equip the affine space R^{n+1} with an additional Euclidean structure, defined by a scalar product

$$<,>: R^{n+1} \times R^{n+1} \to R.$$

Let $x : M \to R^{n+1}$ be a centroaffine hypersurface with μ as its Euclidean unit normal field. As the position vector is transversal, the Euclidean support function

$$\rho(E) := < \mu, -x >$$

is nowhere zero and

$$\rho(E)^{-1}\mu =: x^* : M \to R^{n+1}$$

defines the so called polar hypersurface x^* of x; the mapping $x \to x^*$ is called polarization. Both hypersurfaces satisfy the relations

$$< x^*, x >= -1, \quad < dx^*(v), x >= 0, \quad < x^*, dx(v) >= 0$$

for any tangent field $v \in TM$. This correspondence immediately implies that $(x^*)^* = x$; for this reason we also use the notion polar pair for x, x^*.

It is known in affine hypersurface theory that polarity can be much better studied in terms of centroaffine hypersurface theory; see section 37 in [15], and section 7 in [13]. The bijective correspondence between x and x^* is exactly the correspondence between the hypersurface x and its centroaffine conormal image. The following proposition recalls how fundamental centroaffine invariants behave under polarization (Proposition 7.2.1 and Corollary 7.4.2 in [13]); for our purpose, we add additional trivial consequences for the Tchebychev operator $\mathcal{T}$. Using an obvious notation for the centroaffine invariants of x^*, we have the following list:

Proposition 3.5 *Let x, x^* be a polar pair of centroaffine hypersurfaces. Then*

(i) $h = h^*$; (ii) $A = -A^*$; (ii) $T = -T^*$; (iv) $\mathcal{T} = -\mathcal{T}^*$; (v) the equiaffine support functions satisfy $\rho(e)\rho(e)^* = 1$; (vi) the Tchebychev functions satisfy $\psi\psi^* = 1$.

Corollary 3.2 (i) *x is a proper affine sphere if and only if x^* is a proper affine sphere; (ii) $\mathcal{T} \equiv 0$ if and only if $\mathcal{T}^* \equiv 0$; (iii) x is centroaffine extremal if and only if x^* is centroaffine extremal; (iv) x is a hypersurface in the class Q if and only if x^* belongs to Q; (v) x is centroaffinely complete if and only if x^* is .*

Proof (i) $T = 0$ characterizes proper affine spheres. (iii) Trivial consequence from $\mathcal{T} = -\mathcal{T}^*$. (iv) Apply Proposition 3.5 and Proposition 3.2.

As a consequence, considering the Examples 3.1 and 3.3, polarization preserves the type of such classes of centroaffine extremal hypersurfaces. In particular, the hypersurfaces of both classes have vanishing Tchebychev operator, and from this neither the generalized Calabi construction nor polarization produces examples with non-vanishing Tchebychev operator.

4. Notions of completeness

In the foregoing section we discussed classes of examples of extremal centroaffine hypersurfaces satisfying at least one of the completeness conditions (i) or (ii) from the beginning of section 3. In this section we recall definitions and summarize known results on relations between the two notions of completeness from section 2 in [9].

Definition 4.1 *(i) Euclidean completeness, that is the completeness of the Riemannian metric on M induced from a Euclidean metric on A^{n+1}; this notion is independent of the specific choice of the Euclidean metric on the affine space and thus it is a notion of affine geometry; see [8], p. 110;*

(ii) centroaffine completeness, that is the completeness of the centroaffine metric h.

Observation 4.1 *From Hadamard's theorem it is well known that any locally strongly convex, Euclidean complete hypersurface is the boundary of a convex body; if it is not compact, it can be represented as a convex graph over a plane. In particular, this applies to centroaffine hypersurfaces with hyperbolic normalization.*

The following result is a consequence of a result due to Cheng-Yau (see [9], section 2.3); as on proper affine spheres the completeness of the centroaffine metric and the Blaschke metric are equivalent, we can state

Proposition 4.1 *On proper affine spheres, the Euclidean completeness implies the completeness of the centroaffine metric.*

Definition 4.2 *For any function F, defined on M, we define the k-norm*

$$\|F\|_k := |F| + \|\nabla F\|_h + \cdots + \|\nabla^k F\|_h$$

where ∇ is the covariant differentiation with respect to the centroaffine metric h.

