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ANALYSIS AND NUMERICAL SOLUTION OF LINEAR DELAY DIFFERENTIAL-ALGEBRAIC EQUATIONS*

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Abstract. The analysis and numerical solution of initial value problems for linear delay differential-algebraic equations (DDAEs) is discussed. Characteristic properties of DDAEs are analyzed and the differences between causal and noncausal DDAEs are studied. The method of steps is analyzed and it is shown that it has to be modified for general DDAEs. The classification of ordinary delay differential equations (DDEs) is generalized to DDAEs, and a numerical solution procedure for general retarded and neutral DDAEs is constructed. The properties of the algorithm are studied and the theoretical results are illustrated with a numerical example.

Key words. Delay differential-algebraic equation, differential-algebraic equation, delay differential equations, method of steps, derivative array, classification of DDAEs.

AMS subject classifications. 34A09, 34A12, 65L05, 65H10.

1. Introduction. In this paper we study the analysis and numerical solution of general *linear delay differential-algebraic equations (DDAEs) with variable coefficients* and a single constant delay $\tau > 0$ of the form

$$E(t)\dot{x}(t) = A(t)x(t) + B(t)x(t-\tau) + f(t), \tag{1.1}$$

in a time interval $\mathbb{I} = [0, t_f)$, where $\dot{x}$ denotes the time derivative of the vector valued function x. The desired function x maps from $\mathbb{I}_\tau := [-\tau, t_f)$ to $\mathbb{C}^n$ and the coefficients are matrix functions $E,\ A,\ B : \mathbb{I} \to \mathbb{C}^{m,n}$, and $f : \mathbb{I} \to \mathbb{C}^m$. To achieve uniqueness of solutions of (1.1) one typically has to prescribe initial functions of the form

$$\phi : [-\tau, 0] \to \mathbb{C}^n, \text{ such that } x|_{[-\tau,0]} = \phi. \tag{1.2}$$

For simplicity, we assume that $t_f = \ell\tau$, so that the time interval is $\mathbb{I} = [0, \ell\tau)$, for an integer $\ell \in \mathbb{N}$. We also allow for $\ell = \infty$ and then set $\mathbb{I} = [0, \infty)$.

Most of the results in this paper can also be extended to multiple delays, but here we only discuss the single delay case. DDAEs of the form (1.1) arise as linearization of nonlinear DDAEs $F(t, \dot{x}(t), x(t), x(t-\tau)) = 0$ around a non-stationary nominal solution [10], and they describe the local behavior in the neighborhood of the nominal solution. Here, however, we restrict ourselves to the linear variable coefficient case.

Two important subclasses of (1.1) that occur in various applications are differential-algebraic equations (DAEs) with $B \equiv 0$, and delay differential equations (DDEs), where $m = n$ and E is the identity matrix. A typical viewpoint that is often taken in the analysis and numerical solution of DDEs and DDAEs is to introduce an artificial inhomogeneity $g(t) = B(t)x(t-\tau)+f(t)$ and to consider instead of (1.1) the *associated DAE*

$$E(t)\dot{x}(t) = A(t)x(t) + g(t) \quad \text{for all} \quad t \in \mathbb{I}. \tag{1.3}$$

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If the associated DAE (1.3) is uniquely solvable for all sufficiently smooth inhomogeneities g and appropriate consistent initial vectors, then the solution of (1.1) with initial function (1.2) can be uniquely determined step-by-step by solving a sequence of DAEs on consecutive intervals $[i\tau, (i+1)\tau]$. This is the most common approach for systems with delays, often called the *(Bellman) method of steps*, see e.g., [3–6, 8, 9, 14, 33, 36]. However, we will show in Section 3 that this method is only suitable for *causal systems*, where the solution at a current time depends on the system coefficients at current and past time points but not on future time points. In general, it is possible that the corresponding initial value problem of a noncausal DDAE possesses a unique solution, even though the associated DAE is neither square nor uniquely solvable. The method of steps immediately fails in this situation. Such non-square systems arise in many applications, in particular for dynamical systems which are automatically generated by modeling and simulation software such as [1, 13, 27, 35], due to interface conditions or redundant equations which may make the resulting system over- or under-determined.

Despite the fact that the numerical solution of DAEs has been intensively studied [7, 24], and a similar maturity has been reached for DDEs [5, 17], there are relatively few investigations on the numerical solution of DDAEs, see e.g. [3, 4, 14, 33, 36]. In particular, the numerical solution of general over- and under-determined DDAEs has not been studied so far. To close this gap for linear DDAEs with variable coefficients as the first step towards numerical methods for general DDAEs is the aim of this paper, which is structured as follows. Section 2 reviews some results in the theory of DAEs. In Section 3, we discuss some characteristic properties of DDAEs and the major differences between the two classes of causal and noncausal systems as well. Then we review the classical method of steps and its drawbacks when applying it to general linear DDAEs. To overcome these drawbacks, we suggest a modification of the method of steps and how to implement it in Section 4. As the classical method of steps, our proposed generalization is only efficient for retarded and neutral DDAEs. Finally, we illustrate the theoretical results and the behavior of the numerical method via an example.

2. Notations and Preliminaries. For the DAE (1.3) associated with (1.1) one frequently uses the concept of classical solutions, i.e., functions $x : \mathbb{I} \to \mathbb{C}^n$ that are continuously differentiable and satisfy (1.3) pointwise, see e.g. [7, 24]. However, in the case of DDAEs, there is no reason why $E(0)\dot{x}(0)$ in (1.1) should be equal to $E(0)\dot{\phi}(0^-)$. Moreover, it has been observed, see e.g. [4, 8, 14], that a discontinuity of $\dot{x}$ at $t = 0$ may propagate with time, and then typically $\dot{x}$ is discontinuous at every point $j\tau$. To deal with this difficulty we use the following solution concept.

Definition 2.1.

1. *A function $x : \mathbb{I}_\tau \to \mathbb{C}^n$ is called a* piecewise differentiable solution *of* (1.1)*, if it is continuous, piecewise continuously differentiable and satisfies* (1.1) *almost everywhere.*
2. *An initial function ϕ is called* consistent *if the initial value problem* (1.1)-(1.2) *has at least one piecewise differentiable solution.*
3. *The DDAE* (1.1) *is called* solvable *if it has at least one piecewise differentiable solution. It is called* regular *if in addition, for every consistent initial function, the solution of the initial value problem* (1.1)-(1.2) *is unique.*

In the following, when we speak of a solution, we always mean a piecewise differentiable solution, and when we discuss the numerical solution of (1.1), we assume that the initial value problem (1.1)–(1.2) is regular. Note that for DAEs, if the system

coefficients are sufficiently smooth then the piecewise differentiable solution is exactly the classical solution.

Let us recall some results for the DAE (1.3) that will be used later. It is well-known that the solution of (1.3) may depend on derivatives of g and there may exist hidden constraints for the inhomogeneity g and its derivatives. To classify the regularity requirements for general over- and under-determined DAEs one uses the *strangeness index* introduced in [22, 23], which generalizes the differentiation index [7]. See [24, 28] for a detailed discussion of different index concepts.

Since derivatives are needed, for a practical numerical integration method, one uses so-called *derivative arrays*, [7, 24], i.e., one differentiates (1.3) k times to form the inflated DAE

$$\begin{aligned} E(t)\dot{x}(t) - A(t)x(t) + g(t) &= 0, \\ \left(\frac{d}{dt}\right)\left(E(t)\dot{x}(t) - A(t)x(t) + g(t)\right) &= 0, \\ &\vdots \\ \left(\frac{d}{dt}\right)^k\left(E(t)\dot{x}(t) - A(t)x(t) + g(t)\right) &= 0. \end{aligned} \tag{2.1}$$

Under some smoothness and constant rank assumptions, the minimum number of differentiations that are needed to extract from the derivative arrays a so-called *strangeness-free* DAE of the form

$$\begin{bmatrix} \hat{E}_1(t) \\ 0 \\ 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} \hat{A}_1(t) \\ \hat{A}_2(t) \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} \hat{g}_1 \\ \hat{g}_2 \\ \hat{g}_3 \end{bmatrix}, \quad \begin{matrix} d \\ a \\ v \end{matrix} \tag{2.2}$$

where the matrix function $\begin{bmatrix} \hat{E}_1^T & \hat{A}_2^T \end{bmatrix}^T$ has pointwise full row rank, is called the *strangeness-index* μ of the DAE (1.3) and of the pair of functions (E, A), see e.g. [22–24]. Note that the strangeness-free DAE (2.2) has exactly the same solution set as the DAE (1.3). The procedure of transforming system (1.3) to the strangeness-free system (2.2) is called the *strangeness-free formulation*. The quantities μ, d, a, v and $u := n - d - a$ are called the *characteristic quantities* of the DAE (1.3). In particular, u is *the number of undetermined variables* contained in the state x. The strangeness-free DAE (2.2), from the theoretical viewpoint, reveals the existence of solutions (if $\hat{g}_3 \equiv 0$) and all the algebraic constraints on the state x (via the equation $\hat{A}_2 x + \hat{g}_2 = 0$), and thus determines the consistency conditions that an initial vector x^0 must obey. If $\hat{g}_3 \not\equiv 0$ then one can regularize the system by removing this equation, or by setting $\hat{g}_3 = 0$. If after this regularization, the strangeness-free DAE (2.2) is square and uniquely solvable, then it can be treated by any classical method for DAEs, see [24].

