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Optimal control of planar flow of incompressible non-Newtonian fluids

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Optimal control of planar flow of incompressible non-Newtonian fluids. [1]

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Abstract. We consider an optimal control problem for the evolutionary flow of incompressible non-Newtonian fluids in a two-dimensional domain. The existence of optimal controls is proven. Furthermore, we investigate first-order necessary as well as second-order sufficient optimality conditions. The analysis relies on new results providing solutions with bounded gradients for the flow equations.

Key words. Optimal control, non-Newtonian fluids, necessary optimality conditions, sufficient optimality conditions.

Mathematics Subject Classification: Primary 76D55, Secondary 49K20, 76A05

1 Introduction

We investigate an optimal control problem for the evolutionary flow for incompressible non-Newtonian fluids in a fixed bounded domain $\Omega \subset \mathbb{R}^2$ with a fixed time horizon T. As a model problem we minimize the following quadratic objective functional J

$$J(u,f) = \frac{1}{2} \int_Q \left| u(x,t) - u_d(x,t) \right|^2 \mathrm{d}x \, \mathrm{d}t + \frac{\gamma}{2} \|f\|_{F^s}^2 \tag{1.1}$$

subject to the initial-boundary-value problem for the system of evolutionary equations

$$\begin{aligned}
u_t - \mathrm{div}(\sigma(\mathrm{D}u)) + (u \cdot \nabla)u + \nabla \pi &= f &&\text{in } Q, \\
\mathrm{div}\, u &= 0 &&\text{in } Q, \\
u &= 0 &&\text{on } \Sigma, \\
u(0) &= u_0 &&\text{in } \Omega.
\end{aligned} \tag{1.2}$$

The optimization variables are the control f and the response, that consists of the velocity field u and the pressure π. Moreover, we have denoted $Q := \Omega \times (0,T)$ and $\Sigma := \Gamma \times (0,T)$ with Γ

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being the $C^{2+\mu}$-boundary of Ω, $\mu > 0$. Further, functions $u_d \in L^2(Q)^2$, and $u_0 \in L^2(\Omega)^2$ are given; as to u_0, later we still need some regularity (2.11). The parameter γ is a positive real number. The function space F^s, whose norm occurs in the definition (1.1) of J, will be specified later, see (2.16).

We denote by $\mathrm{D}u$ the symmetric gradient of a function u, i.e. $\mathrm{D}u := \frac{1}{2}\left(\nabla u^T + \nabla u\right)$. The mapping σ is a mapping from $\mathbb{R}^{2\times2}_{\mathrm{sym}}$ to $\mathbb{R}^{2\times2}_{\mathrm{sym}}$, the space of all symmetric $\mathbb{R}^{2\times2}$-matrices. The precise assumptions on σ can be found in Section 2.

The governing equations were first studied by mathematically Ladyzhenskaya [18, 19] and Lions [20], see the discussion in the monograph of Nečas, Málek, Rokyta, and Růžička [23]. The resulting partial differential equations are of quasi-linear type. They generalize Navier-Stokes equations, which are semi-linear and contained as the special case $\sigma(D) = \nu D$, $\nu > 0$.

Optimal control problems for non-Newtonian fluids are rarely investigated. We mention the work of Slawig [24] for the stationary case. Control of a parabolic equation with power-law differential operator was considered by White [27]. An optimal control problem with temperature-dependent viscosity was modeled by Kunisch and Marduel [17]. Numerical studies of shape optimization problems with non-Newtonian fluids are considered by Abraham, Behr, and Heinkenschloss in [2]. For related optimal control problems for the Navier-Stokes equations, we refer to [1, 8, 9, 12, 13, 14, 25]. Necessary optimality conditions for optimal control problems subject to quasilinear elliptic equations are considered by Casas and Fernández [5], Casas and Yong [7], and Lou [21]. Optimal control problems subject to parabolic equations were studied by Casas, Fernández, and Yong [6], and Fernández [10, 11].

We restrict the considerations to the two-dimensional case. This is due the fact that known regularity results, namely by Kaplický [15], guarantee that the coefficients in the main part of the differential operator for the linearized and the adjoint equations are in $L^\infty(Q)$. Such a result is needed for optimality conditions and not known for problems in three dimensions, for which only existence of optimal controls can be proved.

The article is organized as follows. In Section 2 known results are collected. The existence of optimal controls is proven in Proposition 2.1. Section 3 deals with the first-order necessary optimality conditions, which are finally proven in Theorem 3.9. As pre-requisite, the control-to-state mapping and its continuity and differentiability properties are analyzed. Second-order sufficient optimality conditions are then investigated in Section 4, Theorem 4.5. Finally, in Section 5 we will comment on the three-dimensional case and prove existence of optimal controls in a particular situation.

2   Notation and preliminary results

Here, we will summarize assumptions on the non-linearity σ as well as known existence and regularity results for the state equation. We now assume that σ has a potential $\Phi : \mathbb{R}^{2\times2}_{\mathrm{sym}} \to \mathbb{R}^+$, i.e. $\sigma_{ij}(D) = \partial_{ij}\Phi(|D|^2)$ with $\partial_{ij} := \partial/\partial_{D_{ij}}$. We assume further that Φ is a C^3 function with $\Phi(0) = 0$ and $\partial_{ij}\Phi(0) = 0$ for all $i,j \in \{1,2\}$. Moreover, we require that there are positive

constants

$$2 \le p < 4 \tag{2.3}$$

and C_1, C_2, C_3 such that

$$C_1(1 + |D|^2)^{\frac{p-2}{2}}|\tilde{D}|^2 \le \partial_{ij}\sigma_{kl}(D)\tilde{D}_{ij}\tilde{D}_{kl}, \tag{2.4}$$

$$|\partial_{ij}\sigma_{kl}(D)| \le C_2(1 + |D|^2)^{\frac{p-2}{2}} \tag{2.5}$$

$$|\partial_{ij}\partial_{mn}\sigma_{kl}(D)| \le C_3(1 + |D|^2)^{\frac{p-3}{2}} \tag{2.6}$$

hold for all $D, \tilde{D} \in \mathbb{R}^{2\times2}_{\mathrm{sym}}$, $i, j, k, l, m, n \in \{1, 2\}$. These assumptions except (2.6) are frequently used in the literature, see e.g. [15, 22, 23]. For existence of optimal controls it suffices to assume $\Phi \in C^2$ and (2.4)–(2.5). Since we want to deal with second-order derivatives of σ, we assumed in addition that Φ is C^3 and that we have the bound (2.6) of σ''. These assumptions on σ covers a wide range of applications in non-Newtonian fluids, see [23]. For the special choice $\sigma(D) = \nu D$, $p = 2$, the mentioned case of the Navier-Stokes equation with viscosity coefficient $\nu > 0$ is included. The assumptions (2.4)–(2.5) imply the monotonicity of σ

$$(\sigma(D_1) - \sigma(D_2))(D_1 - D_2) \ge 0 \quad \forall D_1, D_2 \in \mathbb{R}^{2\times2}_{\mathrm{sym}}$$

as well as its boundedness

$$|\sigma(D)| \le c|D|^{p-1} \quad \forall D \in \mathbb{R}^{2\times2}_{\mathrm{sym}},$$

see [22, Lemma 2.1]

We will use in the sequel the standard Sobolev spaces. To incorporate the divergence-free condition, we will use

$$V := \{v \in H_0^1(\Omega)^2 : \operatorname{div} v = 0\}, \; H := \{v \in L^2(\Omega)^2 : \operatorname{div} v = 0\}.$$

Many of the quantities occuring in the article are vector-valued functions. For the sake of brevity, we will use occasionally the same notations of function spaces for scalar and vector-valued functions.

2.1 The state equation

We are looking for weak solutions of the intial-boundary value problem (1.2). Let an initial value $u_0 \in V$ and a right-hand side $f \in L^2(0, T; V')$ be given. Then a function $u \in L^p(0, T; V \cap W^{1,p}(\Omega))$ with $u_t \in L^2(0, T; V')$ is called a *weak solution*, if it satisfies $u(0) = u_0$ and

$$\int_0^T \langle u_t, \phi \rangle_{V',V} \, \mathrm{d}t + \int_Q \sigma(\mathrm{D}u)\mathrm{D}\phi + (u \cdot \nabla)u\phi \; \mathrm{d}x \, \mathrm{d}t = \int_0^T \langle f, \phi \rangle_{V',V} \, \mathrm{d}t$$

for all smooth and divergence-free test functions ϕ with $\langle \cdot, \cdot \rangle_{V',V}$ being the duality pairing between V' and V. Here, some implicit summations took place, so let us write the second and third term explicitly:

$$\int_Q \sigma(\mathrm{D}u)\mathrm{D}\phi + (u \cdot \nabla)u\phi \; \mathrm{d}x \, \mathrm{d}t = \int_Q \sum_{i,j=1}^2 \left(\sigma_{ij}(\mathrm{D}u)(\mathrm{D}\phi)_{ij} + u_i \frac{\partial u_j}{\partial x_i}\phi_j \right) \mathrm{d}x \, \mathrm{d}t.$$

Of course, in view of the definition of D, it holds $(\mathrm{D}u)_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$. Note, that the pressure is eliminated in the weak formulation due to the use of divergence-free test functions.