We have the following results about relations between Euclidean completeness and centroaffine completeness

Theorem 4.1(see p.148, p. 151 and p. 156 in [9]) *(i) Let $x : M \longrightarrow R^{n+1}$ be a Euclidean complete, locally strongly convex hypersurface with hyperbolic centroaffine normalization. If ψ satisfies*

$$\|\log\psi\|_2 \leq c_0$$

for some positive constant c_0, then (M, h) is centroaffine-complete.

(ii) Let $x : M \longrightarrow R^{n+1}$ be a centroaffine-complete hypersurface with $\|\log\psi\|_3 \leq c_0$ for some positive constant c_0, then $x : M \longrightarrow R^{n+1}$ is Euclidean complete.

(iii) Let M be a Euclidean complete convex hypersurface with hyperbolic centroaffine normalization. If the Tchebychev function ψ of M satisfies

$$\|log\psi\|_2 \leq c_0$$

for some positive constant c_0, then M is asymptotic to the boundary of a convex cone V.

Remark 4.1 From Theorem 4.1 it follows that, on hyperbolic centroaffine hypersurfaces, the condition on the 3-norm in (ii) implies the equivalence of both completeness conditions as well as the asymptotic property in (iii).

5. Centroaffine Bernstein problems

In section 3 we studied large classes of centroaffine extremal hypersurfaces. All the explicit examples have vanishing Tchebychev operator. Comparing the class of hyperbolic affine spheres and the class of examples given in 3.3, there is only one type of hypersurfaces in the intersection of both classes, namely the hypersurfaces represented by

$$x_1 x_2 \cdots x_{n+1} = c, \quad c > 0.$$

Concerning completeness conditions, the compact case is solved by Wang's theorem. Thus only complete, non-compact centroaffine extremal hypersurfaces are still of interest. The classes in example 3.1 and 3.3 can be represented as graphs over R^n, that means they are Euclidean complete. The hypersurfaces in examples 3.1 and 3.3 are also centroaffine complete. For hyperbolic centroaffine hypersurfaces, Theorem 4.1 clarifies the relations between Euclidean completeness and centroaffine completeness.

Comparing the centroaffine examples with the two versions of the "Affine Bernstein Conjecture" recalled in the introduction, we see that the centroaffine situation is quite different: we have many locally strongly convex, extremal and complete centroaffine hypersurfaces and not just one candidate for the formulation of a centroaffine Bernstein problem. Thus, for classification theorems we need further conditions. There are natural geometric candidates for such conditions: (a) conditions on intrinsic curvature invariants of the centroaffine metric; (b) conditions concerning the Tchebychev field or the associated Tchebychev form, resp., which somehow measure the deviation from a proper affine sphere; as already mentioned above, the associated Tchebychev form also can be expressed in terms of different volume forms.

We already mentioned the highly nonlinear character of the Euler-Lagrange equations in the form (3.12) and (3.14). So far, all our examples satisfy the relation $\mathcal{T} \equiv 0$, which is $\nabla_r \nabla_s \log \psi = 0$ for ψ given by (3.10). It is much more complicated to solve even this system than the well known equation (3.15) for hyperbolic affine spheres.

In the following we list several related versions of centroaffine Bernstein problems for locally strongly convex hypersurfaces; some of the problems are stated in form of conjectures.

Centroaffine Bernstein Problem I: *Let $x : M \to R^{n+1} (n \geq 2)$ be a centroaffine extremal hypersurface satisfying one of the completeness conditions from Definition 4.1. Is $\mathcal{T} \equiv 0$?*

Centroaffine Bernstein Conjecture: *Let $x : M \to R^{n+1} (n \geq 2)$ be a centroaffine extremal hyperbolic hypersurface satisfying one of the completeness conditions from Definition 4.1. If the Ricci curvature of the centroaffine metric is non-negative, then x is centroaffinely equivalent to one of the following hypersurfaces*

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1, \quad \alpha_1 > 0, \cdots, \alpha_{n+1} > 0.$$