Another important observation in the (numerical) solution procedure for DAEs is that the set of all algebraic constraints (including hidden constraints) contained in the DAE (1.3) is exactly the second block row equation of (2.2). In general, these algebraic constraints must be selected from the derivative array (2.1). In contrast to this, the differential equations in the first block row equation of (2.2) can be selected directly from the original DAE (1.3).

In order to understand the effect of this regularization process for DDAEs, we

consider a modification of the DAE (1.3), of the following form

$$E(t)\dot{x}(t) = A(t)x(t) + T(t)\lambda(t) + f(t), \tag{2.3}$$

for all $t \in \mathbb{I}$, together with an initial vector

$$x(0) = x^0. \tag{2.4}$$

Here the function parameter $\lambda : \mathbb{I} \to \mathbb{C}^p$ and the function coefficients E, A, T, f are assumed to be sufficiently differentiable. The smoothness comparison between λ and the state variable x gives rise to the following classification.

DEFINITION 2.2. *The DAE* (2.3) *is called:*

i) retarded *if for any continuous function λ, there exists a solution x to the initial value problem* (2.3)–(2.4)*;*

ii) neutral *if for any continuously differentiable function λ, there exists a solution x to the initial value problem* (2.3)–(2.4)*;*

iii) adavanced*, in the remaining case, where λ must be at least two times continuously differentiable to guarantee the existence of a solution to* (2.3)–(2.4)*.*

This classification has a direct consequence for the resulting DAE (2.2).

LEMMA 2.3. *Suppose that the parameter dependent DAE* (2.3) *is not advanced and the strangeness index μ is well-defined for the function pair (E, A) of* (2.3)*. Considering $T(t)\lambda(t) + f(t)$ as a new inhomogeneity, then the strangeness-free formulation applied to* (2.3) *results in a system*

$$\begin{bmatrix} \hat{E}_1(t) \\ 0 \\ 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} \hat{A}_1(t) \\ \hat{A}_2(t) \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} \hat{T}_1^0(t) \\ \hat{T}_2^0(t) \\ \hat{T}_3^0(t) \end{bmatrix} \lambda(t) + \begin{bmatrix} 0 \\ 0 \\ \hat{T}_3^1(t) \end{bmatrix} \dot{\lambda}(t) + \begin{bmatrix} \hat{f}_1(t) \\ \hat{f}_2(t) \\ \hat{f}_3(t) \end{bmatrix}, \tag{2.5}$$

where the matrix function $\begin{bmatrix} \hat{E}_1^T & \hat{A}_2^T \end{bmatrix}^T$ is of pointwise full row rank. In addition, if (2.3) *is of retarded type then $\hat{T}_2^0 = 0$ and $\hat{T}_3^1 = 0$.*

Proof. When applying the strangeness-free formulation to the DAE (2.3), the assumption that the system is not advanced ensures that all algebraic constraints of (2.3) have the form

$$0 = \widetilde{A}_2(t)x(t) + \widetilde{T}_2(t)\lambda(t) + \tilde{f}_2(t),$$

for some matrix functions $\widetilde{A}_2$, $\widetilde{T}_2$, $\tilde{f}_2$. Furthermore, in the retarded case, since the solution $x(t)$ of (2.3) is differentiable and λ is only continuous, we see that $\widetilde{T}_2$ must be identically zero. On the other hand, all differential equations of (2.3) have the form

$$\widetilde{E}_1(t)\dot{x}(t) = \widetilde{A}_1(t)x(t) + \widetilde{T}_1(t)\lambda(t) + \tilde{f}_1(t),$$

for some matrix functions $\widetilde{E}_1$, $\widetilde{A}_1$, $\widetilde{T}_1$, $\tilde{f}_1$. Moreover, consistency conditions for λ and for the inhomogeneity of the DAE (2.3) can only arise from one of the following three sources:

i) Adding an algebraic equation to another algebraic equation;

ii) adding a differential equation to another differential equation;

iii) adding the derivative of some algebraic equation to a differential equation.

As a consequence, the consistency condition for the inhomogeneity of (2.3) does not contain any derivatives of λ of order bigger than one, and hence, we obtain the resulting DAE (2.5). □

3. Analysis of linear DDAEs. Evidently, DDAEs inherit properties from their subclasses, for example, from the DAE side the structure of the pair (E, A) of matrix functions, or from the DDE side the discontinuity propagation in time and the smoothness requirements of an initial function. However, it has been observed that DDAEs can face further difficulties, which occur neither for DAEs nor for DDEs, see e.g. [4, 8, 11, 15, 16]. One important reason for these difficulties is the potential non-causality of general DDAEs.

In the control context, the concept of *causality* means that the output at a current time t depends only on the input at current and past time points, see e.g. [21, 30]. This causality concept can be adapted to DDAEs as follows.

DEFINITION 3.1. *A time-delayed system is called* causal *if for a consistent initial function the solution $x(t)$ of the corresponding initial value problem at the current time t depends only on the inhomogeneity f at current and past time points (i.e., $s \leqslant t$), but not future time points ($s > t$).*

Even though both DDEs and DAEs are causal, DDAEs are not always causal. For example, the scalar equation

$$0 \cdot \dot{x}(t) = 0 \cdot x(t) + x(t-\tau) - f(t), \quad \text{for all } t \in (0, \infty), \tag{3.1}$$

is noncausal, since the unique solution $x(t) = f(t+\tau)$ depends on f at the future time point $t + \tau$.

3.1. Analysis of linear causal DDAEs. Most of prior studies on DDAEs consider square systems where the associated DAE (1.3) is regular. By Lemma 3.4 below, we will see that restricted to square DDAEs, regularity is equivalent to causality. For this reason, we now discuss the analysis for general causal DDAE systems.

The following theorem presents the resulting system obtained by applying the strangeness-free formulation of DAEs to causal DDAEs.

THEOREM 3.2. *Consider a causal DDAE* (1.1) *and assume that the strangeness-index μ is well-defined for the pair of functions (E, A). Then* (1.1) *has the same solution set as the DDAE*

$$\begin{bmatrix} \hat{E}_1(t) \\ 0 \\ 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} \hat{A}_1(t) \\ \hat{A}_2(t) \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} \hat{B}_{0,1}(t) \\ \hat{B}_{0,2}(t) \\ 0 \end{bmatrix} x(t-\tau) + \sum_{i=1}^{\mu} \begin{bmatrix} 0 \\ \hat{B}_{i,2}(t) \\ 0 \end{bmatrix} x^{(i)}(t-\tau) + \begin{bmatrix} \hat{f}_1(t) \\ \hat{f}_2(t) \\ \hat{f}_3(t) \end{bmatrix} \begin{matrix} d \\ a \\ v \end{matrix} \tag{3.2}$$

where $\begin{bmatrix} \hat{E}_1^T & \hat{A}_2^T \end{bmatrix}^T$ has pointwise full row rank. The sizes of the block row equations are d, a and v. Moreover, some block row equations may not present.