The existence and uniqueness of a weak solution for the two-dimensional case and $p \geq 2$ is due to Ladyzhenskaya [18, 19] and Lions [20]. In particular, in [18] it is proven that for $u_0 \in W^{1,p}(\Omega) \cap V$ and $f \in L^2(Q)$ the unique weak solution of (1.2) satisfies

$$u \in L^p(0, T; W^{1,p}(\Omega) \cap V), \quad u_t \in L^2(Q). \tag{2.7}$$

This directly implies that the optimal control problem (1.1)–(1.2) is solvable in the control space $F = L^2(Q)$.

Proposition 2.1 (Existence of optimal controls.) *Let (2.4)–(2.5) be satisfied with $p \geq 2$. Let an initial value $u_0 \in W^{1,p}(\Omega) \cap V$ be given. We assume $\gamma > 0$ in (1.1) Then for $F = L^2(Q)$ there exists an optimal control $\bar{f} \in F$ of the optimal control problem (1.1)–(1.2).*

Proof. Obviously, the problem is feasible, since $f^0 = 0$ and the associated solution $u^0 = 0$ of (1.2) is an admissible pair. If f^0 is already optimal nothing is to prove.

If f^0 is not an optimal control, there must be controls f with lower values of the objective functional. This allows us to restrict the optimal control problem to the set

$$F_0 = \left\{ f \in L^2(Q) : \|f\|_{L^2} \leq \frac{1}{\gamma} \|u^0 - u_d\|_{L^2}^2 \} \right\}.$$

Since the objective functional is bounded from below, there is a minimizing sequence of controls f_n with associated states u_n, such that $\inf J = \lim_{n \to \infty} J(u_n, f_n)$. After extracting subsequences, we have the existence of weak limits $\bar{f} \in L^2(Q)$ and $\bar{u}$, with $f_n \rightharpoonup \bar{f}$ in L^2 and $u_n \rightharpoonup \bar{u}$ in the spaces given by (2.7). In the following, we will only apply the weak convergences $\nabla u_n \rightharpoonup \nabla \bar{u}$ in $L^p(Q)$ and $u_{n,t} \rightharpoonup \bar{u}_t$ in $L^2(Q)$, respectively. It remains to prove that $\bar{u}$ is the solution of the state equation with control $\bar{f}$.

Let $v \in L^p(0, T; W^{1,p}(\Omega))$ be an arbitrary test function with div $v = 0$. The assumptions (2.4)–(2.5) on σ imply the monotonicity of the associated Nemytskiĭ operator, see [23, Lemma 5.1.19]. Exploiting this monotonicity of σ we get

$$0 \leq \int_Q (\sigma(\mathrm{D}u_n) - \sigma(\mathrm{D}v))\mathrm{D}(u_n - v) \,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_Q (f_n - u_{n,t} - (u_n \cdot \nabla)u_n)(u_n - v) - \sigma(\mathrm{D}v)\mathrm{D}(u_n - v) \,\mathrm{d}x\,\mathrm{d}t. \tag{2.8}$$

Here, we used that u_n is the weak solution of the state equation. By compact embeddings, we have the strong convergence $u_n \to \bar{u}$ in $L^2(Q)$, which gives $\int_Q (f_n - u_{n,t})u_n \to \int_Q (\bar{f} - \bar{u}_t)\bar{u}$. Since u_n and $\bar{u}$ are divergence free, we can pass to the limit in the convective term. So we can pass to the limit in (2.8) and obtain

$$0 \leq \int_Q (\bar{f} - \bar{u}_t - (\bar{u} \cdot \nabla)\bar{u})(\bar{u} - v) - \sigma(\mathrm{D}v)\mathrm{D}(\bar{u} - v) \,\mathrm{d}x\,\mathrm{d}t. \tag{2.9}$$

We finish by Minty's trick. Setting $v := \bar{u} + \varepsilon w$, $\varepsilon > 0$, w smooth with $\operatorname{div} w = 0$, we derive $0 \leq \int_Q (\bar{f} - \bar{u}_t - (\bar{u} \cdot \nabla)\bar{u})(-\varepsilon w) - \sigma(\mathrm{D}(\bar{u} + \varepsilon w))\mathrm{D}(-\varepsilon w)$. Dividing by $-\varepsilon$ and letting $\varepsilon \to 0$, we obtain $0 \geq \int_Q (\bar{f} - \bar{u}_t - (\bar{u} \cdot \nabla)\bar{u})w - \sigma(\mathrm{D}\bar{u})\mathrm{D}(w)$. Here, we applied the continuity of σ, see Lemma 3.1 below. Analogously, we get with $v := \bar{u} - \varepsilon w$ the reverse inequality $0 \leq \int_Q (\bar{f} - \bar{u}_t - (\bar{u} \cdot \nabla)\bar{u})w - \sigma(\mathrm{D}\bar{u})\mathrm{D}(w)$, which proves that $\bar{u}$ is the weak solution to $\bar{f}$ of (1.2), since the test function w was arbitrary. By lower semicontinuity of J, it follows by a standard argument, that $(\bar{u}, \bar{f})$ is indeed optimal. $\qquad\square$

The regularity of weak solutions made it possible to prove the existence of solution. For our purposes however, this so far stated regularity is not sufficient. We will need higher regularity results to derive an optimality system. It turns out that even $\nabla u \in L^\infty(Q)$ is necessary to deal with first-order optimality conditions.

Higher regularity results for non-Newtonian fluids are difficult to obtain in general. For optimal control, we unfortunately need very strong regularity, namely boundedness of the velocity gradient. There are only few such results known up to nowadays. For Dirichlet boundary conditions, we will base our considerations on the following result about regularity of solutions of (1.2) which was proven in [15]. A similar result for space-periodic boundary condition can be found in [16]. As already mentioned, analogous results for the spatially three-dimensional case does not seem to be available.

Theorem 2.2 (Kaplický [15].) *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with $C^{2+\mu}$ boundary. Let the assumptions (2.4)–(2.5) on σ hold with some $p \in [2, 4)$. Let us assume that the right-hand side f and the initial value u_0 fulfill*

$$f \in L^\infty(0, T; L^2(\Omega)), \quad f(0) \in L^2(\Omega), \quad f_t \in L^2(0, T; V') \tag{2.10}$$

and

$$u_0 \in W^{r,2}(\Omega) \cap V, \quad r > 2 \, (r = 2 \text{ if } p = 2). \tag{2.11}$$

Then the unique weak solution u of (1.2) satisfies for $s \in (1, 2)$ $(s = 2$ if $p = 2)$

$$u \in L^\infty(0, T; W^{2,s}(\Omega)), \quad u_t \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; V). \tag{2.12}$$

If moreover there is a $\tilde{q} > 2$ such that

$$f \in L^\infty(0, T; L^{\tilde{q}}(\Omega)), \quad f_t \in L^{\tilde{q}}(0, T; W^{-1,\tilde{q}}(\Omega)), \tag{2.13}$$

then there exist $q > 2$ and $\alpha > 0$ such that for all $\varepsilon \in (0, T)$ it holds

$$\nabla u \in L^\infty(\varepsilon, T; W^{1,q}(\Omega)) \cap C^{0,\alpha}(\bar{\Omega} \times [\varepsilon, T]). \tag{2.14}$$

This theorem provides us with bounded gradients of the solution u. However, the gradients stay bounded only for positive times but not up to the initial time $t = 0$. To overcome this difficulty, let us consider the following initial-boundary value problem

$$\begin{aligned}
u_t - \operatorname{div}(\sigma(\mathrm{D}u)) + (u \cdot \nabla)u + \nabla\pi &= f && \text{in } \Omega \times (-\tau, T), \\
\operatorname{div} u &= 0 && \text{in } \Omega \times (-\tau, T), \\
u &= 0 && \text{on } \Gamma \times (-\tau, T), \\
u(-\tau) &= u_{-\tau} && \text{in } \Omega,
\end{aligned} \tag{2.15}$$

which is the original equation but starts at some time $-\tau < 0$. If we have that $u_{-\tau}$ and f fulfill all requirements of the previous theorem on the larger time interval, then we obtain boundedness of the gradients of u on $Q = \Omega \times (0, T)$. This leads us to the following assumption on the initial value u_0 and the definition of the control space F.

Assumption 1 *The initial value u_0 is equal to $u(0)$ with u being the solution of the problem*

$$u_t - \operatorname{div}(\sigma(\mathrm{D}u)) + (u \cdot \nabla)u + \nabla\pi = 0 \qquad in \ \Omega \times (-\tau, 0),$$
$$\operatorname{div} u = 0 \qquad in \ \Omega \times (-\tau, 0),$$
$$u = 0 \qquad on \ \Gamma \times (-\tau, 0),$$
$$u(-\tau) = u_{-\tau} \qquad in \ \Omega,$$

where the function $u_{-\tau}$ satisfies (2.11) with $u_{-\tau}$ in place of u_0.

