Continuous and discrete flows over time
A general model based on measure theory

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Abstract Network flows over time form a fascinating area of research. They model the temporal dynamics of network flow problems occurring in a wide variety of applications. Research in this area has been pursued in two different and mainly independent directions with respect to time modeling: discrete and continuous time models.

In this paper we deploy measure theory in order to introduce a general model of network flows over time combining both discrete and continuous aspects into a single model. Here, the flow on each arc is modeled as a *Borel measure* on the real line (time axis) which assigns to each suitable subset a real value, interpreted as the amount of flow entering the arc over the subset. We focus on the maximum flow problem formulated in a network where capacities on arcs are also given as Borel measures and storage might be allowed at the nodes of the network. We generalize the concept of cuts to the case of these *Borel Flows* and extend the famous MaxFlow-MinCut Theorem.

Keywords Network Flows · Flows Over Time · Measure Theory · MaxFlow-MinCut

1 Introduction

Network flows over time (also called *dynamic flows* in the literature) are an interesting and challenging area of research. In contrast to classical *static* flows, they include a temporal dimension and consequently provide a more realistic modeling tool for a wide variety of applications. In general, there are two aspects which distinguish flows over time from static flows. Firstly, flow values on arcs are not constant but may change over time due to seasonally altering demands, supplies, and arc capacities. Secondly, flow does not travel instantaneously through a network but requires a certain amount of time to travel through each arc.

The notion of *flows over time* was first introduced by Ford and Fulkerson [7,8]. They study the *maximum flow over time* problem where the aim is to find the maximum

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amount of flow that can be sent from a source node to a sink node within a given time horizon. Ford and Fulkerson show that this problem can be solved efficiently by one minimum cost flow computation on the given network, where transit times of arcs are interpreted as arc costs. Since then, flows over time have become an area of active research and many authors have extensively studied different features of flows over time (see, e.g., [4, 10–12, 17, 18] and the references therein).

In the model studied by Ford and Fulkerson, time is measured in discrete time steps and arc capacities are time independent. In contrast to this, Philpott [14] and Anderson, Nash, and Philpott [2] study the maximum flow over time problem in a network with zero transit times and time-varying transit and storage capacities for the case where time is modeled as a continuum. They extend the concept of cuts to their continuous-time setting and establish a MaxFlow-MinCut theorem (see also [1]). This result was later extended by Philpott [15] to arbitrary transit times on the arcs.

For the case in which the network parameters (e.g., costs, capacities, supplies, and demands) are independent of time, and transit times on arcs as well as the time horizon are integral, Fleischer and Tardos [6] point out a close correspondence between discrete and continuous flows over time. In fact, in this case every continuous flow over time problem can be formulated and solved as a discrete flow over time problem. Fleischer and Tardos [6] show how a number of results and algorithms for the discrete time model can be carried over to the analogous continuous-time model, even if the time horizon is not integral. These results do not remain true for the more general setting where network parameters are subject to fluctuation over time.

Both discrete and continuous models have their advantages and disadvantages. Discrete flow over time problems are considerably easier to solve computationally, but they suffer from a serious drawback: the times at which decisions are being made are fixed in advance before the problem is solved. For many applications, this is by no means a necessary feature of the problem. This is where the continuous-time model comes into play allowing decisions to be made at arbitrary points in time. Although this approach is, in theory, suitable to model various applications such as pipeline systems for transportation (e.g., the problem of pumping water through a water distribution network), it fails to capture the discrete nature of typical applications such as vehicle routing and scheduling (e.g., the scheduling of trains in a railway network).

Contribution of the paper. A precise description of many real-world problems requires a combination of discrete- and continuous-time models. One such example is a crude oil distribution system. There are several methods that are used to transport crude oil: pipelines, tank trucks, railroad tank cars, barges, and tankers. Here, pumping crude oil into pipes naturally requires a continuous time model, whereas scheduling the transport of crude oil by tank trucks, railroad tank cars, barges, and tankers must be done in a discrete time model. As a consequence, it is worthwhile to capture both discrete and continuous aspects of real-world scenarios by means of a single model. Our approach is based on measure theory. The flow on each arc is modeled as a *measure* on the real line (time axis) which assigns to each suitable subset a real value, interpreted as the amount of flow entering the arc over the subset. We thus extend the notion of flows over time from the viewpoint of measure theory.

This approach is novel and has, to the best of our knowledge, never been pursued in the network flow literature so far. The only work taking a similar approach is by Philpott [16] who studies the continuous-time shortest path problem. This problem is an extension of the shortest path problem to networks with time-dependent arc costs.

Moreover, each arc has a fixed transit time and waiting at the nodes of the network is allowed but causes a time-dependent cost. Philpott [16] formulates the problem as a linear program in the space of finite Borel measures over $\mathbb{R}$, introduces a dual program, and proves various structural results including strong duality for the case where the cost functions are all Lipschitz-continuous.

In this paper we study the maximum flow over time problem formulated on a directed network where the flow on each arc is a Borel measure on $\mathbb{R}$ and storage might be allowed at the nodes of the network. Flow on arcs and storage of flow at nodes are subject to upper bounds given by Borel measures and right-continuous functions of bounded variation, respectively. We establish a MaxFlow-MinCut Theorem under the assumption that the arc capacities are *finite* Borel measures. While the basic idea is the same as in the proof of the corresponding theorem for the static case, our general measure-based definition of flows over time imposes quite a few complications and interesting challenges. We generalize the definition of cuts and their capacities as well as the concept of residual networks and reachable nodes to these Borel flows. It turns out that, in order to make this generalization work, a number of new ideas and techniques are required; an illustrative description of one particular problem occurring in this process is, for example, given at the beginning of Section 6 in Examples 1, 2, and 3.

Outline. The paper is organized as follows. We begin our discussion in Section 2 by briefly describing flows over time in a discrete and continuous model. We then explain how these two models can be combined into a single model by using measures and introduce the notion of *Borel flows*. In Section 3, we formulate the maximum Borel flow problem as an infinite-dimensional linear program and prove the existence of a maximum Borel flow.

Section 4 is devoted to the definition of *Borel cuts* and their capacities. A Borel cut is defined by assigning a Borel set to each node, containing the points in time when the node belongs to the source side of the cut. It is shown that the capacity of any Borel cut is an upper bound on the value of each Borel flow.

In Section 5 we define the residual network with respect to a Borel flow. Afterwards, in Section 6, it is shown that the value of a Borel flow can be improved if the sink is reachable from the source in the residual network. To this end, a procedure is presented to compute, for each node, the points in time at which the node is reachable from the source. In general, this procedure is not an algorithm for actual computation, but rather gives a definition for the set of the points in time at which flow can reach a node.

In Section 7 we show that the procedure of Section 5 yields a Borel cut whose capacity equals the value of a maximum Borel flow if it is applied to the residual network of a maximum Borel flow. This constitutes the main result of the paper.

In Section 8 we discuss several promising directions for future research. In Appendix A we briefly review the definitions and results from the area of measure theory which we use in this paper. We suggest that readers who are not familiar with measure theory first read Appendix A in order to follow the paper. In Appendix B we prove some technical lemmas.

2 Borel flows

We consider a directed graph $G = (V, E)$ with finite *node set* V and finite *arc set* E. A single commodity must be routed through G from a *source* $s \in V$ to a *sink* $t \in V$.

We assume that there is an s-v-path and a v-t-path in G for every node $v \in V$. This assumption imposes no loss of generality since nodes which are not contained in any s-t path are useless for routing flow from s to t and can therefore be deleted. An arc $e \in E$ from a node v to a node w is denoted by $e := (v, w)$. In this case, we say that node v is the *tail* of e and w is the *head* of e, and write $\text{tail}(e) := v$ and $\text{head}(e) := w$. Each arc $e \in E$ has an associated *transit time* $\tau_e \in \mathbb{R}$ specifying the required amount of time for traveling from the tail to the head of e. More precisely, if flow leaves node v at time θ along an arc $e = (v, w)$, it arrives at w at time $\theta + \tau_e$. Note that the transit times are not necessarily nonnegative. One particular reason is that we also consider flows in the residual network (see Section 5) and in general, the residual network contains arcs with negative transit times.

In general, the research on flows over time has pursued two main approaches with respect to time modeling, namely discrete and continuous time models. In the discrete model, time is discretized into intervals of unit length. For integral transit times $(\tau_e)_{e \in E}$ and an integral time horizon T, a *discrete flow over time* is defined by a function

$$x_e : \{0, 1, 2, \ldots, T-1\} \longrightarrow \mathbb{R}_{\geq 0} \ ,$$

for each arc $e \in E$. Here, the value $x_e(\theta)$ denotes the amount of flow which is sent at time θ into arc e and arriving at the head of e at time $\theta + \tau_e$. In contrast, a *continuous flow over time* consists of a Lebesgue integrable function

$$x_e : [0, T) \longrightarrow \mathbb{R}_{\geq 0} \ ,$$

for each arc $e \in E$. Here, the value $x_e(\theta)$ represents the rate at which flow enters arc e at time θ.

In what follows we make use of measure theory and introduce a new model of flows over time that encompasses both the discrete and the continuous model. To simplify notation, we consider the entire real line $\mathbb{R}$ instead of the time interval $[0, T)$ and set the initial time to $-\infty$ and the final time to ∞. This is, of course, no restriction since any maximum flow over time problem with time horizon T can be considered on $\mathbb{R}$ by letting all arc capacities be zero outside the interval $[0, T)$. In order to motivate the use of measure theory, we first let $\mathcal{B}$ be the collection of all intervals in $\mathbb{R}$. In order to describe the flow over time on each arc $e \in E$, we assign a value $x_e(I)$ to each time interval I indicating the amount of flow entering arc e over the time interval I. Thus, intuitively, the function $x_e : \mathcal{B} \to \mathbb{R}$ has to satisfy the following properties:

(i) The flow assigned to the empty set is 0, i.e., $x_e(\emptyset) = 0$.
(ii) An amount of flow is always nonnegative, i.e., $x_e(I) \geq 0$ for all $I \in \mathcal{B}$.
(iii) For a countable collection $(I_i)_{i \in \mathbb{N}}$ of pairwise disjoint intervals in $\mathcal{B}$, it holds that

$$x_e\Big(\bigcup_{i \in \mathbb{N}} I_i \Big) = \sum_{i \in \mathbb{N}} x_e(I_i) \ .$$

On closer inspection of property (iii) we observe that $\mathcal{B}$ must be closed under countable unions. Otherwise this property is not well defined. In addition we require that $\mathcal{B}$ is also closed under complement. Therefore we extend the definition of $\mathcal{B}$ to the smallest set containing all (open) intervals which is closed under complements and countable unions. Hence $\mathcal{B}$ is the *Borel σ-algebra on* $\mathbb{R}$ and a member $B \in \mathcal{B}$ is called a *Borel set* or *measurable set*. In this manner properties (i)–(iii) make x_e to a

Borel measure over $\mathbb{R}$. Thus, measure theory provides a realistic and adequate tool for modeling flow distributions over time.