Proof. Reinterpreting the DDAE (1.1) as it associated DAE (1.3) with the new inhomogeneity $g(t) = B(t)x(t-\tau) + f(t)$, and applying the strangeness-free formulation to it, we obtain the system

$$\begin{bmatrix} \hat{E}_1(t) \\ 0 \\ 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} \hat{A}_1(t) \\ \hat{A}_2(t) \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} \hat{B}_{0,1}(t) \\ \hat{B}_{0,2}(t) \\ \hat{B}_{0,3}(t) \end{bmatrix} x(t-\tau) + \sum_{i=1}^{\mu} \begin{bmatrix} 0 \\ \hat{B}_{i,2}(t) \\ \hat{B}_{i,3}(t) \end{bmatrix} x^{(i)}(t-\tau) + \begin{bmatrix} \hat{f}_1(t) \\ \hat{f}_2(t) \\ \hat{f}_3(t) \end{bmatrix} \begin{matrix} d \\ a \\ v \end{matrix} \tag{3.3}$$

where $\begin{bmatrix} \hat{E}_1^T & \hat{A}_2^T \end{bmatrix}^T$ has pointwise full row rank. By shifting forward the last block row equation by τ, one sees that

$$0 = \sum_{i=0}^{\mu} \hat{B}_{i,3}(t+\tau) x^{(i)}(t) + \hat{f}_3(t+\tau),$$

and hence the vector $x(t)$ depends on the function f at the future time $t+\tau$ if there exists at least one function $\hat{B}_{i,3}$ that is non-zero. This violates the causality of (1.1). Therefore, either $v=0$ or in the last block row equation, the functions $\hat{B}_{i,3}$, $i=0,\ldots,\mu$ are identically zero. As a result, from (3.3) we obtain (3.2). □

The solvability of the initial value problem (1.1)–(1.2) for causal DDAEs is given by the following corollary.

COROLLARY 3.3. *Consider the DDAE* (1.1) *and assume that all the requirements of Theorem 3.2 are satisfied, so that system* (3.2) *is well-defined.*

1. *The DDAE* (1.1) *is solvable if and only if either $v=0$ or $\hat{f}_3(t)=0$, for all $t\geq 0$.*
2. *An initial function ϕ is consistent if and only if in addition, ϕ is sufficiently smooth and it satisfies the following consistency condition*

$$0=\hat{A}_2(0)\phi(0)+\sum_{i=0}^{\mu}\hat{B}_{i,3}(0)\phi^{(i)}(-\tau)+\hat{f}_2(0).$$

3. *The DDAE* (1.1) *is regular if and only if in addition, $\begin{bmatrix}\hat{E}_1^T & \hat{A}_2^T\end{bmatrix}^T$ is pointwise invertible, i.e., $d+a=n$.*

We have the following relation between the causality of a DDAE and the regularity of the associated DAE.

LEMMA 3.4. *Consider a square DDAE* (1.1). *Then, it is causal if only if the associated DAE* (1.3) *is regular.*

Proof. Considering the system (3.3), we see that the strangeness-free formulation of the associated DAE (1.3) is

$$\begin{bmatrix}\hat{E}_1(t)\\0\\0\end{bmatrix}\dot{x}(t)=\begin{bmatrix}\hat{A}_1(t)\\\hat{A}_2(t)\\0\end{bmatrix}x(t)+\begin{bmatrix}\hat{g}_1(t)\\\hat{g}_2(t)\\\hat{g}_3(t)\end{bmatrix},\quad\begin{matrix}d\\a\\v\end{matrix}.$$

Since the DDAE (1.1) is square, we see that both the causality of the DDAE (1.1) and the regularity of the associated DAE (1.3) are equivalent to the fact that $d+a=n$ and $v=0$. □

Note that Lemma 3.4 does not hold if the DDAE (1.1) is non-square because then v can be nonzero and still the solution may be unique.

3.2. The method of steps. The numerical solution of initial value problems for DDAEs, until now, has only been considered for causal, square systems, see e.g. [3, 4, 11, 14, 25, 33, 34, 36, 37]. For such systems, the solution is usually computed by the classical (Bellman) *method of steps*, which has been previously used for solving DDEs, as in [5, 6, 21]. In order to prepare for the study of general DDAEs, let us recall this method.

Introducing sequences of matrix and vector valued functions E_i, A_i, B_i, f_i, x_i, for each $i\in\mathbb{N}$, on the time interval $[0,\tau]$ via

$$\begin{aligned}&E_i(t):=E(t+(i-1)\tau),\ A_i(t):=A(t+(i-1)\tau),\ B_i(t):=B(t+(i-1)\tau),\\&f_i(t):=f(t+(i-1)\tau),\ x_i(t):=x(t+(i-1)\tau),\ x_0(t):=\phi(t-\tau),\end{aligned}\tag{3.4}$$

one can rewrite the initial value problem (1.1)–(1.2) as the sequence of DAEs

$$E_i(t)\dot{x}_i(t)=A_i(t)x_i(t)+B_i(t)x_{i-1}(t)+f_i(t),\tag{3.5}$$

for $t \in (0, \tau)$, and for $i = 1,\ 2, \ldots, \ell$, with initial conditions

$$x_i(0) = x_{i-1}(\tau). \tag{3.6}$$

Here (3.5) is a parameter dependent DAE in the variable x_i with the function parameter x_{i-1}. The idea of the method of steps is to compute x_i by solving the initial value problem (3.5)–(3.6), provided that the function x_{i-1} is already determined and thus the solution x of the initial value problem (1.1)–(1.2) is reconstructed step-by-step via

$$x(t) = x_i(t - (i-1)\tau) \ \text{ for every } t \in [(i-1)\tau, i\tau]. \tag{3.7}$$

Clearly, this strategy requires that the initial value problem (3.5)–(3.6) has a unique solution x_i for any sufficiently smooth function $B_i(t)x_{i-1}(t) + f_i(t)$ and for any consistent initial vector $x_i(0)$. Under this condition, numerical integration methods based on the method of steps have been successfully implemented for linear DDAEs of the form (1.1) and also for several classes of nonlinear DDAEs, see e.g. [3, 4, 14, 19, 33]. However, if the DDAE (1.1) is noncausal then this unique solvability condition does not hold, and so this approach is not feasible, consider for example equation (3.1), where the initial value problem (3.5)–(3.6) has multiple solutions, even though the initial value problem (1.1)–(1.2) has a unique solution. The reason for this failure is that the method of steps takes into account only the equation at the current time, which is not enough for general noncausal DDAEs. A modification of the method of steps, therefore, is necessary.

Another important point to consider in the numerical solution of causal DDAEs is the application range of the method of steps. The analysis of the quality of numerical approximations to the solution of dynamical systems (in particular, DDEs) is often based on Taylor expansion, which requires that the analytical solution is sufficiently smooth up to a desired order. This is one of the reasons to divide DDEs into different classes of equations, namely *retarded, neutral, advanced*, see e.g. [6, 18], and the numerical solution of DDEs has been studied mostly for equations of retarded and neutral type. Until now, there is still a lack of a systematic theory for advanced DDEs, the references are rare and limited to only few special applications [12, 20, 31]. Inherited from the theory of DAEs, hidden structures may exist in DDAEs and in fact, even though it looks like of retarded type, the DDAE (1.1) can possess an underlying DDE of neutral or advanced type, [3, 19]. The advanced situation, which may lead to difficulties in the (numerical) solution procedure, has been excluded in prior investigations of DDAEs, [3, 4, 14, 19, 33]. Furthermore, the classification of DDAEs has only been done for very restricted classes of systems, see [3, 19], and does not apply to general linear DDAEs. Because of this, we extend the classification of DDEs to DDAEs based on the type of their underlying DDEs as follows.

Definition 3.5. *The DDAE* (1.1) *is said to be*

i) retarded *if all the scalar, non-redundant equations of* (1.1)*, including all hidden constraints, are of the form*

$$\sum_{\beta=0}^{K_+} a_\beta(t) x^{(\beta)}(t) = \sum_{\alpha=0}^{K_-} b_\alpha(t) x^{(\alpha)}(t-\tau) + \gamma(t), \tag{3.8}$$

where $a_{K_+}(t) \neq 0$ *for all* $t \in \mathbb{I}$, $b_{K_-} \not\equiv 0$ *and* $K_+ > K_-$. *Here we also allow the case that* $K_- = -\infty$, *which means that the factor* $\sum_{\alpha=0}^{K_-} b_\alpha(t) x^{(\alpha)}(t-\tau)$ *is not present in* (3.8).

ii) neutral *if all the scalar, non-redundant equations of* (1.1) *are of the form* (3.8) *with* $K_+ \geq K_-$ *and among them there is at least one equality;*

iii) advanced *if there exists at least one scalar, non-redundant equation of* (1.1) *that is of the form* (3.8) *with* $0 \leqslant K_+ < K_-$.

For the development of a solution procedure for general DDAEs, the most natural idea is to extend the method of steps and consequently, we may expect that this extended method can successfully handle only retarded and neutral systems. The theoretical solvability, however, will be available for any system of any type.