Let us now define the control space F^s by

$$F^s := \{f : \ \exists \tilde{f} \in F^s_{-\tau} \text{ with } \tilde{f}|_Q = f \text{ and } \tilde{f}|_{\Omega \times (-\tau, 0)} = 0\}, \tag{2.16}$$

where $F^s_{-\tau}$ denotes the space

$$F^s_{-\tau} := W^{1+s,2}(-\tau, T; L^2(\Omega)) \cap L^2(-\tau, T; W^{s,2}(\Omega)).$$

For $s \geq 0$, any function $f \in F^s_{-\tau}$ satisfies the requirement (2.10) adapted to the time interval $(-\tau, T)$. For each $s > 0$ there is a $\tilde{q} > 2$ such that even condition (2.13) is satisfied. Hence, the solution u of (2.15) with right-hand sides $f \in F^s_{-\tau}$, $s > 0$, and initial values $u_{-\tau}$ as above will satisfy $\nabla u \in L^\infty(Q)$. Since the orginal control problem was formulated on the time interval $(0, T)$, any admissible control has to be zero on the interval $(-\tau, 0)$. So we defined the control space F^s as the space of functions that have an extension in $F^s_{-\tau}$, which is zero on $(-\tau, 0)$, to obtain the same solution regularity. Let us summarize all these considerations in the following corollary.

Corollary 2.3 *Let u_0 fulfill Assumption 1. Then for every $f \in F^s$, $s \geq 0$ the problem (1.2) admits a unique solution u. For $s > 0$ the regularity $\nabla u \in L^\infty(Q)$ holds.*

3   First-order necessary optimality conditions

Due to the fact that σ is C^2 we can write for $u_1, u_2 \in L^q(0, T; W^{1,q}(\Omega))$ and almost all $\xi := (x, t) \in Q$

$$\sigma(\mathrm{D}u_1(\xi)) - \sigma(\mathrm{D}u_2(\xi)) = \int_0^1 \sigma'(\mathrm{D}u_2(\xi) + s(\mathrm{D}u_1(\xi) - \mathrm{D}u_2(\xi)))(\mathrm{D}u_1(\xi) - \mathrm{D}u_2(\xi)) \, \mathrm{d}s \tag{3.17}$$

and

$$\sigma(\mathrm{D}u_1(\xi)) - \sigma(\mathrm{D}u_2(\xi)) - \sigma'(\mathrm{D}u_2(\xi))(\mathrm{D}u_1(\xi) - \mathrm{D}u_2(\xi))$$
$$= \int_0^1 \int_0^s \sigma''(\mathrm{D}u_2(\xi) + \tau(\mathrm{D}u_1(\xi) - \mathrm{D}u_2(\xi)))(\mathrm{D}u_1(\xi) - \mathrm{D}u_2(\xi))^2 \mathrm{d}\tau \, \mathrm{d}s. \tag{3.18}$$

These representations allow us to investigate the properties of the Nemytskiĭ (or superposition) operator induced by σ. In the sequel, we will denote by $\sigma, \sigma', \sigma''$ also the Nemytskiĭ operators induced by the function $\sigma, \sigma', \sigma''$, respectively.

Lemma 3.1 *The Nemytskiĭ-operator associated to σ and defined by*

$$(\sigma(D))(x,t) = \sigma(D(x,t))$$

is continuous from $L^r(0,T; L^q(\Omega; \mathbb{R}^{2\times 2}_{sym}))$ to $L^{\frac{r}{p-1}}(0,T; L^{\frac{q}{p-1}}(\Omega; \mathbb{R}^{2\times 2}_{sym}))$ for $q, r \geq p-1$.

Moreover, the Nemytskiĭ-operator is Fréchet differentiable from the space $L^r(0,T; L^q(\Omega; \mathbb{R}^{2\times 2}_{sym}))$ to $L^{\frac{r}{p-1}}(0,T; L^{\frac{q}{p-1}}(\Omega; \mathbb{R}^{2\times 2}_{sym}))$ for $p > 3$ and $q, r \geq p-1$. For $2 \leq p \leq 3$, it is Fréchet differentiable from $L^r(0,T; L^q(\Omega; \mathbb{R}^{2\times 2}_{sym}))$ to $L^{\frac{r}{2}}(0,T; L^{\frac{q}{2}}(\Omega; \mathbb{R}^{2\times 2}_{sym}))$ for $q, r \geq 2$.

Its Fréchet derivative is given by the Nemytskiĭ operator induced by the function σ'.

Proof. The assumptions (2.4)–(2.6) imply that the Nemytskiĭ operator σ maps the space $L^r(0,T; L^q(\Omega; \mathbb{R}^{2\times 2}_{\text{sym}}))$ to $L^{\frac{r}{p-1}}(0,T; L^{\frac{q}{p-1}}(\Omega; \mathbb{R}^{2\times 2}_{\text{sym}}))$, see [22, Lemma 2.1]. Thus, it is continuous.

The function σ is C^2 by assumption, hence (3.18) holds for a.a. $(x,t) \in Q$. This representation yields together with (2.6) that

$$\left| \sigma(D(x,t) + \tilde{D}(x,t)) - \sigma(D(x,t)) - \sigma'(D(x,t))\tilde{D}(x,t) \right|$$
$$\leq \frac{C_3}{2} |\tilde{D}(x,t)|^2 \cdot \begin{cases} 1 & \text{for } p < 3, \\ (1 + |D(x,t)| + |\tilde{D}(x,t)|)^{p-3} & \text{for } p \geq 3. \end{cases}$$

Hence we can estimate for $p > 3$

$$\|\sigma(D + \tilde{D}) - \sigma(D) - \sigma'(D + \tilde{D})\tilde{D}\|_{L^{\frac{r}{p-1}}(L^{\frac{q}{p-1}})} \leq c(1 + \|D\| + \|\tilde{D}\|)^{p-3}_{L^r(L^q)} \|\tilde{D}\|^2_{L^r(L^q)}.$$

For $2 \leq p < 3$ we have a uniform bound on σ'', and it holds

$$\|\sigma(D + \tilde{D}) - \sigma(D) - \sigma'(D + \tilde{D})\tilde{D}\|_{L^{\frac{r}{2}}(L^{\frac{q}{2}})} \leq c\|\tilde{D}\|^2_{L^r(L^q)}.$$

Both estimates allow us to proof that appropriate norms of the remainder term $\sigma(D + \tilde{D}) - \sigma(D) - \sigma'(D + \tilde{D})\tilde{D}$ divided by $\|\tilde{D}\|_{L^r(L^q)}$ vanishes as $\|\tilde{D}\|_{L^r(L^q)}$ tends to zero. $\square$

3.1 Lipschitz estimates and linearized equations

In order to prove differentiability of the control-to-state mapping, we first investigate its local Lipschitz properties.

Lemma 3.2 *Let $f_1, f_2 \in F^s$, $s > 0$, be given together with their respective solutions u_1, u_2 of (1.2). Then it holds with some constant c depending on u_1, f_1 but not on u_2, f_2*

$$\|u_1 - u_2\|_{L^2(V)} + \|u_1 - u_2\|_{L^\infty(H)} \leq c\|f_1 - f_2\|_{L^2(V')}.$$

Furthermore it holds

$$\|u_{1,t} - u_{2,t}\|_{L^2(V')} \leq \bar{c}(\|f_1\|_{F^s}, \|f_2\|_{F^s})\|f_1 - f_2\|_{L^2(V')}$$

with a continuous function $\bar{c} : \mathbb{R}_+^2 \to \mathbb{R}_+$.

Proof. The first assertion follows immediately by the strong monotonicity of σ, i.e.

$$(\sigma(D_1) - \sigma(D_2)) : (D_1 - D_2) \geq C_1|D_1 - D_2|^2$$

for any $D_1, D_2 \in \mathbb{R}_{\text{sym}}^{2 \times 2}$ with C_1 from (2.4), and related estimates for the Navier-Stokes equations, see e.g. [14]. Moreover, we can estimate the time derivative of the difference $u_1 - u_2$ by writing

$$u_{1,t} - u_{2,t} =$$
$$= f_1 - f_2 + \text{div}(\sigma(Du_1) - \sigma(Du_2)) - (u_1 \cdot \nabla)(u_1 - u_2) - ((u_1 - u_2) \cdot \nabla)u_2$$
$$= f_1 - f_2 + \text{div}\,\sigma_{Du_1}^{Du_2}(Du_1 - Du_2) - (u_1 \cdot \nabla)(u_1 - u_2) - ((u_1 - u_2) \cdot \nabla)u_2,$$

where $\sigma_{Du_1}^{Du_2}(Du_1 - Du_2) = \int_0^1 \sigma'(Du_2 + s(Du_1 - Du_2))(Du_1 - Du_2)\,ds$ is given by (3.17). Now, we apply the assumption (2.5) on σ' to estimate

$$\left|(\sigma_{Du_1}^{Du_2}(Du_1 - Du_2), D\phi)\right| \leq \|\sigma_{Du_1}^{Du_2}\|_{L^\infty}\|u_1 - u_2\|_{L^2(V)}\|\phi\|_{L^2(V)}$$
$$\leq c\left(1 + \|Du_1\|_{L^\infty}^{p-2} + \|Du_2\|_{L^\infty}^{p-2}\right)\|u_1 - u_2\|_{L^2(V)}\|\phi\|_{L^2(V)}.$$

Regarding the convective terms we do the following estimation

$$|((u_1 \cdot \nabla)(u_1 - u_2) - ((u_1 - u_2) \cdot \nabla)u_2,\ \phi)| \leq \|u_1\|_{L^\infty}\|u_1 - u_2\|_{L^2(V)}\|\phi\|_{L^2(V)}$$
$$+ \|u_1 - u_2\|_{L^2(V)}\|\phi\|_{L^2(V)}\|u_2\|_{L^\infty}.$$

Altogether, we find for the $L^2(0, T; V')$-norm of the difference of the time derivatives the estimate

$$\|u_{1,t} - u_{2,t}\|_{L^2(V')} \leq c\left(1 + \|Du_1\|_{L^\infty}^{p-2} + \|Du_2\|_{L^\infty}^{p-2} + \|u_1\|_{L^\infty} + \|u_2\|_{L^\infty}\right)\|f_1 - f_2\|_{L^2(V')}.$$

$\square$

In the proof, it was essential to use the regularity $\nabla u_i \in L^\infty(Q)$. If the controls f_1, f_2 are only in F^0 then this regularity is not available, and one gets a Lipschitz estimate for the time derivatives in weaker norms, i.e. only with respect to $W^{-1-\varepsilon,2}$-norms, $\varepsilon > 0$.