Following the above observations, a *Borel flow* x is defined by a family of Borel measures

$$x_e : \mathcal{B} \longrightarrow \mathbb{R} \qquad\qquad \forall e \in E\ .$$

Here, the value $x_e(B)$ gives the amount of flow entering arc e over the Borel set B.

Moreover, with each arc $e \in E$ we associate a Borel measure $u_e : \mathcal{B} \to \mathbb{R}_{\geq 0}$ which denotes its capacity. That is, $u_e(B)$ is an upper bound on the amount of flow that is able to enter arc e over the Borel set B. We require that a Borel flow x fulfills *arc capacity constraints*

$$x_e(B) \leq u_e(B) \qquad\qquad \forall e \in E, B \in \mathcal{B}\ .$$

The flow x induces a *storage* function Y_v on $\mathbb{R}$ at each node v by the following *flow conservation constraint*

$$Y_v(\theta) := \sum_{e\in\delta^-(v)} x_e\big((-\infty, \theta - \tau_e]\big) - \sum_{e\in\delta^+(v)} x_e\big((-\infty, \theta]\big) \qquad \forall \theta \in \mathbb{R}\ . \tag{1}$$

Here and throughout the rest of the paper, $\delta^+(v)$ and $\delta^-(v)$ are used to denote the sets of arcs leaving node v and entering node v, respectively. Note that Y_v is the difference between two right continuous, monotonic increasing functions and thus is a right continuous function of bounded variation. In (1), the first sum represents the total amount of flow arriving at node v up to time θ. Analogously, the second sum represents the total amount of flow leaving node v up to time θ. Thus, the value $Y_v(\theta)$ gives the amount of flow stored at node v at the point in time θ. Flow originates at the source s and terminates at the sink t. Thus we must have

$$Y_s(\theta) \leq 0 \qquad \text{and} \qquad Y_t(\theta) \geq 0 \qquad \forall \theta \in \mathbb{R}\ .$$

We suppose that the storage of flow at a node $v \in V$ is bounded from above by a function $U_v : \mathbb{R} \to \mathbb{R}_{\geq 0}$. The value $U_v(\theta)$ is an upper bound on the amount of flow that can be stored at node v at time θ. We assume that U_v is of bounded variation and continuous from the right for each node v. This imposes no restriction since Y_v is a right continuous function of bounded variation. Further, with each node $v \in V \setminus \{s, t\}$ we also associate a right continuous function $L_v : \mathbb{R} \to \mathbb{R}_{\geq 0}$ of bounded variation. The value $-L_v(\theta)$ is a lower bound on the storage at node v at time θ. Note that we explicitly allow "negative" storage. We assume that the lower bound L_v is zero for all nodes v when talking about the original network. The reason for introducing lower bounds $(L_v)_{v\in V\setminus\{s,t\}}$ is to unify the notation later when we introduce the concept of residual networks. In a residual network $L_v(\theta)$ can be nonzero for some node v and some $\theta \in \mathbb{R}$, which indicates the maximum amount of flow that can be reduced from the available storage at node v at time θ.

We assume that there is no initial storage at any node and flow must not remain at any node except s and t. This means that the values $Y_v(-\infty) := \lim_{\theta\to-\infty} Y_v(\theta)$ and $Y_v(\infty) := \lim_{\theta\to\infty} Y_v(\theta)$ must be zero for each node $v \in V \setminus \{s, t\}$. Notice that both limits exist since Y_v is of bounded variation. Therefore, for each $v \in V \setminus \{s, t\}$, we require $U_v(-\infty) = U_v(\infty) = L_v(-\infty) = L_v(\infty) = 0$.

A Borel flow x with corresponding storage Y fulfills the *node capacity constraint* at node $v \in V \setminus \{s,t\}$ if

$$-L_v(\theta) \leq Y_v(\theta) \leq U_v(\theta) \tag{2}$$

for all $\theta \in \mathbb{R}$. The Borel flow x is called an *s-t Borel flow* if it satisfies the node capacity constraint at all nodes $v \in V \setminus \{s,t\}$. The *value* $\mathrm{val}(x)$ of an s-t Borel flow x is defined as the total net flow out of node s, that is,

$$\mathrm{val}(x) := \sum_{e \in \delta^+(s)} |x_e| - \sum_{e \in \delta^-(s)} |x_e| \ .$$

Here and subsequently, $|x_e|$ denotes the total amount of flow entering arc e over time, i.e., $|x_e| := x_e(\mathbb{R})$. Notice that, due to flow conservation, $\mathrm{val}(x)$ is equal to the total net inflow into node t. An s-t Borel flow is called *maximum* if it has maximum value among all s-t Borel flows.

The problem which we analyze in this paper is:

Maximum Borel Flow Problem (MBFP)	
Input:	A network consisting of a directed graph $G := (V,E)$, a source $s \in V$, a sink $t \in V$, arc capacities $u_e : \mathcal{B} \to \mathbb{R}_{\geq 0}$ for $e \in E$, and node capacities $U_v : \mathbb{R} \to \mathbb{R}_{\geq 0}$ and $L_v : \mathbb{R} \to \mathbb{R}_{\geq 0}$ for $v \in V \setminus \{s,t\}$.
Task:	Find a maximum s-t Borel flow x.

A natural question arising here is whether there exists a maximum s-t Borel flow. As we see in the next section we are able to prove the existence of such a Borel flow if u_e is finite (i.e., $u_e(\mathbb{R}) < \infty$) for each arc $e \in E$. Therefore we assume that u_e and, hence, x_e are finite for each arc e throughout the rest of the paper.

In what follows, we briefly illustrate how MBFP includes the maximum flow over time problem in both discrete and continuous models as special cases. Since these models are defined on a time interval $[0,T)$ we have to set up the arc capacities such that no flow can be sent along an arc e outside the interval $[0,T)$, i.e., $u_e(\mathbb{R} \setminus [0,T)) = 0$. If arc capacities $(u_e)_{e \in E}$ are discrete measures, concentrated on a finite set $\Omega = \{\theta_1, \ldots, \theta_m\}$ (e.g., $\Omega = \{0,1,\ldots,T-1\}$), then, for each arc $e \in E$, x_e must be a measure concentrated on Ω. Here $x_e(\{\theta\})$ gives the amount of flow entering arc e at time step $\theta \in \Omega$ which is bounded from above by $u_e(\{\theta\})$.

We turn now to the continuous model. Let the arc capacities $(u_e)_{e \in E}$ be absolutely continuous with respect to the Lebesgue measure. It follows from the Radon–Nikodym Theorem (see, e.g., [5]) that for each arc $e \in E$ there exists a Lebesgue measurable function $u'_e : [0,T) \to \mathbb{R}_{\geq 0}$ such that $u_e(B) = \int_B u'_e \, d\theta$ for each Borel set B. Since $x_e(B) \leq u_e(B)$ for each arc $e \in E$ and each Borel set B, the measure x_e is also absolutely continuous, and hence there exists a Lebesgue measurable function $x'_e : [0,T) \to \mathbb{R}_{\geq 0}$ such that $x_e(B) = \int_B x'_e \, d\theta$ for each B. It is well-known that $0 \leq x_e(B) \leq u_e(B)$ for each Borel set B implies $0 \leq x'_e(\theta) \leq u'_e(\theta)$ for almost every $\theta \in [0,T)$. Hence, the value $x'_e(\theta)$ can be interpreted as the rate of flow (i.e., amount of flow per time unit) entering arc e at the point in time θ and the value $u'_e(\theta)$ as an upper bound on the flow rate into arc e at time θ.

3 An Infinite-dimensional Linear Program for MBFP

As mentioned previously, a natural question for MBFP is whether there exists a maximum s-t Borel flow. In order to answer this question, we provide a mathematical formulation of MBFP and prove the existence of an optimal solution for the corresponding problem. For it and throughout this paper, we use the following notation and we also refer to Appendix A for readers unfamiliar with measure theory. For convenience, we denote the measures by small letters (such as μ, ν, f, y, z, u, h) and their corresponding distribution functions by capital letters (such as M, N, F, Y, Z, U, H). Moreover, for a real value τ and a Borel measure μ the *shifted measure* $\mu - \tau$ is defined by $(\mu - \tau)(B) = \mu(B - \tau)$ for each $B \in \mathcal{B}$, where $B - \tau := \{\theta - \tau \mid \theta \in B\}$. In this case, the distribution function of $\mu - \tau$ is denoted by $M - \tau$.

Defining the mathematical program, we recall that for each node $v \in V \setminus \{s, t\}$, the storage function Y_v is a right continuous function of bounded variation and moreover $Y_v(-\infty) = 0$ (because of (2) and the assumption $L_v(-\infty) = U_v(-\infty) = 0$). This implies that Y_v is a distribution function for each $v \in V \setminus \{s, t\}$ and thus there is a corresponding *signed* Borel measure y_v derived from the formula $y_v\big((-\infty, \theta]\big) = Y_v(\theta)$. For a Borel set B, the value $y_v(B)$ can be interpreted as the overall change in storage at v over the Borel set B. Then the flow conservation constraint (1) at node $v \in V \setminus \{s, t\}$ can be written as

$$\sum_{e \in \delta^+(v)} x_e - \sum_{e \in \delta^-(v)} (x_e - \tau_e) + y_v = 0 \ . \tag{3}$$

Following our above discussion, a mathematical formulation of MBFP is given by the following infinite-dimensional linear program:

$$\begin{aligned} \max \quad & \sum_{e \in \delta^+(s)} |x_e| - \sum_{e \in \delta^-(s)} |x_e| \\ \text{s.t.} \quad & \sum_{e \in \delta^+(v)} x_e - \sum_{e \in \delta^-(v)} (x_e - \tau_e) + y_v = 0 \qquad \forall v \in V \ , \\ & \qquad\qquad 0 \le x_e \le u_e \qquad \forall e \in E \ , \\ & \qquad\qquad -L_v \le Y_v \le U_v \qquad \forall v \in V \setminus \{s, t\} \ , \\ & \qquad\qquad -Y_s \ge 0 \ , \\ & \qquad\qquad Y_t \ge 0 \ . \end{aligned} \tag{MBFP}$$

For every Borel flow $x = (x_e)_{e \in E}$ a signed measure $y = (y_v)_{v \in V \setminus \{s,t\}}$ with corresponding distribution function $Y = (Y_v)_{v \in V \setminus \{s,t\}}$ is uniquely determined by (3). If x and Y satisfy the constraints of (MBFP), we say that x (with corresponding storage Y) is *feasible*. The optimum value of (MBFP) is defined as the supremum of $\text{val}(x)$ over all feasible Borel flows x. The following theorem shows that there exists a feasible Borel flow which achieves the optimum value of (MBFP) and thus the maximum in (MBFP) is well defined. Note that we call such an x maximum Borel flow.