3.3. Characteristic properties of general linear DDAEs. In the following, we discuss several characteristic properties of general linear DDAEs, which lead to important consequences for both the theoretical and the numerical solution of the corresponding initial value problems.

First, for noncausal DDAEs, some constraints of a current state $x(t)$ may be hidden in the system at future times such as $t+\tau$, $t+2\tau$, ... Thus, in order to determine a current state, one may have to utilize the system at multiple future time points. This can be easily seen from the following example.

EXAMPLE 3.6. Consider the DDAE

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} x(t-\tau) + \begin{bmatrix} 1 \\ -t \end{bmatrix}, \tag{3.9}$$

for all $t \in (0,\infty)$ and an initial function $x(t) = \phi(t)$ for $t \in [-\tau, 0]$. Considering any fixed $t \in (0,\tau)$ and inserting $x(t-\tau) = \phi(t-\tau)$ into (3.9) yields an underdetermined system, which does not uniquely determine the second component of $x(t)$. However, the DDAE (3.9) at the future point $t+\tau$ gives rise to the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x}(t+\tau) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -t-\tau \end{bmatrix}, \tag{3.10}$$

which contains the algebraic constraint $0 = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) - t - \tau$. The coupled system (3.9)–(3.10) uniquely determines $x(t)$, which means that one needs to utilize (3.9) at least at two time points t and $t+\tau$ to determine $x(t)$.

As a result, in order to determine the current state $x(t)$, one can use at least two different operators:

i) The *shift (forward) operator* $\Delta_{-\tau}$ that maps the equation (1.1) into the equation

$$E(t+\tau)\dot{x}(t+\tau) = A(t+\tau)x(t+\tau) + B(t+\tau)x(t) + f(t+\tau),$$

provided that the point t satisfies $t < t_f - \tau$.

ii) The *differentiation operator* that maps the equation (1.1) into the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E(t)\dot{x}(t) - A(t)x(t)\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(B(t)x(t-\tau) + f(t)\right). \tag{3.11}$$

For the theoretical determination of $x(t)$ at an arbitrary point t, the use of the differentiation operator is not relevant, since the DDAE (3.11) is only a consequence of the DDAE (1.1). On the other hand, the shift operator presents a critical restriction to the solution space of the DDAE (1.1). We illustrate this fact by revisiting the DDAE (3.1) at one point $t \in \mathbb{I}$. Let $g(t) := x(t-\tau) - f(t)$. Applying the differentiation operator to (3.1) leads to the system

$$\begin{aligned} 0 \cdot \dot{x}(t) &= 0 \cdot x(t) + g(t), \\ 0 \cdot \ddot{x}(t) &= 0 \cdot \dot{x}(t) + \dot{g}(t), \end{aligned}$$

which is not enough to uniquely determine $x(t)$. On the other hand, applying the shift operator to (3.1) leads to the system

$$\begin{aligned} 0 \cdot \dot{x}(t) &= 0 \cdot x(t) + g(t), \\ 0 \cdot \dot{x}(t+\tau) &= 0 \cdot x(t+\tau) + x(t) - f(t+\tau), \end{aligned}$$

that uniquely determines $x(t)$.

REMARK 3.7. It is important to note that in general the two operators $\frac{\mathrm{d}}{\mathrm{d}t}$ and $\Delta_{-\tau}$ do not commute, since the derivatives of the functions E, A, B, x, f may exist at the point $t+\tau$ but do not exist at the point t, or vice versa. Finding an optimal way to combine the differentiation and shift operator, in order to fully understand the solvability of the DDAE (1.1) and to compute the solution of the initial value problem (1.1)-(1.2) is still an open problem, see [15, 16] for some partial results.

The second characteristic property of DDAEs is that the system can have not only hidden algebraic constraints but also further hidden differential equations, as demonstrated in the following example.

EXAMPLE 3.8. Consider the DDAE

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \end{bmatrix} + \begin{bmatrix} -t \\ -1-e^{t-1} \\ 1 \end{bmatrix}, \tag{3.12}$$

on the time interval $\mathbb{I} = [0, \infty)$. To deduce all the hidden equations for the state $x(t)$ in the DDAE (3.12), we proceed as follows.

1) Differentiate the second equation to obtain $\dot{x}_3(t)$ and insert it into the first equation to eliminate $\dot{x}_3(t)$, then one obtains an algebraic constraint for $x_2(t)$

$$0 = x_2(t) + \dot{x}_1(t-1) - t - e^{t-1}. \tag{3.13}$$

2) Differentiate (3.13) and insert it into the third equation of (3.12) to eliminate $\dot{x}_2(t)$. This leads to

$$0 = \ddot{x}_1(t-1) - e^{t-1}. \tag{3.14}$$

This equation gives a consistency condition for an initial function ϕ when $t \in [0, 1]$. Furthermore, by shifting (3.14) forward by 1, we obtain a hidden second order differential equation.

This characteristic property implies that for general DDAEs, in contrast to the case of DAEs, one cannot not select all the differential equations that describe the dynamics of the system from the original DDAE.

The third characteristic property is that the underlying DDE of the DDAE (1.1) can contain arbitrarily high order derivatives of $x(t)$ and $x(t-\tau)$. Revisiting system (3.12), we see that under the consistency condition $0 = \ddot{\phi}_1(t-1) - e^{t-1}$, for all $t \in (0, 1)$, the DDAE (3.12) has the same solution set as the system

$$\begin{aligned} -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{x}_2(t) \\ \ddot{x}_3(t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \end{bmatrix} \\ &\quad + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t-1) \\ \dot{x}_2(t-1) \\ \dot{x}_3(t-1) \end{bmatrix} + \begin{bmatrix} -t-e^{t-1} \\ -1-e^{t-1} \\ -e^{t} \end{bmatrix}. \end{aligned}$$

An implicit formulation of the underlying DDE then is

$$-\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}\ddot{x}_1(t)\\\ddot{x}_2(t)\\\ddot{x}_3(t)\end{bmatrix}=\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}\ddot{x}_1(t-1)\\\ddot{x}_2(t-1)\\\ddot{x}_3(t-1)\end{bmatrix}+\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1^{(3)}(t-1)\\x_2^{(3)}(t-1)\\x_3^{(3)}(t-1)\end{bmatrix}+\begin{bmatrix}-e^{t-1}\\-e^{t-1}\\-e^{t}\end{bmatrix}.$$

In general, if the DDAE (1.1) is noncausal, then the strangeness-free formulation and the underlying DDE can contain high order derivatives of $x(t)$ and $x(t-\tau)$. If this is the case, then applying numerical methods like Runge-Kutta or BDF methods to the strangeness-free formulation will be complicated or may not even be feasible.

3.4. Generalization of the method of steps for DDAEs. Similar to the method of steps, the main task in the solution procedure for DDAEs is to compute the solution x of the initial value problem (1.1)-(1.2) in the time interval $[(i-1)\tau, i\tau]$, $1 \leqslant i \leqslant \ell$, or equivalently, to determine the function x_i, provided that functions $x_{i-1}, \ldots, x_0$ are already known. Then the sequence of DAEs

$$E_j(t)\dot{x}_j(t) = A_j(t)x_j(t) + B_j(t)x_{j-1}(t) + f_j(t), \quad j = 1, \ldots, i-1.$$

contains only redundant equations, which do not contribute to the determination of x_i, but the solvability of x_i is governed by the sequence of DAEs

$$E_{i+j}(t)\dot{x}_{i+j}(t) = A_{i+j}(t)x_{i+j}(t)+B_{i+j}(t)x_{i+j-1}(t)+f_{i+j}(t), \quad j = 0, \ldots, \ell-i. \tag{3.15}$$

Note that depending on the time interval, the sequence of DAEs (3.15) may have finitely many equations ($\ell < \infty$) or infinitely many equations ($\ell = \infty$). In the second case, one certainly cannot use the whole set (3.15) to determine x_i, and this fact motivates the *shift index* concept in the next definition.

