

IMPROVED APPROXIMATIONS FOR  
MINIMUM CARDINALITY  
QUADRANGULATIONS OF FINITE  
ELEMENT MESHES

by

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# Improved Approximations for Minimum Cardinality Quadrangulations of Finite Element Meshes<sup>†</sup>

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## Abstract

Conformal mesh refinement has gained much attention as a necessary preprocessing step for the finite element method in the computer-aided design of machines, vehicles, and many other technical devices. For many applications, such as torsion problems and crash simulations, it is important to have mesh refinements into quadrilaterals. In this paper, we consider the problem of constructing a minimum-cardinality conformal mesh refinement into quadrilaterals. However, this problem is  $\mathcal{NP}$ -hard, which motivates the search for good approximations. The previously best known performance guarantee has been achieved by a linear-time algorithm with a factor of 4. We give improved approximation algorithms. In particular, for meshes without so-called folding edges, we now present a 1.867-approximation algorithm. This algorithm requires  $\mathcal{O}(nm \log n)$  time, where  $n$  is the number of polygons and  $m$  the number of edges in the mesh. The asymptotic complexity of the latter algorithm is dominated by solving a  $T$ -join, or equivalently, a minimum-cost perfect  $b$ -matching problem in a certain variant of the dual graph of the mesh. If a mesh without foldings corresponds to a planar graph, the running time can be further reduced to  $\mathcal{O}(n^{3/2} \log n)$  by an application of the planar separator theorem.

## 1 Introduction

In recent years, the conformal refinement of *finite element meshes* has gained much attention as a necessary preprocessing step for the finite element method in the computer-aided design of machines, vehicles, and many other technical devices. Much work has been done on decompositions into *triangles*; see [Ho88] for a survey. However, for many applications, such as torsion problems and crash simulations, it is important to have mesh refinements into quadrilaterals [ZT89]. See also [Tou95] for a systematic survey on quadrangulations.

A *polygon* is a closed and connected region in the plane or, more generally, of a smooth surface in the three-dimensional space, bounded by a finite, closed sequence of straight line segments (*edges*). The endpoints of the line segments or curves are the *vertices*. A polygon is *convex* if the internal angle at each vertex is at most  $\pi$ . A *mesh* is a set of openly disjoint, convex polygons (Fig. 1). A mesh may contain *folding edges*, that is, edges incident to more than two polygons (Fig. 5). We call a mesh *homogeneous* if it does not contain folding edges.

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Figure 1: A coarse mesh modeling a chassis of a car. This mesh has been constructed by a German car company.



Figure 2: The conformal refinement produced by the algorithm in [MMW95, MMW96].

In a *conformal refinement* of a mesh, each polygon is decomposed into strictly convex quadrilaterals, and if two quadrilaterals share more than a corner, they share exactly one edge as a whole (Fig. 2).

Workpieces are modeled interactively as meshes; see Fig. 1 for an example of an instance taken from practice. However, such meshes are usually very coarse and not conformal. To be suitable for the finite element method, the mesh has to be refined into a conformal mesh in a preprocessing step. Previous work puts emphasis on the shape of the quadrilaterals (angles should neither be too small nor too large; the *aspect ratio*, i.e. the ratio between the largest and the smallest side of a quadrilateral, should be small). This is important for the numerical accuracy in the later iterations of the cyclic design process, when the model has become mature and exact results are required for fine-tuning.

In this paper now, we focus on the early stages of this process, where the model is designed only roughly, and the numerical accuracy must only suffice to indicate the general tendency. Hence, the development time is crucial, which in turn is determined by the run time of the finite element method. This raises the following problem: Given a mesh, find a conformal refinement with a *minimum* number of quadrilaterals.

Until recently, work on this problem (cf. [MMW95, MMW96] and [TA93]) has considered the number of quadrilaterals only heuristically or not at all. Usually, a *template model* is used, which restricts the possibilities of decomposing a single polygon to a few classes of *templates*. These templates are designed to achieve good angles and aspect ratios heuristically. For example, the most important template for quadrangular polygons is a  $(p \times q)$ -grid, where  $p$  and  $q$  are variable. However, this template uses  $p \cdot q$  quadrilaterals, which is quadratic in size compared with minimal quadrangulations of size  $\mathcal{O}(p + q)$  (easy to see). Therefore, algorithms often refine workpieces into too many quadrilaterals, which makes the finite element method very costly or even infeasible.

Unfortunately, it is hard to find conformal decompositions into a minimum number of quadrilaterals:

**Theorem 1.1** [MW96] *The minimum cardinality conformal mesh refinement problem is  $\mathcal{NP}$ -hard even for homogeneous meshes.*

For single polygons, however, this problem is efficiently solvable. More precisely, two variants of the problem can be solved in linear time, namely the case which allows to insert additional vertices to arbitrary positions and the case which allows additional vertices only into the *interior* of the polygon, but not on its boundary, i.e. it forbids to subdivide edges.

In the mesh refinement problem, the polygons cannot be refined independently since we have to ensure that the mesh is conformal. Hence, we carefully distinguish between *conformal refinements*, where vertices can be inserted at arbitrary positions, and conformal decompositions (see Fig. 3 for an example). By a *conformal decomposition* of a single polygon we will always mean the variant which does not allow to subdivide edges but to place vertices into the *interior* of the polygon (see Fig. 4). The following theorem holds:

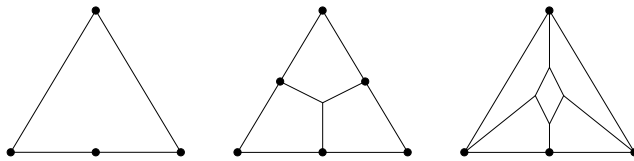


Figure 3: A triangular-shaped convex polygon with four vertices (left); an optimal *refinement*, which places two extra vertices on the boundary (middle); and an optimal *decomposition*, where no additional vertices on the boundary are allowed (right).

**Theorem 1.2** [MW96] *There is a linear-time algorithm which constructs a minimal conformal decomposition of a polygon into strictly convex quadrilaterals.*

In Section 3 we extend the work of [MW96] and give a characterization of the structure of minimal conformal refinements. Insight into this structure enables us to design also a linear-time algorithm for minimal conformal refinements of single polygons. This is not only interesting from the structural point of view, it also allows us to compute lower bounds for the mesh refinement problem in an efficient way.

There is also a well-known (see, for example [Joe95]), but important characterization of those polygons which can be decomposed into strictly convex quadrilaterals:

**Lemma 1.3** *A simple, not necessarily convex polygon  $P$  admits a conformal decomposition if and only if the number of vertices of  $P$  is even.*

Lemma 1.3 and Theorem 1.2 give rise to the following two-stage approach: First, subdivide a couple of edges such that each polygon achieves an even number of vertices; second, refine each polygon separately according to the algorithm mentioned in Theorem 1.2. Clearly, the first stage determines the approximation factor. In [MW96], each edge of the mesh is subdivided exactly once, which trivially makes all polygons even. It is also proved in [MW96] that this simple strategy already yields a 4-approximation. The analysis of this simple strategy is tight: for a conformal mesh of quadrilaterals, this algorithm obviously takes four times as many quadrilaterals as the optimum. To improve upon this performance guarantee, we apply a more sophisticated strategy.

A few related problems have found some attention. Note, for example, that it is important that polygons are by definition convex polygons, as Lubiw [Lub85] has shown that both problems, minimum refinement and minimum decomposition, are  $\mathcal{NP}$ -hard for single, but *non-convex* polygons with holes. To the best of our knowledge, the complexity status of the refinement problem for non-convex polygons *without* holes is still open. Everett et al. [ELOSU92] give lower and upper bounds on the number of quadrilaterals in a conformal refinement of simple, not necessarily convex polygons (with and without holes), but not on decompositions. Refs. [Sac82, ST81] investigate perfect decompositions of (star-shaped) rectilinear polygons into *non-strictly* convex quadrilaterals, and [Lub85] considers perfect decompositions of non-convex polygons but even allows overlapping internal edges. See [Tou95] for a systematic survey.

In Sect. 5, we will present the main results of this paper:

- There is a linear-time approximation algorithm which exceeds ratio 2 by an additive term of at most  $\Delta(\mathcal{M})$ . This parameter  $\Delta(\mathcal{M})$  (to be defined below in Def. 2.1) depends on the mesh structure, but for all practical instances that we know of,  $\Delta(\mathcal{M})$  is significantly smaller than the minimal number of quadrilaterals in a conformal refinement. Hence, for such instances, this yields a 3-approximation. (For general instances, this algorithm always guarantees a 4-approximation.)

- As an immediate consequence, this yields a linear-time 3-approximation for homogeneous meshes. (This is not true for the algorithm in [MW96].)
- For homogeneous meshes, we can even do better, namely, we get a 1.867-approximation algorithm which runs in  $\mathcal{O}(nm \log n)$  time, where  $n$  ( $m$ ) is the number of polygons (edges) in the mesh. If a homogeneous mesh corresponds to a planar graph, the running time can be further reduced to  $\mathcal{O}(n^{3/2} \log n)$  by an application of the planar separator theorem.

The asymptotic complexity of the algorithm for homogeneous meshes is dominated by solving a *T-join problem*, or equivalently, a *minimum-cost perfect b-matching problem* (see the monograph by Derigs [Der88] or the survey by Gerards [Ger95] for matching problems) in a certain variant on the dual graph of the mesh. In our application, the algorithm from [Gab83] requires  $\mathcal{O}(nm \log n)$  time. Usually, (homogeneous) meshes are sparse, i.e. they have only  $m = \mathcal{O}(n)$  edges.

All our results also carry directly over to the following, slightly more general variant on the minimum mesh refinement problem. Suppose that the given mesh is too coarse to expect reasonable results from the finite element method, but a finite element error estimation gives lower bounds on the mesh density which should be achieved. More precisely, suppose that these lower bounds on the mesh density are expressed as lower bounds on the number of vertices which have to be placed on the original edges in a feasible refinement. (There are CAD packages which pursue this strategy.) The general problem is to find a conformal refinement which respects these lower bounds, but minimizes the number of quadrilaterals.

The rest of the paper is organized as follows. In Section 2, we start with some preliminaries and introduce further terminology. Then, in Section 3 we review a characterization of minimal decompositions of polygons. Based on that, as mentioned above, we also give a new characterization of the structure of minimal refinements of polygons.

In Section 4 we present two combinatorial results (cf. Lemma 4.1 and Lemma 4.3). Roughly speaking, these results mean that the minimum number of quadrilaterals needed for a decomposition of a polygon does not increase exorbitantly, if each edge is subdivided at most once. The proofs of Lemmas 4.1 and 4.3 are quite involved and somewhat technical. In Section 5, we present the new approximation algorithms and prove their performance guarantees. Finally, we conclude with further remarks.

## 2 Preliminaries and Further Definitions

Let  $\mathcal{P} = \{P_1, P_2, \dots, P_n\}$  be the set of polygons forming the mesh. These polygons are convex, but not necessarily strictly convex. Two polygons are  *neighbored*  if they have an interval of the boundary in common which has strictly positive length. These neighborhood relationships induce an undirected graph  $G = (V, E)$ , which is embedded on the surface approximated by the mesh and whose faces are the polygons. More precisely,  $V$  consists of the corners of the polygons. If a corner of a polygon also belongs to the interior of a side of another polygon, it subdivides this side. Hence, we may identify common intervals of neighbored sides of polygons with each other, and  $E$  consists of these intervals after identification.

Note that the graph  $G$  of a mesh need not be planar; for example, a mesh approximating the surface of a torus has genus one. The set of all folding edges that are incident to exactly the same homogeneous components is called a *folding*.

For an edge  $e_i \in E$ , let  $E_i$  be the set of all those polygons which are incident to  $e_i$ . A *combinatorial description* of a mesh consists of the graph  $G$  and the hypergraph  $H = (\mathcal{P}, \{E_1, \dots, E_m\})$  with vertex set  $\mathcal{P}$  and edge set  $\{E_1, \dots, E_m\}$ . We will often identify a mesh with its combinatorial description.

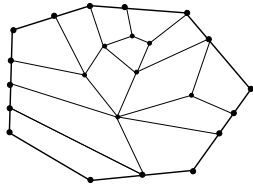


Figure 4: A convex polygon with 7 corners and 16 vertices and a conformal decomposition with 7 additional, internal vertices. (The decomposition is not minimal.)

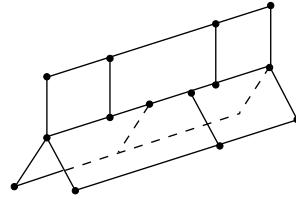


Figure 5: A small mesh with three homogeneous components and one folding, which consists of five folding edges. The corresponding hypergraph has 13 edges of degree one (boundary edges), 4 hyperedges of degree 2, and 5 hyperedges of degree 3.

A *non-folding path* in  $H$  is a path between two polygons  $P_1, P_2 \in \mathcal{P}$  which contains only hyperedges of cardinality two, i.e. only such hyperedges which belong to exactly two polygons. Being connected by a non-folding path is an equivalence relation on the set of polygons. Its equivalence classes are exactly the *homogeneous components* of a mesh. For a mesh  $G = (V, E)$  let  $G_1, \dots, G_{c(\mathcal{P})}$  denote the homogeneous components, and  $c(\mathcal{P})$  the number of components. The *degree* of an edge in  $E$  is the number of incident polygons. The *boundary* of a mesh is the set of all hyperedges with degree one (the *boundary edges*).

Since all folding edges within a folding are incident to exactly the same homogeneous components, the degree of a folding is well defined. Let  $D(\mathcal{M})$  denote the total sum of the degrees of all foldings that consist of an *odd* number of folding edges each. This allows us to define the parameter  $\Delta(\mathcal{M})$  which appears in the performance guarantee we can achieve for meshes, in general:

**Definition 2.1**  $\Delta(\mathcal{M}) := D(\mathcal{M}) + |\mathcal{P}| - c(\mathcal{P})$ .

Empirically, the mesh parameter  $\Delta(\mathcal{M})$  is fairly small. In a whole bunch of real-world examples, which stem from the German car industry, the average number of odd foldings per homogeneous component is less than three, and always smaller than the minimum number of quadrilaterals needed for that component. (This means that we guarantee a 3-approximation for such instances from practice.) In fact, it seems hard to imagine a non-pathological instance, where  $\Delta(\mathcal{M})$  is larger than the minimum number of quadrilaterals in an optimal mesh refinement.

A vertex of a convex polygon is a *corner* if its internal angle is strictly less than  $\pi$ . An *interval* of a polygon  $P$  is a path of edges on its boundary. A *segment*  $S$  is an interval between two successive corners of  $P$ .

A conformal decomposition of  $P$  is usually identified with the planar, embedded graph  $G_P = (V, E)$  whose outer face is  $P$  and whose internal faces are the quadrilaterals. Let  $q(G)$  denote the number of internal, quadrangular faces. Let  $G^* = (V^*, E^*)$  be the variant on the *dual graph* which arises by removing the vertex corresponding to the outer face of  $G_P$ . We call a conformal decomposition of a convex polygon  $P$  *perfect* if it has no vertices other than  $P$ .