Theorem 1 *There exists a maximum Borel flow for the problem* (MBFP)*.*

Proof Let R be the feasible region of (MBFP), that is, the set of all feasible Borel flows. The feasible region R is nonempty since the zero flow is feasible. Moreover, R is bounded since by our assumption u_e is finite for each $e \in E$ and hence, for any feasible Borel flow x we have

$$||x|| := \sum_{e \in E} |x_e| \leq \sum_{e \in E} |u_e| < \infty \ . \tag{4}$$

By the Riesz Representation Theorem (see, e.g., [5]), the space $M(\mathbb{R})$ of finite Borel measures on $\mathbb{R}$ is the topological dual of the space $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$. We can show by a similar argument as in [16] that R is closed in the weak topology $\sigma(M(\mathbb{R}), C(\mathbb{R}))$. On the other hand, we know by Alaoglu's Theorem (see again [5]) that the closed unit ball of the space $M(\mathbb{R})$ is compact in the weak topology. Hence, we can conclude that R is compact in the weak topology on $M(\mathbb{R})$. This establishes the result since the objective function of (MBFP) is a linear $\sigma(M(\mathbb{R}), C(\mathbb{R}))$-continuous functional and hence attains its maximum over a compact set. For a detailed treatment of the methodology we refer to [1, Chapter 3]. □

4 Borel cuts

In the static framework of network flows, an *s*-*t* cut is defined as a subset $S \subseteq V$ of nodes with $s \in S$ and $t \in V \setminus S$. The capacity $\text{cap}(S) := \sum_{e \in \delta^+(S)} u_e$ is defined as the sum of the capacities of arcs going from the *s*-side S to the *t*-side $V \setminus S$. It is a famous result that the value of a maximum *s*-*t* flow equals the minimum capacity of an *s*-*t* cut. This is well-known as the MaxFlow-MinCut Theorem which is due to Ford and Fulkerson [8]. We wish to develop a similar result for Borel flows. The first step is to extend the definition of an *s*-*t* cut and its capacity to the case of Borel flows in an elaborate way.

We define a *Borel cut* $S := (S_v)_{v \in V}$ by measurable sets S_v, one for each $v \in V$. A Borel cut $S = (S_v)_{v \in V}$ is called an *s*-*t* Borel cut if $S_s = \mathbb{R}$ and $S_t = \emptyset$. We denote with $S_v^c := \mathbb{R} \setminus S_v$ the complement of S_v. We say that node v belongs to the *s-side* of S for the points in time $\theta \in S_v$ and to the *t-side* of S for the points in time $\theta \in S_v^c$. Thus, an arc $e = (v, w)$ connects the *s*-side to the *t*-side for all times in $S_v \cap (S_w - \tau_e)^c$.

Since we want to find a tight upper bound on the maximum value of an *s*-*t* Borel flow we are interested in the capacity of an *s*-*t* Borel cut. For technical reasons we restrict the definition of *s*-*t* Borel cuts (the reasons are discussed below before Lemma 1). For this and the remainder of the paper, let $M^{\succ 0}$ be the set of all points $\theta \in \mathbb{R}$ where a distribution function M or its left limit is positive at θ. More precisely:

$$M^{\succ 0} := \left\{\theta \in \mathbb{R} \mid M(\theta-) > 0 \text{ or } M(\theta) > 0\right\} \ , \tag{5}$$

where $M_v(\theta-)$ denotes the limit of M_v at θ from left, i.e., $M_v(\theta-) := \lim_{\vartheta \nearrow \theta} M_v(\vartheta)$. Since M is right continuous the set $M^{\succ 0}$ is the countable union of pairwise disjoint intervals. Now an *s*-*t* Borel cut $S = (S_v)_{v \in V}$ has to satisfy the following additional property: For every node $v \in V \setminus \{s, t\}$ the set

$$\Gamma_v := S_v \cap U_v^{\succ 0} \tag{6}$$

is a countable union of pairwise disjoint intervals.

Let $S = (S_v)_{v \in V}$ be an *s*-*t* Borel cut and consider a node v. By definition, Γ_v is expressible as $\bigcup_{i \in J_v} I_{v,i}$, where J_v is a countable set of indices and $I_{v,i}$, $i \in J_v$, are pairwise disjoint intervals. Each interval $I_{v,i}$ is supposed to be inclusion-wise maximal, i.e., there is no interval $I \subseteq \Gamma_v$ strictly containing $I_{v,i}$. We keep this assumption throughout the paper; whenever we express Γ_v as a countable union of intervals we suppose that the intervals are inclusion-wise maximal. Let $\alpha_{v,i}$ and $\beta_{v,i}$ be the left and right boundary of the interval $I_{v,i}$, respectively. An interval $I_{v,i}$ can be of the form $(\alpha_{v,i}, \beta_{v,i})$, $[\alpha_{v,i}, \beta_{v,i})$, $(\alpha_{v,i}, \beta_{v,i}]$, or $[\alpha_{v,i}, \beta_{v,i}]$. Therefore we partition the set J_v of indices into four subsets. Let J_v^1 (J_v^2, J_v^3, and J_v^4) be the set of indices i for which $I_{v,i}$ is open (left-closed & right-open, right-closed & left-open, and closed, respectively). With these constructions, the *capacity* $\operatorname{cap}(S)$ of S is defined by

$$\begin{aligned} \operatorname{cap}(S) := & \sum_{e=(v,w)\in E} u_e\big(S_v \cap (S_w - \tau_e)^c\big) + \\ & \sum_{v\in V\setminus\{s,t\}} \Big(\sum_{i\in J_v^1\cup J_v^2} U_v(\beta_{v,i}-) + \sum_{i\in J_v^3\cup J_v^4} U_v(\beta_{v,i}) \Big) . \end{aligned} \tag{7}$$

We set the capacity $\operatorname{cap}(S)$ to ∞ if any infinite sum does not converge. The first sum indicates the contribution of capacities of arcs at the points in time when the arcs cross the cut; the second one represents the contribution of the storage capacities at points in time when node v passes the cut from the s-side to the t-side. Note that points in time at which the capacity is zero do not contribute any value to the capacity of the cut. Therefore it is sufficient to consider only Γ_v when considering the contribution of node capacities to the capacity of the cut. We refer to an s-t Borel cut whose capacity is minimum among all s-t Borel cuts as a *a minimum Borel cut*.

In the following we shortly explain why we restrict the definition of an s-t Borel cut S to those cuts such that the each set $\Gamma_v = S_v \cap U^{\succ 0}$, $v \in V$ is the countable union of pairwise disjoint intervals. First notice that, in general, S_v can be any measurable set (e.g., the set of irrational numbers, the Cantor set and so on). If we do not consider the restriction on the sets Γ_v, then the contribution of node capacities to the capacity of S becomes unclear. In particular, there is no obvious definition of the points in time at which a node v passes the cut from the s-side to the t-side. For overcoming this problem one could require that S_v, instead of the set Γ_v, is a countable union of intervals for each node $v \in V$. But as we observe below in Example 1, there might be no s-t Borel cut whose capacity equals the value of a maximum Borel flow. On the other hand, we will show that there always exists a minimum Borel cut S in which the sets Γ_v, $v \in V$, are countable union of intervals. First we prove that the capacity of an s-t Borel cut is an upper bound on the value of any s-t Borel flow. For the proof, we require the following technical result, the proof of which can be found in Appendix B.

Lemma 1 *Let μ be a finite signed Borel measure on $\mathbb{R}$ with a nonnegative distribution function M. Let $A := \mathbb{R} \setminus M^{\succ 0}$ be the set of points $\theta \in \mathbb{R}$ for which M is continuous and zero at θ. Then $\mu|_A = 0$, i.e., the set A is a strict μ-null set.*

Lemma 2 *The capacity of any s-t Borel cut is an upper bound on the value of any s-t Borel flow.*

Proof Let x be an s-t Borel flow with corresponding storage y and let S be an s-t Borel cut. Since $S_s = \mathbb{R}$ and $\mathbb{R} - \tau = \mathbb{R}$ for each $\tau \in \mathbb{R}$, we can express the value of x as

$$\operatorname{val}(x) = \sum_{e\in\delta^+(s)} x_e(S_s) - \sum_{e\in\delta^-(s)} (x_e - \tau_e)(S_s) . \tag{8}$$

Moreover, as $S_t = \emptyset$, we have

$$\sum_{e\in\delta^+(t)} x_e(S_t) - \sum_{e\in\delta^-(t)} (x_e - \tau_e)(S_t) = 0 \ . \tag{9}$$

On the other hand, the flow conservation constraint at node $v \in V$ is separately valid for all $B \in \mathcal{B}$ and hence for its corresponding set S_v. By summing up these equations over $v \in V \setminus \{s,t\}$, we get

$$\sum_{v\in V\setminus\{s,t\}} \left(\sum_{e\in\delta^+(v)} x_e(S_v) - \sum_{e\in\delta^-(v)} (x_e - \tau_e)(S_v) \right) + \sum_{v\in V\setminus\{s,t\}} y_v(S_v) = 0 \ . \tag{10}$$

Thus, adding (8), (9), and (10) leads to

$$\mathrm{val}(x) = \sum_{v\in V} \left(\sum_{e\in\delta^+(v)} x_e(S_v) - \sum_{e\in\delta^-(v)} (x_e - \tau_e)(S_v) \right) + \sum_{v\in V\setminus\{s,t\}} y_v(S_v) \ . \tag{11}$$

In the first term on the right hand side of the above equation, each arc appears exactly once with a positive sign and exactly once with a negative sign. Therefore the first term is equal to sum of $x_e(S_v) - (x_e - \tau_e)(S_w)$ over all arcs $e = (v,w) \in E$. Since $(x_e - \tau_e)(S_w) = x_e(S_w - \tau_e)$ for each arc $e = (v,w) \in E$, a single term of this sum is bounded by $u_e\big(S_v \cap (S_w - \tau_e)^c\big)$ as follows:

$$\begin{aligned} x_e(S_v) - x_e(S_w - \tau_e) &= x_e\big(S_v \cap (S_w - \tau_e)^c\big) - x_e\big(S_v^c \cap (S_w - \tau_e)\big) \\ &\le u_e\big(S_v \cap (S_w - \tau_e)^c\big) \ . \end{aligned}$$

Hence, the first term is bounded by $\sum_{e=(v,w)\in E} u_e\big(S_v \cap (S_w - \tau_e)^c\big)$.