Now let us investigate the linearized equation. To this end, let $\bar{u}$ be a solution of the nonlinear equation (1.2) that fulfills the regularity assertions of Corollary 2.3, e.g. $\nabla u \in L^\infty(Q)$. Then we are looking for solutions of the following initial-boundary value problem with a given right-hand side h

$$\begin{aligned}
u_t - \text{div}(\sigma'(D\bar{u})Du) + (\bar{u} \cdot \nabla)u + (u \cdot \nabla)\bar{u} + \nabla\pi &= h &&\text{in } Q, \\
\text{div}\,u &= 0 &&\text{in } Q, \\
u &= 0 &&\text{on } \Sigma, \\
u(0) &= 0 &&\text{in } \Omega.
\end{aligned} \tag{3.19}$$

In the weak formulation of this problem appears now the term $\sigma'(D\bar{u})DuD\phi$, which is to be understood as

$$\sigma'(D\bar{u})DuD\phi = \sum_{i,j,k,l=1}^{2} \frac{\partial \sigma_{ij}(D\bar{u})}{\partial_{kl}}(Du)_{kl}(D\phi)_{ij}. \tag{3.20}$$

Lemma 3.3 *Let us assume $\nabla\bar{u} \in L^{\infty}(Q)$. Then for all $h \in L^2(0,T;V')$ the linearized equation (3.19) admits a unique weak solution $u \in L^2(0,T;V)$ with $u_t \in L^2(0,T;V')$. Moreover, there is a constant $c > 0$ independent of u such that it holds*

$$\|u_t\|_{L^2(V')} + \|u\|_{L^2(V)} \leq c\|h\|_{L^2(V')}. \tag{3.21}$$

Proof. The proof is carried out by a standard Galerkin procedure. Let u^N be the solution of the approximate problem. It fulfills

$$\|u^N\|_{L^2(V)} + \|u^N\|_{L^{\infty}(H)} \leq c\|h\|_{L^2(V')}, \tag{3.22}$$

with a constant c independent of N and h. Here, we used assumption (2.4) on the strong monotonicity of $\sigma'(D\bar{u}) : \mathbb{R}_{\text{sym}}^{2\times2} \to \mathbb{R}_{\text{sym}}^{2\times2}$. With the same arguments as in Lemma 3.2 above, one can prove for the time derivative

$$\|u_t^N\|_{L^2(V')} \leq c\|h\|_{L^2(V')}. \tag{3.23}$$

Hence, there exists a weak limit $u \in L^2(0,T;V)$ with $u_t \in L^2(0,T;V')$ such that, after extracting a subsequence if necessary, $u^N \rightharpoonup u$ in $L^2(0,T;V)$ and $u_t^N \rightharpoonup u_t$ in $L^2(0,T;V')$. By the Aubin-Lions theorem the space $L^2(0,T;V) \cap W^{1,2}(0,T;V')$ is compactly embedded in $L^r(0,T;H)$ for every $r < \infty$. Hence, we have the strong convergence $u^N \to u$ in $L^r(0,T;H)$, and we can pass to the limit in the weak formulation. The solution u inherits the desired estimates from (3.22) and (3.23). $\square$

Here again, the regularity $\nabla\bar{u} \in L^{\infty}(Q)$ was crucial. If this is not fulfilled then the estimate of the time derivative u_t in $L^2(0,T;V')$ is not available, which implies that the time derivative is not in duality with the solution itself. Hence, we cannot test the equation (3.19) by the solution to prove uniqueness.

Exploiting the regularity $\nabla\bar{u} \in L^{\infty}(Q)$ allows us to apply a result of Kaplický [15] for generalized Stokes equations. Consider the following system, which coincides with the linearized system (3.19) except for the missing convective terms:

$$\begin{aligned}
u_t - \text{div}(\sigma'(D\bar{u})Du) + \nabla\pi &= h && \text{in } Q, \\
\text{div}\, u &= 0 && \text{in } Q, \\
u &= 0 && \text{on } \Sigma, \\
u(0) &= 0 && \text{in } \Omega.
\end{aligned} \tag{3.24}$$

For the existence and regularity of solutions to that equation, we have the following.

Proposition 3.4 *There are positive constants C, L depending on Ω such that if for $q \in (2, 2+\delta)$, $\delta = LC_1/\{C_2(1 + \|\nabla\bar{u}\|_{L^\infty})^{p-2}\}$, the right-hand side fulfills $h \in L^q(0, T; W^{-1,q}(\Omega))$, then the unique weak solution u of (3.24) satisfies*

$$\|u\|_{L^q(W^{1,q})} + \|u\|_{L^\infty(L^q)} \leq C \frac{C_2^{\frac{1}{q}}}{C_1}(1 + \|\nabla\bar{u}\|_{L^\infty})^{\frac{p-2}{q}} \|h\|_{L^q(W^{-1,q})}.$$

Here, C_1, C_2 are given by (2.4)–(2.5).

Proof. The proof follows immediately from [15, Proposition 2.4] using (2.4)–(2.5) to compute uniform bounds of the smallest and largest eigenvalue of σ'. $\qquad\square$

With the previous result at hand, we can prove regularity of solutions of (3.19) as well as a Lipschitz continuity result stronger than in Lemma 3.2.

Lemma 3.5 *Let right-hand sides $f_1, f_2 \in F^s$, $s > 0$ be given. Then for the associated solutions u_1, u_2 of the nonlinear equation (1.2) there is a constant L depending on Ω and a constant δ given by*

$$\delta = \min\left\{2, LC_1C_2^{-1}(1 + \|\nabla u_1\|_{L^\infty} + \|\nabla u_2\|_{L^\infty})^{-(p-2)}\right\}$$

such that for every $q \in (2, 2+\delta)$ it holds

$$\|u_1 - u_2\|_{L^q(W^{1,q})} + \|u_1 - u_2\|_{L^\infty(L^q)} \leq c(u_1, u_2)\|f_1 - f_2\|_{L^q(W^{-1,q})},$$

where the constant c depends continuously on $\|\nabla u_1\|_{L^\infty}, \|\nabla u_2\|_{L^\infty}, \|u_1\|_{L^\infty}$, and $\|u_2\|_{L^\infty}$.

Proof. Obviously, the right-hand sides f_1, f_2 are in $L^q(0, T; W^{-1,q}(\Omega))$ by assumption. Let us denote by d the difference of u_1 and u_2, i.e. $d := u_1 - u_2$. By construction, d fulfills the equation

$$\begin{aligned} d_t - \operatorname{div}\sigma_{Du_1}^{Du_2}(Du_1 - Du_2) + \nabla\pi &= f_1 - f_2 - (u_1 \cdot \nabla)d - (d \cdot \nabla)u_2 && \text{in } Q, \\ \operatorname{div} d &= 0 && \text{in } Q, \\ d &= 0 && \text{on } \Sigma, \\ d(0) &= 0 && \text{in } \Omega \end{aligned}$$

with $\sigma_{Du_1}^{Du_2}$ as in the proof of Lemma 3.2. Now, we use again the regularity result [15, Proposition 2.4] for the generalized Stokes system with L^∞-coefficients. To apply this result, we have to derive uniform lower and upper eigenvalue bounds γ_1 and γ_2 of $\sigma_{Du_1}^{Du_2}$. By assumption (2.4), we have $\gamma_1 = C_1$ as a lower bound. For the upper bound we use (2.5) and get $\gamma_2 = C_2(1 + \|\nabla u_1\|_{L^\infty} + \|\nabla u_2\|_{L^\infty})^{p-2}$. Hence, we obtain $\delta := L\gamma_1/\gamma_2 = LC_1/(C_2(1 + \|\nabla u_1\|_{L^\infty} + \|\nabla u_2\|_{L^\infty})^{p-2})$ as upper bound for the integrability exponent. Then the mentioned result of [15] yields for $q \in (2, 2+\delta)$ the estimate

$$\begin{aligned} \|d\|_{L^q(W^{1,q})} + \|d\|_{L^\infty(L^q)} &\leq C(1 + \|\nabla u_1\|_{L^\infty} + \|\nabla u_2\|_{L^\infty})^{\frac{p-2}{q}} \cdot \\ &\quad \cdot \left(\|f_1 - f_2\|_{L^q(W^{-1,q})} + \|(u_1 \cdot \nabla)d + (d \cdot \nabla)u_2\|_{L^q(W^{-1,q})}\right). \end{aligned}$$