We will denote a polygon by the counterclockwise sequence of the lengths of its segments. For example,  $(1, 1, 1, 1)$  denotes the strictly convex quadrilateral,  $(1, 1, 2) = (1, 2, 1) = (2, 1, 1)$  the quadrilateral degenerated to a triangle (see Fig. 3), and  $(4, 1, 2, 3, 2, 2, 2) = (1, 2, 3, 2, 2, 2, 4) = \dots$  the polygon in Fig. 4. This is justified by the following observation (cf. Lemma 3.4 in

[MW96]): If two polygons  $P_1$  and  $P_2$  have the same such sequence (up to cyclic shifts), then every graph of a conformal decomposition for  $P_1$  is also the graph of some conformal decomposition for  $P_2$  and vice versa. For brevity, we say that a polygon is *even* (*odd*) if it has an even (odd) number of edges.

For a convex polygon  $P$  with an even number of vertices,  $\min(P)$  denotes the minimum number of quadrilaterals required by any conformal decomposition of  $P$ . For an arbitrary convex polygon  $P$  with edge set  $E_P$ , a mapping  $X_P : E_P \rightarrow \mathbb{N}_0$  is called *feasible* if  $\sum_{e \in E_P} X_P(e)$  has the same parity as  $|E_P|$ . In particular, if  $|E_P|$  is even,  $X_P \equiv 0$  is possible, too. For simplicity, we will usually write  $X$  instead of  $X_P$ , as the dependence from the polygon  $P$  should be clear from the context.

The polygon  $P_X$  is constructed from  $P$  by subdividing each edge  $e \in E_P$  exactly  $X(e)$  times. Hence,  $X$  feasible means that  $P_X$  admits a conformal decomposition. Moreover,  $\text{Min}(P)$  denotes the minimum number of quadrilaterals in any conformal decomposition of any polygon  $P_X$ ,  $\text{Min}(P) := \min\{q(G) \mid G \text{ conformal decomposition of } P_X, X : E_P \rightarrow \mathbb{N}_0 \text{ feasible}\}$ . In other words,  $\text{Min}(P)$  is the minimum number of quadrilaterals in any conformal refinement of the polygon  $P$ . For a (feasible) mapping  $X$ , we denote  $|X| = \sum_{e \in E_P} X(e)$ .

### 3 The Structure of Minimal Decompositions and Refinements of Polygons

This section first briefly reviews a characterization of the structure of minimal decompositions of polygons, given in [MW96]. Based on these results we can also characterize the structure of minimal refinements. Finally, the knowledge of this structure enables us to design a linear-time algorithm for minimal refinements.

We need some additional terminology. For a conformal decomposition  $G = G_P = (V, E)$  of polygon  $P$ , recall the definition of  $G^*$  from Section 2. Each degree-one vertex  $v^*$  of  $G^*$  points to a trivial segment of  $P$ . We will sometimes identify such a vertex with this trivial segment.

Let  $i(G)$  denote the number of internal vertices, that is, the members of  $V$  that do not lie on  $P$ . With the help of Euler's formula it is easy to see, that  $q(G)$  and  $i(G)$  are related via

$$q(G) = i(G) + |E_P|/2 - 1, \quad (1)$$

where  $|E_P|$  is the number of edges of  $P$ .

The graph  $K_{1,3}$  is the complete bipartite graph on 1+3 vertices. We use the term *subdivision* of  $K_{1,3}$  when each edge of the  $K_{1,3}$  is replaced by a path of arbitrary length.

An *interval* on a polygon  $P$  is a path on its boundary. An interval is *trivial* if it consists of exactly one edge of  $P$ . A *segment*  $S$  is an interval between two successive corners of  $P$ . Let  $e_1$  and  $e_2$  be two different edges of  $P$ . Then  $I[e_1, e_2]$  denotes the interval counterclockwise from  $e_1$  to  $e_2$ , including neither  $e_1$  nor  $e_2$ . The length  $L(I)$  of an interval  $I$  is the number of its edges. Moreover,  $K(I)$  denotes the maximum size of a choice of strictly convex internal vertices of  $I$  such that no two of them are neighbored on  $P$ . We often denote  $(L - 2K)(I) := L(I) - 2 \cdot K(I)$ . Note that  $(L - 2K)(I)$  is always nonnegative.

Lemmas 3.1 and 3.3 first characterize minimal decompositions of perfect polygons, whereas Lemma 3.6 treats the general case. Recall that we call a polygon perfect if it has a decomposition without additional vertices. This implies that the polygon is even.

**Lemma 3.1** [MW96] *Let  $P$  be an even polygon with exactly two trivial segments  $e_1$  and  $e_2$ , and let  $I_1 := I[e_1, e_2]$  and  $I_2 := I[e_2, e_1]$ . Without loss of generality we have  $L(I_1) \geq L(I_2)$ . Then  $P$  is perfectly decomposable if and only if  $(L - 2K)(I_1) \leq L(I_2)$ . The dual graph  $G^*$  of a perfect decomposition is a path with leaves  $e_1$  and  $e_2$ .*

In Lemma 3.3, we assume the following scenario.

**Scenario 3.2** *Let  $P$  be an even polygon with at least three trivial segments. Let  $e_1$ ,  $e_2$ , and  $e_3$  be three trivial segments such that the counterclockwise order around  $P$  is  $e_1 \prec e_2 \prec e_3 \prec e_1$ . Let  $I_1 := I[e_1, e_2]$ ,  $I_2 := I[e_2, e_3]$ ,  $I_3 := I[e_3, e_1]$ , and w.l.o.g.  $L(I_1) \geq L(I_2)$  and  $L(I_1) \geq L(I_3)$ . Assume that  $L(I_1)$  is minimum subject to all these conditions.*

**Lemma 3.3** [MW96] *In Scenario 3.2,  $P$  is perfectly decomposable if and only if  $(L - 2K)(I_1) \leq L(I_2) + L(I_3) + 1$ . In this case, there is a perfect decomposition such that either  $G^*$  is a path from  $e_1$  to  $e_2$ , or  $G^*$  is a subdivision of  $K_{1,3}$  with leaves  $e_1$ ,  $e_2$ , and  $e_3$ .*

**Remark 3.4** *If all conditions of Scenario 3.2 are fulfilled, with the only exception that  $L(I_1)$  is not minimum, but  $(L - 2K)(I_1) \leq L(I_2) + L(I_3) + 1$  holds, then  $P$  is perfectly decomposable and has a decomposition of the structure in Lemma 3.3. For that purpose, it is not necessary that  $L(I_1)$  is minimum.*

It is useful to extend the notion of perfectness also to odd polygons: If  $|E_P|$  is odd, the polygon  $P$  is said to be *perfect* if one additional vertex on the boundary suffices to allow for a perfect decomposition of the resulting polygon  $P'$ .

**Lemma 3.5** *There is a linear-time algorithm that tests whether a given polygon  $P$  is perfect. Moreover, if the polygon is odd, we can determine in the same time complexity all those edges for which a single subdivision allows a perfect decomposition.*

**Proof:** Lemma 3.1 immediately translates into a linear-time algorithm for polygons with exactly two trivial segments. (If the polygon is odd, one additional point is placed on the shorter interval between the two trivial segments.)

So consider the case that  $P$  has more than two trivial segments, and first the case that  $P$  is even. Because of Lemma 3.3, we have to find trivial segments  $e_1$ ,  $e_2$  and  $e_3$  as described in Scenario 3.2. In [MW96] it is shown how this can be done in linear time. Moreover, the proof of Lemma 3.3 also shows how to construct the perfect decomposition if  $P$  is even and if the precondition  $(L - 2K)(I_1) \leq L(I_2) + L(I_3) + 1$  is fulfilled.

This establishes the lemma if  $P$  is even. Hence, assume now that  $P$  is odd, and that we have determined  $e_1$ ,  $e_2$  and  $e_3$  according to Scenario 3.2. As the polygon  $P = (1, 1, 1)$  is obviously not perfect, we may assume that  $|E_P| > 4$ .

First we check whether subdividing one of  $e_1$ ,  $e_2$  or  $e_3$  allows for a perfect decomposition. Certainly this can be done in linear time. To check the other edges, we apply a case distinction.

*Case I:  $L(I_1) = L(I_2)$  or  $L(I_1) = L(I_3)$ .*

Assume without loss of generality that  $L(I_1) = L(I_2)$ . If we place one additional point on  $I_1$  to get the interval  $I'_1$  in  $P'$ , then  $L(I'_1) \geq L(I_2)$  and  $L(I'_1) \geq L(I_3)$ . In addition, we certainly have

$$(L - 2K)(I'_1) \leq L(I'_1) = L(I_1) + 1 \leq L(I_2) + L(I_3) + 1.$$

Hence, by Lemma 3.3 and the following remark,  $P'$  is perfect. A similar argument holds if one additional point is placed either on  $I_2$  or on  $I_3$ . In any case, the resulting polygon is perfect (even if  $L(I_2) > L(I_1)$  or  $L(I_3) > L(I_1)$  afterwards).

*Case II:  $L(I_1) > L(I_2)$  and  $L(I_1) > L(I_3)$ .*

In this case, we test whether  $(L - 2K)(I_1) \leq L(I_2) + L(I_3) + 2$  holds. In the affirmative case, we may place one additional point on any edge of  $I_2$  or  $I_3$  and the resulting polygon is perfect.



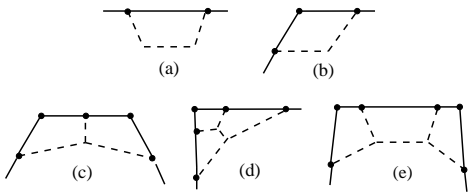


Figure 6: The first class of cut components in Lemma 3.6(2). The solid lines belong to  $P$ , and the dashed lines are internal edges. Only the structure of the graph matters; the concrete lengths and angles are only exemplary.

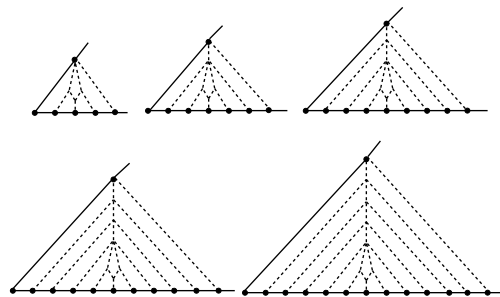


Figure 7: The five smallest cut components of the second class. The definition of the whole (infinite) class might be obvious.

Otherwise, the resulting polygon is certainly not perfect if one edge of  $I_2$  or  $I_3$  is subdivided. (Note that the triple  $e_1, e_2$ , and  $e_3$  according to Scenario 3.2 would be the same and still  $L(I_1) \geq L(I_2), L(I_3)$ .) In particular, we have that  $L(I_1) > L(I_2) + L(I_3) + 2$ . This implies that there is no trivial segment  $e_4 \subset I_1$ , as otherwise the choice  $e_1, e_4$  and  $e_2$  would lead to a strictly shorter largest interval and therefore to a contradiction to the choice of our triple of segments. Hence, any placement of an additional point on  $I_1$  to get  $I'_1$  would not change the choice of our triple  $e_1, e_2$  and  $e_3$ , and  $L(I'_1)$  is still minimal for  $P'$ . As  $K(I'_1) \leq K(I_1) + 1$ , we have

$$(L - 2K)(I'_1) \geq (L - 2K)(I_1) - 1 > L(I_2) + L(I_3) + 1.$$

This implies that  $P'$  is not perfect either if we subdivide an edge of  $I_1$ .

It remains to consider the possibilities to subdivide an edge of  $I_1$  if  $(L - 2K)(I_1) \leq L(I_2) + L(I_3) + 2$  holds. In a single pass along  $I_1$  we have to check for each edge of  $I_1$  individually whether we have  $(L - 2K)(I'_1) \leq L(I_2) + L(I_3) + 1$  if  $I'_1$  is obtained from  $I_1$  by subdividing this edge. (Note that we can evaluate  $K(I'_1)$  in constant time after a linear time preprocessing for  $K(I_1)$ .) In the affirmative case, we know that this subdivision makes the polygon perfect. In the negative case, the same arguments as in the previous paragraph show that this edge cannot be subdivided to yield a perfect decomposition. This completes the case distinction.  $\square$

Let  $G = (V, E)$  be an undirected planar, embedded graph. An *area component* of  $G$  is a subgraph  $G'$  induced by a connected component of  $G^*$ . More precisely,  $G'$  consists of all vertices and edges incident to the polygons that correspond to this component of  $G^*$ . An *area decomposition* of  $G$  is a collection of area components such that the inducing components of  $G^*$  partition all vertices in  $V^*$ . Intuitively, this means that the internal faces of  $G$  are partitioned and covered by closed, but openly disjoint, connected areas.

**Lemma 3.6 [MW96]** *For even  $P \notin \{(2, 1, 1), (4, 2, 2), (4, 3, 3), (3, 3, 3, 3)\}$ , there is a conformal decomposition  $G$  with minimum  $q(G)$  such that there is an area decomposition of  $G$  with the following properties:*

1. *The area decomposition consists of at most four area components.*
2. *All area components except one are isomorphic to one of the components depicted in Figs. 6 and 7. These area components are henceforth called the cut components.*
3. *The remaining area component is outerplanar. This area component is henceforth called the core component.*
4. *No two cut components share an edge.*
5. *All cut components except at most one are of type (c), (d), or (e) in Fig. 6.*

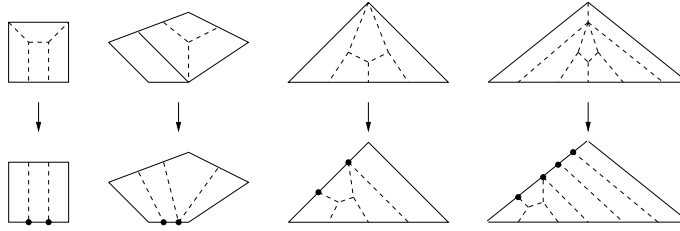


Figure 8: How to remove cut components of type (a) and (b) in Fig. 6 or Fig. 7 by insertion of additional vertices on the boundary.

6. If a cut component of type (a) or (b) in Fig. 6 or a cut component in Fig. 7 occurs, the core component admits a path decomposition.
7. If a cut component of the type in Fig. 7 occurs, the area decomposition contains at most two cut components; if a cut component of type (a) or (b) in Fig. 6 occurs, this is the only cut component.

The structure of minimal refinements is quite similar to that of minimal decompositions:

**Lemma 3.7** For  $P \notin \{(1, 1, 1), (2, 1, 1), (4, 2, 2), (4, 3, 3), (3, 3, 3, 3)\}$ , there is a conformal refinement  $G$  with minimum  $q(G)$  such that there is an area decomposition of  $G$  with the following properties:

1. The area decomposition consists of at most four area components.
2. All area components except one are isomorphic to one of type (c), (d), or (e) as depicted in Fig. 6. These area components are henceforth called the cut components.
3. The remaining area component is outerplanar. This area component is henceforth called the core component.
4. No two cut components share an edge.
5. If more than one additional vertex is placed on the boundary of  $P$ , then the core component admits a path decomposition.
6. If the core component admits a path decomposition with leaves  $e_1$  and  $e_2$ , then the additional vertices which are placed on the core component either all belong to the interval  $I_1 := I[e_1, e_2]$  or they belong all to  $I_2 := I[e_2, e_1]$ .