It remains to bound the second term on the right hand side of (11). Since $Y_v \le U_v$ we have $Y_v^{\succ 0} \subseteq U_v^{\succ 0}$ and thus $S_v \setminus U_v^{\succ 0} \subseteq \mathbb{R} \setminus U_v^{\succ 0} \subseteq \mathbb{R} \setminus Y_v^{\succ 0}$. On the other hand, Lemma 1 shows that $\mathbb{R} \setminus Y_v^{\succ 0}$ is a strict y_v-null set. So we get $y_v(S_v \setminus U_v^{\succ 0}) = 0$ and as a consequence $y_v(S_v) = y_v(S_v \cap U_v^{\succ 0}) = y_v(\Gamma_v)$. Hence, we can bound a single term of the sum in the second term of (11) as follows:

$$\begin{aligned} y_v(S_v) = y_v(\Gamma_v) &= \sum_{i=1}^{\infty} y_v(I_{v,i}) \\ &= \sum_{i\in J_v^1} \big(Y_v(\beta_{v,i}-) - Y_v(\alpha_{v,i})\big) + \sum_{i\in J_v^2} \big(Y_v(\beta_{v,i}-) - Y_v(\alpha_{v,i}-)\big) \\ &\quad + \sum_{i\in J_v^3} \big(Y_v(\beta_{v,i}) - Y_v(\alpha_{v,i})\big) + \sum_{i\in J_v^4} \big(Y_v(\beta_{v,i}) - Y_v(\alpha_{v,i}-)\big) \\ &\le \sum_{i\in J_v^1\cup J_v^2} U_v(\beta_{v,i}-) + \sum_{i\in J_v^3\cup J_v^4} U_v(\beta_{v,i}) \ . \end{aligned}$$

Hence, the second term is bounded by $\sum_{v\in V\setminus\{s,t\}} \Big(\sum_{i\in J_v^1\cup J_v^2} U_v(\beta_{v,i}-) + \sum_{i\in J_v^3\cup J_v^4} U_v(\beta_{v,i}) \Big)$.
This concludes the proof. □

5 Residual networks

As already mentioned before, we wish to develop a MaxFlow-MinCut Theorem for our setting of flows over time, that is, the existence of an s-t Borel flow x and an s-t Borel cut S for which $\mathrm{val}(x) = \mathrm{cap}(S)$ holds. Once this is proved, we can conclude that x is a maximum Borel flow and S is a minimum Borel cut because of Lemma 2. The existence of a maximum s-t Borel flow is guaranteed by Theorem 1. Thus, to derive a MaxFlow-MinCut theorem, it suffices to construct an s-t Borel cut whose capacity equals the value of a maximum s-t Borel flow. One approach is to go along the same lines as in the static case. More precisely, starting from a given maximum s-t Borel flow x, we try to come up with an s-t Borel cut whose capacity equals the value of x. Thus, we need the concept of residual networks as well as augmentation for Borel flows.

For an arc $e = (v, w) \in E$ we denote the corresponding *backward arc* (w, v) by $\overleftarrow{e} := (w, v)$. If for some nodes $v, w \in V$ both arcs (v, w) and (w, v) belong to E we have to introduce backward arcs for each of them leading to an inconsistence notation. This conflict could be resolved by introducing an artificial node on one of the arc. So we assume without loss of generality that this problem never occur. The transit time of a backward arc $\overleftarrow{e}$ with $e \in E$ is defined by $\tau_{\overleftarrow{e}} := -\tau_e$. Notice that the transit time of a backward arc is in general negative. We denote the set of all backward arcs by $\overleftarrow{E}$ and set $E^r := E \cup \overleftarrow{E}$.

With respect to a given Borel flow x, we introduce the following definitions. For each arc $e \in E$ we define the *residual capacity* of e and the corresponding backward arc $\overleftarrow{e}$ as $u^r_e := u_e - x_e$ and $u^r_{\overleftarrow{e}} := x_e - \tau_e$, respectively. For each $B \in \mathcal{B}$, $u^r_e(B)$ and $u^r_{\overleftarrow{e}}(B)$ represent the maximum amount by which flow can be increased and reduced, respectively, on arc e over B without violating the constraints $0 \leq x_e \leq u_e$.

Let Y be the storage function induced by x. For each node $v \in V \setminus \{s, t\}$, we define the *upper* and *lower residual capacity* of v as $U^r_v := U_v - Y_v$ and $L^r_v := L_v + Y_v$, respectively. For any point in time $\theta \in \mathbb{R}$, $U^r_v(\theta)$ gives the maximum additional amount of flow that can be stored at node v at time θ and $L^r_v(\theta)$ represents the maximum amount of flow that can be reduced from the available storage at node v at time θ without violating the constraints $-L_v \leq Y_v \leq U_v$.

The network consisting of the residual graph $G^r := (V, E^r)$ and the residual capacities $(u^r_e)_{e \in E^r}$, $(U^r_v)_{v \in V}$, and $(L^r_v)_{v \in V}$ is called the *residual network* of G with respect to the Borel flow x. An s-t Borel flow f in G^r with $\mathrm{val}(f) > 0$ is called an *augmenting s-t Borel flow.*

Lemma 3 *Let x be an s-t Borel flow in G. If there is an augmenting s-t Borel flow f, then x is not maximum.*

Proof We define the *augmented Borel flow* x^f by

$$x^f_e := x_e + f_e - (f_{\overleftarrow{e}} - \tau_{\overleftarrow{e}}) \qquad \text{for all } e \in E \ .$$

We prove that x^f is a feasible Borel flow of value $\mathrm{val}(x^f) = \mathrm{val}(x) + \mathrm{val}(f)$.

First we show that $0 \leq x^f_e \leq u_e$ for all arcs $e \in E$. Because of the definition of residual capacities, for each arc $e \in E$ we get

$$x^f_e = x_e + f_e - (f_{\overleftarrow{e}} - \tau_{\overleftarrow{e}}) \leq x_e + f_e \leq x_e + (u_e - x_e) = u_e \qquad \text{and}$$
$$x^f_e = x_e + f_e - (f_{\overleftarrow{e}} - \tau_{\overleftarrow{e}}) \geq x_e - (f_{\overleftarrow{e}} - \tau_{\overleftarrow{e}}) \geq x_e - \big((x_e - \tau_e) + \tau_e\big) = 0 \ .$$

Let z and y^f be the storage induced by f and x^f, respectively, in the residual network G^r and the original network G. For all $v \in V1$ we show that $y_v^f = y_v + z_v$ in the following. Note that x and f satisfy (MBFP) on G and G^r, respectively, and that G is a subgraph of G^r. By definition of storage y_f we get

$$\begin{aligned} y_v^f &= \sum_{e\in\delta^-(v)} (x_e^f - \tau_e) - \sum_{e\in\delta^+(v)} x_e^f \\ &= \sum_{e\in\delta^-(v)} \Big(\big(x_e + f_e - (f_{\overleftarrow{e}} - \tau_{\overleftarrow{e}})\big) - \tau_e\Big) - \sum_{e\in\delta^+(v)} \big(x_e + f_e - (f_{\overleftarrow{e}} - \tau_{\overleftarrow{e}})\big) \ . \end{aligned}$$

Further, we know that $\big(-(f_{\overleftarrow{e}} + \tau_e)\big) - \tau_e$ is equal to $-f_{\overleftarrow{e}}$ and *not* equal to $-f_{\overleftarrow{e}} - 2\tau_e$ since the subtraction of a real number is a (horizontal) shifting of the measure. Thus, expanding the shifting in the first sum leads to

$$y_v^f = y_v + \sum_{e\in\delta^-(v)} (f_e - \tau_e) + \sum_{e\in\delta^+(v)} (f_{\overleftarrow{e}} - \tau_{\overleftarrow{e}}) - \sum_{e\in\delta^+(v)} f_e - \sum_{e\in\delta^-(v)} f_{\overleftarrow{e}} \ .$$

If an arc e is contained in $\delta^+(v)$ and $\delta^-(v)$, then the backward arc $\overleftarrow{e}$ is contained in $\delta^-_{G^r}(v)$ and $\delta^+_{G^r}(v)$, respectively. Hence, we obtain

$$y_v^f = y_v + \sum_{e\in\delta^-_{G^r}(v)} (f_e - \tau_e) - \sum_{e\in\delta^+_{G^r}(v)} f_e = y_v + z_v \ .$$

For the feasibility, it remains to show $-L_v \le Y_v^f \le U_v$ and $|y_v^f| = 0$ for all $v \in V\setminus\{s,t\}$. This is obtained as follows:

$$\begin{aligned} Y_v^f &= Y_v + Z_v \ge Y_v - (L_v + Y_v) = -L_v \ , \\ Y_v^f &= Y_v + Z_v \le Y_v + (U_v - Y_v) = U_v \ , \\ |y_v^f| &= |y_v + z_v| = |y_v| + |z_v| = 0 \ . \end{aligned}$$

Thus x^f is a feasible Borel flow. In particular, this means that x_e^f is nonnegative for each arc $e \in E$. Further, shifting does not influence the norm of a measure. Therefore we get for the flow value of x^f:

$$\begin{aligned} \mathrm{val}(x^f) &= \sum_{e\in\delta^+(s)} |x_e^f| - \sum_{e\in\delta^-(v)} |x_e^f| \\ &= \sum_{e\in\delta^+(s)} \big|x_e + f_e - (f_{\overleftarrow{e}} - \tau_{\overleftarrow{e}})\big| - \sum_{e\in\delta^-(s)} \big|x_e + f_e - (f_{\overleftarrow{e}} - \tau_{\overleftarrow{e}})\big| \\ &= \sum_{e\in\delta^+(s)} \big(|x_e| + |f_e| - |f_{\overleftarrow{e}}|\big) - \sum_{e\in\delta^-(s)} \big(|x_e| + |f_e| - |f_{\overleftarrow{e}}|\big) \\ &= \mathrm{val}(x) + \sum_{e\in\delta^+_{G^r}(s)} f_e - \sum_{e\in\delta^-_{G^r}(s)} f_e \\ &= \mathrm{val}(x) + \mathrm{val}(f) \ . \end{aligned}$$

This completes the proof since x^f is a feasible Borel flow with a strictly larger flow value than x. □

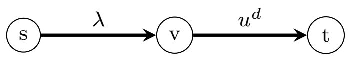


Fig. 1 Network for Example 1. The capacities are shown on the arcs and all transit times are 0.