Definition 3.9. *For a fixed $i \leqslant \ell$, consider the sequence of DAEs* (3.15)*. The minimum integer $k \geq 0$ such that the so-called* shift-inflated system

$$E_{i+j}(t)\dot{x}_{i+j}(t) = A_{i+j}(t)x_{i+j}(t) + B_{i+j}(t)x_{i+j-1}(t) + f_{i+j}(t), \quad j = 0, \ldots, k, \tag{3.16}$$

has a unique solution x_i, provided a function x_{i-1} and assumed that the initial vector $x_i(0) = x_{i-1}(\tau)$ is consistent, is called the shift index *with respect to i, and denoted by $\kappa(i)$.*

The reason for failure of the method of steps is that it uses only equation (3.5) (which is the first equation of system (3.16)) to determine x_i. However, all the further constraints on x_i are also contained in (3.16) and these need to be extracted as well, so that x_i can be computed in a unique way from (3.16). Except for the constraint that is directly available in the first equation of (3.16), the other constraints for x_i can be filtered out of the DAE system

$$\mathfrak{E}_i\dot{y} = \mathfrak{A}_i y + \mathfrak{B}_i x_i + \mathfrak{g}_i, \tag{3.17}$$

with

$$\mathfrak{E}_i := \begin{bmatrix} E_{i+1} & & & \\ & E_{i+2} & & \\ & & \ddots & \\ & & & E_{i+k} \end{bmatrix}, \; \mathfrak{A}_i := \begin{bmatrix} A_{i+1} & & & \\ B_{i+2} & A_{i+2} & & \\ & \ddots & \ddots & \\ & & B_{i+k} & A_{i+k} \end{bmatrix},$$

$$\mathfrak{B}_i := \begin{bmatrix} B_{i+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \; y := \begin{bmatrix} x_{i+1} \\ x_{i+2} \\ \vdots \\ x_{i+k} \end{bmatrix}, \; \mathfrak{g}_i := \begin{bmatrix} f_{i+1} \\ f_{i+2} \\ \vdots \\ f_{i+k} \end{bmatrix},$$

which contains x_i as a function parameter. For notational simplicity, from now on we omit the argument t in all matrix-valued and vector-valued functions.

Redundant equations in (3.17) lead to constraints for x_i. To extract these constraints, we determine a strangeness-free formulation of the parameter dependent DAE (3.17) given by

$$\begin{bmatrix} \mathfrak{E}_{i,1} \\ 0 \\ 0 \end{bmatrix} \dot{y} = \begin{bmatrix} \mathfrak{A}_{i,1} \\ \mathfrak{A}_{i,2} \\ 0 \end{bmatrix} y + \sum_{j=0}^{\mu} \begin{bmatrix} \mathfrak{B}_{i,j} \\ \mathfrak{C}_{i,j} \\ \mathfrak{D}_{i,j} \end{bmatrix} x_i^{(j)} + \begin{bmatrix} \mathfrak{g}_{i,1} \\ \mathfrak{g}_{i,2} \\ \mathfrak{g}_{i,3} \end{bmatrix}, \tag{3.18}$$

where $\begin{bmatrix} \mathfrak{E}_{i,1}^T & \mathfrak{A}_{i,2}^T \end{bmatrix}^T$ has pointwise full row rank, and μ is the strangeness-index of the pair of functions $(\mathfrak{E}_i, \mathfrak{A}_i)$ as in (3.17), which we assume to be well-defined. The constraints for x_i hidden in the DAE (3.17) are then represented by the last block row of (3.18) and we have the following lemma.

LEMMA 3.10. *If the strangeness-index μ of the pair of functions $(\mathfrak{E}_i, \mathfrak{A}_i)$ as in* (3.17) *is well defined, then the set of equations for x_i arising from* (3.16) *is represented by the differential-algebraic system*

$$\begin{aligned} E_i \dot{x}_i &= A_i x_i + B_i x_{i-1} + f_i, \\ 0 &= \sum_{j=0}^{\mu} \mathfrak{D}_{i,j} x_i^{(j)} + \mathfrak{g}_{i,3}. \end{aligned} \tag{3.19}$$

Furthermore, the function x_i is uniquely determined from (3.16) *if and only if x_i is uniquely determined from* (3.19) *together with the initial conditions*

$$x_i^{(j)}(0) = x_{i-1}^{(j)}(\tau^-), \; j = 0, \ldots, \mu - 1. \tag{3.20}$$

REMARK 3.11. If $\mu > 1$ and $\mathfrak{D}_{i,\mu} \not\equiv 0$ then the DAE (3.19) is of order higher than one. Then, to compute x_i, one first needs to perform an order reduction by introducing new variables that represent derivative components of x_i. The resulting system can then be solved by any numerical method for first order DAEs, see [24], and from this solution one can extract the solution x_i if sufficiently many initial functions are given. The order reduction method, however, is not unique and may also lead to further difficulties in the numerical solution, see e.g. [2, 32]. For this reason, in [15, 29] new methods are proposed to handle initial value problems for high-order DAEs directly.

Even though in general μ depends on i, for notational convenience, we shall write μ instead of $\mu(i)$.

Restricted to DDAEs of retarded and neutral type, the DAE (3.19) can be significantly simplified.

LEMMA 3.12. *Assume that the DDAE* (1.1) *is of either retarded or neutral type. Under the conditions of Lemma 3.10, the DAE* (3.19) *becomes*

$$\begin{bmatrix} E_i \\ -\mathfrak{D}_{i,1} \end{bmatrix} \dot{x}_i = \begin{bmatrix} A_i \\ \mathfrak{D}_{i,0} \end{bmatrix} x_i + \begin{bmatrix} B_i \\ 0 \end{bmatrix} x_{i-1} + \begin{bmatrix} f_i \\ \mathfrak{g}_{i,3} \end{bmatrix}. \tag{3.21}$$

In addition, if the strangeness index is well-defined for (3.21)*, then x_i is also the solution of the strangeness-free DAE*

$$\begin{bmatrix} \widetilde{E}_{i,1} \\ 0 \\ 0 \end{bmatrix} \dot{x}_i = \begin{bmatrix} \widetilde{A}_{i,1} \\ \widetilde{A}_{i,2} \\ 0 \end{bmatrix} x_i + \begin{bmatrix} \widetilde{B}_{i,1} \\ \widetilde{B}_{i,2} \\ \widetilde{B}_{i,3} \end{bmatrix} x_{i-1} + \begin{bmatrix} 0 \\ 0 \\ \widetilde{B}_{i,4} \end{bmatrix} \dot{x}_{i-1} + \begin{bmatrix} \tilde{\mathfrak{g}}_{i,1} \\ \tilde{\mathfrak{g}}_{i,2} \\ \tilde{\mathfrak{g}}_{i,3} \end{bmatrix}, \quad \begin{matrix} \hat{d}_i \\ \hat{a}_i \\ \hat{v}_i \end{matrix} \tag{3.22}$$

where the matrix function $\begin{bmatrix} \widetilde{E}_{i,1}^T & \widetilde{A}_{i,2}^T \end{bmatrix}^T$ *is pointwise nonsingular.*

Proof. Since the DDAE (1.1) is not of advanced type, it follows that $x_{i+1}, \ldots, x_{i+k}$ are at least as smooth as x_i. Hence, the DAE (3.17) is not of advanced type either, and Lemma 2.3 applied to (3.17) results in the system

$$\begin{bmatrix} \mathfrak{E}_{i,1} \\ 0 \\ 0 \end{bmatrix} \dot{y} = \begin{bmatrix} \mathfrak{A}_{i,1} \\ \mathfrak{A}_{i,2} \\ 0 \end{bmatrix} y + \begin{bmatrix} \mathfrak{B}_{i,0} \\ \mathfrak{C}_{i,0} \\ \mathfrak{D}_{i,0} \end{bmatrix} x_i + \begin{bmatrix} 0 \\ 0 \\ \mathfrak{D}_{i,1} \end{bmatrix} \dot{x}_i + \begin{bmatrix} \mathfrak{g}_{i,1} \\ \mathfrak{g}_{i,2} \\ \mathfrak{g}_{i,3} \end{bmatrix},$$

which yields the desired system (3.21).

Analogously, we see that the parameter dependent DAE (3.21) in the variable x_i with the function parameter x_{i-1} is also a non-advanced DAE, and hence Lemma 2.3 applied to (3.21) implies system (3.22). □

From Lemma 3.10, we can deduce that the shift index is well-defined, even if $\ell = \infty$.