**Remark 3.8** Note that four area components are sometimes necessary, consider for example  $P = (8, 2, 8, 2, 8, 2)$ , which uses three components of type (c) in both its optimal decomposition and refinement.

**Proof of Lemma 3.7:** Let  $Y : E_P \rightarrow \mathbb{N}_0$  be optimal, that is,  $\text{Min}(P) = \min(P_Y)$ , and let  $|Y|$  be maximal among all minimal refinements. It is easily checked that  $P_Y = P$ , for  $P_Y \in \{(4, 2, 2), (4, 3, 3), (3, 3, 3, 3)\}$ . Moreover,  $P = (1, 1, 1)$  or  $P = (2, 1, 1)$  have optimal decompositions with  $P_Y = (2, 2, 2)$ , and are explicitly mentioned as exceptions. Hence, in the following, we can assume that none of the exceptions in Lemma 3.6 occurs. In particular, there is an optimal decomposition  $G$  of  $P_Y$  which has an area decomposition of the form claimed in Lemma 3.6. This immediately establishes the properties (1), (3) and (4).

**Property (2):** If  $G$  has an area component of type (a), (b) or as depicted in Fig. 7, this either contradicts optimality or the choice of  $Y$ , as the following modifications will show (see also Fig. 8).

Obviously,  $G$  has no area component of type (a), as otherwise we may place the two internal vertices on the boundary of  $P$  and get a strictly better solution. Next consider an area component of type (b). By Lemmas B.8 and B.9 in [MW96], the internal vertex of such a component can be assumed to have degree three. But with the help of two additional vertices on the boundary, we can always avoid the internal vertex, and thereby get an optimal solution with  $|Y'| = |Y| + 2$ , unless  $P_Y = (2, 2, 1, 1)$ . But note that for the optimal solution to  $P_Y = (2, 2, 1, 1)$ , there is an area decomposition with a single cut component of type (c). Similarly, a cut component as depicted in Fig. 7 can be converted into a cut component of type (d) and a path using  $2k - 2$  additional vertices for some  $k > 1$ . Altogether, these arguments yield property (2).

**Property (5):** Suppose that  $|Y| > 1$  and the core component, denoted by  $G'$ , with outer face  $P'$  does not allow for a path decomposition. Hence,  $G'^*$  must be a subdivision of  $K_{1,3}$ . We may assume that the trivial segments  $e_1, e_2$  and  $e_3$  are the leaves in counterclockwise order around  $P'$ , and  $I_1 := I[e_1, e_2]$ ,  $I_2 := I[e_2, e_3]$ ,  $I_3 := I[e_3, e_1]$  on  $P'$ , fulfilling Scenario 3.2. Hence, we have  $L(I_1) \geq L(I_2)$  and  $L(I_1) \geq L(I_3)$ . Lemma 3.3 implies  $(L - 2K)(I_1) \leq L(I_2) + L(I_3) + 1$ . Moreover, we even have  $L(I_1) < L(I_2) + L(I_3) + 1$ , as otherwise a path solution with leaves  $e_1$  and  $e_2$  exists. Clearly,  $G'^*$  being a subdivision of  $K_{1,3}$  and our assumption that there is no path solution for  $P'$  implies that  $L(I_1), L(I_2), L(I_3) \geq 2$ .

Note that an additional vertex can never be a corner of  $P_Y$ . Moreover, it would be strictly suboptimal if an additional vertex creates the segment of length 2 (i.e. the horizontal segment in Fig. 6) for a cut component of type (c), or the segment of length 3 (i.e. the horizontal segment in Fig. 6) for a cut component of type (e). (Here we use again, that by Lemmas B.8 and B.9 in [MW96], the internal vertex of such components can be assumed to have degree three.) We claim that we may assume that at least two additional vertices belong to  $P'$ , call them  $v_1$  and  $v_2$ .

To see this claim, observe that we can avoid to place an additional vertex adjacent to a corner which is cut away by a component of type (c) or (d). This can be done by an exchange with a vertex from  $P'$  of the same segment, unless all other vertices of this segment are already additional vertices. Both possibilities yield the claim.

So consider now the polygon  $P''$  which we had obtained if neither  $v_1$  nor  $v_2$  had been inserted. Let  $I'_1, I'_2$  and  $I'_3$  be the intervals of  $P''$  corresponding to  $I_1, I_2$  and  $I_3$ , respectively. If we show that  $P''$  is still perfect, this contradicts optimality of the refinement  $P_Y$ , and property (5) follows.

We consider three different cases separately.

*Case I:*  $v_1$  and  $v_2$  belong to  $I_1$ .

We have  $L(I'_1) = L(I_1) - 2$ ,  $L(I'_2) = L(I_2)$  and  $L(I'_3) = L(I_3)$ . If  $L(I'_1) \geq L(I'_2)$  and  $L(I'_1) \geq L(I'_3)$ , then

$$(L - 2K)(I'_1) \leq L(I'_1) < L(I_1) < L(I_2) + L(I_3) + 1 = L(I'_2) + L(I'_3) + 1$$

implies that  $P''$  is perfect by Lemma 3.3.

Otherwise, we may assume that  $L(I'_2) > L(I'_1)$  and  $L(I'_2) \geq L(I'_3)$ . But then

$$(L - 2K)(I'_2) \leq L(I'_2) = L(I_2) \leq L(I_1) = L(I'_1) + 2 \leq L(I'_1) + L(I'_3) + 2.$$

As  $L(I'_1) + L(I'_2) + L(I'_3)$  is odd, the inequality can be strengthened to  $(L - 2K)(I'_2) \leq L(I'_1) + L(I'_3) + 1$ . Hence,  $P''$  is perfect.

*Case II:*  $v_1$  but not  $v_2$  belongs to  $I_1$ .

Now we may assume that  $L(I'_1) = L(I_1) - 1$ ,  $L(I'_2) = L(I_2) - 1$  and  $L(I'_3) = L(I_3)$ . Then we have  $L(I'_1) \geq L(I'_2)$ . If we also have  $L(I'_1) \geq L(I'_3)$ , then  $P''$  is perfect because of  $L(I'_1) < L(I'_2) + L(I'_3) + 1$ .

Otherwise, we have  $L(I'_1) = L(I'_3) - 1$ . But then  $L(I'_3) \leq L(I'_2) + L(I'_1) + 2$ . As  $L(I'_1) + L(I'_2) + L(I'_3)$  is odd, we even obtain  $L(I'_3) \leq L(I'_2) + L(I'_1) + 1$ . This implies that  $P''$  is perfect.

*Case III:* Neither  $v_1$  nor  $v_2$  belong to  $I_1$ .

In this case, we have  $L(I'_1) = L(I_1)$  and  $L(I'_2) + L(I'_3) = L(I_2) + L(I_3) - 2$ .

Thus we have  $L(I'_1) = L(I_1) < L(I_2) + L(I_3) + 1 = L(I'_2) + L(I'_3) + 3$ . The same parity argument as above yields  $L(I'_1) \leq L(I'_2) + L(I'_3) + 1$ , and so  $P''$  is perfect. This finishes the case distinction.

**Property (6):** The statement is trivially fulfilled if  $|Y| < 2$ . Assume without loss of generality that  $L(I_1) \geq L(I_2)$ . By Lemma 3.1, we have  $(L - 2K)(I_1) \leq L(I_2)$ .

Let  $|Y_1|$  be the number of additional points on  $I_1$ . We are done if  $|Y_1| = 0$ . Again, we apply a case distinction:

*Case I:*  $|Y_1| \geq L(I_1) - L(I_2)$ .

Let  $k := L(I_1) - L(I_2)$ . Delete  $k$  of the additional vertices from  $I_1$  to get  $I'_1$ . Then  $L(I'_1) = L(I_2)$ , which means that there is a path solution with a strictly smaller number of quadrilaterals. This contradicts optimality of  $P_Y$ .

*Case II:*  $2 \cdot |Y_1| \geq L(I_1) - L(I_2) > |Y_1|$ .

Define  $k := L(I_1) - |Y_1| - L(I_2) > 0$ . In this case, delete all  $|Y_1|$  additional vertices from  $I_1$  to get  $I'_1$ . As  $k \leq |Y_1|$ , we may reinsert  $k$  of these vertices into  $I_2$  to get  $I'_2$ . By the choice of  $k$ , we now have  $L(I'_1) = L(I'_2)$ . Hence, the modified core polygon has a path solution with not more quadrilaterals than the original one, but has no additional vertex placed on the interval  $I'_1$ .

*Case III:*  $2 \cdot |Y_1| < L(I_1) - L(I_2)$ .

In this case, we also delete all  $|Y_1|$  additional vertices from  $I_1$  to get  $I'_1$ , and reinsert all of them into  $I_2$  to get  $I'_2$ . Hence, we have  $L(I'_1) = L(I_1) - |Y_1|$  and  $L(I'_2) = L(I_2) + |Y_1|$ . The inequality defining Case III yields that  $L(I'_1) \geq L(I'_2)$ . Furthermore, we have

$$\begin{aligned} L(I'_1) - 2 \cdot K(I'_1) &= L(I_1) - |Y_1| - 2 \cdot K(I'_1) \\ &\leq L(I_2) + 2 \cdot K(I_1) - |Y_1| - 2 \cdot K(I'_1) \\ &\leq L(I'_2) + 2 \cdot K(I_1) - 2 \cdot |Y_1| - 2 \cdot K(I'_1) \\ &\leq L(I'_2). \end{aligned}$$

The last inequality follows from the fact that  $K(I_1) \leq K(I'_1) + |Y_1|$ . By Lemma 3.1, the modified core polygon admits a path solution. This establishes property (6).  $\square$

The characterization of the structure of minimal refinements enables us to give an algorithm for this problem with linear running time. Hence, this algorithm has asymptotically optimal running time.

**Theorem 3.9** *There is a linear-time algorithm that constructs a conformal refinement  $G$  that minimizes  $q(G)$ .*

**Proof:** The algorithm is a slight variation of that for minimal decompositions given in [MW96].  $\square$

## 4 Subdivisions of Polygons

In this section, we present two combinatorial results which relate the optimal conformal refinements of a polygon  $P$  to optimal refinements of those polygons, which arise if some of the edges of  $P$  are subdivided by one additional vertex. This result will be useful for conformal refinements of meshes in Sect. 5. In fact, Lemma 4.1 and Lemma 4.3 are the most difficult parts of the proofs of Theorems 5.3 and 5.5, respectively.

**Lemma 4.1** *For a polygon  $P$  and  $X : E_P \rightarrow \{0, 1\}$  we have  $\min(P_X) \leq 2 \cdot \text{Min}(P) + |X| - 1$ , and we even have  $\min(P_X) \leq 2 \cdot \text{Min}(P) + |X| - 2$  except for the following cases:*

1.  $P = (1, 1, 1, 1)$  and  $P_X = (1, 1, 1, 1)$ ;
2.  $P = (1, 1, 1, 1)$  and  $P_X = (2, 2, 1, 1)$ ;
3.  $P = (2, 1, 1, 1)$  and  $P_X = (3, 1, 1, 1)$ ;
4.  $P = (2, 1, 2, 1)$  and  $P_X = (4, 1, 2, 1)$ ;
5.  $P = (2, 1, 1, 1, 1)$  and  $P_X = (4, 1, 1, 1, 1)$ ;
6.  $P = (2, 1, 1)$  and  $P_X = (2, 1, 1)$ .

Obviously,  $\text{Min}(P)$  is a lower bound for  $\min(P_X)$ . However, the gap between this trivial lower bound and the number of quadrilaterals in an optimal decomposition for  $P_X$  can be quite large. Therefore, we introduce *penalty functions*, which give improved lower bounds.

For a polygon  $P$ , the map  $W_P : \{X\} \rightarrow \mathbb{R}_0^+$ , defined on the set of feasible mappings  $X : E_P \rightarrow \{0, 1\}$ , is a *penalty function* for the subdivision of the polygon  $P$  if

$$\text{Min}(P) + W_P(X) \leq \min(P_{X'}) \quad (2)$$

for all  $X' \geq X$ , where  $X' : E_P \rightarrow \mathbb{N}_0$  is a feasible mapping. ( $X' \geq X$  means component-wise greater or equal, i.e.  $X'(e) \geq X(e)$  for all  $e \in E_P$ .) In particular,  $\text{Min}(P) + W_P(X)$  is a lower bound for the number of quadrilaterals in an optimal decomposition for  $P_X$ .

Note that in some cases  $\min(P_{X'}) < \min(P_X)$ , for  $X' > X$ . Examples are  $P_X = (2, 1, 1)$  and  $P_{X'} = (2, 2, 2)$ , or  $P_X = (3, 1, 1, 1)$  and  $P_{X'} = (3, 1, 3, 1)$ . Therefore, the weaker requirement  $\text{Min}(P) + W_P(X) \leq \min(P_X)$  instead of Inequality (2) would not suffice to yield a valid lower bound for our approximation algorithms.

The next lemma gives a penalty function for perfect polygons.

**Lemma 4.2** *Let  $P$  be a perfect polygon. Then  $\widetilde{W}_P(X) := \frac{|X|}{2}$ , if  $|E_P|$  is even, and  $\widetilde{W}_P(X) := \frac{|X|-1}{2}$ , if  $|E_P|$  is odd, is a penalty function for  $P$ .*

**Proof:** Let  $P$  be a perfect polygon where  $|E_P|$  is even. For some given  $X$ , consider a feasible subdivision  $X'$  with  $X' \geq X$ , and denote by  $G_{X'}$  some optimal decomposition of  $P_{X'}$  with  $i(G_{X'})$  internal vertices. Then we have (by Equation (1))

$$\min(P_{X'}) = \frac{|E_P| + |X'|}{2} - 1 + i(G_{X'}) \geq \frac{|E_P| + |X|}{2} - 1 = \text{Min}(P) + \widetilde{W}_P(X),$$

because  $\text{Min}(P) = \frac{|E_P|}{2} - 1$ . The case where  $|E_P|$  is odd, is proved analogously.  $\square$

For polygons of certain types (see Fig. 9) we introduce special penalty functions  $\widetilde{W}_P$  which are encoded by means of an associated auxiliary graph  $G_{\text{aux}}^P = (V_{\text{aux}}^P, E_{\text{aux}}^P)$  with edge weights. In particular, these graphs contain a unique dual edge for each edge of  $P$ . To evaluate  $\widetilde{W}_P(X)$  for a given  $X$ , we have to select edges according to the following rules:

1. an even number of edges has to be chosen for each vertex of  $G_{\text{aux}}^P$ , except those indicated by an arrow in Fig. 9, for which we have to select an odd number of edges.
2. a dual edge has to be chosen if and only if  $X(e) = 1$  for the corresponding edge of  $P$ .
3. the sum of the edge weights should be minimal subject to the first two conditions.