Centroaffine Bernstein Problem II *Does the class of centroaffine extremal hyperbolic graphs over R^n contain other examples as the ones given in examples 3.1 and 3.3 ?*

Centroaffine Bernstein Problem III *Do there exist extremal centroaffine hypersurfaces with complete centroaffine metric which can not be represented as graphs over R^n?*

Centroaffine Bernstein Problem IV *Do there exist extremal centroaffine hypersurfaces satisfying one of the completness conditions such that the Tchebychev field does not have constant norm?*

Centroaffine Bernstein Problem V *Do there exist extremal elliptic centroaffine hypersurfaces satisfying one of the completness conditions which are not hyperellipsoids?*

6 Statement of the results

Theorem 6.1 *Let $x : M \to R^3$ be a noncompact, hyperbolic extremal centroaffine surface with complete centroaffine metric. If the Gaussian curvature K of the centroaffine metric and the length $|T|$ of the Tchebychev vector field satisfy*
(1) $K \geq 0$,
(2) $|T| < \infty$,
then x is centroaffinely equivalent to one of the following surfaces

$$x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} = 1, \quad \alpha_1 > 0, \quad \alpha_2 > 0, \quad \alpha_3 > 0. \tag{6.1}$$

Corollary 6.1 *Let $x : M \to R^3$ be an affine complete hyperbolic affine sphere. If the Gaussian curvature K of the centroaffine metric is nonnegative, then x is affinely equivalent to the following surface*

$$x_1 x_2 x_3 = 1. \tag{6.2}$$

Theorem 6.2 *Let $x : M \to R^{n+1} (n \geq 2)$ be a complete non-compact hyperbolic extremal centroaffine hypersurface with complete centroaffine metric. If the Ricci curvature of the centroaffine metric and the length $|T|$ of the Tchebychev vector field satisfy*
(1) $\mathrm{Ric} \geq 0$,
(2) $|T| = \text{constant}$,
then x is centroaffinely equivalent to one of the following hypersurfaces

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1, \quad \alpha_1 > 0, \cdots \alpha_{n+1} > 0. \tag{6.3}$$

Corollary 6.2 *Let $x : M \to R^{n+1} (n \geq 2)$ be a complete hyperbolic affine hypersphere. If the Ricci curvature of the centroaffine metric is non-negative, then x is affinely equivalent to the following hypersurface*

$$x_1 x_2 \cdots x_{n+1} = 1. \tag{6.4}$$

Theorem 6.3 *Let $x : M \to R^{n+1} (n \geq 2)$ be a metrically complete, non-compact extremal centroaffine hypersurface. If the Ricci curvature of the centroaffine metric and the length $|T|$ of the Tchebychev vector field satisfy*
(1) $\mathrm{Ric} \geq 0$,
(2) $|T| \in L^p(M)$, for some $p > 1$,
then x is centroaffinely equivalent to the following hypersurface

$$x_1 x_2 \cdots x_{n+1} = 1.$$

Theorem 6.4 *Let $x : M \to R^{n+1} (n \geq 2)$ be a metrically complete, non-compact extremal centroaffine hypersurface. If the Ricci curvature of the centroaffine metric is non-negative and* $\log\psi$ *is bounded, then x is centroaffinely equivalent to the following hypersurface*

$$x_1 x_2 \cdots x_{n+1} = 1.$$

Remark 6.1 A hyperboloid $H(c, n)$ satisfies (see Example 3.1)

1. the centroaffine metric is complete and centroaffine extremal,

2. the Tchebychev function is a constant function and the Tchebychev vector field vanishes.

On the other hand its Ricci curvature is a negative constant (see (3.3)). Thus the assumption in Theorems 6.1-6.4 that the " Ricci curvature is nonnegative" is necessary.

Remark 6.2 For the centroaffine hypersurfaces

$$x_1^{\alpha_1}\cdots x_{n+1}^{\alpha_{n+1}} = c,\ c>0,\ (\alpha_1,\cdots,\alpha_{n+1})\neq(1,\cdots,1),\ \alpha_i>0,\ 1\leq i\leq n+1,$$

using (3.17), it is easy to check that $\log\psi$ is not bounded. Thus the assumption in Theorem 6.4 that "$\log\psi$ is bounded" is essential.