THEOREM 3.13. *Suppose that the DDAE* (1.1) *is not of advanced type and consider the shift inflated system* (3.16)*. If the strangeness indices are well-defined for the two DAEs* (3.17) *and* (3.21)*, then there exists a unique shift index $\kappa(i)$ with respect to i. Furthermore, with $k = \kappa(i)$, the sizes of the block row equations in the strangeness-free DAE* (3.22) *satisfy $\hat{d}_i + \hat{a}_i = n$.*

Proof. From Lemma 3.12, we see that for each $k \geq 0$, the sequence of DAEs (3.16) has the same solution x_i as the DAE (3.22). Moreover, $u_k = n - \hat{d}_i - \hat{a}_i$ is the number of undetermined variables contained in the solution x_i of the DAE (3.22). With $k = 0, \ldots, \ell - i$ we obtain a sequence $\{u_k\}_{k\geq 0}$ of nonnegative integers. Introducing

$$\begin{aligned} \mathbb{M}_k &:= \{x_i : [0,\tau] \to \mathbb{C}^n \mid \text{ there exist functions } x_{i+1}, \ldots, x_{i+k} \text{ that satisfy (3.16)}\}, \\ \mathbb{N}_k &:= \{x_i : [0,\tau] \to \mathbb{C}^n \mid x_i \text{ solves the DAE (3.22)}\}, \end{aligned}$$

we see that $\mathbb{M}_k = \mathbb{N}_k$ for every $k \geq 0$ due to Lemmata 3.10 and 3.12.

Since the sequence $\{\mathbb{M}_k\}_{k\geq 0}$ is decreasing, so is the sequence $\{\mathbb{N}_k\}_{k\geq 0}$ and hence the sequence $\{u_k\}_{k\geq 0}$ is also decreasing. The boundedness from below of the sequence

$\{u_k\}_{k\geq 0}$ implies that this sequence becomes stationary. Moreover, since the DDAE (1.1) is regular, the sequence of DAEs (3.16) has a unique solution x_i for $k = \ell - i$, no matter whether ℓ is finite or infinite. Thus, $\lim_{k\uparrow(\ell-i)} u_k = 0$ and hence, the stationarity of the sequence $\{u_k\}_{k\geq 0}$ implies that there exists a finite number k such that $u_k = 0$. Then $\kappa(i) := \min\{k \geq 0 \mid u_k = 0\}$ is the unique shift index. Clearly, since $u_{\kappa(i)} = 0$, we have $\hat{d}_i + \hat{a}_i = n$. □

Changing back the time variable $t \mapsto t+(i-1)\tau$, the first two block row equations of the strangeness-free DAE (3.22) give the system

$$\begin{bmatrix} \hat{E}_{i,1}(t) \\ 0 \end{bmatrix} \dot{x}(t) = \begin{bmatrix} \hat{A}_{i,1}(t) \\ \hat{A}_{i,2}(t) \end{bmatrix} x(t) + \begin{bmatrix} \hat{B}_{i,1}(t) \\ \hat{B}_{i,2}(t) \end{bmatrix} x(t-\tau) + \begin{bmatrix} \hat{\mathfrak{g}}_{i,1}(t) \\ \hat{\mathfrak{g}}_{i,2}(t) \end{bmatrix}, \quad \begin{matrix} \hat{d}_i \\ \hat{a}_i \end{matrix} \tag{3.23}$$

for $t \in [(i-1)\tau, i\tau]$, where $\begin{bmatrix} \hat{E}_{i,1}^T(t) & \hat{A}_{i,2}^T(t) \end{bmatrix}^T$ is pointwise nonsingular. Hence, the corresponding initial value problem for the DDAE (3.23) uniquely determines $x_i = x|_{[(i-1)\tau, i\tau]}$. We call system (3.23) the *regular, strangeness-free formulation* of the DDAE (1.1) on $[(i-1)\tau, i\tau]$. For a numerical solution of the initial value problem (1.1)-(1.2), it is necessary to extract this formulation pointwise. We address this topic in the next section.

4. Numerical solution of linear retarded or neutral DDAEs. This section is devoted to the numerical solution of linear DDAEs with variable coefficients. As discussed in the previous section, we aim to use the generalized method of steps for solving general linear DDAEs of either retarded or neutral type.

For $k \in \mathbb{N}$, differentiating the DDAE (1.1) k times we obtain the *derivative array*

$$M(t)z(t) = P(t)z(t-\tau) + g(t), \tag{4.1}$$

where

$$M := \begin{bmatrix} -A & E & & & & \\ -\dot{A} & \dot{E} - A & E & & & \\ -\ddot{A} & \ddot{E} - 2\dot{A} & 2\dot{E} - A & E & & \\ & \vdots & & \ddots & \ddots & \\ -A^{(k)} & E^{(k)} - kA^{(k-1)} & \dots & \dots & k\dot{E} - A & E \end{bmatrix},$$

$$P := \begin{bmatrix} B & & & & & 0 \\ \dot{B} & B & & & & 0 \\ \ddot{B} & 2\dot{B} & B & & & 0 \\ \vdots & & \ddots & \ddots & & \vdots \\ B^{(k)} & kB^{(k-1)} & \dots & k\dot{B} & B & 0 \end{bmatrix}, \quad z := \begin{bmatrix} x \\ \dot{x} \\ \vdots \\ x^{(k+1)} \end{bmatrix}, \quad g := \begin{bmatrix} f \\ \dot{f} \\ \vdots \\ f^{(k+1)} \end{bmatrix}.$$

In the following by the subscript $j\tau$, $j \in \mathbb{Z}$, we denote the evaluation of a function at the point $t + j\tau$. For a fixed $t \in \mathbb{I}$, let i be such that $t \in ((i-1)\tau, i\tau]$ and assume that the shift index $\kappa = \kappa(i)$ with respect to i is well-defined. Thus, the shift-inflated

system (3.16) is then given by

$$\begin{array}{rcl} E_{0\tau}\dot{x}_{0\tau} &=& A_{0\tau}x_{0\tau} + B_{0\tau}x_{-\tau} + f_{0\tau}, \\ E_{\tau}\dot{x}_{\tau} &=& A_{\tau}x_{\tau} + B_{\tau}x_{0\tau} + f_{\tau}, \\ E_{2\tau}\dot{x}_{2\tau} &=& A_{2\tau}x_{2\tau} + B_{2\tau}x_{\tau} + f_{2\tau}, \\ & \vdots & \\ E_{\kappa\tau}\dot{x}_{\kappa\tau} &=& A_{\kappa\tau}x_{\kappa\tau} + B_{\kappa\tau}x_{(\kappa-1)\tau} + f_{\kappa\tau}. \end{array} \tag{4.2}$$

In order to build the derivative arrays one needs to rewrite (4.2) as

$$\begin{aligned} &\begin{bmatrix} E_{0\tau} & & & & \\ & E_{\tau} & & & \\ & & E_{2\tau} & & \\ & & & \ddots & \\ & & & & E_{\kappa\tau} \end{bmatrix} \begin{bmatrix} \dot{x}_{0\tau} \\ \dot{x}_{\tau} \\ \dot{x}_{2\tau} \\ \vdots \\ \dot{x}_{\kappa\tau} \end{bmatrix} \\ &= \begin{bmatrix} A_{0\tau} & & & & \\ B_{\tau} & A_{\tau} & & & \\ & B_{2\tau} & A_{2\tau} & & \\ & & \ddots & \ddots & \\ & & & B_{\kappa\tau} & A_{\kappa\tau} \end{bmatrix} \begin{bmatrix} x_{0\tau} \\ x_{\tau} \\ x_{2\tau} \\ \vdots \\ x_{\kappa\tau} \end{bmatrix} + \begin{bmatrix} B_{0\tau}x_{-\tau}+f_{0\tau} \\ f_{\tau} \\ f_{2\tau} \\ \vdots \\ f_{\kappa\tau} \end{bmatrix}. \end{aligned} \tag{4.3}$$

Suppose that for the DAE (4.3), the strangeness index $\hat{\mu}(t)$ is well-defined and is constant in a sufficiently small neighborhood of t. Then we can build the derivative arrays with $k = \hat{\mu}(t)$. However, to reduce the cost in the determination of the shift index κ, it would be better to build the derivative arrays (with $k = \hat{\mu}(t)$) for the equations of system (4.2) instead of building derivative arrays for the entire system (4.3). If we proceed in this way, then we obtain the so-called *double-inflated system*

$$\begin{bmatrix} M_{0\tau} & & & & \\ -P_{\tau} & M_{\tau} & & & \\ & -P_{2\tau} & M_{2\tau} & & \\ & & \ddots & \ddots & \\ & & & -P_{\kappa\tau} & M_{\kappa\tau} \end{bmatrix} \begin{bmatrix} z_{0\tau} \\ z_{\tau} \\ z_{2\tau} \\ \vdots \\ z_{\kappa\tau} \end{bmatrix} = \begin{bmatrix} P_{0\tau} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} z_{-\tau} + \begin{bmatrix} g_{0\tau} \\ g_{\tau} \\ g_{2\tau} \\ \vdots \\ g_{\kappa\tau} \end{bmatrix}. \tag{4.4}$$

We denote the matrix coefficients of (4.4) as

$$\mathcal{M} := \begin{bmatrix} M_{0\tau} & & & & \\ -P_{\tau} & M_{\tau} & & & \\ & -P_{2\tau} & M_{2\tau} & & \\ & & \ddots & \ddots & \\ & & & -P_{\kappa\tau} & M_{\kappa\tau} \end{bmatrix}, \quad \mathcal{P} := \begin{bmatrix} P_{0\tau} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathcal{G} := \begin{bmatrix} g_{0\tau} \\ g_{\tau} \\ g_{2\tau} \\ \vdots \\ g_{\kappa\tau} \end{bmatrix}.$$

We discuss now how to derive the regular, strangeness-free formulation (3.23) from the double-inflated system (4.4).