The function value of  $\widetilde{W}_P(X)$  is exactly the sum of the chosen edge weights. Obviously, we can evaluate  $\widetilde{W}_P(X)$  in linear time. It is tedious but easy to verify that for all types of polygons given in Fig. 9 the functions are indeed penalty functions.

In the next lemma we use the following penalty functions: For all types of polygons given in Fig. 9, we use the penalty function defined in that figure. If a polygon is perfect but not among those listed in Fig. 9, we take the weight function as defined in Lemma 4.2. For all other types of polygons, we simply take  $\widetilde{W}_P \equiv 0$ . With respect to these penalty functions, we can show the following:

**Lemma 4.3** *For a polygon  $P$  and the penalty function  $\widetilde{W}_P : \{X\} \rightarrow \mathbb{R}_0^+$  (as defined in the previous paragraph) the following holds:*

$$\min(P_X) \leq \frac{5}{3} \cdot (\text{Min}(P) + \widetilde{W}_P(X)) + |X| - 2 \quad \text{if } |X| > 0, \quad (3)$$

$$\min(P_X) \leq \frac{5}{3} \cdot \text{Min}(P) - \frac{2}{3} \quad \text{if } |X| = 0, \quad (4)$$

except for the following cases:

1.  $P = (1, 1, 1)$  and  $P_X = (2, 1, 1)$ .
2.  $P = (2, 1, 1)$  and  $P_X = (2, 1, 1)$ .

The proofs of Lemma 4.1 and 4.3 are divided into several steps. We first use the characterizations from Lemma 3.1 and prove in Lemmas 4.5, 4.6, 4.7 and 4.8 the correctness of Lemma 4.1 and Lemma 4.3 for the case that a minimal refinement of the polygon admits a perfect decomposition such that  $G^*$  is a path or a subdivision of  $K_{1,3}$ , respectively. Finally, we treat the non-perfect case in Lemma 4.9. We even show slightly stronger inequalities than those required for Lemma 4.3, if the minimal refinement has no perfect path decomposition.

**Assumption 4.4** *Let  $Y : E_P \rightarrow \mathbb{N}_0$  be optimal, that is,  $\text{Min}(P) = \min(P_Y)$ .*

As a warm-up for the stronger inequalities to come, we first prove the following lemma:

**Lemma 4.5** *If  $P_Y$  admits a perfect decomposition such that  $G^*$  is a path then Lemma 4.1 holds.*

**Proof:** Lemma 4.1 is easy to see for the exceptional cases. Hence, we have to show

$$\min(P_X) \leq 2 \cdot \min(P_Y) + |X| - 2. \quad (5)$$

for all other situations.

Let  $G'$  be such a decomposition of  $P_Y$ , and let  $e_1$  and  $e_2$  be the leaves of the corresponding variant  $G^*$  on the dual graph where the vertex corresponding to the outer face is removed. Clearly,  $e_1$  and  $e_2$  are trivial segments of  $P$ , too, but not necessarily of  $P_X$ .

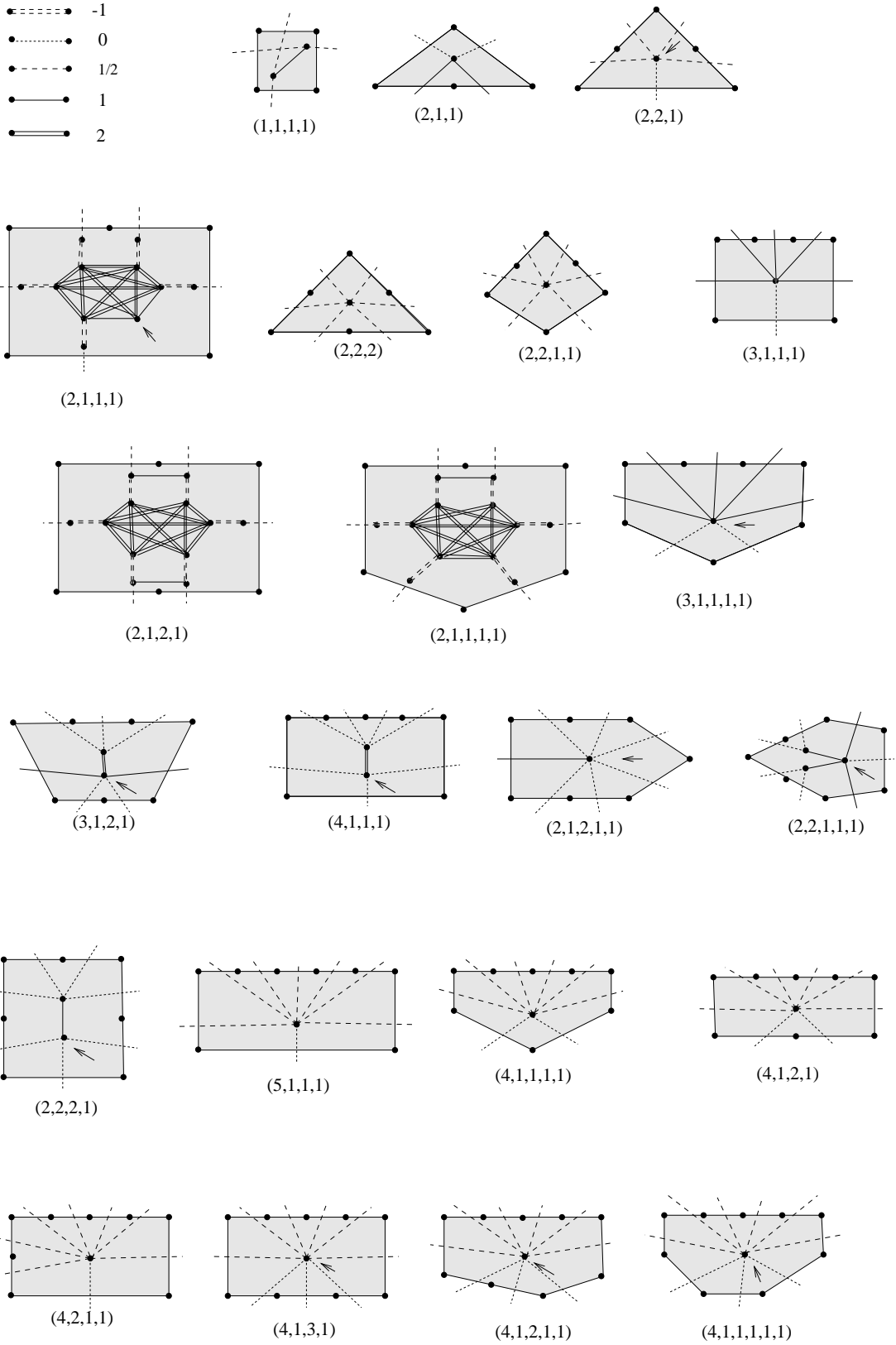


Figure 9: The auxiliary graphs  $G_{\text{aux}}^P$  for polygons of different types which encode our penalty functions. The weight of an edge is indicated by its line style. For all vertices inside a polygon we have to select an even number of edges, except for those indicated by an arrow, for which an odd number of edges is required.

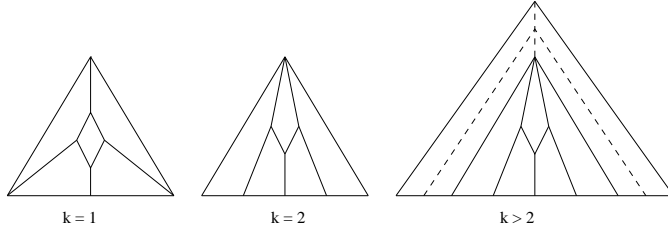


Figure 10: Illustration of optimal conformal decompositions of  $(2k, 1, 1)$ .

For  $P$ , let  $I_1 := I[e_1, e_2]$  and  $I_2 = I[e_2, e_1]$ . Let  $I'_1$  and  $I'_2$  denote the corresponding intervals of  $P_Y$  and  $I''_1$  and  $I''_2$  the corresponding intervals of  $P_X$ . Let  $|Y| := \sum_{e \in E_P} Y(e)$ . As  $G'$  is perfect, we have

$$q(G') = \frac{L(I'_1) + L(I'_2)}{2} = \frac{L(I_1) + L(I_2) + |Y|}{2}. \quad (6)$$

The decomposition we construct for  $P_X$  is denoted by  $G''$ . To prove Ineq. (5), it suffices to show

$$q(G'') \leq L(I_1) + L(I_2) + |Y| + |X| - 2. \quad (7)$$

W.l.o.g., we have  $L(I''_1) \geq L(I''_2)$ . Let

$$\delta := \min \left\{ K(I''_1), \left\lceil \frac{1}{2} [L(I''_1) - L(I''_2)] \right\rceil \right\}.$$

Fig. 11 shows the different cases for  $G''$  provided  $\delta = 0$  (explanations below).

For  $\delta > 0$ , we modify the procedure as follows. Let  $\mathcal{K}$  be an arbitrary set of internal corners of  $I''_1$  such that  $|\mathcal{K}| = \delta$  and no two vertices in  $\mathcal{K}$  are neighbored on  $P_X$ . Then we construct  $P'_X$  from  $P_X$  by shrinking each edge that is incident to a vertex in  $\mathcal{K}$ ; in other words, each vertex in  $\mathcal{K}$  is identified with its two neighbors. Each such “supervertex” is treated as a non-corner, so that it is incident to an internal edge in any conformal decomposition of  $P'_X$ .

Let  $G''' = (V''', E''')$  be the conformal decomposition of  $P'_X$  according to Fig. 11. Then the decomposition  $G''$  for  $P_X$  is constructed from  $G'''$  as follows: Let  $v \in \mathcal{K}$  and let  $v_1$  and  $v_2$  be the neighbors of  $v$  on  $I''_1$ . Then we choose an arbitrary internal edge  $\{v, w\} \in E'''$  and replace it by  $\{v_1, w\}$  and  $\{v_2, w\}$ . This yields  $\delta$  additional quadrilaterals.

Let  $X_1 := X \cap I_1$ ,  $X_2 := X \cap I_2$  and  $\bar{X} := X \cap \{e_1, e_2\}$ . Then we have  $X = X_1 \cup X_2 \cup \bar{X}$  and  $0 \leq |\bar{X}| \leq 2$ . Now we are going to consider the individual cases in Fig. 11.

*Case I:*  $(L - 2K)(I''_1) - L(I''_2) \leq 0$ .

The following equation is easy to see for  $|\bar{X}| = 0, 1, 2$ :

$$q(G'') = \frac{L(I''_1) + L(I''_2) + |\bar{X}|}{2} + |\bar{X}|. \quad (8)$$

Therefore, we have to show

$$\frac{L(I''_1) + L(I''_2) + |\bar{X}|}{2} + |\bar{X}| \leq L(I'_1) + L(I'_2) + |X| - 2. \quad (9)$$



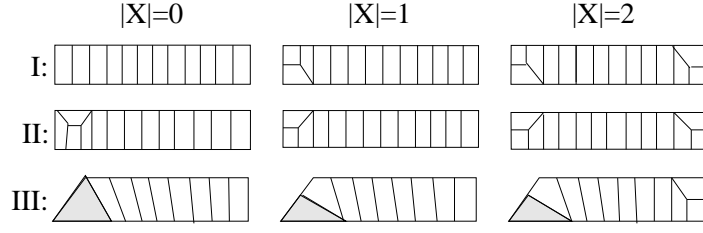


Figure 11:  $G''$  in the different cases in the proof of Lemma 4.1 for  $\delta = 0$ . In each case,  $I_1''$  is the horizontal line below. The grey triangles in Case III indicate decompositions according to Fig. 10, respectively.

However, since  $L(I_1'') + L(I_2'') + |\bar{X}| = L(I_1) + L(I_2) + |X|$  and  $L(I_1') + L(I_2') = L(I_1) + L(I_2) + |Y|$ , this is fulfilled whenever

$$|\bar{X}| + 4 \leq L(I_1) + L(I_2) + |X_1| + |X_2| + 2|Y|. \quad (10)$$

We next consider all cases for which Ineq. (10) is not immediate. (In particular, this means  $L(I_1) + L(I_2) < 6$ .)

The case  $L(I_1) + L(I_2) = 2$  (i.e.,  $P = (1, 1, 1, 1)$ ) is easily checked “by hand.” (Note that this includes the first and second exceptions of Lemma 4.1). Next consider the case  $L(I_1) + L(I_2) = 3$ , that is,  $P \in \{(2, 1, 1, 1), (1, 1, 1, 1, 1)\}$ . Then we have  $|Y| = 1$ , so Ineq. (10) reduces to  $|\bar{X}| + 2 \leq L(I_1) + L(I_2) + |X_1| + |X_2|$ . This is fulfilled if  $|\bar{X}| < 2$ . However,  $|X|$  is odd. Hence,  $|\bar{X}| = 2$  implies  $|X_1| + |X_2| > 0$ , and Ineq. (10) is fulfilled again.

Now assume  $L(I_1) + L(I_2) = 4$ . Then Ineq. (10) is true unless  $|X_1| = |X_2| = |Y| = 0$ , because  $|\bar{X}| \leq 2$ , and  $|\bar{X}|$  has the same parity as  $|X_1| + |X_2|$ . If  $L(I_1) = 3$  and  $L(I_2) = 1$  or vice versa, the dual path  $G^*$  does not end with  $e_1$  and  $e_2$ . However, the case  $L(I_1) = L(I_2) = 2$  and  $|X_1| = |X_2| = |Y| = 0$  is easily checked by hand again.

Finally assume  $L(I_1) + L(I_2) = 5$ . Then  $|Y|$  is odd, and Ineq. (10) is fulfilled.

*Case II:*  $1 \leq (L - 2K)(I_1'') - L(I_2'') \leq 2$ .

Let  $\Delta := (L - 2K)(I_1'') - L(I_2'')$ . Now we easily obtain for  $|\bar{X}| = 0, 1, 2$ :

$$q(G'') = \frac{L(I_1'') + L(I_2'') + |\bar{X}|}{2} + \Delta, \quad (11)$$

Therefore, Ineq. (7) is fulfilled whenever

$$\Delta + 2 \leq \left( \frac{L(I_1)}{2} + \frac{L(I_2)}{2} + \frac{|Y|}{2} \right) + \frac{|Y|}{2} + \frac{|X|}{2}. \quad (12)$$

Recall that  $q(G') = [L(I_1) + L(I_2) + |Y|]/2$ . As  $\Delta \leq 2$  in Case II, Ineq. (12) is fulfilled whenever

$$q(G') \geq 4. \quad (13)$$

If one of  $|X|$  or  $|Y|$  is strictly positive, we have  $(|X| + |Y|)/2 \geq 1$ , because  $|X| = 1$  implies  $|Y| > 0$  and vice versa. Hence, Ineq. (13) can be strengthened to  $q(G') \geq 3$  except for the trivial case  $|X| = |Y| = 0$ .