6 Reachability

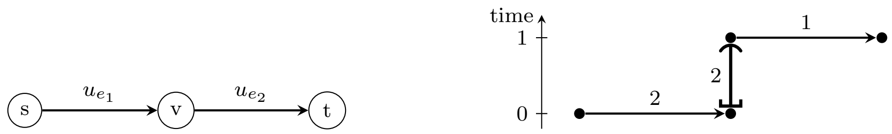
The next step is to find a Borel cut whose capacity is equal to the value of a maximum Borel flow. As already mentioned, it is quite natural to carry over the classical static approach. In the static case, a minimum cut can be defined by the nodes that are reachable from source s in the residual network of a maximum flow. For the case of Borel flows this means that a minimum Borel cut can be defined by the times when a node is reachable from s. It turns out, however, that Borel flows require a somewhat more intricate definition, as the following three examples indicate. Example 1 shows that we must exclude certain points in time and Example 2 shows that we also must add certain points in time. Moreover, Example 3 shows that a careful treatment is required in adding or excluding certain points in time.

Example 1 Consider the network depicted in Figure 1 which consists of three nodes s, v, and t and of two arcs $e_1 = (s, v)$ and $e_2 = (v, t)$. All transit times are 0. The capacity of arc e_1 is set to the Lebesgue measure λ (i.e., $\lambda([a, b]) = b - a$ for all real $a \leq b$) and the capacity of arc e_2 is set to some discrete measure u^d concentrated on the rational numbers (i.e., $\text{supp}(u^d) = \mathbb{Q}$). Further, storage of flow is not permitted at the intermediate node v, i.e., $U_v = L_v = 0$. Hence, all flow arriving at v must immediately enter arc e_2. Thus, no measurable amount of flow can be routed from s to t, i.e., the zero flow is a maximum Borel flow. Therefore the original network and the residual network coincide.

Let us consider the points in time at which node v is reachable as one would expect these points to appear in a minimum Borel cut. It is obvious that flow is able to reach node v at every point in time $\theta \in \mathbb{R}$ since the capacity of e_1 is equal to the Lebesgue measure. So we would expect the cut S defined by $S_s := \mathbb{R}$, $S_v := \mathbb{R}$, and $S_t := \emptyset$ to be a minimum s-t Borel cut. But we have $\text{cap}(S) = u^d(\mathbb{R})$ which is far away from the maximum Borel flow value 0. Actually, setting $S_v := \mathbb{R}$ means that flow can enter arc e_2 at points in time $\theta \in \text{supp}(u^d)$ since the support of u^d is trivially contained in S_v. But this is not really true since no flow can enter arc e_2 at any point in time. We therefore exclude the support of u^d and set $S_v := \mathbb{R} \backslash \text{supp}(u^d) = \mathbb{R} \backslash \mathbb{Q}$, which leads to a minimum Borel cut of capacity 0. Note that $\text{supp}(u^d)$ is a λ-null set, i.e., $\lambda(\text{supp}(u^d)) = 0$. Hence, the exclusion of $\text{supp}(u^d)$ has no impact on the flow behavior on arc e_1.

Example 1 shows also that continuous and discrete flows can be mixed only if there is some storage capacity that allows to convert one quantity into the other. Also note that every Borel cut S where S_v is a countable union of intervals has a capacity strictly greater than zero. More precisely, we have $\text{cap}(S) = \lambda(S_v^c) + u^d(S_v)$; if S_v is a countable union of intervals that does not contain any rational point, then it excludes a subset of $\mathbb{R}$ with nonzero Lebesgue measure (i.e., $\lambda(S_v^c) > 0$), and if S_v contains some rational points, then $u^d(S_v) > 0$. Hence, the restriction that Γ_v (in Example 1 $\Gamma_v = \emptyset$) is the countable union of intervals is not extendable to the whole set S_v.

Example 2 Consider the network depicted in Figure 2(a) with three nodes s, v, and t and two arcs $e_1 = (s, v)$ and $e_2 = (v, t)$. On arc e_1 we can route two units of flow at



(a) Original network. (b) Time expanded network.

Fig. 2 Network for Example 1. The capacities are shown on the arcs and all transit times are 0.

time 0 and arc e_2 allows routing one unit of flow at time 1. Further, storage of two flow units is allowed at the intermediate node v within the time interval $[0, 1)$. The corresponding time expanded network is shown in Figure 2(b). In particular, we have

$$u_{e_1}(B) = \begin{cases} 2 & \text{for } 0 \in B \\ 0 & \text{for } 0 \notin B \end{cases} \quad \text{and} \quad u_{e_2}(B) = \begin{cases} 1 & \text{for } 1 \in B \\ 0 & \text{for } 1 \notin B \end{cases} \quad \text{for all } B \in \mathcal{B} ,$$

$$U_v(\theta) = \begin{cases} 2 & \text{for } \theta \in [0, 1) \\ 0 & \text{for } \theta \notin [0, 1) \end{cases} \quad \text{and} \quad L_v(\theta) = 0 \quad \text{for all } \theta \in \mathbb{R} .$$

In this example, a maximum Borel flow routes one unit of flow from s to t as follows: One unit of flow enters arc e_1 at time 0 and arrives at v at time 0. After waiting at v within the time interval $[0, 1)$, the flow unit enters arc e_2 at time 1 and reaches t at the same time. Let x be this Borel flow. Then x is given as follows:

$$x_{e_1}(B) = \begin{cases} 1 & \text{for } 0 \in B \\ 0 & \text{for } 0 \notin B \end{cases} \quad \text{and} \quad x_{e_2}(B) = \begin{cases} 1 & \text{for } 1 \in B \\ 0 & \text{for } 1 \notin B \end{cases} \quad \text{for all } B \in \mathcal{B} , \tag{12}$$

$$\text{with storage function} \qquad Y_v(\theta) = \begin{cases} 1 & \text{for } \theta \in [0, 1) \\ 0 & \text{for } \theta \notin [0, 1) \end{cases} . \tag{13}$$

Thus the signed measure y_v is discrete, concentrated on the set $\{0, 1\}$ with $y_v(\{0\}) = 1$ and $y_v(\{1\}) = -1$.

Now consider the residual network G^r with respect to x. It is not hard to see that flow is able to reach node v at every point in the time interval $[0, 1)$ since an additional flow unit can reach node v at time 0 and then wait at node v within this time interval. Therefore we would expect that the cut S defined by $S_s := \mathbb{R}$, $S_v := [0, 1)$, and $S_t := \emptyset$ is a minimum s-t Borel cut. Let us compute the capacity of S. The arc capacities have no contribution to $\text{cap}(S)$ (i.e., the first sum in (7) is zero) and the node capacity U_v contributes a value of 2 to $\text{cap}(S)$ since $\Gamma_v = [0, 1)$ and $U_v(-1) = 2$. Thus $\text{cap}(S) = 2$, which is not equal to the value of x. We now observe that $U_v^r(1-) > 0$ and flow can reach node v within the interval $[0, 1)$. In fact, we have $U_v^{r,\succ 0} = [0, 1]$. It is thus reasonable to consider the point 1 as an element of S_v and define a new s-t Borel cut S' by $S'_v := S_v \cup \{1\} = [0, 1]$. Recall that right-open and right-closed intervals are treated differently in computing the capacity of an s-t Borel cut. Here we have $\text{cap}(S') = 1$ and get a minimum Borel cut of capacity 1.

Example 3 Consider Example 2 but with the node capacity U_v given as:

$$U_v(\theta) = \begin{cases} 1 - \theta & \text{for } \theta \in [0, 1) \\ 1 & \text{for } \theta \notin [0, 1) \end{cases} \qquad \text{for all } \theta \in \mathbb{R} .$$

Here one unit of flow can reach node v at time 0. This flow has to wait at node v until time 1, before routing towards the sink. On the other hand, the node capacity at v decreases to zero when the time tends to 1. Therefore, intuitively, no flow can be sent from s to t. Hence, the maximum Borel s-t flow is zero and, consequently, the original network and the residual network coincide.

We are now interested in computing a Borel cut of capacity zero. We have $U_v^{\succ 0} = \mathbb{R}$ and flow is able to reach node v at time 0. So a natural candidate S for a minimum Borel s-t cut can be given by $S_s := \mathbb{R}$, $S_v := [0, \infty)$, and $S_t := \emptyset$. The arc capacity u_{e_1} has no contribution to $\text{cap}(S)$, the arc capacity u_{e_2} has a contribution of value 1 to $\text{cap}(S)$, and the node capacity U_v contributes a value of 1 to $\text{cap}(S)$ since $\Gamma_v = [0, \infty)$ and $U_v(\infty) = 1$. Thus we have $\text{cap}(S) = 2$ which is far away from the maximum Borel flow value 0. On the other hand, although $U_v^{\succ 0} = \mathbb{R}$, flow can not reach node v within the interval $[1, \infty)$ as $U_v(1-) = 0$. Hence, it is reasonable to restrict S_v to the time interval $[0, 1)$, which leads to a Borel cut of capacity 0. It is also worth to mention that despite the similarities to Example 2, the point in time 1 must be excluded from S_v in this example.

In Example 3, the minimum Borel cut is obtained by setting $S_v := [0, 1)$ as flow can reach node v at time 0 and $[0, 1) \subset U_v^{\succ 0}$. However, although $U_v^{\succ 0} = \mathbb{R}$, no point in $[1, \infty)$ is reachable as $U_v^r(1-) = 0$. In fact, $U_v^{\succ 0}$ is the union of the intervals $(-\infty, 1)$ and $[1, \infty)$ where for all a, b in one of these two intervals with $a < b$ there exists an $\epsilon > 0$ such that $U_v^{\succ 0}|_{[a,b)} > \epsilon$. Intuitively, this ensures that some flow arriving at v at time a can be stored in v until time b. This motivates the following concept of *positive intervals*.