It remains to investigate the last addend on the right-hand side. We obtain with integration by parts

$$\int_Q ((u_1 \cdot \nabla)d + (d \cdot \nabla)u_2) \cdot \phi \, dx \, dt = \int_Q -(u_1 \cdot \nabla)\phi \cdot d - (d \cdot \nabla)\phi \cdot u_2 \, dx \, dt.$$

Then for $q \leq 4$, $q' = q/(q-1) \geq 4/3$ we can estimate

$$\left| \int_Q (u_1 \cdot \nabla)\phi \cdot d + (d \cdot \nabla)\phi \cdot u_2 \, dx \, dt \right| \leq c \left(\|u_1\|_\infty + \|u_2\|_\infty \right) \|\phi\|_{L^{q'}(W^{1,q'})} \|d\|_{L^4}.$$

By Lemma 3.2, we have already $\|d\|_{L^4} \leq c \|f_1 - f_2\|_{L^2(V')}$, and the claimed Lipschitz inequality is proven. $\qquad\square$

Corollary 3.6 *Let $\nabla\bar{u} \in L^\infty(Q)$ be satisfied. Then for every h in the space $L^q(0, T; W^{-1,q}(\Omega))$ with $q \in [2, 2+\delta)$, where δ is as in Proposition 3.4, the system (3.19) has a unique solution u that satisfies*

$$\|u\|_{L^q(W^{1,q})} + \|u\|_{L^\infty(L^q)} \leq c\|h\|_{L^q(W^{-1,q})}$$

with a constant $c > 0$ depending on $\bar{u}$ but not on h.

Proof. By Lemma 3.3, we get the existence of a unique weak solution u in the space $L^2(0, T; V)$ with $u_t \in L^2(0, T; V')$. Now we put the terms $(\bar{u} \cdot \nabla)u + (u \cdot \nabla)\bar{u}$ on the right-hand side, and estimate their $L^q(0, T; W^{-1,q}(\Omega))$ as in the proof of the previous lemma. Then the claim follows from Proposition 3.4. $\qquad\square$

3.2   Differentiability of the control-to-state mapping

We already know that for each control right-hand side f the nonlinear state equation admits a unique solution. Let us denote by S the underlying mapping from controls to states, $S(f) = u$. In the previous sections we studied continuity properties of that mapping. In order to prove necessary optimality conditions, we have to investigate the differentiability of S. Although it would suffice for first-order optimality conditions to have Gâteaux differentiability, we prove Fréchet differentiability of S. We will show that the Fréchet derivative $S'(\bar{f})h$ is the unique weak solution of the following system with $\bar{u} = S(\bar{f})$

$$
\begin{aligned}
u_t - \operatorname{div}(\sigma'(\mathrm{D}\bar{u})\mathrm{D}u) + (\bar{u} \cdot \nabla)u + (u \cdot \nabla)\bar{u} + \nabla\pi &= h && \text{in } Q, \\
\operatorname{div} u &= 0 && \text{in } Q, \\
u &= 0 && \text{on } \Sigma, \\
u(0) &= 0 && \text{in } \Omega.
\end{aligned}
\tag{3.25}
$$

Here, we heavily rely on the fact that the coefficients of σ' in this linearized equation are in $L^\infty(Q)$.

Lemma 3.7 *Let the parameter s be greater than zero. Then the control-to-state mapping $S : f \mapsto u$ is Fréchet differentiable from F^s, see (2.16), to $L^2(0, T; V) \cap L^\infty(0, T; L^2(\Omega))$. Moreover, for each $\bar{f} \in F^s$ there is $\bar{\delta} > 0$ such that the mapping S is Fréchet differentiable at $\bar{f}$ from F^s to $L^q(0, T; W^{1,q}(\Omega) \cap V) \cap L^\infty(0, T; L^q(\Omega))$ for all $q \in [2, 2+\bar{\delta})$.*

Proof. Let us only prove the local differentiability result. Fréchet differentiability of S into $L^2(0, T; W^{1,2}(\Omega)) \cap L^\infty(0, T; L^2(\Omega))$ follows by embedding arguments.

Let us take $\bar{f} \in F^s$ and fix an open and bounded neighborhood $B(\bar{f})$ in F^s and set $B = B(\bar{f}) \cap F^s$. By Theorem 2.2, we have that the $L^\infty(0,T;W^{1,\infty}(\Omega))$-norms of the functions in $S(B)$ are bounded. Let us denote this bound by M, e.g. $\|\nabla u\|_{L^\infty(0,T;L^\infty)} < M$ for all $u \in S(B)$. Furthermore, we define $\bar{\delta}$ by

$$\bar{\delta} := \min \left\{ 2,\, LC_1C_2^{-1}(1+2M)^{-(p-2)} \right\},$$

compare with the expressions for δ in Proposition 3.4 and Lemma 3.5.

Now let us take f from the neighborhood B and set $h = f - \bar{f}$. Let $\bar{u} = S(\bar{f})$, $u = S(\bar{f} + h)$ be the weak solutions of the nonlinear equation. Let $d = S'(\bar{f})h$ be the solution of (3.25), which exists and is unique by Corollary 3.6. Then the remainder $r = u - \bar{u} - d$ fulfills the equation

$$\begin{aligned}
r_t - \operatorname{div}(\sigma'(\mathrm{D}\bar{u})\mathrm{D}r) + \nabla\pi &= -(d \cdot \nabla)d - \operatorname{div}\sigma'^{\mathrm{D}u}_{\mathrm{D}\bar{u}}\mathrm{D}(u-\bar{u})\mathrm{D}(u-\bar{u}) &&\text{in } Q, \\
\operatorname{div} r &= 0 &&\text{in } Q, \\
r &= 0 &&\text{on } \Sigma, \\
r(0) &= 0 &&\text{in } \Omega,
\end{aligned} \tag{3.26}$$

with $\sigma'^{\mathrm{D}u}_{\mathrm{D}\bar{u}} = \int_0^1 \int_0^{s_1} \sigma''(\mathrm{D}\bar{u}+s_2\mathrm{D}(u-\bar{u}))\,ds_2\,ds_1$, cf. (3.18). In order to apply Proposition 3.4, we have to estimate the terms on the right-hand side. With that proposition, we obtain a maximal integrability of the solution with respect to some L^q-norms, $q \in (2, 2+\delta)$, where δ depends on bounds of the coefficients of the differential operator. The above defined constant $\bar{\delta}$ fulfills the requirements of Proposition 3.4. Hence, we can take $\tilde{q} \in (2, 2+\bar{\delta})$ and set $\bar{q} = \frac{1}{2}\left(\tilde{q} + 2 + \bar{\delta}\right) \in (\tilde{q}, 2+\bar{\delta})$.

We estimate the convective term on the right-hand side of (3.26) using by-part integration. Applying the result of Corollary 3.6 we have with $\bar{q} > \tilde{q}$

$$\|(d \cdot \nabla)d\|_{L^{\tilde{q}}(W^{-1,\tilde{q}})} \leq c\|d\|^2_{L^{2\tilde{q}}} \leq c\|d\|_{L^\infty(L^{\bar{q}})}\|d\|_{L^{\bar{q}}(L^\infty)} \leq c\|h\|^2_{L^{\bar{q}}(W^{-1,\bar{q}})}.$$

The addend involving the second-order remainder term $\sigma'^{\mathrm{D}u}_{\mathrm{D}\bar{u}}$ is then estimated by

$$\begin{aligned}
\left| \int_Q \left(\int_0^1 \int_0^{s_1} \sigma''(\mathrm{D}\bar{u}+s_2\mathrm{D}(u-\bar{u}))\,ds_2\,ds_1 \right) \mathrm{D}(u-\bar{u})\mathrm{D}(u-\bar{u})\mathrm{D}\phi\,dx\,dt \right| & \\
\leq c\left(1 + \|\mathrm{D}\bar{u}\|^{p-3}_{L^\infty} + \|\mathrm{D}u\|^{p-3}_{L^\infty}\right)\|\mathrm{D}(u-\bar{u})\|^2_{L^{2\tilde{q}}}\|\mathrm{D}\phi\|_{L^{\tilde{q}'}} & \\
\leq c\left(1 + \|\mathrm{D}\bar{u}\|^{p-3}_{L^\infty} + \|\mathrm{D}u\|^{p-3}_{L^\infty}\right)\|\mathrm{D}(u-\bar{u})\|^{2\theta}_{L^{\bar{q}}}\|\mathrm{D}(u-\bar{u})\|^{2-2\theta}_{L^\infty}\|\mathrm{D}\phi\|_{L^{\tilde{q}'}} &
\end{aligned} \tag{3.27}$$

with $\theta = \frac{1}{2}\frac{\bar{q}}{\tilde{q}} > \frac{1}{2}$ by construction. Since $\bar{u}$, u are solutions of the nonlinear equation, the factors on the right-hand side of (3.27) are bounded. This proves that the right-hand side of (3.26) is in $L^{\tilde{q}}(0,T;W^{-1,\tilde{q}}(\Omega))$. Furthermore, Lemma 3.5 yields the Lipschitz-type estimate

$$\|\mathrm{D}(u-\bar{u})\|_{L^{\bar{q}}} \leq c\|h\|_{L^{\bar{q}}(W^{-1,\bar{q}})}.$$