Obviously,  $q(G') = 1$  is impossible in Case II. The remaining case  $q(G') = 2$  is easily checked by hand. (Note that this includes the exceptions 3. - 5. of Lemma 4.1.)

*Case III:*  $(L - 2K)(I_1'') - L(I_2'') \geq 3$  and  $P \neq (2, 1, 1, 1, 1)$ .

In the third row of Fig. 11, the number of quadrilaterals in the white area is  $L(I_2'')$  for  $|\bar{X}| \leq 1$  and  $L(I_2'') + 2$  for  $|\bar{X}| = 2$ . On the other hand, the grey area is decomposed according to Fig. 10. Like in Fig. 10, let  $2k$  denote the number of horizontal edges below the grey area. Then we have  $2k = L(I_1'') - L(I_2'')$ , if  $|\bar{X}|$  is even, and  $2k = L(I_1'') - L(I_2'') + 1$ , if  $|\bar{X}|$  is odd. So the number of quadrilaterals in the grey area is  $L(I_1'') - L(I_2'') + 1$  and  $L(I_1'') - L(I_2'') + 2$ , respectively. Recall that  $\delta = 0$  is assumed in Fig. 11. If  $\delta > 0$ , restoring the shrunken edges yields  $K(I_1'')$  additional quadrilaterals, but now we have  $2k = L(I_1'') - L(I_2'') - 2K(I_1'')$  and  $2k = L(I_1'') - L(I_2'') - 2K(I_1'') + 1$ , respectively. In any case, we obtain for  $|\bar{X}| = 0, 1, 2$ :

$$q(G'') = L(I_1'') - K(I_1'') + |\bar{X}| + 1. \quad (14)$$

Hence, Ineq. (7) is fulfilled whenever

$$3 \leq L(I_2) + |Y| + |X_2| + K(I_1''). \quad (15)$$

So assume  $L(I_2) + |Y| + |X_2| + K(I_1'') \leq 2$  in the remainder. Note that  $K(I_1'') \geq K(I_1)$ . Hence, we have  $K(I_1) \leq 1$ , because otherwise we had  $L(I_2) = 0$ , and the dual path  $G^*$  would not point to  $e_1$  and  $e_2$ . From the proof of Lemma 3.1, it is easy to see that  $(L - 2K)(I_1) \leq L(I_2) + |Y|$ . Therefore,  $K(I_1) = 0$  implies  $L(I_1) \leq 2$ , which is impossible in Case III.

So consider the case  $K(I_1) = 1$ . Then we also have  $L(I_2) + |Y| = 1$ , that is,  $L(I_2) = 1$  and  $|Y| = 0$ . Therefore,  $L(I_1)$  is odd. Since  $L(I_1) = 1$  is impossible in Case III, this means  $L(I_1) = 3$ . However, this means  $P = (2, 1, 1, 1, 1)$ , and again  $G^*$  would not point to  $e_1$  and  $e_2$ .  $\square$

For Lemma 4.6, we need some more terminology. Note that constructing  $P_Y$  from  $P$  may be seen as replacing each edge  $e \in E_P$  by a segment  $S(e)$  of length  $Y(e) + 1$ . We extend  $X$  from  $P$  to  $P_Y$  as follows: For  $e \in E_P$ , we choose an arbitrary  $e' \in S(e)$  and define  $X(e') := X(e)$ . For all other  $e' \in S(e)$ , we set  $X(e') = 0$ .

**Lemma 4.6** *Lemma 4.1 is true if  $P_Y$  is perfect.*

**Proof:** Because of Lemmas 3.1, 3.3, and 4.5, it suffices to consider the case that  $P_Y$  admits a perfect decomposition  $G$  such that  $G^*$  is a subdivision of  $K_{1,3}$ . Let  $v^*$  be the branching vertex of  $G^*$ , let  $e_1^*$ ,  $e_2^*$ , and let  $e_3^*$  be the edges of  $G^*$  incident to  $v^*$ . Note that removing one of the edges  $e_1^*$ ,  $e_2^*$ , and  $e_3^*$  decomposes  $G^*$  into two paths. The basic idea of the proof is to apply Lemma 4.5 to both paths.

For  $j = 1, 2, 3$ , let  $e_j$  be the primal edge of  $G$  corresponding to  $e_j$ . The primal operation corresponding to the removal of  $e_j^*$  is cutting  $G$  along  $e_j$  and inserting  $e_j$  in both connected components. Moreover, let  $X_j$  be the subset of edges in  $X$  that are incident to quadrilaterals in the  $j$ th branch. Next we apply a case distinction.

*Case I:* at least one  $|X_j|$  is even.

Let  $G'$  and  $G''$  denote the subgraphs of  $G$  resulting from cutting the  $j$ th branch as described above. In particular, let  $G'$  correspond to this branch and  $G''$  to the rest. Let  $P'$  and  $P''$  denote the outer faces of  $G'$  and  $G''$ , respectively. It is easy to see that  $\min(P_Y) = \min(P') + \min(P'') = \text{Min}(P') + \text{Min}(P'')$ . By Lemma 4.5, we further have  $\min(P'_{X_j}) \leq 2 \cdot \text{Min}(P') + |X_j| - 1$  and

$\min(P''_{X \setminus X_j}) \leq 2 \cdot \text{Min}(P'') + |X \setminus X_j| - 1$ . In summary, we obtain  $\min(P_X) \leq \min(P'_X) + \min(P''_X) \leq 2 \cdot \text{Min}(P') + 2 \cdot \text{Min}(P'') + |X| - 2 = 2 \cdot \text{Min}(P) + |X| - 2$ .

*Case II:* all  $|X_j|$  are odd.

Now we choose  $j \in \{1, 2, 3\}$  arbitrarily and construct  $G'$ ,  $G''$ ,  $P'_Y$ , and  $P''$  by cutting the  $j$ th branch as described above. Let  $X' := X_j \cup \{v_j, \widetilde{W}_j\}$  and  $X'' := X \setminus X_j \cup \{v_j, \widetilde{W}_j\}$ . From Lemma 4.5, we conclude  $\min(P''_{X''}) \leq 2 \cdot \text{Min}(P'') + |X''| - 2$ , because the trivial segment  $\{v_j, \widetilde{W}_j\}$  of  $P''$  belongs to  $X''$  and hence none of the exceptions in Lemma 4.1 applies to  $P''$  and  $X''$  (not even the second one, since obviously  $P'' \neq (1, 1, 1, 1)$ ). Analogously, the second exception in Lemma 4.1 is the only one that may apply to  $P'$  and  $X'$ . Hence, if the second exception does not apply either, we further conclude  $\min(P'_{X'}) \leq 2 \cdot \text{Min}(P') + |X'| - 2$  from Lemma 4.5, which gives  $\min(P_X) \leq 2 \cdot \text{Min}(P) + |X'| + |X''| - 4 = 2 \cdot \text{Min}(P) + |X| - 2$ .

Now assume that the second exception of Lemma 4.1 does apply. Then we can only conclude  $\min(P_X) \leq 2 \cdot \text{Min}(P) + |X| - 1$  at this point of the argumentation. Let  $w_1^*$  be the (unique) quadrilateral in the  $j$ th branch, and for some optimal decomposition of  $P''$ , let  $w_2^*, \dots, w_i^*$  be the quadrilaterals incident to the vertex between the two copies of  $e_j$  in  $P''$ . Let  $P'''$  be the polygon comprising  $w_1^*, \dots, w_i^*$ . It is easy to see that the current decomposition of  $P'''$  can be replaced by another decomposition such that at least one quadrilateral is saved by that. This proves the claim for Case II, too.  $\square$

The following lemma holds only for polygons with more than 9 edges. However, an easy although quite extensive case analysis for all types of polygons with up to 9 edges shows the correctness of Lemma 4.3 for such polygons. The corresponding details are omitted.

**Lemma 4.7** *If  $P_Y$  admits a perfect decomposition such that  $G^*$  is a path, then the following holds for all feasible  $X$  and for all polygons  $P$  with  $|E_P| \geq 10$ :*

$$\min(P_X) \leq \frac{5}{3} \cdot (\text{Min}(P) + \widetilde{W}_P(X)) + |X| - 2 \quad \text{if } |X| > 0, \quad (16)$$

$$\min(P_X) \leq \frac{5}{3} \cdot \text{Min}(P) - 1 \quad \text{if } |X| = 0. \quad (17)$$

**Proof:** The general idea of this proof follows that of Lemma 4.5. Let  $G'$  be a decomposition of  $P_Y$  such that  $G^*$  is a path, and let  $e_1$  and  $e_2$  be the leaves of the corresponding variant  $G^*$  on the dual graph where the vertex corresponding to the outer face is removed. Clearly,  $e_1$  and  $e_2$  are trivial segments of  $P$ , too, but not necessarily of  $P_X$ .

Note that the case  $X = Y$  is trivial, hence we assume in the following  $X \neq Y$ .

For  $P$ , let  $I_1 := I[e_1, e_2]$  and  $I_2 = I[e_2, e_1]$ . Let  $I'_1$  and  $I'_2$  denote the corresponding intervals of  $P_Y$  and  $I''_1$  and  $I''_2$  the corresponding intervals of  $P_X$ . Let  $|Y| := \sum_{e \in E_P} Y(e)$ . As  $G'$  is perfect, we have

$$q(G') = \frac{L(I'_1) + L(I'_2)}{2} = \frac{L(I_1) + L(I_2) + |Y|}{2}. \quad (18)$$

The decomposition we construct for  $P_X$  is denoted by  $G''$ . Denote by  $i(G'')$  the number of internal vertices in  $G''$ . Then we have

$$q(G'') = \frac{L(I_1) + L(I_2) + |X|}{2} + i(G''). \quad (19)$$

To prove Ineqs. (16) and (17), it suffices to show

$$i(G'') \leq \frac{L(I_1) + L(I_2) - 6}{3} + \frac{5}{6}|Y| + \frac{|X|}{2} + \frac{5}{3}\widetilde{W}_P(X) \quad \text{if } |X| > 0, \quad (20)$$

$$i(G'') \leq \frac{L(I_1) + L(I_2) - 3}{3} + \frac{5}{6}|Y| \quad \text{if } |X| = 0. \quad (21)$$

Note that  $|E_P| \geq 10$  implies  $L(I_1) + L(I_2) \geq 8$ .

Without loss of generality, we have  $L(I_1'') \geq L(I_2'')$ . Let

$$\delta := \min \left\{ K(I_1''), \left\lceil \frac{1}{2}[L(I_1'') - L(I_2'')] \right\rceil \right\}.$$

See again Fig. 11 for the different cases for  $G''$  provided  $\delta = 0$ .

Let  $X_1 := X \cap I_1$ ,  $X_2 := X \cap I_2$  and  $\bar{X} := X \cap \{e_1, e_2\}$ . Then we have  $X = X_1 \cup X_2 \cup \bar{X}$  and  $0 \leq |\bar{X}| \leq 2$ . Now we are going to consider the individual cases in Fig. 11.

*Case I:*  $(L - 2K)(I_1'') - L(I_2'') \leq 0$ .

If  $|\bar{X}| = 0$ , we have  $i(G'') = 0$ , and Ineqs. (20) and (21) are immediate. If  $|\bar{X}| = 1$ , then  $i(G'') \leq 1$ . As either  $|X| \geq 2$  or  $|X| = 1$  and  $|Y| \geq 1$ , the Ineq. (20) is fulfilled again. Now assume  $|\bar{X}| = 2$ . This implies  $i(G'') \leq 2$ . If  $|Y| = 0$ , then  $P$  is perfect, which means  $\widetilde{W}_P(X) \geq 1$  and  $|X| \geq 2$ . If  $|Y| = 1$ , then  $|X| \geq 3$ . Otherwise,  $|Y| \geq 2$  and  $|X| \geq 2$ . In all these cases, Ineq. (20) is fulfilled.

*Case II:*  $1 \leq (L - 2K)(I_1'') - L(I_2'') \leq 2$ .

Let  $\Delta := (L - 2K)(I_1'') - L(I_2'')$ .

*Subcase IIa:*  $|\bar{X}| = 0$  or  $|\bar{X}| = 2$ .

As  $L(I_1'') + L(I_2'')$  is even in this case, we have that  $\Delta = 0$  by parity arguments, and  $i(G'') = 2$  (cf. Figure 11). If  $|Y| = 0$ , then  $|X| \geq 2$  and  $\widetilde{W}_P(X) \geq 1$ , which implies Ineq. (20). Assume now that  $|Y| = 1$ . If  $|X| \geq 3$ , the Ineq. (20) is immediate. So let  $|X| = 1$ . As  $E_P$  is odd, we have  $L(I_1) + L(I_2) \geq 9$ . This suffices to yield Ineq. (20).

Assume next that  $|Y| = 2$ . If  $|X| = 0$ , then Ineq. (21) is fulfilled, because  $L(I_1) + L(I_2) \geq 6$ . Otherwise we have  $|X| \geq 2$ , and Ineq. (20) is certainly fulfilled.

Finally, Ineqs. (20) and (21) are immediate, if  $|Y| \geq 3$ .

*Subcase IIb:*  $|\bar{X}| = 1$ .

Now we have  $\Delta = 1$  and  $i(G'') = 1$  (cf. Figure 11). If  $|Y| = 0$ , then  $|X| \geq 2$ . Otherwise  $|Y| \geq 1$  and  $|X| \geq 1$ . In both cases, Ineq. (20) is immediate.

*Case III:*  $(L - 2K)(I_1'') - L(I_2'') \geq 3$ .

In the third row of Fig. 11, the number of quadrilaterals in the white area is  $L(I_2'')$  for  $|\bar{X}| \leq 1$  and  $L(I_2'') + 2$  for  $|\bar{X}| = 2$ . On the other hand, the grey area is decomposed according to Fig. 10. Recall that  $\delta = 0$  is assumed in Fig. 11. Let  $2k$  denote the number of horizontal edges below the grey area, after restoring the shrunken edges, if  $\delta > 0$ . Then we have

$$2k = (L - 2K)(I_1'') - L(I_2''), \quad \text{if } |\bar{X}| \text{ is even, and} \quad (22)$$

$$2k = (L - 2K)(I_1'') - L(I_2'') + 1, \quad \text{if } |\bar{X}| \text{ is odd.} \quad (23)$$

Assume first that  $L(I_1) < L(I_2)$ . Combining  $L(I_1'') = L(I_1) + |X_1|$  and  $L(I_2'') = L(I_2) + |X_2|$  with Eq. (22) and (23), respectively, yields  $2k \leq |X_1| - 1$ , if  $|\bar{X}|$  is even, and  $2k \leq |X_1|$  otherwise.