Let μ be a Borel measure with corresponding distribution function M. Recall that $M^{\succ 0}$, defined by (5), denotes the set of all points in time where M or its left limit is positive at θ. Further, we call an interval I *positive* if for all $a, b \in I$ with $a < b$ there exists an $\epsilon > 0$ such that $M|_{[a,b)} > \epsilon$. The following lemma shows that $M^{\succ 0}$ is expressable as a countable union of such intervals, the proof of which can be found in Appendix B. Note that, as already mentioned, $M^{\succ 0}$ is the countable union of pairwise disjoint intervals. Hence, each interval out of a countable union can be assumed to be positive. Further, defining the capacity of a Borel cut via the countable union of inclusionwise maximal intervals as in (7) leads to the same value as considering inclusionwise maximal positive intervals instead.

Lemma 4 *Let M be a distribution function. Then the set $M^{\succ 0}$ can be written as a countable union of pairwise disjoint positive intervals.*

At this point, let us introduce some more notations which are used in the the rest of the paper. We define $M^{>0}$ and $M^{<|\mu|}$ to denote the set of all points $\theta \in \mathbb{R}$ such that $M(\theta) > 0$ and $M(\theta-) < |\mu|$, respectively. More precisely,

$$M^{>0} := \{\theta \in \mathbb{R} \mid M(\theta) > 0\} \qquad \text{and} \qquad M^{<|\mu|} := \{\theta \in \mathbb{R} \mid M(\theta-) < |\mu|\} \ .$$

Notice that these two sets are intervals.

In the following we construct a procedure for deriving the points in time when a node is reachable from s. The procedure gets as input any network and produces as output sets S_v^i for each $i \in \mathbb{N}$ and each node $v \in V \setminus \{s, t\}$. A set S_v^i contains all points in time at which flow is able to arrive at node v using exactly i arcs. These sets are computed inductively over i. In order to compute S_v^i for a fixed v and a fixed i, we first consider the sets S_w^{i-1} of times at which flow is able to arrive at any predecessor node w of v (i.e., $(w,v) \in E$) along exactly $i-1$ arcs. From these times we derive the points in time at which v is reachable using exactly i arcs (including arc (w,v) as the last arc) and waiting occurs at v. It follows from Example 2 that we must add certain points in time for the latter case. We next consider the case where no waiting is allowed at v and obtain the set S_v^i. Example 1 shows us that we must exclude certain points in time. A Borel cut $S := (S_v)_{v \in V}$ is then given by $S_v := \bigcup_{i \in \mathbb{N}} S_v^i$ for $v \in V$.

Reachability Procedure

Input:	A network consisting of a directed graph G, transit times τ_e, arc capacities u_e, and node capacities U_v and L_v.
Output:	Sets S_v^i for $v \in V$ and $i \in \mathbb{N}$ determining at which times a node is reachable from s using *exactly* i arcs.

(1) Initialize $i := 0$ and $S_v^j := \begin{cases} \mathbb{R} & \text{if } v = s \text{ and } j = 0 \\ \emptyset & \text{otherwise} \end{cases}$ for $v \in V$ and $j \in \mathbb{N}$.

(2) For each arc $e \in E$, let $g_e := u_e|_{S_v^i}$ where $v = \text{tail}(e)$.

(3) For each node $v \in V$ do:

(a) Define $\mu_1 := \sum_{e \in \delta^-(v)} (g_e - \tau_e)$ and $\mu_2 := \sum_{e \in \delta^+(v)} u_e$.

(b) Let $U_v^{\succ 0} = \bigcup_{k \in J} I_k$ be the disjoint union of positive intervals where $J \subseteq \mathbb{N}$.
Set $h_k := \mu_1|_{I_k}$ for each $k \in J$.
Set $S_+ := \bigcup_{k \in J} \big(H_k^{>0} \cap I_k\big)$.

(c) Let $L_v^{\succ 0} = \bigcup_{k \in J} I_k$ be the disjoint union of positive intervals where $J \subseteq \mathbb{N}$.
Set $h_k := \mu_1|_{I_k}$ for each $k \in J$.
Set $S_- := \bigcup_{k \in J} \big(H_k^{<|h_k|} \cap I_k\big)$.

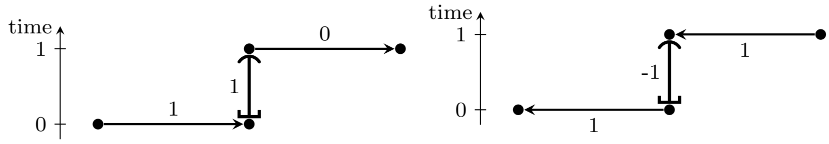
(d) Use the Lebesgue Decomposition Theorem in order to find ν^{ac} and ν^{s} such that:
- $\mu_2 = \nu^{\text{ac}} + \nu^{\text{s}}$,
- ν^{ac} is absolutely continuous with respect to μ_1,
- ν^{s} and μ_1 are mutually singular.

Find a set $A \subseteq \mathbb{R}$ such that $\mu_1(A) = 0$ and $\nu^{\text{s}}(A^c) = 0$.
Set $\bar{A} := A \cup (U_v^{\succ 0} \setminus S_+) \cup (L_v^{\succ 0} \setminus S_-)$.
Set $S_0 := \text{supp}(\mu_1) \setminus \bar{A}$.

(e) Set $S_v^{i+1} := S_0 \cup S_+ \cup S_-$.

(4) Set $i := i + 1$ and go to (2).

As mentioned already, the positive intervals in Steps (3b) and (3c) are supposed to be inclusion-wise maximal. It is also worth to mention that for each $v \in V$, the sequence S_v^i of sets is not necessarily monotonic with respect to inclusion. Further, note that in general, the Reachability Procedure never terminates and even requires infinite memory. But this causes no problem in theory since the Reachability Procedure is not meant as an algorithm but rather as a definition of the sets S_v^i. From these sets we deduce a Borel cut $S := (S_v)_{v \in V}$ by $S_v := \bigcup_{i \in \mathbb{N}} S_v^i$ for $v \in V$.



(a) Forward arcs and upper storage bounds. (b) Backward arcs and lower storage bounds.

Fig. 3 Residual network for Example 5. The capacities are shown and all transit times are 0.

Hence, S can be seen as an output of the procedure. Nevertheless, it is of great interest, whether the Reachability Procedure terminates in finite time or not. We discuss briefly some arising questions in the conclusion. We next illustrate the Reachability Procedure using the instances of Example 1 and 2.

Example 4 Consider the instance of Example 1. Since the zero flow is a maximal Borel flow the sink t should not be reachable. Since the source s has no incoming arc the sets S_s^i are never changed after initialization, i.e, we have $S_s^0 = \mathbb{R}$ and $S_s^i = \emptyset$ for $i \geq 1$. When processing the intermediate node v in the first iteration, we obtain $\mu_1 = \lambda$ and $\mu_2 = u^d$. As λ and u^d are mutually singular, we have $\nu^{\text{ac}} = 0$ and $\nu^{\text{s}} = u^d$. Since $\lambda(\operatorname{supp}(u^d)) = 0$ and $u^d(\mathbb{R} \setminus \operatorname{supp}(u^d)) = 0$ we set $A := \operatorname{supp}(u^d)$. Hence, this implies $S_0 = \operatorname{supp}(\lambda) \setminus \operatorname{supp}(u^d) = \mathbb{R} \setminus \operatorname{supp}(u^d)$. Since the storage of flow at node v is not permitted, we have $U_v^{\succ 0} = L_v^{\succ 0} = \emptyset$, which implies $S_+ = S_- = \emptyset$ as well. This leads to $S_v^1 = \mathbb{R} \setminus \operatorname{supp}(u^d)$. In each of the following iterations $i \geq 1$, we have $S_v^i = \emptyset$ since $\mu_1 = \lambda|_{S_s^i} = \lambda|_{\emptyset} = \emptyset$, and as a result $S_v = \mathbb{R} \setminus \operatorname{supp}(u^d)$.

Next we consider sink t. In each iteration, we have $\mu_1 = \mu_2 = 0$, which leads to $S_t = \emptyset$. Hence, the Borel cut constructed by the Reachability Procedure equals $S_s = \mathbb{R}$, $S_v = \mathbb{R} \setminus \operatorname{supp}(u^d)$ and $S_t = \emptyset$, which has a capacity of zero.

Example 5 In this example, we consider Example 2 and the residual network with respect to the Borel flow x given by (12). Let ν be the measure concentrated on $\{0\}$ with $\nu(\{0\}) = 1$ and $C : \mathbb{R} \to \mathbb{R}$ be the function defined by $C|_{[0,1)} = 1$ and $C|_{\mathbb{R}\setminus[0,1)} = 0$. Then the residual arc and node capacities are given as follow (see also Figure 3):

$$u^r_{e_1} = u^r_{\overleftarrow{e}_1} = \nu \ , \qquad u^r_{e_2} = 0 \ , \qquad u^r_{\overleftarrow{e}_2} = \nu - 1 \ , \qquad \text{and} \qquad U^r_v = L^r_v = C \ .$$

We consider the process of the procedure in the first iteration. For node v we have

$$\mu_1 = u^r_{e_1}|_{S_s^0} + u^r_{\overleftarrow{e}_2}|_{S_t^0} = \nu \qquad \text{and} \qquad \mu_2 = u^r_{e_2} + u^r_{\overleftarrow{e}_1} = \nu \ .$$

In Step (3b), as $U_v^{r \succ 0} = [0,1]$, we get $h_1 = \mu_1|_{[0,1]} = \nu$. Thus, we have $H_1^{>0} = [0,\infty)$ and consequently $S_+ = H_1^{>0} \cap [0,1] = [0,1]$. In Step (3c), we have $L_v^{r \succ 0} = [0,1]$ and $h_1 = \nu$ as in Step (3b). Since $H_1^{<|h_1|} = (-\infty,0]$ we get $S_- = H_1^{<|h_1|} \cap [0,1] = \{0\}$. Since it holds that $\mu_2 = \mu_1$, we have $\nu^{\text{ac}} = \mu_2$, $\nu^{\text{s}} = 0$, and set $A := \emptyset$. This gives us $\bar{A} = (-\infty, 0)$ and consequently $S_0 = \operatorname{supp}(u^r_{e_1}) = \{0\}$. By the union of S_0, S_+ and S_-, we obtain $S_v^1 = [0,1]$. For node t, we have $\mu_1 = u^r_{e_2}|_{S_v^0} = 0$ implying $S_t^1 = \emptyset$.