Here again, the constant $\bar{\delta}$ fulfills the assumptions of that lemma. Now, we can apply Proposition 3.4 to get

$$\begin{aligned}
\|r\|_{L^{\tilde{q}}(W^{1,\tilde{q}})} + \|r\|_{L^\infty(L^{\tilde{q}})} \leq c\Big(&\|h\|^2_{L^{\bar{q}}(W^{-1,\bar{q}})} + c\left(1 + \|\mathrm{D}\bar{u}\|^{p-3}_{L^\infty} + \|\mathrm{D}u\|^{p-3}_{L^\infty}\right)\cdot \\
&\cdot \|\mathrm{D}(u-\bar{u})\|^{2-2\theta}_{L^\infty}\|h\|^{2\theta}_{L^{\bar{q}}(W^{-1,\bar{q}})}\Big).
\end{aligned}$$

The constants involved in this estimate stay bounded as $h \to 0$ in F. Hence it holds

$$\frac{\|r\|_{L^{\tilde{q}}(W^{1,\tilde{q}})} + \|r\|_{L^{\infty}(L^{\tilde{q}})}}{\|h\|_{L^{\bar{q}}(W^{-1,\bar{q}})}} \to 0 \text{ as } \|h\|_{L^{\bar{q}}(W^{-1,\bar{q}})} \to 0.$$

Thus, we proved Fréchet differentiability of S at $\bar{f}$ from F to the space $L^{\tilde{q}}(0,T;W^{1,\tilde{q}}) \cap L^{\infty}(0,T;L^{\tilde{q}})$ for all $\tilde{q} \in (2, 2+\bar{\delta})$. $\qquad\square$

Let us remark, that the proof for Fréchet differentiability of S mapping to $L^2(0,T;W^{1,2}(\Omega)) \cap L^{\infty}(0,T;L^2(\Omega))$ can not be proven directly using the Lipschitz estimate for $\|\mathrm{D}(u-\bar{u})\|_{L^2}$ of Lemma 3.3. Then (3.27) holds only with $\theta = 1/2$, which is not enough to prove that the remainder term vanishes as $h \to 0$. Hence, the detour via L^q-spaces was necessary.

It remains to investigate the adjoint operator of $S'(\bar{f})$. By Corollary 3.6 it is continuous from $L^2(0,T;V')$ to $L^2(0,T;V) \cap L^{\infty}(0,T;L^2)$. The adjoint operator $S'(\bar{f})^*$ is then linear and continuous from the dual space of $L^2(0,T;V) \cap L^{\infty}(0,T;L^2)$ to $L^2(0,T;V)$. By transposition arguments as in [14, Prop. 3.3], one finds that it is the solution operator of the so-called adjoint system

$$
\begin{aligned}
-w_t - \mathrm{div}(\sigma'(\mathrm{D}\bar{u})^{\top}\mathrm{D}w) - (\bar{u} \cdot \nabla)w + (\nabla\bar{u})^{\top}w + \nabla\mu &= z && \text{in } Q, \\
\mathrm{div}\, w &= 0 && \text{in } Q, \\
w &= 0 && \text{on } \Sigma, \\
w(T) &= 0 && \text{in } \Omega,
\end{aligned}
\tag{3.28}
$$

given in the very weak formulation

$$\int_0^T \langle w, \phi_t \rangle_{V',V}\, \mathrm{d}t + \int_Q \sigma'(\mathrm{D}\bar{u})^{\top}\mathrm{D}w\mathrm{D}\phi + (\bar{u} \cdot \nabla)\phi w + (\phi \cdot \nabla)\bar{u}w\ \mathrm{d}x\, \mathrm{d}t = \int_0^T \langle z, \phi \rangle_{V',V}\, \mathrm{d}t$$

for all $\phi \in L^2(0,T;V)$ with $\phi_t \in L^2(0,T;V')$ and $\phi(0) = 0$. Likewise (3.20), the term involving σ' is to be understood as

$$\sigma'(\mathrm{D}\bar{u})^{\top}\mathrm{D}w\mathrm{D}\phi = \sum_{i,j,k,l=1}^{2} \frac{\partial\sigma_{ij}(\mathrm{D}\bar{u})}{\partial_{kl}}(\mathrm{D}\phi)_{kl}(\mathrm{D}w)_{ij} = \sigma'(\mathrm{D}\bar{u})\mathrm{D}\phi\mathrm{D}w.$$

Let us finally consider the solvability of the system (3.28) and the regularity of its solution.

Corollary 3.8 *Let $\nabla\bar{u} \in L^{\infty}(Q)$ be given. Then, for each right-hand side $z \in L^q(0,T;W^{-1,q}(\Omega))$ with $q \in [2, 2+\delta)$, where δ is as in Proposition 3.4, the system (3.28) has a unique solution w that satisfies*

$$\|w\|_{L^q(W^{1,q})} + \|w\|_{L^{\infty}(L^q)} \le c\|z\|_{L^q(W^{-1,q})}$$

with a constant $c > 0$ depending on $\bar{u}$ but not on z.

The proof is identical to the proof of Corollary 3.6, since Kaplický's result [15] works also for the 'transposed' coefficients $\sigma(\mathrm{D}\bar{u})^{\top}$ in the differential operator. Here, again the smoothness $\nabla\bar{u} \in L^{\infty}(Q)$ is essential.

3.3 Necessary optimality conditions

Now, we have everything at hand to investigate necessary optimality conditions. Let us define the reduced cost functional using the control-to-state mapping S by

$$\Phi(f) = J(S(f), f).$$

Obviously, the minimization of J subject to the state equation is equivalent to minimize Φ over all admissible controls.

Now, let $\bar{f}$ be a locally optimal control in F^s, $s > 0$, with associated state $\bar{u} = S(\bar{f})$. Then $\bar{f}$ is also a local minimum of the reduced cost function Φ. The first-order necessary optimality condition is given by

$$\Phi'(\bar{f})h = 0 \quad \forall h \in F^s.$$

Let $S'(\bar{f})h$ be the solution of the linearized equation (3.25) with right-hand side h. Further, let us denote the embedding $F^s \to L^2(0, T; V')$ by E. Then the derivative Φ' can be written explicitly as

$$\Phi'(\bar{f})h = (\bar{u} - u_d, S'(\bar{f})Eh)_{L^2(Q)} + (\bar{f}, h)_{F^s} = 0.$$

Using the method of transposition, we can write

$$(S'(\bar{f})^*(\bar{u} - u_d), Eh)_{F^s} = (w, Eh) = \int_Q wh \, \mathrm{d}x \, \mathrm{d}t,$$

where w is the very weak solution of (3.28) with right-hand side $z = \bar{u} - u_d$. Summarizing these arguments, we proved the following.

Theorem 3.9 *Let $\bar{f}$ be a locally optimal control in F^s, $s > 0$, with associated state $\bar{u} = S(\bar{f})$. Then there is an adjoint state $\bar{w} \in L^2(0, T; V)$ as the unique very weak solution of the adjoint system*

$$\begin{aligned}
-w_t - \mathrm{div}(\sigma'(\mathrm{D}\bar{u})^\top \mathrm{D}w) - (\bar{u} \cdot \nabla)w + (\nabla\bar{u})^\top w + \nabla\mu &= \bar{u} - u_d & in\ Q, \\
\mathrm{div}\, w &= 0 & in\ Q, \\
w &= 0 & on\ \Sigma, \\
w(T) &= 0 & in\ \Omega,
\end{aligned} \tag{3.29}$$

such that

$$\int_Q wh \, \mathrm{d}x \, \mathrm{d}t + (\bar{f}, h)_{F^s} = 0$$

is fulfilled for all $h \in F^s$.