Assume now that  $L(I_1) \geq L(I_2)$ . By Lemma 3.7 (6), we can further assume that all additional vertices in  $Y$  (if any) are placed on  $I_2$ . This implies  $L(I_1) - 2K(I_1) \leq L(I_2) + |Y|$ . Therefore, we get

$$2k \leq |Y| + |X_1| - |X_2| + 2K(I_1) - 2K(I_1'') \leq |Y| + |X_1|,$$

if  $|\bar{X}|$  even, and  $2k \leq |Y| + |X_1| + 1$ , otherwise. (The last two inequalities are weaker than those for the case  $L(I_1) < L(I_2)$ , hence, we use them in the following.)

*Subcase IIIa:*  $|\bar{X}| = 0$ .

In this case, we have  $i(G'') \leq k + 1$  since  $k \geq 2$  (otherwise we were in Case I or II). Hence, to prove Ineqs. (20) and (21) it suffices to show

$$k + 1 \leq \frac{L(I_1) + L(I_2) - 6}{3} + \frac{5}{6}|Y| + \frac{|X|}{2} + \frac{5}{3}\widetilde{W}_P(X)$$

(the last two terms vanish if  $|X| = 0$ ). As  $k \leq \frac{|Y| + |X|}{2}$ , we only need to show

$$1 \leq \frac{L(I_1) + L(I_2) - 6}{3} + \frac{1}{3}|Y| + \frac{5}{3}\widetilde{W}_P(X). \quad (24)$$

If  $|Y| = 0$ , then  $P$  is perfect and  $\widetilde{W}_P(X) \geq 1$ . If  $|Y| = 1$ , then  $E_P$  is odd, and hence  $L(I_1) + L(I_2) \geq 9$ . Otherwise,  $|Y| \geq 2$ . As  $L(I_1) + L(I_2) \geq 8$ , these facts imply Ineq. (24) in any case.

*Subcase IIIb:*  $|\bar{X}| = 1$ .

Now we have  $i(G'') \leq k + 1$ , and  $k \leq \frac{|Y| + |X_1| + 1}{2}$ . Therefore, it suffices to show

$$\frac{3}{2} \leq \frac{L(I_1) + L(I_2) - 6}{3} + \frac{1}{3}|Y| + \frac{5}{3}\widetilde{W}_P(X) + \frac{\bar{X}}{2}. \quad (25)$$

Exactly the same case distinction as in Subcase IIIa shows the validity of Ineq. (25).

*Subcase IIIc:*  $|\bar{X}| = 2$ .

In this subcase, we have  $i(G'') \leq k + 2$ . Hence, it suffices to show

$$2 \leq \frac{L(I_1) + L(I_2) - 6}{3} + \frac{1}{3}|Y| + \frac{5}{3}\widetilde{W}_P(X) + \frac{\bar{X}}{2}. \quad (26)$$

As  $\frac{\bar{X}}{2} = 1$ , the latter inequality reduces to Ineq. (24), and so holds for the same reasons as those given in Subcase IIIa.  $\square$

For a decomposition  $G$  of the polygon  $P$ , we have introduced in Sec. 3 the notion of an area decomposition of  $G$  into cut and core components. If the decomposition of  $P$  is not completed, we take a different view on the cut components. We say that a *cut-operation* is applied to the polygon  $P$ , if one of the subgraphs illustrated in Fig. 6 and 7 is used to *reduce* the given polygon  $P$  to a polygon  $P'$ . In particular, operation (c) is called a *1-cut*, whereas operations (d) and (e) are *2-cuts*. Finally, the cuts described in Fig. 7 are called *(2k, 1, 1)-cuts*.

**Lemma 4.8** *If  $P_Y$  is perfect, but does not admit a perfect decomposition such that  $G^*$  is a path, then the following holds for all feasible  $X$ :*

$$\min(P_X) \leq \frac{3}{2} \min(P_Y) + |X| - 2. \quad (27)$$

**Proof:** Because of Lemma 3.3,  $P_Y$  has a perfect decomposition  $G$  such that  $G^*$  is a subdivision of  $K_{1,3}$ . Lemma 3.7 (item 5) implies that  $|Y| \leq 1$ . We only need to consider that  $|E_P| \geq 15$ , as otherwise  $P_Y$  would also admit a perfect decomposition such that  $G^*$  is a path or  $P = (4, 1, 4, 1, 3, 1)$  or  $P = (3, 1, 3, 1, 3, 1)$ . The latter two cases are easily checked by hand.

Denote by  $G'$  the graph of an optimal decomposition for  $P_X$ . As  $q(G') = \frac{|E_P| + |X|}{2} - 1 + i(G')$  and  $q(G) = \frac{|E_P| + |Y|}{2} - 1$ , it suffices to show

$$i(G') \leq \frac{|E_P| - 10}{4} + \frac{|X|}{2} + \frac{3}{4}|Y|. \quad (28)$$

If  $|X| = 0$ , then also  $|Y| = 0$ , as otherwise  $|Y| \geq 2$ . This implies  $P_X \equiv P_Y$ , and  $i(G') = 0$ . Therefore, Ineq. (28) holds in this case.

So assume that  $|X| > 0$ . The idea of the remaining part of this proof is constructive and works as follows: We start from  $P_Y$  and its perfect decomposition  $G$  and insert step by step an even number of additional points from  $X$ . Denote the polygon in step  $i$  by  $P_i$ . We continue until we end up with  $P_f = P_X$ , for some  $f$ . (If  $|Y| = 1$ , we possibly remove the corresponding vertex of  $P_Y$  in the very first step and select only one vertex from  $X$ .) In each step, we rebuild a decomposition for the intermediate polygon  $P_i$  and call it  $G_i$ . The final decomposition  $G_f$  for  $P_X$  might be suboptimal, but will satisfy  $i(G_f) \leq \frac{|E_P| - 10}{4} + \frac{|X|}{2} + \frac{3}{4}|Y|$ , which clearly suffices.

Let  $e_1, e_2$ , and  $e_3$  be the three trivial segments which correspond to the leaves of the subdivision of  $K_{1,3}$  such that the counterclockwise order around  $P_Y$  is  $e_1 \prec e_2 \prec e_3 \prec e_1$ . Let  $I_1 := I[e_1, e_2]$ ,  $I_2 := I[e_2, e_3]$ ,  $I_3 := I[e_3, e_1]$ .

Note that we may assume that each vertex of  $P_Y$  which is not incident to one of the leaves  $e_1, e_2$ , and  $e_3$  has a degree of three or four in  $G$ .

Let us first consider the case, that one of the leaves, say  $e_1$ , is subdivided by  $X$ , i.e.  $X(e_1) = 1$ . If  $|Y| = 1$ , then we take as  $P_1$  the polygon  $P$  plus one additional point on  $e_1$ . Obviously,  $P_1$  is an even polygon. We build a decomposition for  $P_1$  as follows. A cut operation of type (c) in Fig. 6 is applied to the segment of length two which arises from the trivial segment  $e_1$  by the insertion of the additional point. This reduces  $P_1$  to the polygon  $P'_1$ . Using the decomposition of  $P_Y$  into a subdivision of  $K_{1,3}$ , it is now easy to see that  $P'_1$  is perfect and allows for a perfect decomposition with leaves among  $e_2, e_3$  and one of the newly inserted internal edges. This perfect decomposition may either be a subdivision of  $K_{1,3}$  or a path, but it maintains the property, that each vertex which is not incident to one of the corresponding leaves has a degree of at least three.

If  $|Y| = 0$ , then there must be another edge which is subdivided by  $X$ . If another leaf, say  $e_2$ , is subdivided, then we take as  $P_1$  the polygon  $P$  plus one additional point on  $e_1$  and one on  $e_2$ . In this case, we apply two cut operations of type (c) in Fig. 6, one for each segment of length two created by the insertion of the additional points at  $e_1$  and  $e_2$ . Similarly as in the previous case, this reduces  $P_1$  to the polygon  $P'_1$ , and  $P'_1$  is perfect. It allows for a perfect decomposition with leaves among  $e_3$  and the newly created internal edges.

If no other leaf is subdivided, we just choose one additional point of  $X$  placed on  $I_1, I_2$  or  $I_3$ . This again gives us a polygon  $P_1$ . One cut operation of type (c) in Fig. 6 suffices in this case, and the reduced polygon  $P'_1$  is perfect.

It might be the case that there are still one, say  $e_3$ , or two leaves, say  $e_2$  and  $e_3$ , which are subdivided by  $X$ . Then the analogous cases of step one are repeated in the second step.

Now we may assume that all points of  $X$  corresponding to leaves in  $G$  have been inserted. If  $|Y| = 1$  but no leaf has been subdivided by  $X$ , then we are still in step one. Then we may select any additional point from  $X$  to create  $P_1$ . It is easy to see that  $P_1$  is still perfect and has a decomposition with leaves among  $e_1, e_2$  and  $e_3$ .

Otherwise, we select, if existing, a pair of additional points from  $X$ , one from  $I_i$  and one from  $I_j$ , for  $i \neq j$ . This gives a new polygon  $P_{k+1}$ . It obviously has a decomposition  $G_{k+1}$  with just one more quadrilateral than  $G_k$ , and its core component has the same leaves as before. Hence, we can continue in this way until all additional points from  $X$  which have not been introduced so far belong to the same interval, say to  $I_1$ .

The number of remaining points must be even, say  $2c$ . In the final step, we introduce these  $2c$  points all at once. But now we can apply a  $(2c, 1, 1)$ -cut operation as depicted in Fig. 7, to reduce the interval  $I_1 \cup X_1$  by the same length as it was enlarged by the insertion of additional points. Thus, we get a decomposition for  $G_f$  which uses  $c + 1$  more internal vertices than  $G_{f-1}$ .

In summary, we have built a decomposition for  $P_X$  which uses exactly these  $c + 1$  internal vertices plus as many internal vertices as cut operations of type (c). If a cut operation of type (c) and a  $(2c, 1, 1)$ -cut are applied to the same segment of  $P_X$ , this is clearly suboptimal. Hence, we can save one internal vertex in these cases. This means that there is always a decomposition for  $P_X$  with no more than  $c + 2$  internal vertices, if  $c > 0$ , and no more than three internal vertices, otherwise. As  $|E_P| \geq 15$  and  $2c \leq |X_1|$ , this clearly suffices to prove Ineq. (28).  $\square$

**Lemma 4.9** *If  $P_Y$  is not perfect, then the following holds for all feasible  $X$ :*

$$\min(P_X) \leq \frac{3}{2} \min(P_Y) + |X| - 2, \quad (29)$$

unless  $P \in \{(1, 1, 1), (2, 1, 1), (2, 2, 1), (4, 2, 1, 1)\}$ .

**Proof:** The cases  $P_Y \in \{(4, 2, 2), (4, 3, 3), (3, 3, 3, 3)\}$  can be checked by hand. For all other cases, we can apply Lemma 3.7.

Denote by  $G$  the graph of an optimal decomposition of  $P_Y$ , and by  $G'$  the graph of an optimal decomposition for  $P_X$ . As  $q(G') = \frac{|E_P| + |X|}{2} - 1 + i(G')$  and  $q(G) = \frac{|E_P| + |Y|}{2} - 1 + i(G)$  it suffices to show

$$i(G') \leq \frac{|E_P| - 10}{4} + \frac{|X|}{2} + \frac{3}{4}|Y| + \frac{3}{2}i(G). \quad (30)$$

The general idea of the proof is very similar to that of Lemma 4.8. The main difference is that now  $|Y| > 1$  is possible. If  $|Y| > 1$ , we first construct a decomposition for  $P$  with a new  $Y'$  for which  $|Y'| \leq 1$  holds, and  $\min(P_{Y'}) = \min(P_Y)$ . Then, we proceed as in the proof of Lemma 4.8, i.e. we insert in several steps points from  $X$  and rebuild a decomposition each time.

So assume first that  $|Y| \geq 2$ . Denote by  $P'$  the outer face of the core component of  $G$ . By Lemma 3.7, items (5) and (6), the core component of the decomposition is a path in  $G^*$  with leaves  $e_1$  and  $e_2$ , say, and all vertices in  $Y \cap P'$  belong to the same interval of  $P'$ , say to  $I_1 := I[e_1, e_2]$ . Note that  $Y \setminus P'$  can only be non-empty if there is a cut component of type (d). But then an original vertex from  $I_1$  and such an additional one can exchange their roles (as being original and additional, respectively) unless  $I_1$  contains only additional vertices. In the latter case all additional vertices belong to the same segment of  $P$ .

Let  $\mathcal{K}$  be the set of corners of  $I_2 := I[e_2, e_1]$  which has degree 2 in the path decomposition. We construct  $P''$  from  $P$  by shrinking each edge that is incident to a vertex in  $\mathcal{K}$ . This shrinking operation may be necessary for one reason: in some cases, we later want to apply a  $2k$ -cut operation with a base segment on  $I_2$ . Without shrinking we possibly reduce by that operation

the available set of corners on  $I_2$  which are not neighbored. Hence, in such cases it might be impossible to reconstruct a decomposition with the same number of quadrilaterals.

Furthermore, we build a decomposition for  $P''$  as follows. Let  $k = \lfloor \frac{|Y|}{2} \rfloor$ . Select a cut operation which has been applied in the decomposition for  $P_Y$ . It must be of type (c), (d) or (e) in Fig. 6. With respect to  $P''$ , this cut operation is replaced (at the corresponding corner) by a  $2k$ -cut if it was of type (c), and by a  $(2k+2, 1, 1)$ -cut otherwise. If a second cut operation has been applied to  $P_Y$ , it is also applied to  $P''$ . If  $|Y|$  is odd, we place one additional point arbitrarily on some edge of  $I_1$ . This defines  $Y'$ . If  $|Y|$  is even, we choose  $Y' \equiv 0$ . In any case, we have  $|Y'| \leq 1$ .

Clearly, the remaining part of  $P''_{Y'}$  has a path decomposition, and reversing the shrinking process from  $P''_{Y'}$  back to  $P_{Y'}$  yields a decomposition for  $P_{Y'}$  with no more quadrilaterals than in the decomposition for  $P_Y$  (we have only exchanged  $2k$  vertices on the boundary of  $P_Y$  by  $k$  more internal vertices in  $P_{Y'}$ ).

Hence, it suffices to show Ineq. (30) for the case that  $|Y| \leq 1$ . If the core component of  $G$  only has a decomposition as a subdivision of  $K_{1,3}$ , then Ineq. (30) has already been proven in Lemma 4.8.

However, the case that the core component of  $G$  has a path decomposition can be handled in just the same way as in Lemma 4.8. The only slight difference is that we first apply the same shrinking operation to  $P$  as above in the case for  $|Y| > 1$  (and for the reasons given there), and reverse this operation in the very end. All further details are easy to see.  $\square$

## 5 Approximation of Minimal Conformal Mesh Refinements

In this section, we describe the improved approximation algorithms. In the following, we will need a certain variant  $G^d = (V^d, E^d)$  on the dual graph of the graph  $G = (V, E)$  of a homogeneous mesh  $\mathcal{M}$ .