7. Lemmas and Proofs of Theorem 6.1 and Theorem 6.2

We will apply the following well known Bochner-Lichnerowicz formula as a tool.

$$\frac{1}{2}\Delta(|T|^2) = \frac{1}{2}\Delta(\sum_i (T_i)^2) = \sum_{i,j}(T_{i,j})^2 + \sum_{i,j} R_{ij}T_iT_j + \sum_i T_i(\sum_k T_{k,k})_i. \tag{7.1}$$

If we assume that the trace of the Tchebychev operator is constant, i.e., $\sum_k T_{k,k} =$ constant, then (7.1) becomes

$$\frac{1}{2}\Delta(|T|^2) = \frac{1}{2}\Delta(\sum_i (T_i)^2) = \sum_{i,j}(T_{i,j})^2 + \sum_{i,j} R_{ij}T_iT_j. \tag{7.2}$$

Lemma 7.1 *Let $x : M \to R^3$ be a metrically complete, noncompact centroaffine surface with constant trace of the Tchebychev operator. If the Gaussian curvature K of the centroaffine metric and the length $|T|$ of the Tchebychev vector field satisfy*

(1) $K \geq 0$,

(2) $|T| < \infty$,

then the Tchebychev vector field is parallel, i.e., $T_{i,j} = 0$.

Proof As we assume $K \geq 0$, we conclude from the Riemann mapping theorem that either M is conformally equivalent to the Riemannian sphere S^2, or M is conformally equivalent to the Euclidean space R^2. From the assumption the surface is complete, but non-compact, thus we know that M is conformally equivalent to the Euclidean space R^2.

We apply (7.2), $Ric = Kh$ and the assumption $K \geq 0$:

$$\frac{1}{2}\Delta(|T|^2) \geq \sum_{i,j}(T_{i,j})^2 \geq 0,$$

that is, $|T|^2$ is a subharmonic function on M. The assumption $|T|^2 < \infty$ gives $|T|^2 =$ constant (see Leon Karp [4]), and (7.2) implies $T_{i,j} = 0$, i.e., $\mathcal{T} = 0$.

Lemma 7.2 *Let $x : M \to R^{n+1}$ be a complete noncompact centroaffine hypersurface with* trace$\mathcal{T}$ = constant. *If the Ricci curvature of the centroaffine metric and the length $|T|$ of the Tchebychev vector field satisfy*

(1) Ric ≥ 0,

(2) $|T| =$ constant,

then $\mathcal{T} \equiv 0$.

The proof follows again from (7.2).

We need the following generalized maximum principle

Lemma 7.3 (Omori-Yau [14], [22]). *Let M be a complete Riemannian manifold with Ricci curvature bounded from below. Let f be a C^2-function which is bounded from below on M. Then there is a sequence of points $\{p_k\}$ in M such that*

$$\lim_{k\to\infty} f(p_k) = \inf(f), \quad \lim_{k\to\infty}|\mathrm{grad}(f)|(p_k) = 0, \quad \lim_{k\to\infty}\Delta f(p_k) \geq 0.$$

Proposition 7.1 *Let $x : M \to R^{n+1}$ be a complete, noncompact hyperbolic centroaffine hypersurface with $\mathcal{T} \equiv 0$. If the Ricci curvature of the centroaffine metric is non-negative, then x is centroaffinely equivalent to one of the following hypersurfaces*

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1, \quad \alpha_1 > 0, \cdots, \alpha_{n+1} > 0.$$

For the proof we need the following lemma

Lemma 7.4 *Let $x : M \to R^{n+1}$ be a centroaffine hypersurface with* Ric ≥ 0 *and $\mathcal{T} \equiv 0$. Then the normalized scalar curvature satisfies*

$$\Delta\kappa \geq 4\kappa(\kappa - \epsilon). \tag{7.3}$$

Proof. By use of (2.15), (2.23) and (2.26), we have the following calculation (c.f. [6], [8])