This necessary optimality conditions can be expressed equivalently in terms of the *Langrangian functional*, which we define by

$$L(u, f, w) = J(u, f) - \int_0^T \langle u_t, w \rangle_{V', V} \, \mathrm{d}t - \int_Q (\sigma(\mathrm{D}u)\mathrm{D}w + (u \cdot \nabla)uw) \, \mathrm{d}x \, \mathrm{d}t$$

$$+ \int_Q fw \, \mathrm{d}x \, \mathrm{d}t. \tag{3.30}$$

The adjoint state w plays now the role of a Lagrangian multiplier to the state equation constraint. Then the statement of Theorem 3.9 is equivalent to:

Corollary 3.10 *Let $(\bar{u}, \bar{f})$ be a pair of locally optimal control and state. Then it is necessary that there exists a multiplier $w \in L^2(0, T; V)$ such that*

$$\frac{\partial L}{\partial u}(\bar{u}, \bar{f}, \bar{w})\phi = 0 \quad \forall \phi \in L^2(0, T; V) \cap H^1(0, T; V')$$

and

$$\frac{\partial L}{\partial f}(\bar{u}, \bar{f}, \bar{w})h = 0 \quad \forall h \in F^s.$$

4 Second-order sufficient optimality conditions

In this section, we will briefly discuss sufficient optimality conditions. Let $\bar{f} \in F^s$, $s > 0$ be given such that $(\bar{u}, \bar{f}, \bar{w})$ fulfill optimality system of Theorem 3.9. Additionally, let us assume the following coercivity condition on the second derivative of the Lagrangian: there exists $\alpha > 0$ such that for all $h \in F^s$ with associated $z = S'(\bar{f})h$ it holds

$$L_{u,f}''(\bar{u}, \bar{f}, \bar{w})[(z, h)^2] \geq \alpha \|h\|_{F^s}^2. \tag{4.31}$$

For convenience we write this second derivative explicitly as

$$L_{u,f}''(\bar{u}, \bar{f}, \bar{w})[(z, h)^2] = \|z\|_{L^2(Q)}^2 + \gamma \|h\|_{F^s}^2 + \int_Q \sigma''(\mathrm{D}\bar{u})[\mathrm{D}z, \mathrm{D}z]\mathrm{D}w + 2(z \cdot \nabla)zw \, \mathrm{d}x \, \mathrm{d}t$$

with

$$\sigma''(\mathrm{D}\bar{u})[\mathrm{D}z_1, \mathrm{D}z_2]\mathrm{D}w = \sum_{i,j,k,l,m,n=1}^{2} \frac{\partial^2 \sigma_{ij}(\mathrm{D}\bar{u})}{\partial_{kl}\partial_{mn}} \cdot (\mathrm{D}z_1)_{kl}(\mathrm{D}z_2)_{mn}(\mathrm{D}w)_{ij}.$$

Here, one can see that new difficulties arise: the integral of this quantities must exist. Hence, we need higher regularity of solutions of the linearized as well as the adjoint equations. On Q the gradient $\mathrm{D}\bar{u}$ is essentially bounded. Thus, the regularity $\mathrm{D}z, \mathrm{D}w \in L^3(Q)$ would be sufficient to obtain $\sigma''(\mathrm{D}\bar{u})[\mathrm{D}z, \mathrm{D}z]\mathrm{D}w \in L^1(Q)$. If $\mathrm{D}w \in L^\infty(Q)$ holds, we can estimate for instance

$$\left| \int_Q \sigma''(\mathrm{D}\bar{u})[\mathrm{D}z_1, \mathrm{D}z_2]\mathrm{D}w \, \mathrm{d}x \, \mathrm{d}t \right| \leq c(\|\mathrm{D}\bar{u}\|_{L^\infty})\|\mathrm{D}z_1\|_{L^2}\|\mathrm{D}z_2\|_{L^2}\|\mathrm{D}w\|_{L^\infty}. \tag{4.32}$$

4.1 Higher regularity results

To prove higher regularity of the solutions of the linearized and of the adjoint system, we will rely on a recently published result by Bothe and Prüss [4] concerning maximal regularity of generalized Stokes systems. The key assumption is that the coefficients in the main part of the differential operator are continuous. This is indeed satisfied in our case: Theorem 2.2 gives $\nabla \bar{u} \in C(\bar{Q})$ and hence $\sigma'(\mathrm{D}\bar{u}) \in C(\bar{Q})$.

Lemma 4.1 *Let $\nabla \bar{u} \in C(\bar{Q})$ be given. Then the solution u of the linearized system* (3.25) *for $h \in F^s$, $s \geq 0$, satisfies $\nabla u \in L^4(Q)$.*

Proof. For right-hand sides in $L^2(Q)$ the solutions of (3.24) are in $W^{1,2}(0,T;L^2(\Omega))$ and in $L^2(0,T;H^2(\Omega) \cap V)$ by [4, Theorem 4.1]. This space is continuously embedded in $L^\infty(0,T;V)$ and in $L^4(0,T;W^{1,4}(\Omega))$. Hence it suffices to investigate the right-hand side $h - (\bar{u} \cdot \nabla)u - (u \cdot \nabla)\bar{u}$ of (3.25) in $L^2(Q)$. From the regularity of u provided by Corollary 3.6 and $\nabla \bar{u} \in L^\infty(Q)$ it follows $(\bar{u} \cdot \nabla)u + (u \cdot \nabla)\bar{u} \in L^2(Q)$ immediately. Then the claim follows using the maximal regularity result [4, Theorem 4.1]. $\qquad\square$

This result can be directly transferred to the case of the adjoint system (3.29).

Corollary 4.2 *Let $u_d \in L^2(Q)$ be given, and let $\bar{u}$ satisfy all regularity provided by Theorem 2.2. Then the solution w of the adjoint system* (3.28) *satisfies* $\mathrm{D}w \in L^4(Q)$.

Under higher regularity requirements on the right-hand sides or on the control space F, one can proven even a $W^{1,\infty}$-result for the linearized equation.

Lemma 4.3 *Let $\nabla \bar{u} \in C(\bar{Q})$ be given. Then the solution u of the linearized system* (3.25) *for $h \in F^s$, $s > 2/3$, satisfies $\nabla u \in L^\infty(Q)$.*

Proof. We want to show that the solution u belongs to the function space of maximal regularity $W^{1,q}(0,T;L^q(\Omega)) \cap L^q(0,T;W^{2,q}(\Omega))$ for some $q > 2$. Then w is also continuous on $[0,T]$ with values in $W^{2-2/q,q}(\Omega)$. The latter space is continuously imbedded in $W^{1,\infty}(\Omega)$ for $q > 4$, which gives us $\nabla u \in L^\infty(Q)$.

The maximal solution regularity is provided by [4, Theorem 4.1] under the assumption that $h - (\bar{u} \cdot \nabla)u - (u \cdot \nabla)\bar{u}$ belongs to $L^q(Q)$, $q > 4$.

Let us argue that for $s > \frac{2}{3}$ every $h \in F^s$ is also in $L^q(0,T;L^q(\Omega))$ for $q > 4$ with continuous (and compact) embedding. The space F^s is compactly embedded in $L^q(0,T;W^{s\theta,2}(\Omega))$ with $\theta \in (0,1)$, $-\frac{1}{q} < \frac{1}{2} - \theta$, see [3]. To allow $q > 4$ we have to choose $\theta < \frac{3}{4}$. The space $W^{s\theta,2}(\Omega)$ itself is embedded in $L^r(\Omega)$ with $r \le \frac{2}{1-s\theta}$ provided $s\theta < 1$. If $s\theta > 1$ holds, r can be made arbitrary large. That is, $W^{s\theta,2}(\Omega)$ is embedded in $L^r(\Omega)$ with $r > 4$ if $s\theta > \frac{1}{2}$ or $\theta > \frac{1}{2s}$ holds. Since $s > \frac{2}{3}$ by assumption, we can choose θ from the open and non-empty interval $(\frac{1}{2s}, \frac{3}{4})$ to allow for $q > 4$ and $r > 4$ in the above embeddings, which proves the embedding of F in $L^q(Q)$ for $q > 4$.

It remains to investigate the L^q-norm of $(\bar{u} \cdot \nabla)u + (u \cdot \nabla)\bar{u}$. At this point, we refer to the proof of [26, Theorem 2.7], where a bootstrapping procedure is carried out to prove a similar result for the linearized Navier-Stokes equations. In each step one has to apply the above mentioned result of [4] instead of the analogon for the Stokes system, which was used in [26]. $\qquad\square$

In the analysis of the second-order optimality conditions it will turn out that the regularity $\mathrm{D}w \in L^\infty(Q)$ would be benefitial, see below in the proof of Theorem 4.5 the discussion of the remainder term $\tilde{r}_2$.

Corollary 4.4 *Let $\bar{u}$ with $\nabla \bar{u} \in C(\bar{Q})$ and $u_d \in L^q(Q)$ with some $q > 4$ and be given. Then the solution w of the adjoint system satisfies $\mathrm{D}w \in L^\infty(Q)$.*

Proof. Since u fulfills homogeneous boundary conditions, the right-hand side $u - u_d$ of the adjoint equations is in $L^q(Q)$ with $q > 4$. By a similar bootstrapping technique, we can prove

the result following the lines of the analogous result [26, Theorem 3.3]. □

Let us observe that the regularity requirement on $\bar{u}$ of that corollary is already fulfilled if $\bar{u}$ is a strong solution according to Theorem 2.2 and (2.12).

4.2 Sufficiency

Finally, we state and prove that the coercivity condition (4.31) is sufficient for local optimality.

Theorem 4.5 *Let $(\bar{u}, \bar{f}, \bar{w})$ fulfill the optimality system of Theorem 3.9. Suppose further that there is a constant $\alpha > 0$ such that the coercivity assumption (4.31) is satisfied. Moreover, let us assume that σ is of class C^3 in addition to the assumptions of Section 2. Let the desired state u_d be in $L^q(Q)$, $q > 4$.*

Then there are constants $\rho > 0$ and $\beta > 0$ such that the quadratic growth conditions

$$J(u, f) \geq J(\bar{u}, \bar{f}) + \beta \|f - \bar{f}\|_{F^s}^2$$

holds for all $f \in F^s$ with $\|f - \bar{f}\|_{F^s} < \rho$ and $u = S(f)$, which implies that the control $\bar{f}$ is locally optimal.