**Definition 5.1** *For a homogeneous mesh  $\mathcal{M}$  and its corresponding graph  $G = (V, E)$ , the graph  $G^d = (V^d, E^d)$  (multiple edges allowed) has a vertex for each polygon of  $\mathcal{M}$ . If the homogeneous mesh  $\mathcal{M}$  has a non-empty boundary, exactly one more vertex is added to  $V^d$ . Two dual vertices that correspond to polygons of  $\mathcal{M}$  are connected by an edge of  $E^d$  if and only if they share an edge in  $E$ . For each boundary edge (if existing) the additional vertex is connected with the dual vertex whose corresponding polygon is incident to the boundary.*

We will need the next fact, Lemma 5.2, for all algorithmic results to follow. Let  $G = (V, E)$  be an undirected graph, and let each vertex be either labeled *odd* or labeled *even*. This *odd/even* labeling is called *feasible* if the number of vertices labeled *odd* is even. A subgraph  $G'$  of a graph  $G$  is called *feasible* if the following holds: The degree of a vertex is odd in  $G'$  if and only if its label is “*odd*.” In the literature, such a subgraph is often called a *T-join* (see, for example [Ger95]).

**Lemma 5.2** *There is a linear-time algorithm that produces a feasible acyclic subgraph  $F$  of a connected graph  $G$  with respect to a feasible odd/even labeling.*

**Proof:** Let  $T$  be a spanning tree of  $G$  and let  $F$  be the forest comprising all edges  $e$  of  $T$  that divide  $T - e$  into two subtrees, each with an odd number of vertices labeled *odd*. It is easy to see that  $F$  is feasible.  $\square$

**Theorem 5.3** *There is a linear-time algorithm that constructs a conformal refinement of a mesh  $G$  such that the number of quadrilaterals exceeds twice the optimum by at most  $\Delta(\mathcal{M})$ .*



**Proof:** First we describe the algorithm and prove that it constructs a conformal refinement. Recall from Lemma 1.3 that we have to ensure that every polygon becomes even. For each folding that consists of an odd number of folding edges, we select one of these edges and subdivide it once. We will see that this suffices to refine all homogeneous components separately. So let  $G_i = (V_i, E_i)$  be a homogeneous component. First consider the edges in  $E_i$  that have degree one but are not folding edges of the original mesh. In other words, consider the the boundary edges of  $G$  which belong to  $G_i$ . If the number of these edges is odd, we select exactly one of these edges and subdivide it once, too. After this procedure, the number of edges of degree one in  $E_i$  is even. (Note that all other edges have degree 2, because  $G_i$  is homogeneous.)

Let  $G_i^d = (V_i^d, E_i^d)$  denote the variant on the dual graph of  $G_i$  as in Definition 5.1. Then the vertex of  $G_i^d$  added for the boundary edges has even degree in  $G_i^d$ . Therefore, the number of odd vertices of  $G_i^d$  that correspond to polygons (and hence the number of odd polygons themselves) in  $G_i$  is even. Consequently, we may apply the algorithm of Lemma 5.2 to construct a feasible acyclic subgraph  $F_i$  of  $G_i^d$ , where a vertex is labeled *odd/even* according to the parity of its degree. Next each edge of  $E_i$  that corresponds to an edge in  $F_i$  is subdivided exactly once. Obviously, every polygon is now even, and we apply the algorithm from [MW96] to decompose each polygon separately.

It remains to show that this construction leads to a refinement that exceeds twice the optimum by at most  $\Delta(\mathcal{M})$ .

For a polygon  $P$  of the homogeneous component  $G_i$ ,  $i = 1, \dots, k$ , let  $X_P : E_P \rightarrow \{0, 1\}$  be defined such that  $X_P(e) = 1$  means that  $e$  corresponds to a dual edge of  $F_i$ . Analogously, let  $Y_P : E_P \rightarrow \{0, 1\}$  attain 1 exactly on the edges of  $P$  that are selected in the algorithm to make all foldings even. Moreover, let  $Z_P : E_P \rightarrow \{0, 1\}$  attain 1 on an edge if and only if this edge is selected to make the number of boundary edges of  $G_i$  even. Let  $P' := P_{(X_P + Y_P + Z_P)}$ . Of course, we have  $X_P(e) + Y_P(e) + Z_P(e) \leq 1$  for each edge  $e$ . Therefore, Lemma 4.1 gives

$$\min(P') \leq 2 \cdot \text{Min}(P) + |X_P| + |Y_P| + |Z_P| - 1 . \quad (31)$$

Let  $\mathcal{P}_i$  denote the set of polygons in  $G_i$ . Since the feasible subgraph  $F_i$  of  $G_i^d$  constructed by the algorithm is acyclic, we have  $\sum_{P \in \mathcal{P}_i} |X_P| \leq 2 \cdot (|\mathcal{P}_i| - 1)$ , and since  $\sum_{P \in \mathcal{P}_i} |Z_P| \leq 1$ , Ineq. (31) sums up to

$$\sum_{P \in \mathcal{P}_i} \min(P') \leq \sum_{P \in \mathcal{P}_i} \left[ 2 \cdot \text{Min}(P) + |Y_P| \right] + |\mathcal{P}_i| - 1 . \quad (32)$$

Note that  $\sum_{i=1}^k \sum_{P \in \mathcal{P}_i} |Y_P| = D(\mathcal{M})$ . Hence, Ineq. (32) sums up to

$$\sum_{P \in \mathcal{P}} \min(P') \leq 2 \cdot \sum_{P \in \mathcal{P}} \text{Min}(P) + D(\mathcal{M}) + |\mathcal{P}| - k = 2 \cdot \sum_{P \in \mathcal{P}} \text{Min}(P) + \Delta(\mathcal{M}) .$$

□

As an immediate consequence, we obtain for the special cases of meshes without foldings of odd degree, i.e. where  $D(\mathcal{M})$  vanishes, the following corollary:

**Corollary 5.4** *There is a linear-time algorithm that yields a 3-approximation for the minimum conformal refinement problem for the special cases where  $D(\mathcal{M}) = 0$ . This includes, in particular, the homogeneous meshes.*

For meshes without foldings, we can even find significantly better approximations using a nice application of matching techniques:

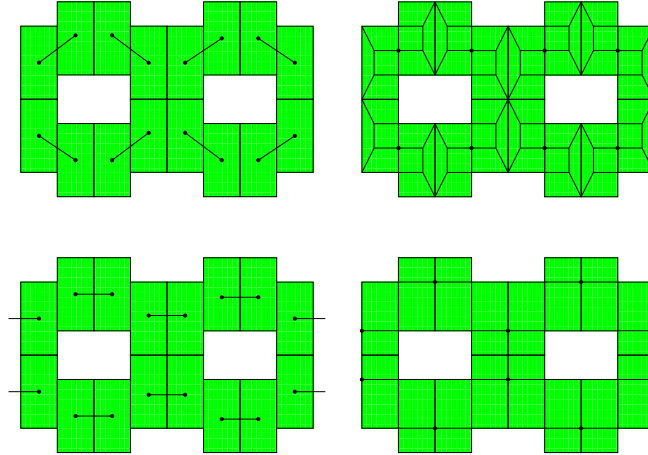


Figure 12: A class of instances where the feasible subgraph with the minimum number of edges (top, left) only yields a 2-approximation (top, right). The optimal solution (bottom, right) corresponds to a larger feasible subgraph (bottom, left).

**Theorem 5.5** *For homogeneous meshes, there is an  $\mathcal{O}(nm \log n)$  algorithm that constructs a conformal refinement with a performance guarantee of 1.867.*

Like in the proof of Theorem 5.3, we construct a feasible acyclic subgraph  $F$ , but now we use penalty functions to find subgraphs which allow for a better analysis. The idea is to choose edge weights in such a way that we get an improved lower bound if some of the expensive edges are chosen in a feasible subgraph of minimum weight. Note that, in general, it is not true that a feasible subgraph with a smaller number of edges gives a better result. Fig. 12 shows an example where the feasible subgraph with the minimum number of edges only yields a 2-approximation.

We determine  $F$  in an auxiliary graph  $G_{\text{aux}}^d = (V_{\text{aux}}^d, E_{\text{aux}}^d)$ . The graph  $G_{\text{aux}}^d$  is constructed from  $G^d$  as follows. Each polygon  $v^d \in V^d$  of one of the types in Fig. 9 is replaced by a couple of vertices and edges, which respectively form subgraphs as shown in Fig. 9. Each edge  $e \in E_{\text{aux}}^d$  is assigned a weight  $w(e^d)$ , derived from the penalty functions introduced in Sec. 4. If an edge  $e^d \in E_{\text{aux}}^d$  corresponds to an edge  $e$  which belongs to two polygons  $P_1$  and  $P_2$ , then the penalty functions for both polygons contribute to the weight  $w(e^d)$ : we just take the sum of the corresponding weights. The contribution to the weights  $w(e^d)$  of the edges in Fig. 9 are introduced there, too. If a polygon  $P$  is perfect, then all edges  $e^d$ , for which  $e$  belongs to  $P$ , have a contribution of  $1/2$  to the weight of  $w(e^d)$ . For all polygons  $P$  which are perfect,  $|E_P|$  is odd and which are not contained in Fig. 9, the  $G_{\text{aux}}^d$  is slightly modified. For each such polygon with corresponding dual vertex  $v^d$ , we add a new vertex  $v_a^d$  and an edge  $e_a^d = (v^d, v_a^d)$  with weight  $w(e_a^d) = -\frac{1}{2}$ . Observe that this realizes the penalty functions for perfect polygons as defined in Sec. 4. (Recall that we can test for perfectness of a polygon in linear time.) All other edges  $e^d \in E_{\text{aux}}^d$  have weight  $w(e^d) = 0$ .

We say that a subgraph  $F_{\text{aux}}$  of  $G_{\text{aux}}^d$  is feasible if the degree of every vertex outside these four kinds of subgraphs has the same parity in  $G_{\text{aux}}^d$  and  $F_{\text{aux}}$ , and all vertices inside these subgraphs have even degree in  $F_{\text{aux}}$  except for the vertex indicated by an arrow in Fig. 9, which must have odd degree in  $F_{\text{aux}}$ . The vertices  $v_a^d$  introduced for perfect polygons with  $|E_P|$  odd must have odd degree, whereas  $v^d$  must have even degree.

So far, this leads only to a performance guarantee with a factor of 2:

**Lemma 5.6** *Let  $F_{\text{aux}}$  be a feasible subgraph of  $G_{\text{aux}}^d$  such that the sum of all weights  $w(\cdot)$  of edges in  $F_{\text{aux}}$  is minimum. Let  $F$  be a feasible acyclic subgraph of  $G^d$  such that  $F$  is constructed from  $F_{\text{aux}}$  by shrinking all subgraphs in Fig. 9 and removing all cycles from the shrunk  $F_{\text{aux}}$ . Then subdividing once all edges of  $G$  that correspond to dual edges in  $F$  yields a 2-approximation.*

**Proof:** First observe that there are no negative cycles in  $G_{\text{aux}}^d$  with respect to the edge weights  $w(\cdot)$ . This is easy to see from Fig. 9.

Denote by  $w(F_{\text{aux}})$  the total weight of an optimal  $F_{\text{aux}}$ . We claim that  $\sum_{P \in \mathcal{P}} \text{Min}(P) + w(F_{\text{aux}})$  is a lower bound on the number of quadrilaterals in any feasible mesh refinement.

To see this, take an optimal conformal refinement. For  $e \in E$ , let  $Y(e)$  denote how often  $e$  is subdivided in this refinement. Let  $Y'(e) \in \{0, 1\}$  be the remainder of  $Y(e)/2$ . Then subdividing each edge  $e \in E$  exactly  $Y'(e)$  times makes all polygons even.

Hence,  $Y'$  corresponds to a feasible subgraph  $F'_{\text{aux}}$  in  $G_{\text{aux}}^d$ . As  $w(F_{\text{aux}})$  is minimum, we have  $w(F_{\text{aux}}) \leq w(F'_{\text{aux}})$ . The edge weights are derived from penalty functions. Thus, we have for each polygon  $\text{Min}(P) + \widetilde{W}_P(Y') \leq \min(P_Y)$ , because  $Y' \leq Y$ . Summing up over all polygons, establishes the lower bound  $\text{low} := \sum_{P \in \mathcal{P}} \text{Min}(P) + w(F_{\text{aux}})$ .

For a polygon  $P \in \mathcal{P}$ , let  $X_P : E_P \rightarrow \{0, 1\}$  denote which edges are subdivided in the refinement induced by  $F$ . Now it suffices to show

$$\sum_{P \in \mathcal{P}} \min(P_{X_P}) \leq 2 \cdot \sum_{P \in \mathcal{P}} (\text{Min}(P) + \widetilde{W}_P(X_P)) .$$

We distinguish between two cases: If  $|X_P| = 0$ , then we have  $\min(P_X) \leq 2 \cdot \text{Min}(P)$ , by Lemma 4.3. Otherwise, we have  $\min(P_X) \leq 2 \cdot (\text{Min}(P) + \widetilde{W}_P(X_P)) + |X_P| - 2$ .

Hence, we get

$$\sum_{P \in \mathcal{P}} \min(P_{X_P}) \leq 2 \cdot \sum_{P \in \mathcal{P}} (\text{Min}(P) + \widetilde{W}_P(X_P)) + \sum_{P \in \mathcal{P}, |X_P| > 0} (|X_P| - 2).$$

As  $F$  is a forest on  $|\{P \in \mathcal{P}, |X_P| > 0\}|$  vertices, the sum  $\sum_{P \in \mathcal{P}, |X_P| > 0} |X_P|$  cannot exceed that number.

This implies the 2-approximation.  $\square$

With a slight modification of the procedure which led to the 2-approximation, we can further improve our approximation guarantee. The key observation is that we could not fully exploit what we have showed in Lemma 4.3 because of polygons of type  $P = (1, 1, 1)$ . If the mesh has no such triangles then Lemma 4.3 would imply a  $\frac{5}{3}$ -approximation. The idea, therefore, is to treat triangles in a special way, namely, we glue single triangles which are neighbored to larger components. More precisely, two triangles belong to the same *triangle component* if there is a path in  $G^d$  which contains only vertices which correspond to triangles or to the special boundary vertex. Roughly speaking, such triangle components are treated as if they were single polygons. The notion of a feasible mapping  $X$ , defined on the edge set of a polygon, is extended in the obvious way to the boundary edges of a triangle component.

For our algorithm, the modification is very simple: In the auxiliary graph  $G_{\text{aux}}^d$ , which has been built up as before, we repeatedly identify vertices by contraction of edges in  $G_{\text{aux}}^d$ . An edge is contracted if and only if both its endpoints are among the vertices which correspond to triangles, the special vertex corresponding to the boundary and those which have previously been identified in the process of repeated contractions.