$$\begin{aligned} \Delta A_{ijk} &= \sum_l A_{ijk,ll} = \sum_l A_{ijl,kl} \\ &= \sum_l A_{ijl,lk} + \sum_{r,l} A_{ijr}R_{rlkl} + \sum_{r,l} A_{ril}R_{rjkl} + \sum_{r,l} A_{rjl}R_{rikl} \\ &= nT_{i,jk} + \sum_r A_{ijr}R_{rk} + + \sum_{r,l} A_{ril}R_{rjkl} + \sum_{r,l} A_{rjl}R_{rikl} \\ &= \sum_r A_{ijr}R_{rk} + \sum_{r,l} A_{ril}R_{rjkl} + \sum_{r,l} A_{rjl}R_{rikl}, \end{aligned} \tag{7.4}$$

where we used $\mathcal{T} \equiv 0$. (7.4) and (2.22) give

$$\begin{aligned} \tfrac{1}{2}n(n-1)\Delta J &= \Delta\Big(\sum_{i,j,k}(A_{ijk})^2\Big) \\ &= \sum_{i,j,k,l}(A_{ijk,l})^2 + \sum_{i,j,k,l} A_{ijk}A_{ijk,ll} \\ &= \sum_{i,j,k,l}(A_{ijk,l})^2 + \sum A_{ijk}A_{ijr}R_{rk} + \sum A_{ijk}A_{ril}R_{rjkl} + \sum A_{ijk}A_{rjl}R_{rikl} \\ &= \sum_{i,j,k,l}(A_{ijk,l})^2 + \sum A_{ijk}A_{ijr}R_{rk} + \sum (A_{ijk}A_{ril} - A_{ijl}A_{irk})R_{rjkl} \\ &= \sum_{i,j,k,l}(A_{ijk,l})^2 + \sum (R_{rjkl})^2 + A_{ijk}A_{ijr}R_{rk} - 2\epsilon R \\ &\geq \sum (R_{rjkl})^2 - 2\epsilon R \\ &\geq \frac{2}{n(n-1)}R^2 - 2\epsilon R, \end{aligned} \tag{7.5}$$

where we used Ric ≥ 0 and the following well known estimate

$$\sum (R_{rjkl})^2 \geq \frac{2}{n-1}\sum (R_{rk})^2 \geq \frac{2}{n(n-1)}R^2. \tag{7.6}$$

From (2.25), we have

$$n(n-1)J = n(n-1)(\kappa - \epsilon) + n^2|T|^2. \tag{7.7}$$

The assumption $\mathcal{T} \equiv 0$ implies that $|T|^2$ is constant; we insert (7.7) into (7.5)

$$\begin{aligned} \tfrac{1}{2}n(n-1)\Delta\kappa &= \Delta(\sum_{i,j,k}(A_{ijk})^2) \\ &\geq \tfrac{2}{n(n-1)}R^2 - 2\epsilon R = 2n(n-1)\kappa(\kappa - \epsilon). \end{aligned} \tag{7.8}$$

Proof of Proposition 7.1. For any given positive constant δ, define the positive smooth function u on M by

$$u := \frac{1}{\sqrt{\kappa + \delta}}. \tag{7.9}$$

Through a direct calculation, by use of (7.3) and $\epsilon = -1$, the Laplacian Δu of u satisfies

$$\begin{aligned} u\Delta u &= 3|\text{grad}(u)|^2 - \tfrac{1}{2(\kappa+\delta)^2}\Delta\kappa \\ &\leq 3|\text{grad}(u)|^2 - \tfrac{2}{(\kappa+\delta)^2}\kappa(\kappa+1). \end{aligned} \tag{7.10}$$

We have $u \geq 0$; as we assumed that the Ricci curvature is non-negative, we can apply the generalized maximum principle (Lemma 7.3) of Omori and Yau to the function u on M. Then there is a sequence of points $\{p_k\}$ on M such that

$$\text{lim}_{k\to\infty} u(p_k) = \inf(u), \quad \text{lim}_{k\to\infty}|\text{grad}(u)|(p_k) = 0, \quad \text{lim}_{k\to\infty}\Delta u(p_k) \geq 0.$$