In the second iteration, we observe for node v that $\mu_1 = u^r_{e_1}|_{S_s^1} + u^r_{\overleftarrow{e}_2}|_{S_t^1} = 0$. Thus, we get $S_v^2 = \emptyset$. For node t, we have $\mu_1 = u^r_{e_2}|_{S_v^1} = 0$ implying again $S_t^2 = \emptyset$. In the third iteration, we get again $S_v^3 = \emptyset$ and $S_t^3 = \emptyset$.

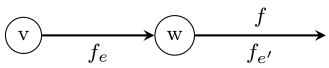


Fig. 4 The setting of Lemma 6.

We terminate the procedure after the third iteration since S_v and S_t remain the same. Summarizing, we get the Borel cut $S_s = \mathbb{R}$, $S_v = [0,1]$ and $S_t = \emptyset$, whose capacity is 1.

In the remainder of this section we show that if t is reachable in the residual network, i.e., $\bigcup_{i\in\mathbb{N}} S_t^i \neq \emptyset$, then the corresponding Borel flow is not maximal. The proof will be carried out via a sequence of lemmas. The first one states that excluding the set $\bar{A}$ from supp(μ_1) in Step (3d) does not change the measure μ_1, i.e., $\mu_1|_{\bar{A}} = 0$. This result will be used later.

Lemma 5 *In each iteration of the procedure, $\bar{A}$ is a strict μ_1-null set, that is, $\mu_1|_{\bar{A}} = 0$.*

Proof Recall that $\bar{A}$ is defined as $\bar{A} := A \cup (U_v^{\succ 0} \setminus S_+) \cup (L_v^{\succ 0} \setminus S_-)$. From the construction of A we know that $\mu_1(A) = 0$. Since μ_1 is nonnegative, we get $\mu_1|_A = 0$. Thus it is sufficient to prove that both $U_v^{\succ 0} \setminus S_+$ and $L_v^{\succ 0} \setminus S_-$ are μ_1-null sets.

In order to prove that $U_v^{\succ 0} \setminus S_+$ is a μ_1-null set, first observe that $U_v^{\succ 0} = \bigcup_{k\in J} I_k$ is the countable union of pairwise disjoint positive intervals where $J \subseteq \mathbb{N}$. Hence, it is enough to show that $I_k \setminus S_+$ is a μ_1-null set for each $k \in J$. Fix a $k \in J$. Because of the definition of S_+ in Step (3b) we have $I_k \setminus S_+ = I_k \setminus \big(H_k^{>0} \cap I_k\big) = I_k \cap (H_k^{>0})^c$ where H_k is the distribution function of the (nonnegative) measure $h_k := \mu_1|_{I_k}$. Further we have $\mu_1(I_k \cap (H_k^{>0})^c) = h_k\big((H_k^{>0})^c\big)$. The definition of $H_k^{>0}$ implies $h_k((-\infty,\theta]) = 0$ for each $\theta \in (H_k^{>0})^c$. Since $(H_k^{>0})^c = \bigcup_{\theta \notin H_k^{>0}} (-\infty,\theta]$ holds, we have $h_k\big((H_k^{>0})^c\big) = 0$ proving $\mu_1(I_k \setminus S_+) = 0$.

It remains to show that $L_v^{\succ 0} \setminus S_-$ is a μ_1-null set. Let $L_v^{\succ 0} = \bigcup_{k\in J} I_k$ be the countable union of pairwise disjoint positive intervals where $J \subseteq \mathbb{N}$. Hence, it is sufficient to show that $I_k \setminus S_-$ is a μ_1-null set for each $k \in J$. Fix a $k \in J$. It follows from the definition of S_- in Step (3c) that $I_k \setminus S_- = I_k \setminus \big(H_k^{<|h_k|} \cap I_k\big) = I_k \cap (H_k^{<|h_k|})^c$ where H_k is the distribution function of the (nonnegative) measure $h_k := \mu_1|_{I_k}$. Further we have $\mu_1(I_k \cap (H_k^{<|h_k|})^c) = h_k\big((H_k^{<|h_k|})^c\big)$. Moreover, from the definition of $H_k^{<|h_k|}$, we get $h_k([\theta,\infty)) = |h_k| - H_k(\theta-) = 0$ for each $\theta \in (H_k^{<|h_k|})^c$. This implies $h_k\big((H_k^{<|h_k|})^c\big) = 0$ since $(H_k^{<|h_k|})^c = \bigcup_{\theta\notin H_k^{<|h_k|}} [\theta,\infty)$ holds. Hence, $\mu_1(I_k \setminus S_+) = 0$. □

Roughly speaking, the next lemma deals with the following situation. Suppose we are able to route flow out of a certain node w with departure times in S_w^i for some i. Then some of this flow can be obtained by routing flow from a predecessor node v with departure times in S_v^{i-1} along arc (v,w) and then out of w. In the proof we have to resolve, among other things, the conflicts established in Examples 1 and 2. We refer to Figure 4 where the scenario of Lemma 6 is depicted.

Lemma 6 *Let $w \in V \setminus \{s\}$ be a node and f be a nonzero measure with $f \leq u_{e'}|_{S_w^n}$ for some arc $e' \in \delta^+(w)$ and some $n \in \mathbb{N}$. Then there exists an arc $e = (v,w)$ and nonzero measures f_e and $f_{e'}$ such that*

$$|f_e| = |f_{e'}|, \quad f_e \leq u_e|_{S_v^{n-1}}, \quad f_{e'} \leq f, \quad \textit{and} \quad -L_w \leq (F_e - \tau_e) - F_{e'} \leq U_w\ .$$

Proof Consider the state of the procedure where i is equal to $n-1$ and node w is processed in the loop of Step (3). Here, S_w^n is computed and we have (using the notation of the procedure)

$$S_w^n = S_0 \cup S_+ \cup S_- \ .$$

Since f is a nonzero measure we obtain $0 < f(\mathbb{R}) = f(\mathbb{R} \setminus S_w^n) + f(S_w^n)$. On the other hand $f \leq u_{e'}|_{S_w^n}$ implies $f(\mathbb{R} \setminus S_w^n) = 0$ because $u_{e'}|_{S_w^n}(\mathbb{R} \setminus S_w^n) = u_{e'}(\emptyset) = 0$. Hence, it holds that $0 < f(S_w^n) \leq f(S_0) + f(S_+) + f(S_-)$. Therefore, at least one measure of $f|_{S_0}$, $f|_{S_+}$, and $f|_{S_-}$ is nonzero. Consequently, we use the following case distinction:
Case 1: We first consider the case that $f|_{S_0}$ is nonzero. Using the notation of the procedure, we observe that $\mu_2|_{S_0}$ is absolutely continuous with respect to μ_1. To see this, let B be a Borel set for which $\mu_1(B) = 0$. We write

$$\mu_2|_{S_0}(B) = \nu^{\mathrm{ac}}(S_0 \cap B) + \nu^{\mathrm{s}}\big(\mathrm{supp}(\mu_1) \cap B \cap \bar{A}^c\big) \ .$$

The first summand on the right hand side is 0 because ν^{ac} is absolutely continuous with respect to μ_1 and we have $\mu_1(B) = 0$. Further, the second summand is zero because $0 \leq \nu^{\mathrm{s}}(\bar{A}^c) \leq \nu^{\mathrm{s}}(A^c) = 0$ as $A \subseteq \bar{A}$. Hence, $\mu_2|_{S_0}$ is absolutely continuous with respect to μ_1. This implies that $f|_{S_0}$ is absolutely continuous with respect to μ_1 because $f \leq \mu_2$ (to see this, observe $f \leq u_{e'}|_{S_w^n} \leq u_{e'} \leq \sum_{e \in \delta^+(w)} u_e = \mu_2$). Therefore $\min\{\mu_1, f|_{S_0}\}$ is a nonzero measure (see Appendix A for a discussion on the minimum of two measures). Hence, there exists a Borel set $B \in \mathcal{B}$ such that

$$\begin{aligned} 0 < \min\{\mu_1(B), f|_{S_0}(B)\} &= \min\{\sum_{e=(v,w)\in\delta^-(w)} (u_e|_{S_v^{n-1}} - \tau_e)(B), f|_{S_0}(B)\} \\ &\leq \sum_{e=(v,w)\in\delta^-(w)} \min\{(u_e|_{S_v^{n-1}} - \tau_e)(B), f|_{S_0}(B)\} \end{aligned}$$

This ensures the existence of an arc $e = (v, w)$ such that $f' := \min\{(u_e|_{S_v^{n-1}} - \tau_e), f|_{S_0}\}$ is a nonzero measure. Hence, setting $f_e := f' + \tau_e$ and $f_{e'} := f'$ leads to the desired result.
Case 2: We next consider the case that $f|_{S_+}$ is nonzero. From Step (3b) of the procedure we know that

$$S_+ = \bigcup_{k \in J} \big(H_k^{>0} \cap I_k\big)$$

for some $J \subseteq \mathbb{N}$. Hence, there exists some $k \in J$ such that f restricted to $I := H_k^{>0} \cap I_k$ is a nonzero measure. Note that I is an interval and let a_I and b_I be the left and the right boundary of I, respectively. Further, we can exclude the case $f|_I$ is concentrated on $\{a_I\}$ since this case is already resolved in Case 1. This can be seen as follows: Having in mind the definitions of restricted and concentrated measures we know that I is left closed. Hence, Step (3b) shows $\mu_1(\{a_I\}) > 0$. This implies $a_I \in S_0$ because of Lemma 5 and thus, Case 1 is applicable since $f(\{a_I\}) > 0$.

In the following we show that there exists $a, b, c \in I$ with $a < b < c$ such that $\mu_1|_{[a,b]}$ and $f|_{[b,c]}$ are nonzero measures and $U_w|_{[a,c)} \geq \epsilon$ for some $\epsilon > 0$. Informally, this ensures that, without violating the node capacity at w, we can send a small amount of flow into node w over the time interval $[a, b]$ which leaves v over the time interval $[b, c]$.

Note that $f|_I$ is not concentrated on $\{a_I\}$ due to our assumption. Assuming also that $f|_I$ is not concentrated on $\{b_I\}$ there exist $b, c \in (a_I, b_I)$ with $b < c$ such that $f|_{[b,c]}$

is a nonzero measure. If $f|_I$ is concentrated on b_I we have $b_I \in I$. Moreover, for $c = b_I$ and any $b \in (a_I, b_I)$ it holds that $f|_{[b,c]}$ is a nonzero measure. Further, the definition of I shows that $\mu_1|_{[a_I,b]}$ is a nonzero measure such that its distribution function is strictly positive on (a_I, b). Thus, there exists an $a \in [a_I, b) \cap I$ such that $\mu_1|_{[a,b]}$ is a nonzero measure. Note that if $\mu_1|_I$ is concentrated on $\{a_I\}$, then I is left closed and we must set $a := a_I \in I$ (otherwise a is taken out of $(a_I, b] \subset I$). Finally, since I is a positive interval with respect to U_v there exists an $\epsilon > 0$ such that $U_v|_{[a,c)} \geq \epsilon$.