In the modified auxiliary graph  $\tilde{G}_{\text{aux}}^d$  we seek for a feasible subgraph  $F_{\text{aux}}$  of minimum weight as before. Given  $F_{\text{aux}}$ , we also shrink all subgraphs in Fig. 9 and remove all cycles from the shrunken  $F_{\text{aux}}$ . Then subdividing once all edges of  $G$  that correspond to dual edges in  $F$  makes the boundary of all triangle components and all other polygons even. For each triangle component, we afterwards use the linear-time algorithm of Lemma 5.2 to extend the feasible subgraph to the whole mesh. If an edge between two triangles is not subdivided by that procedure so far, then we subdivide such an edge twice, as long as none of these two triangles possesses a different edge which has been subdivided twice in that process.

**Lemma 5.7** *The modified procedure yields a performance guarantee of 1.867.*

**Proof:** As in the proof of Lemma 5.6, we obtain a lower bound of

$$low := \sum_{P \in \mathcal{P}} \text{Min}(P) + w(F_{\text{aux}}).$$

Consider a triangle component  $TC$ . Denote by  $q(TC_X)$  the number of quadrilaterals used for the triangle component. Recall that  $X$  is first extended to all triangles of that component by the algorithm of Lemma 5.2, and second, for edges between triangles with  $X(e) = 0$ , we change  $X$  and set  $X(e) = 2$ , if both triangles still have a different edge with  $X(e') = 0$ . Furthermore, let  $\text{Min}(TC)$  denote the number of quadrilaterals used for the triangle component in an optimal refinement. Certainly, we always have  $q(TC_X) \leq \frac{5}{3}\text{Min}(TC)$ .

We claim that we even have

$$q(TC_X) \leq \frac{5}{3}\text{Min}(TC) + |X_{TC}| - 2, \quad \text{if } |X_{TC}| > 0,$$

except for the cases of odd triangle components which are single triangles. Clearly, we have  $\text{Min}(TC) = 3|TC|$ , and  $q(TC_X) \leq 5|TC|$ . Hence, the claim is trivially fulfilled, if  $|X_{TC}| \geq 2$ . Thus, assume  $|X_{TC}| = 1$  and that the triangle component consists of more than just one single triangle. Then, either there is a triangle in that component which is refined to a polygon of type  $(2, 2, 2)$  or there must be an edge for which  $X(e)$  has been changed to 2. The latter means that this triangle component contains a triangle which has been refined to a polygon of type  $(3, 2, 1)$ . In both cases, we need strictly less than  $5|TC|$  quadrilaterals for such a component.

For a single triangle  $T$ , however, we have that  $\min(T_{X_T}) \leq \frac{5}{3}\text{Min}(T) + |X_T| - 1$  holds. Let  $|\mathcal{T}|$  be the number of triangle components which are single triangles.

For all other polygons, we distinguish between two cases: If  $|X_P| = 0$ , then we have  $\min(P_X) \leq \frac{5}{3}\text{Min}(P)$ , by Lemma 4.3. Otherwise, we have  $\min(P_X) \leq \frac{5}{3}(\text{Min}(P) + \widetilde{W}_P(X_P)) + |X_P| - 2$ .

Hence, we get

$$\sum_{P \in \mathcal{P}} \min(P_{X_P}) \leq \frac{5}{3} \sum_{P \in \mathcal{P}} (\text{Min}(P) + \widetilde{W}_P(X_P)) + \sum_{P \in \mathcal{P}, |X_P| > 0} (|X_P| - 2) + |\mathcal{T}|.$$

As  $F$  is a forest on as many as  $|\{P \in \mathcal{P}, |X_P| > 0\}|$  vertices, the sum  $\sum_{P \in \mathcal{P}, |X_P| > 0} |X_P|$  cannot exceed twice that number.

So far, this yields a performance guarantee of  $\frac{5}{3} + \frac{|\mathcal{T}|}{low}$ . Thus, it suffices to show  $\frac{|\mathcal{T}|}{low} \leq \frac{1}{5}$  in order to get the claimed performance guarantee of  $\frac{28}{15} < 1.867$ .

Therefore, we are going to express  $low$  in terms of  $|\mathcal{T}|$ . Consider an optimal mesh refinement with the feasible subgraph  $F_{opt}$  and let  $Y_P$  be the corresponding subdivision for each polygon. Denote by  $T_1$  the number of single triangle components for which  $|Y_P| = 1$ , and by  $T_2$  the number of triangles with  $|Y_P| \geq 3$ . This means that the optimal refinement needs at least  $5T_1 + 3T_2$  quadrilaterals to refine all odd triangle components.

Moreover, the optimal subdivision induces in total  $4T_1 + 6T_2$  edges on the boundary of all single triangle components in an optimal refinement. As all odd triangle components are by definition isolated, all these edges also belong to polygons which are not triangles. Hence, there must be at least  $4T_1 + 6T_2$  edges in the refinement which belong to the boundary of such polygons.

Denote by  $\mathcal{P}_1$  the polygons with degree zero in  $F_{opt}$  (i.e. the polygons with  $P_Y = P$ ), and by  $\mathcal{P}_2$  all other polygons or odd triangle components which are not single triangles. Suppose that  $k$

of the  $4T_1 + 6T_2$  edges belong to polygons in  $\mathcal{P}_1$ , and denote this edge set by  $E_1$ . In particular, this implies  $k \leq 2T_1$ .

A polygon  $P_i \in \mathcal{P}_1$  with  $k_i$  edges from  $E_1$  and  $\ell_i$  other edges needs at least  $\frac{k_i + \ell_i}{2} - 1$  quadrilaterals in any refinement. (Here, we use that for each polygon  $P$ , the lower bound  $\frac{|E_P|}{2} - 1 \leq \text{Min}(P)$  holds.) As each polygon in  $\mathcal{P}_1$  has at least 4 edges, this implies that  $\frac{k_i + \ell_i}{2} - 1 \geq \frac{k_i}{4}$ . In total, we need at least  $\frac{k}{4}$  quadrilaterals for the polygons in  $\mathcal{P}_1$ .

Furthermore, there are  $4T_1 + 6T_2 - k$  edges which belong to polygons in  $\mathcal{P}_2$ . If a polygon is in  $\mathcal{P}_2$ , then it must have at least 6 edges. Hence, similarly as in the other case, we obtain  $\frac{4T_1 + 6T_2 - k}{3}$  as a lower bound on the number of quadrilaterals used for polygons in  $\mathcal{P}_2$ .

Summing up, we finally get

$$\text{low} \geq 5T_1 + 3T_2 + \frac{k}{4} + \frac{4T_1 + 6T_2 - k}{3} \geq 5(T_1 + T_2) = 5|\mathcal{T}|,$$

which finishes the proof.  $\square$

**Proof of Theorem 5.5:** Because of Lemma 5.7, it remains to show how to construct an optimal feasible graph  $F_{\text{aux}}$ . We solve this problem by a reduction to a *capacitated minimum-cost perfect  $b$ -matching problem* [Der88]. In order to introduce this reduction, we first state the problem we want to reduce in more general terms. So let  $G = (V, E)$  be an undirected graph, let  $w(\cdot)$  be a weighting of  $E$ , and for  $v \in V$  let  $\text{equal}(v)$  be a logical flag. We call a subgraph  $F$  of  $G$  feasible if the following holds: The degree of each  $v \in V$  in  $F$  has the same parity as the degree in  $G$  if and only if  $\text{equal}(v)$  is true. The problem is to find a feasible subgraph that minimizes the sum of the edge weights  $w(\cdot)$ .

The reduction is as follows (and was first proposed by Edmonds and Johnson [EJ73]). For  $v \in V$ , let  $b(v)$  equal the degree of  $v$  if  $\text{equal}(v)$  is true, otherwise let  $b(v)$  equal the degree plus one. Let  $\bar{G} = (V, \bar{E})$  denote  $G$  with all loops  $\{v, v\}$ ,  $v \in V$ , added to  $E$ . The weight of such a loop is  $w(\{v, v\}) := 0$ . Moreover, we set  $\ell(e) := 0$  for all  $e \in \bar{E}$ ,  $u(e) := 1$  for  $e \in E$ , and  $u(\{v, v\}) := \lfloor b(v)/2 \rfloor$  for  $\{v, v\} \in \bar{E} \setminus E$ .

There is a straightforward one-to-one correspondence between feasible subgraphs of  $G$  and perfect  $b$ -matchings in  $\bar{G}$  with lower bounds  $\ell(\cdot)$  and upper bounds  $u(\cdot)$ . Moreover, the cost of a  $b$ -matching with respect to  $w(\cdot)$  equals the sum of edge weights of the corresponding feasible subgraph.

Note that the graph of the  $b$ -matching instance is essentially as dense as  $G$ , i.e. it has  $\mathcal{O}(m)$  edges. Apart from pathological constructions, we even have  $m = \mathcal{O}(n)$ . In particular, this is always the case, if the number of corners of each polygon is bounded by some constant. Hence, in such cases, the  $b$ -matching instance runs on  $\mathcal{O}(n)$  edges. However, the underlying graph is not planar, in general.

This establishes the reduction, and by an application of Gabow's [Gab83] algorithm the time bound claimed in the theorem follows.  $\square$

If the graph  $G$  of the homogeneous mesh  $\mathcal{M}$  is planar, the running time of the minimum  $T$ -join algorithm can be slightly improved by an application of the famous planar separator theorem of Lipton and Tarjan.

**Theorem 5.8 (planar separator) [LT79]** *Let  $G$  be a planar graph on  $n$  vertices. Then the vertices of  $G$  can be partitioned into three sets  $A, B, C$ , such that no edge joins a vertex in  $A$  with a vertex in  $B$ , neither  $A$  nor  $B$  contains more than  $\frac{2}{3}n$  vertices, and  $C$  contains no more than  $2\sqrt{2n}$  vertices. Furthermore, the sets  $A, B, C$  can be found in  $\mathcal{O}(n)$  time.*

Following [MNS86], we use the notion of a *good separator*.

**Definition 5.9** A graph  $G$  on  $n$  vertices has a good separator if there exist two constants  $c_1 < 1$  and  $c_2$  satisfying: The vertices of  $G$  can be partitioned into three sets  $A, B, C$  such that no edge joins a vertex in  $A$  with a vertex in  $B$ , neither  $A$  nor  $B$  contains more than  $c_1 n$  vertices, and  $C$  contains no more than  $c_2 \sqrt{n}$  vertices.

**Lemma 5.10** If the graph  $G$  of the homogeneous mesh  $\mathcal{M}$  is planar, then an optimal feasible subgraph can be computed in time  $\mathcal{O}(n^{3/2} \log n)$ .

**Proof:** Barahona[Bar90] and Matsumoto et al. [MNS86] have shown how to solve the minimum  $T$ -join problem in  $\mathcal{O}(n^{3/2} \log n)$  for planar graphs using the planar separator theorem.

We cannot directly use their result, as the graph  $G_{\text{aux}}^d$  on which we have to solve the  $T$ -join problem is not planar, in general. However, with a slight modification of the technique used by Matsumoto et al. [MNS86] we can show in the following that  $G_{\text{aux}}^d$  has a good separator.

Let  $G^d$  be the variant of the dual graph as in Definition 5.1 and  $v_b \in V^d$  be the vertex corresponding to the boundary of the mesh.  $G^d$  need not be planar, if the boundary of the mesh is not connected. However, after deletion of  $v_b$ , the graph  $G^d \setminus \{v_b\}$  is certainly planar, if  $G$  is planar. Hence, we can apply the planar separator theorem to  $G^d \setminus \{v_b\}$ . This means that we can partition the graph  $G^d \setminus \{v_b\}$  into sets  $A, B, C$ , such that no edge joins a vertex in  $A$  with a vertex in  $B$ , neither  $A$  nor  $B$  contains more than  $\frac{2}{3}n$  vertices and  $C$  contains no more than  $2(2n)^{1/2}$  vertices. If we put  $v_b$  into the set  $C$ , we clearly also have a partition for  $G^d$  with the required properties.

The partition  $A, B, C$  naturally induces a partition of the vertices of  $G_{\text{aux}}^d$  into  $A', B'$  and  $C'$ : If a vertex  $v_d \in A$  ( $v_d \in B$ ,  $v_d \in C$ ) is replaced by a subgraph in  $G_{\text{aux}}^d$ , then all vertices of the subgraph belong to  $A'$  ( $B'$ ,  $C'$ , respectively).

We have to show that  $A', B', C'$  yields a good separator. Let  $k$  be the number of vertices of the largest subgraph introduced for a polygon  $v^d$ . It is important that  $k$  is some constant number, namely  $k = 12$  in Fig. 9. Denote by  $n'$  the number of vertices in  $G_{\text{aux}}^d$ . Hence, we have  $n' \leq kn$ . As  $|A| \leq \frac{2}{3}n$ , at least  $\frac{n}{3}$  vertices of  $n'$  cannot belong to  $|A'|$ . This means  $|A'| \leq n' - \frac{n}{3} \leq (1 - \frac{1}{3k})n' = c_1 n'$ , with  $c_1 := 1 - \frac{1}{3k} < 1$ . By symmetry, we can also bound the number of vertices in  $B'$  by  $|B'| \leq c_1 n'$ . Certainly,  $C'$  contains no more than  $|C'| \leq k|C| \leq 2k(2n)^{1/2} \leq 2k(2n')^{1/2}$  vertices. Hence, we can choose  $c_2 := 2k\sqrt{2}$ . Thus, there is a good separator for  $G_{\text{aux}}^d$  which can be found in linear time.

Very similarly, we can also show that in the whole separation tree the subgraphs partitioned by their separators all have good separators.

Now exactly the same analysis as in [Bar90] yields the claimed result.  $\square$

## 6 Concluding Remarks

We conjecture that our performance guarantee of 1.867 for homogeneous meshes is not tight. Indeed, it remains an open question whether examples exist where our approximation is really worse than a  $\frac{5}{3}$ -approximation.

We would like to emphasize that all lower bounds cannot only be used for the proof of the corresponding performance guarantee, they are also efficiently computable — in the same time complexity as the approximation itself.

Our subdivision lemmas, Lemmas 4.1 and 4.3, can – in principle – be somewhat strengthened, at the cost of an increasing number of exceptional cases. This would also involve an even by far more extensive case distinction than the one we had to go through for the results presented in that chapter.

Up to now, we decided not to go on into that direction, because the bottleneck for further improved approximations of homogeneous meshes are triangles. It seems that a different strategy is needed to get a better lower bound.

This is even more important for meshes with foldings where the current worst case guarantee with a factor of 4 seems to be extremely pessimistic. Perhaps, it would help if we could find an approximation of the  $T$ -join problem in hypergraphs.

For the asymptotic complexity of our algorithms, parallel dual edges do not hurt, as we can always modify the  $T$ -join problem into an equivalent one without parallel dual edges. As to the running time of our algorithms it remains an interesting open question how many dual edges are possible in our variant of the dual graph of a homogeneous mesh, if we do not count parallel dual edges? We do not know of homogeneous meshes with more than  $\mathcal{O}(n)$  non-parallel dual edges. Note that a mesh consisting of a single polygon can have an arbitrarily large number of edges, but all corresponding dual edges are parallel.

Finally, we would like to mention that our application of the planar separator can also be extended in a straightforward way to meshes with bounded genus [GHT84].

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