We claim that $\inf(u) \neq 0$. Otherwise, from the definition of u, the assumption $\inf(u) = 0$ gives $\sup(\kappa) = \infty$. Considering the limit for both sides of the inequality (7.10), we get

$$0 = \inf(u) \cdot \text{lim}_{k\to\infty}\Delta u(p_k) \leq -2,$$

which gives a contradiction. Thus $\inf(u) \neq 0$ and then $0 \leq \text{lim}_{k\to\infty}\kappa(p_k) = \sup(\kappa) < \infty$. Considering again the limit for both sides of the inequality (7.10), we get

$$\begin{aligned} 0 &\leq \inf(u) \cdot \text{lim}_{k\to\infty}\Delta u(p_k) \\ &\leq 3 \cdot \text{lim}_{k\to\infty}|\text{grad}(u)|^2(p_k) - \tfrac{2\sup(\kappa)}{(\sup(\kappa)+\delta)^2}(\sup(\kappa) + 1) \\ &= -\tfrac{2\sup(\kappa)}{(\sup(\kappa)+\delta)^2}(\sup(\kappa)+1). \end{aligned} \tag{7.11}$$

(7.11) implies

$$\sup(\kappa) \leq 0,$$

that is

$$\kappa \leq 0.$$

Thus we conclude that $\kappa \equiv 0$ (because we assumed $Ric \geq 0$). From (7.7) and $\epsilon = -1$, we get $J = 1 + \frac{n}{n-1}|T|^2 =$ constant and then (7.5) gives

$$R_{ijkl} \equiv 0, \qquad A_{ijk,l} = 0, \quad 1 \leq i,j,k,l \leq n. \tag{7.12}$$

Thus $x(M)$ has a flat centroaffine metric and its centroaffine Pick form is parallel with respect to its centroaffine metric. The assertion of Proposition 7.1 now follows from Proposition 3.3.

Proofs of Theorem 6.1 and Theorem 6.2: Theorem 6.1 comes from Lemma 7.1 and Proposition 7.1. Theorem 6.2 comes from Lemma 7.2 and Proposition 7.1.

Remark 7.1 We also can get the following local uniqueness results, which generalize the result of Li-Wang (see Proposition 3.3).

Proposition 7.2 *Let $x : M \to R^{n+1}$ be a centroaffine hypersurface with $\mathcal{T} \equiv 0$. If the Ricci curvature of the centroaffine metric is non-negative (or non-positive), then x is locally centroaffinely equivalent to a proper affine sphere or one of the following hypersurfaces*

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1, \quad \alpha_1 > 0, \cdots, \alpha_{n+1} > 0.$$

Proof. Because we assume $\mathcal{T} \equiv 0$, we have from (7.2)

$$\sum_{i,j} R_{ij} T_i T_j \equiv 0. \tag{7.13}$$

From the assumption $R_{ij} \geq 0$ (resp. $R_{ij} \leq 0$) we have either $|T| \equiv 0$, or $R_{ij} \equiv 0$. If $|T| \equiv 0$ then $x : M \to R^{n+1}$ is a proper affine sphere. If $R_{ij} \equiv 0$, we get from (7.5)

$$R_{ijkl} \equiv 0, \qquad A_{ijk,l} = 0, \quad 1 \leq i, j, k, l \leq n.$$

Thus $x(M)$ has a flat centroaffine metric and its centroaffine Pick form is parallel with respect to its centroaffine metric. Proposition 7.2 now follows from Proposition 3.3.

Corollary 7.1 *Let $x : M \to R^{n+1} (n \geq 2)$ be an n-dimensional complete elliptic centroaffine hypersurface with $\mathcal{T} \equiv 0$. If the Ricci curvature of the centroaffine metric is non-negative (resp. non-positive), then x is centroaffinely equivalent to a hyperellipsoid (resp. there does not exist such a hypersurface).*

Proof. From Proposition 7.2, it follows that x is an elliptic affine sphere, thus x is centroaffinely equivalent to a hyperellipsoid (resp. there does not exist such a hypersurface).