Remark 4.1. It should be noted, due to the potential non-causality of the DDAE (1.1), that differential equations of the regular, strangeness-free DDAE (3.23) must be selected from the double-inflated system (4.4), instead of from the original DDAE (1.1). This is in contrast to both cases of non-delayed DAEs and causal DDAEs. We illustrate this fact in the next example.

EXAMPLE 4.2. Consider the initial value problem consisting of the DDAE

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\end{bmatrix} = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}x(t)\\ y(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x(t-\tau)\\ y(t-\tau)\end{bmatrix} + \begin{bmatrix}e^t\\ -e^t+t-\tau\end{bmatrix}, \tag{4.5}$$

for $t \in [0,\infty)$, $\tau = 1$, with an initial function $\phi(t) := \begin{bmatrix}e^t\\ t\end{bmatrix}$ for $t \in [-\tau, 0]$.
By directly checking, we obtain $\kappa = 1$ and the regular, strangeness-free DDAE (3.23) is

$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\end{bmatrix} = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}x(t)\\ y(t)\end{bmatrix} + \begin{bmatrix}0 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x(t-\tau)\\ y(t-\tau)\end{bmatrix} + \begin{bmatrix}1\\ -e^t+1\end{bmatrix}.$$

Clearly, the differential equation $\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\end{bmatrix} = 1$ cannot be selected from the original DDAE (4.5).

From Remark 4.1, we see that it is necessary to select all the constraints for $x(t)$ and $\dot{x}(t)$ contained in (4.4). It is worth to note that $x(t)$ and $\dot{x}(t)$ are only present in $z_{0\tau}$ but not in $z_\tau, \ldots, z_{\kappa\tau}$. For convenience, in the following we will use Matlab notation, [26].

Let the matrix U be such that its columns span the space corange $\mathcal{M}(:, (2n+1) : end)$, i.e.,

$$U^T\mathcal{M}(:, (2n+1) : end) = 0. \tag{4.6}$$

By scaling (4.4) with U^T, we obtain the system

$$U^T\mathcal{M}(:, 1 : 2n)\begin{bmatrix}x(t)\\ \dot{x}(t)\end{bmatrix} = U^T\mathcal{P}z_{-\tau} + U^T\mathcal{G}, \tag{4.7}$$

that contains all the constraints for $x(t)$ and $\dot{x}(t)$ in (4.4).

Note that in the DDAE (3.23) only $x(t-\tau)$ occurs even though $z_{-\tau}$ contains not only $x(t-\tau)$ but also its derivatives $\dot{x}(t-\tau)$, $\ddot{x}(t-\tau)$, ... Thus, the matrix U should be chosen to satisfy the additional condition

$$U^T\mathcal{P}(:, (n+1) : end) = 0. \tag{4.8}$$

In the case that $U^T\mathcal{P}(:, (n+1) : end) \neq 0$, this implies that the DDAE (1.1) is of advanced type which we have excluded by assumption. Denote by

$$\tilde{\mathcal{M}} := U^T\mathcal{M}(:, (n+1) : 2n),\ \tilde{\mathcal{N}} := U^T\mathcal{M}(:, 1 : n),\ \tilde{\mathcal{P}} := U^T\mathcal{P}(:, 1 : n),\ \tilde{\mathcal{G}} := U^T\mathcal{G}, \tag{4.9}$$

and $\tilde{m}$ be the number of rows of $\tilde{\mathcal{M}}$.

With a matrix U that satisfies (4.6) and (4.8), the DDAE (4.7) becomes

$$\tilde{\mathcal{M}}\dot{x}(t) + \tilde{\mathcal{N}}x(t) = \tilde{\mathcal{P}}x(t-\tau) + \tilde{\mathcal{G}}, \tag{4.10}$$

which contains all the algebraic equations and first order differential equations for only the function $x(t)$, but not $x(t+\tau), \ldots, x(t+\kappa\tau)$ in the double-inflated system (4.4). The remaining work now is to select the equations of the regular, strangeness-free formulation (3.23) from (4.10). To do that we consider matrices and associated

spaces spanned by their columns,

$$\begin{array}{lll} Z_2 & \text{basis of} & \ker(\tilde{\mathcal{M}}^T), \\ T_2 & \text{basis of} & \ker(Z_2^T\tilde{\mathcal{N}}), \\ Y_2 & \text{basis of} & \operatorname{range}(Z_2^T\tilde{\mathcal{N}}), \\ Z_1 & \text{basis of} & \operatorname{range}(\tilde{\mathcal{M}}T_2). \end{array} \tag{4.11}$$

The regular, strangeness-free DDAE (3.23) is derived in the following lemma.

LEMMA 4.3. *Consider the double-inflated system* (4.4) *and the DDAE* (4.10) *at the point* $t \in \mathbb{I}$. *With the matrices* Z_1, Z_2, T_2, Y_2 *be defined as in* (4.11), *we have* $\operatorname{rank}(\tilde{\mathcal{M}}T_2) + \operatorname{rank}(Z_2^T\tilde{\mathcal{N}}) = n$. *Furthermore, the DDAE* (3.23) *at the point* t *becomes*

$$\begin{bmatrix} Z_1^T\tilde{\mathcal{M}} \\ 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} Z_1^T\tilde{\mathcal{N}} \\ Y_2^T Z_2^T\tilde{\mathcal{N}} \end{bmatrix} x(t) = \begin{bmatrix} Z_1^T\tilde{\mathcal{P}} \\ Y_2^T Z_2^T\tilde{\mathcal{P}} \end{bmatrix} x(t-\tau) + \begin{bmatrix} Z_1^T\tilde{\mathcal{G}} \\ Y_2^T Z_2^T\tilde{\mathcal{G}} \end{bmatrix}, \tag{4.12}$$

where $\begin{bmatrix} Z_1^T\tilde{\mathcal{M}} \\ Y_2^T Z_2^T\tilde{\mathcal{N}} \end{bmatrix}$ *is nonsingular.*

Proof. First, by the definition of Z_1 we see that $Z_1^T\tilde{\mathcal{M}}T_2$ has full row rank and $\operatorname{rank}(Z_1^T\tilde{\mathcal{M}}T_2) = \operatorname{rank}(\tilde{\mathcal{M}}T_2)$. Since the regular, strangeness-free DDAE (3.23) is contained in the DDAE (4.10), it follows that $\operatorname{rank}\left(\begin{bmatrix} \tilde{\mathcal{M}} \\ Z_2^T\tilde{\mathcal{N}} \end{bmatrix}\right) = n$. Considering the singular value decomposition of $Z_2^T\tilde{\mathcal{N}}$

$$\begin{bmatrix} Y_2^T \\ Y_{2,\perp}^T \end{bmatrix} Z_2^T\tilde{\mathcal{N}} \begin{bmatrix} T_{2,\perp} & T_2 \end{bmatrix} = \begin{bmatrix} \Sigma_N & 0 \\ 0 & 0 \end{bmatrix},$$

where $\begin{bmatrix} Y_2 & Y_{2,\perp} \end{bmatrix}$ and $\begin{bmatrix} T_{2,\perp} & T_2 \end{bmatrix}$ are unitary matrices, we see that

$$\begin{bmatrix} I_{\tilde{m}} & 0 \\ 0 & Y_2^T \\ 0 & Y_{2,\perp}^T \end{bmatrix} \begin{bmatrix} \tilde{\mathcal{M}} \\ Z_2^T\tilde{\mathcal{N}} \end{bmatrix} \begin{bmatrix} T_{2,\perp} & T_2 \end{bmatrix} = \begin{bmatrix} \tilde{\mathcal{M}}T_{2,\perp} & \tilde{\mathcal{M}}T_2 \\ \Sigma_N & 0 \\ 0 & 0 \end{bmatrix},$$

and hence

$$\operatorname{rank}(\tilde{\mathcal{M}}T_2) + \operatorname{rank}(Z_2^T\tilde{\mathcal{N}}) = \operatorname{rank}\left(\begin{bmatrix} \tilde{\mathcal{M}} \\ Z_2^T\tilde{\mathcal{N}} \end{bmatrix}\right) = n. \tag{4.13}$$