Proof. Let $(\bar{u}, \bar{f}, \bar{w})$ be given according to the assumptions. Let us choose a positive radius $\rho_0 > 0$. Let $f \in F^s$ be another feasible control with $\|f - \bar{f}\|_{F^s} < \rho_0$. Define $u := S(f)$. We then have $J(\bar{u}, \bar{f}) = L(\bar{u}, \bar{f}, \bar{w})$ and $J(u, f) = L(u, f, \bar{w})$, since both $\bar{u}$ and u are solutions of the state equation. Taylor expansion of the Lagrangian then yields

$$
\begin{aligned}
L(u, f, \bar{w}) - L(\bar{u}, \bar{f}, \bar{w}) = L_u'(\bar{u}, \bar{f}, \bar{w})(u - \bar{u}) + L_{F^s}'(\bar{u}, \bar{f}, \bar{w})(f - \bar{f}) \\
+ \frac{1}{2} L_{u,f}''(\bar{u}, \bar{f}, \bar{w})[(u - \bar{u}, f - \bar{f})^2] + r_2. \quad (4.33)
\end{aligned}
$$

Due to the optimality conditions, the first two addends vanish, see e.g. Corollary 3.10.

The remainder term in the expansion above is given by

$$r_2 = \int_Q \int_0^1 \int_0^{s_1} \int_0^{s_2} \sigma'''(\mathrm{D}\bar{u} + s_3\mathrm{D}(u - \bar{u}))(\mathrm{D}(u - \bar{u}))^3 \, \mathrm{d}s_3 \, \mathrm{d}s_2 \, \mathrm{d}s_1 \, \mathrm{d}x \, \mathrm{d}t.$$

The argument of σ''' is in $L^\infty(Q)$, since $f, \bar{f}$ and thus $\mathrm{D}\bar{u}, \mathrm{D}u$ lie in bounded sets in F and $L^\infty(Q)$, respectively. Since σ is of class C^3, we have $|\sigma'''(\mathrm{D}\bar{u} + s_3\mathrm{D}(u - \bar{u}))| < M$ for all $s_3 \in (0, 1)$ a.e. on Q. Thus, we can estimate the remainder term r_2 as

$$|r_2| \leq \frac{1}{6} M \|\mathrm{D}(u - \bar{u})\|_{L^3}^3.$$

Analogously to the discussion in Lemma 3.7, there is a $\delta > 0$ such that the Lipschitz estimate of Lemma 3.5 holds for all $q \in (2, 2 + \delta)$ and all f in the neigborhood of $\bar{f}$. This allows us to estimate for some $q > 2$

$$|r_2| \leq \frac{M}{6} \|\mathrm{D}(u - \bar{u})\|_{L^q}^q \|\mathrm{D}(u - \bar{u})\|_{L^\infty}^{3-q} \leq c\|f - \bar{f}\|_{F^s}^q.$$

Let us denote by v the solution of the linearized equation with the right-hand side $f - \bar{f}$, i.e. $v = S'(\bar{f})(f - \bar{f})$. If we use v instead of $u - \bar{u}$, we will introduce an additional error $r_1 := (u - \bar{u}) - v = S(f) - S(\bar{f}) - S'(\bar{f})(f - \bar{f})$. The solution mapping S is Fréchet differentiable from F^s to $L^2(0, T; V)$ by Lemma 3.7, which yields

$$\|r_1\|_{L^2(V)} = o(\|f - \bar{f}\|_{F^s}) \text{ for } \|f - \bar{f}\|_{F^s} \to 0. \tag{4.34}$$

Now, let us replace the argument $u - \bar{u}$ of L'' in (4.33) by $v + r_1$. We obtain

$$\frac{1}{2} L''_{u,f}(\bar{u}, \bar{f}, \bar{w})[(u - \bar{u}, f - \bar{f})^2] = \frac{1}{2} L''_{u,f}(\bar{u}, \bar{f}, \bar{w})[(v, f - \bar{f})^2]$$
$$+ L''_u(\bar{u}, \bar{f}, \bar{w})[v, r_1] + \frac{1}{2} L''_u(\bar{u}, \bar{f}, \bar{w})[r_1, r_1].$$

Then the first addend fulfills the coercivity requirement (4.31). We know $\mathrm{D}\bar{u}, \mathrm{D}w \in L^\infty(Q)$ by Theorem 2.2 and Corollary 4.4, respectively. By Corollary 3.8, we know $\|v\|_{L^2(V)} \leq c\|f - \bar{f}\|_{F^s}$. Hence it follows from the property (4.34) of r_1 and the bound on L'' in (4.32)

$$|\tilde{r}_2| := \left| \frac{\partial^2 L}{\partial u^2}(\bar{u}, \bar{f}, \bar{w})[v, r_1] + \frac{1}{2} \frac{\partial^2 L}{\partial u^2}(\bar{u}, \bar{f}, \bar{w})[r_1, r_1] \right| = o(\|f - \bar{f}\|_{F^s}^2) \text{ for } \|f - \bar{f}\|_{F^s} \to 0.$$

Collecting all these estimates, we finally obtain

$$J(u, f) - J(\bar{u}, \bar{f}) \geq \frac{\alpha}{2} \|f - \bar{f}\|_{F^s}^2 - |r_2| - |\tilde{r}_2|.$$

Since both $|r_2|$ and $|\tilde{r}_2|$ are of size $o(\|f - \bar{f}\|_{F^s}^2)$, there is $\rho_1 > 0$ such that $|r_2| + |\tilde{r}_2| \leq \frac{\alpha}{4} \|f - \bar{f}\|_{F^s}^2$ holds for all $\|f - \bar{f}\|_{F^s} < \rho_1$. Thus, the quadratic growth holds with $\beta := \frac{\alpha}{4}$ and $\rho := \min\{\rho_0, \rho_1\}$. $\square$

5 Concluding remarks

We investigated optimal control problems for non-Newtonian fluids. The existence of optimal controls was proven. Here, it was important to be able to pass to the limit in the state equation. For the development of optimality conditions, it was essential that a solution theory providing $\mathrm{D}u \in L^\infty$ was available.

Let us now comment on two other situations: the case of periodic boundary condition and a possible extension of our work to the three-dimensional case.

5.1 Space-periodic boundary conditions

Based on the regularity results of Kaplický, Málek, Stará [16] we could reformulate our results for spatially periodic boundary conditions on a square domain even for controlling the system from the origin, i.e. $0 = 0$ could be allowed. Furthermore, this result is available for a wider range of exponents p in the assumptions on the nonlinearity, namely it was proven for $p > \frac{4}{3}$. Under similar assumptions on the controls f as in Theorem 2.2, they prove the regularity $u \in C^{1,\alpha}(\bar{Q})$. Here, again the parameter s in the definition of the control space is required to be positive. With these results at hand, one can prove existence of optimal controls as well as necessary and sufficient optimality conditions following the lines of the proofs of Proposition 2.1 and Theorems 3.9 and 4.5, respectively.

5.2 The spatially three-dimensional case

Existence of a unique solution of the equation holds for $p \geq \frac{9}{4}$, cf. [23]. This is a tremendous benefit from considering non-Newtonian fluids, because such uniqueness result is not available for Newtonian fluids, where it was selected by Clay Mathematical Institute as one out of seven most challenging mathematical "Millennium problems". In time of writing this article this problem was still waiting for its (affirmative or not) answer. On the other hand, although some regularity results are available for $\frac{9}{4} \leq p < 3$, see [22], even first-order optimality conditions are still not available. It seems that the Gâteaux differentiability of the solution mapping requires $L^\infty(Q)$-estimate for ∇u or ∇w (not available upto nowadays knowledge), a strategy we applied to the two-dimensional case in this article. Rather it indicates that the control-to-state mapping is not differentiable in the three-dimensional case, and some non-smooth methods are to be applied.

On the other hand, we can easily prove existence of optimal controls in $L^2(Q)$ for $p \geq \frac{9}{4}$. Let us fix the assumptions for the rest of this section: Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a C^3-boundary. Additionally, we assume that (2.4)–(2.5) is satisfied for

$$\frac{9}{4} = 2.25 \leq p.$$

Furthermore, we define the control space

$$F := L^2(Q).$$

The existence proof works also with pointwise constraints on the control as long as the resulting set of admissible controls is closed in L^2, convex, and of course non-empty.

Proposition 5.1 *Let the parameter p of σ in (2.4)–(2.5) satisfy $p \geq \frac{9}{4}$. Let an initial value $u_0 \in W^{1,p}(\Omega) \cap V$ be given. Furthermore, assume that the parameter γ is positive. Then there exists an optimal control $\bar{f} \in L^2(Q)$ for the optimal control problem (1.1)-(1.2).*

Proof. The proof follows the lines of the proof of Proposition 2.1. The uniqueness of solutions of (1.2) for $p \geq \frac{5}{2}$ goes back to Ladyzhenskaya [18]. This result was later improved to allow for $p \in [\frac{9}{4}, 3)$ in [22]. $\qquad \square$

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