Proposition 7.3 *Let $x : M \to R^{n+1} (n \geq 2)$ be an n-dimensional hyperbolic centroaffine extremal hypersurface. If the Ricci curvature of the centroaffine metric is nonnegative, the scalar curvature is constant, and the length of the Tchebychev vector field is constant, then x is centroaffinely equivalent to one of the following hypersurfaces*

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_{n+1}^{\alpha_{n+1}} = 1, \quad \alpha_1 > 0, \cdots \alpha_{n+1} > 0.$$

Proof. As we assume that $x : M \to R^{n+1}$ is a centroaffine extremal hypersurface with $|T| =$ constant, we have from (7.2) that

$$T_{i,j} = 0.$$

Our assumptions imply $J =$ constant and $\epsilon = -1$. From (7.5) we get

$$R_{ijkl} \equiv 0, \qquad A_{ijk,l} = 0, \quad 1 \leq i, j, k, l \leq n.$$

Thus $x(M)$ has a flat centroaffine metric and its centroaffine Pick form is parallel with respect to its centroaffine metric. Proposition 7.3 now follows from Proposition 3.3.

8. Proofs of Theorem 6.3 and Theorem 6.4

We need the following lemmas

Lemma 8.1 ([22]) *Let (M, g) be a complete Riemannian manifold with non-negative Ricci curvature, then any bounded (from below or from above) harmonic function on M must be a constant.*

Lemma 8.2 ([23]) *Let (M, g) be a complete noncompact Riemannian manifold with non-negative Ricci curvature. If for some $p > 1$*

$$\Delta u \geq 0, \qquad u \geq 0, \qquad u \in L^p(M),$$

then u is constant.

Proof of Theorem 6.3 Under the assumptions of Theorem 6.3, we have from (7.2)

$$\frac{1}{2}\Delta(|T|^2) = \sum_{i,j}(T_{i,j})^2 + \sum_{ij} R_{ij}T_iT_j \geq \sum_{i,j}(T_{i,j})^2. \tag{8.1}$$

Noting

$$\frac{1}{2}\Delta(|T|^2) = |T|\Delta|T| + \sum_i(|T|_i)^2, \tag{8.2}$$

we have from (8.1) and (8.2),

$$|T|\Delta|T| \geq \sum_{i,j}(T_{i,j})^2 - \sum_i(|T|_i)^2. \tag{8.3}$$

From (8.3) and

$$\begin{aligned} |T|^2\sum_i(|T|_i)^2 &= \sum_i(|T||T|_i)^2 = \tfrac{1}{2}\sum_i((|T|^2)_i)^2 \\ &= \sum_i(\sum_j T_iT_{i,j})^2 \\ &\leq |T|^2 \cdot \sum_{i,j}(T_{i,j})^2, \end{aligned} \tag{8.4}$$

we conclude that $\Delta|T| \geq 0$, i.e. $|T|$ is a non-negative subharmonic function. From Lemma 8.2, our assumption $|T| \in L^p(M)$ $(p > 1)$ implies that $|T|$ is constant. Thus we get $T_{i,j} = 0$ from (8.1). In this case, as the volume of M is infinite (see [17] or [18]) and as we assume $|T| \in L^p(M)$, we necessarily have $|T| = 0$. Since a complete elliptic affine hypersphere is a hyperellipsoid (compact), Theorem 6.3 then directly follows from Proposition 7.1 and the remarks in Example 3.3.

Proof of Theorem 6.4. Let $x : M \to R^{n+1}$ be an n-dimensional centroaffine extremal hypersurface; then we have

$$\Delta(\log\psi) = 0,$$

where ψ is the Tchebychev function of x. From Lemma 8.1 it follows that $\log\psi$ is constant and that the Tchebychev vector field vanishes. Since a complete elliptic affine hypersphere is a hyperellipsoid (compact), Theorem 6.4 follows from Proposition 7.1 and the remarks in Example 3.3.

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A.-M. Li, H. Li and U. Simon
Technische Universität Berlin,
Institut für Mathematik, MA 8-3,
Straße des 17. Juni 136,
10623 Berlin, Germany
Email: amli@math.tu-berlin.de
hli@sfb288.math.tu-berlin.de
simon@math.tu-berlin.de

A.-M. Li
Department of Mathematics
Sichuan University
Chengdu, 610064
People's Republic of China
Email: math_li@yahoo.com.cn

H. Li
Department of Mathematical Sciences
Tsinghua University
Beijing, 100084
People's Republic of China
Email: hli@math.tsinghua.edu.cn