For the second claim, it suffices to prove that $\begin{bmatrix} Z_1^T\tilde{\mathcal{M}} \\ Y_2^T Z_2^T\tilde{\mathcal{N}} \end{bmatrix}$ is pointwise nonsingular. We observe that

$$\begin{bmatrix} Z_1^T\tilde{\mathcal{M}} \\ Y_2^T Z_2^T\tilde{\mathcal{N}} \end{bmatrix} \begin{bmatrix} T_{2,\perp} & T_2 \end{bmatrix} = \begin{bmatrix} Z_1^T\tilde{\mathcal{M}}T_{2,\perp} & Z_1^T\tilde{\mathcal{M}}T_2 \\ \Sigma_N & 0 \end{bmatrix},$$

and hence, (4.13) implies that $\begin{bmatrix} Z_1^T\tilde{\mathcal{M}} \\ Y_2^T Z_2^T\tilde{\mathcal{N}} \end{bmatrix}$ has full row rank n, which follows that $\begin{bmatrix} Z_1^T\tilde{\mathcal{M}} \\ Y_2^T Z_2^T\tilde{\mathcal{N}} \end{bmatrix}$ is pointwise nonsingular. □

In summary, the reformulation of the DDAE (1.1) as a regular, strangeness-free DDAE (4.12), is given by Algorithm 1 below.

Algorithm 1

1: **Input:** The initial value problem (1.1).
2: **Return:** The regular, strangeness-free DDAE (3.23) pointwise.
3: Consider an arbitrary point $t \in \mathbb{I}$.
4: Let $\kappa = 0$.
5: Construct the double-inflated system (4.4) with the coefficients $\mathcal{M}$, $\mathcal{P}$, $\mathcal{G}$.
6: Determine the matrix U such that the following condition holds

$$U^T \left[\mathcal{M}(:, (2n+1) : end) \quad \mathcal{P}(:, (n+1) : end)\right] = 0.$$

7: Compute the matrices $\tilde{\mathcal{M}}$, $\tilde{\mathcal{N}}$ as in (4.9) and Z_2, T_2, Y_2 as in (4.11).
8: **if** $\operatorname{rank}(\tilde{\mathcal{M}}T_2) + \operatorname{rank}(Z_2^T\tilde{\mathcal{N}}) = n$ **then** determine Z_1 as in (4.11) to derive the DDAE (4.12), which is exactly the regular, strangeness-free DDAE (3.23) at the point t
9: **else** $\kappa := \kappa + 1$, go back to 4
10: **end if**

REMARK 4.4. For linear time invariant DDAEs, the functions $\mathcal{M}$, $\mathcal{P}$, $\mathcal{G}$, Z_2, T_2, Y_2, Z_1 can be chosen to be constant matrices on the whole interval. In this way, we only need to compute the matrices at the initial point 0 and can use them in the whole integration process.

We illustrate Algorithm 1 by the following example.

EXAMPLE 4.5. Consider the IVP consisting of the DDAE

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{bmatrix} + \begin{bmatrix} 1 - e^{\frac{t-\tau}{10}} \\ -t \end{bmatrix}, \tag{4.14}$$

for $t \in [0, \infty)$, $\tau = 1$, with an initial function $\phi(t) := \begin{bmatrix} t \\ e^{\frac{t}{10}} \end{bmatrix}$, for $t \in [-\tau, 0]$.

Applying Algorithm 1 to (4.14), we proceed as follows. With $\kappa = 0$ we obtain

$$\tilde{\mathcal{M}} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ -1 & 0 \end{bmatrix}, \ \tilde{\mathcal{N}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \ \tilde{\mathcal{P}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \end{bmatrix}, \ Z_2 = \begin{bmatrix} 0.7071 & 0.7071 \\ 0.5000 & -0.5000 \\ -0.5000 & 0.5000 \end{bmatrix},$$
$$T_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \tilde{\mathcal{M}}T_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ Z_2^T\tilde{\mathcal{N}} = \begin{bmatrix} 0.7071 & 0 \\ 0.7071 & 0 \end{bmatrix}.$$

Thus, $\operatorname{rank}(\tilde{\mathcal{M}}T_2) + \operatorname{rank}(Z_2^T\tilde{\mathcal{N}}) = 1 < 2$, which implies that the shift index is bigger than 0 (equivalently, the DDAE (4.14) is noncausal). With $\kappa = 1$ we obtain

$$\tilde{\mathcal{M}} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix}, \ \tilde{\mathcal{N}} = \begin{bmatrix} 0 & -0.7071 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}, \ \tilde{\mathcal{P}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}, \ Z_2 = \begin{bmatrix} 0.7071 & 0.7071 & 0 \\ 0.5000 & -0.5000 & 0 \\ -0.5000 & 0.5000 & 0 \\ 0 & 0 & 1.0000 \end{bmatrix},$$
$$T_2 = [\,]^{2,0}, \ \tilde{\mathcal{M}}T_2 = [\,]^{4,0}, Z_2^T\tilde{\mathcal{N}} = \begin{bmatrix} 0 & -0.5000 \\ 0 & -0.5000 \\ -1.0000 & 0 \end{bmatrix}, \ Z_1 = [\,]^{4,0}, \ Y_2 = \begin{bmatrix} 0 & -0.7071 \\ 0 & -0.7071 \\ 1.0000 & 0 \end{bmatrix}.$$

Here by $[\,]^{i,j}$ we denote the empty matrix of the size i by j. In this case $\operatorname{rank}(\tilde{\mathcal{M}}T_2)+\operatorname{rank}(Z_2^T\tilde{\mathcal{N}})=2$ and therefore the shift index is $\kappa=1$. The regularized DDAE (4.12) is

$$\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}(t)\\ \dot{y}(t)\end{bmatrix}+\begin{bmatrix}-1 & 0\\ 0 & 0.7071\end{bmatrix}\begin{bmatrix}x(t)\\ y(t)\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}x(t-\tau)\\ y(t-\tau)\end{bmatrix}+\begin{bmatrix}-t\\ 0.7071\ e^{\frac{t}{10}}\end{bmatrix}. \quad (4.15)$$

Existing solvers such as RADAR5 [14], fail to handle system (4.14) due to its noncausality. However, the same solver successfully handles the regularized DDAE (4.15), which is pointwise computed automatically by Algorithm 1. To compute the numerical solution of the corresponding initial value problem for (4.15), we follow [14], which uses the Radau IIA collocation method with three stages and Lagrange interpolation to approximate the term $x(t-\tau)$. The numerical solution and the absolute error are presented in Figure 4.1. Both the absolute and relative tolerances for rank decisions and matrix computations are set to be 10^{-5}.

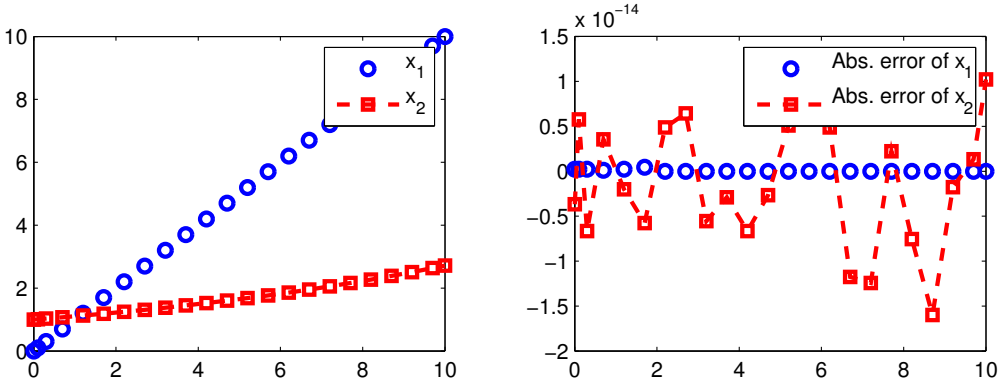

FIG. 4.1. *Numerical solution and absolute error for* (4.14).

In conclusion, to construct a solver for general DDAEs, where the system can be noncausal, remodeling the system before applying a numerical method is important, and sometimes it is indispensable. This reformulation procedure can be performed pointwise in a stable and robust way by the proposed Algorithm 1.

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