Because of the definition of μ_1, there exists an arc $e = (v, w) \in \delta^-(w)$ such that the measure $\bar{f}_e := u_e|_{S_v^{n-1}} - \tau_e$ restricted to $[a, b]$ is a nonzero measure. We let

$$\alpha := \min\Big\{\big|\bar{f}_e|_{[a,b]}\big|, \big|f|_{[b,c]}\big|, \epsilon\Big\}$$

and define f_e and $f_{e'}$ as follows:

$$f_e := \frac{\alpha}{\big|\bar{f}_e|_{[a,b]}\big|}\bar{f}_e|_{[a,b]} + \tau_e \qquad \text{and} \qquad f_{e'} := \frac{\alpha}{\big|f|_{[b,c]}\big|} f|_{[b,c]}\ .$$

This yields the desired result.

Case 3: It remains to consider the case that $f|_{S_-}$ is nonzero which is similar to the previous case. Recall from Step (3c) that

$$S_- := \bigcup_{k \in J} \big(H_k^{<|h_k|} \cap I_k\big)$$

for some $J \subseteq \mathbb{N}$. Hence, there is some $k \in J$ such that f restricted to $I := H_k^{<|h_k|} \cap I_k$ is a nonzero measure. Note that I is an interval and let a_I and b_I be the left and the right boundary of I, respectively. Further, we can assume without loss of generality that $f|_I$ is not concentrated on $\{b_I\}$. Otherwise this case is resolved in Case 1 which can be seen as follows: Because of the definitions of restricted and concentrated measures we know that I is right closed in this case. Hence, Step (3c) shows $\mu_1(\{b_I\}) > 0$ (Note that $H_k^{<|h_k|}$ is defined via limits from left). This implies $b_I \in S_0$ because of Lemma 5 and thus, Case 1 is applicable since $f(\{b_I\}) > 0$.

In the following we show that there exists $a, b, c \in I$ with $a < b < c$ such that $\mu_1|_{[b,c]}$ and $f|_{[a,b]}$ are nonzero measures and $L_w|_{[a,c)} \geq \epsilon$ for some $\epsilon > 0$. Informally, this ensures that, without violating the node capacity at w, we can send a small amount of flow into node w over the time interval $[b, c]$ which leaves v over the time interval $[a, b]$. That is, we route a small portion of flow back in time.

Note that $f|_I$ is not concentrated on $\{b_I\}$ due to our assumption. Assuming also that $f|_I$ is not concentrated on $\{a_I\}$ there exist $a, b \in (a_I, b_I)$ with $a < b$ such that $f|_{[a,b]}$ is a nonzero measure. If $f|_I$ is concentrated on a_I we have $a_I \in I$. Moreover, for $a = a_I$ and any $b \in (a_I, b_I)$ it holds that $f|_{[a,b]}$ is a nonzero measure. Further, the definition of I shows that $\mu_1|_{[b,b_I]}$ is a nonzero measure such that its distribution function is strictly less than $|h_k|$ on (b, b_I). Thus, there exists an $c \in (b, b_I] \cap I$ such that $\mu_1|_{[b,c]}$ is a nonzero measure. Note that if $\mu_1|_I$ is concentrated on $\{b_I\}$, then I is right closed and we must set $c := b_I \in I$ (otherwise c is taken out of $[b, b_I) \subset I$). Finally, since I is a positive interval with respect to L_v there exists an $\epsilon > 0$ such that $L_v|_{[a,c)} \geq \epsilon$.

Because of the definition of μ_1, there exists an arc $e = (v, w) \in \delta^-(w)$ such that the measure $\bar{f}_e := u_e|_{S_v^{n-1}} - \tau_e$ restricted to $[b, c]$ is a nonzero measure. We let

$$\alpha := \min\Big\{\big|\bar{f}_e|_{[b,c]}\big|, \big|f|_{[a,b]}\big|, \epsilon\Big\}$$

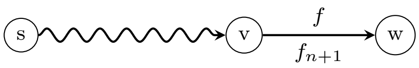


Fig. 5 The setting of Lemma 8.

and define f_e and $f_{e'}$ as follows:

$$f_e := \frac{\alpha}{\left|\bar{f}_e|_{[b,c]}\right|}\bar{f}_e|_{[b,c]} + \tau_e \qquad \text{and} \qquad f_{e'} := \frac{\alpha}{\left|f|_{[a,b]}\right|} f|_{[a,b]} \ .$$

This yields the desired result. □

For the next lemma we need the definition of a flow-carrying path. Consider a sequence $P = (e_1, \dots, e_n)$ of arcs such that the head of each arc is the tail of the next. Notice that $e_1, \dots, e_n$ are not necessarily pairwise distinct. Let v be the tail of e_1 and w be the head of e_n. The arcs sequence P is called a *flow-carrying v-w-path* if there exist flows $f_1, \dots, f_n$ associated to arcs $e_1, \dots, e_n$, respectively, so that $|f_1| = \ldots = |f_n|$ and further $(\sum_{i|e_i=e} f_i)_{e\in E}$ is a v-w Borel flow.

Informally, the next lemma considers the following situation. Suppose we are able to route flow out of a certain node v with departure times in S_v^i for some i. Then some of this flow can be obtained by routing flow first from s to v along a flow carrying path with exactly i arcs and subsequently out of v. The scenario of Lemma 8 is shown in Figure 5. To prove this, we require the following result whose proof is given in Appendix B.

Lemma 7 *Let μ_1, μ_2, and ν_1 be finite Borel measures on $\mathbb{R}$ with corresponding distribution functions M_1, M_2, and N_1, respectively. Further, assume that $|\mu_1| \geq |\mu_2|$ and $\nu_1 \leq \mu_1$. Then there exists a (finite) Borel measure $\nu_2 \leq \mu_2$ with distribution function N_2 such that $|N_1(\theta) - N_2(\theta)| \leq |M_1(\theta) - M_2(\theta)|$ for each $\theta \in \mathbb{R}$, i.e, the vertical distance between the distribution functions does not increase when replacing μ_1 and μ_2 with ν_1 and ν_2, respectively.*

Lemma 8 *Let f be a nonzero measure and $f \leq u_e|_{S_v^n}$ for some arc $e = (v, w)$ and some n. Then there exists a flow-carrying s-w-path $P = (e_1, \dots, e_{n+1})$ with corresponding flows $f_1, \dots, f_{n+1}$ containing $n+1$ arcs for which $e_{n+1} = e$ and $f_{n+1} \leq f$.*

Proof The proof is by induction over n. Obviously, this lemma holds for $n = 0$. Thus we assume that the assertion holds for $n-1$ ($n > 0$) and proceed to show that the lemma is true for n.

Suppose that f is a nonzero measure and $f \leq u_e|_{S_v^n}$ for some arc $e = (v, w)$. Lemma 6 implies the existence of some $e' = (w', v)$ and nonzero measures $f_{e'}$ and f_e such that

$$|f_{e'}| = |f_e| \ , \qquad f_{e'} \leq u_{e'}|_{S_{w'}^{n-1}} \ , \qquad f_e \leq f \ , \qquad -L_v \leq (F_{e'} - \tau_{e'}) - F_e \leq U_v \ .$$

By the induction hypothesis there is a flow-carrying s-v-path $P = (e_1, \dots, e_n)$ with corresponding Borel flows $g_1, \dots, g_n$ for which $e_n = e'$ and $g_n \leq f_{e'}$. Then it follows that $P' = (e_1, \dots, e_n, e)$ with corresponding Borel flows $\frac{1}{2}g_1, \dots, \frac{1}{2}g_n$, $\frac{1}{2}g_{n+1}$ is a flow-carrying s-w-path where g_{n+1} is the result of Lemma 7 with respect to $f_{e'}$, f_e and $g_n \leq f_{e'}$. Note that Lemma 7 ensures that the node capacity constraint remains valid at node v. Also note that the division by 2 is needed in case that $e = e_i$ for some $i = 1, \dots, n$. This concludes the proof of the lemma. □

Corollary 1 *Suppose that the sink t is reachable in the residual network with respect to some s-t Borel flow x. Then x is not a maximum Borel flow.*

Proof We have $\bigcup_{i\in\mathbb{N}} S_t^i \neq \emptyset$ since the sink t is reachable. Thus there exists an $n \in \mathbb{N}$ for which $S_t^n \neq \emptyset$ and we can conclude that $u_e^r|_{S_v^{n-1}}$ is nonzero for some arc $e \in \delta^-(t)$. It then follows from Lemma 8 that there is a flow-carrying s-t path containing a finite number of arcs. Lemma 3 implies that x is not a maximum Borel flow. □

7 MaxFlow-MinCut Theorem

In this section we prove the MaxFlow-MinCut Theorem for Borel flows and Borel cuts. The basic idea of the proof is similar to the static case. Given a maximum Borel flow x, we compute, for each $v \in V$, the set S_v of points in time for which node v is reachable in the corresponding residual network. We apply the Reachability Procedure on the residual network and set $S_v := \bigcup_{i\in\mathbb{N}} S_v^i$. In the following two lemmas we show that $S := (S_v)_{v\in V}$ is a well defined s-t Borel cut and that its capacity equals the value of x.

Lemma 9 *Suppose that x is a (maximum) s-t Borel flow and $S = (S_v)_{v\in V}$ is the corresponding s-t Borel cut computed by the* Reachability Procedure. *For each $v \in V$, the set $\Gamma_v := S_v \cap U_v^{\succ 0}$ can be expressed as a countable union of pairwise disjoint intervals.*

Proof Let $v \in V \setminus \{s, t\}$ be some node. Recalling the definition of the residual network first observe that $U_v = U_v^r + L_v^r$. Since U_v, U_v^r, and L_v^r are functions of bounded variation the left limit exists everywhere. Hence, we have $U_v^{\succ 0} = U_v^{r,\succ 0} \cup L_v^{r,\succ 0}$.

Next, consider a certain iteration $i-1$ of the procedure and let v be processed in the loop of Step (3). Using the notation of the procedure, we show that $S_v^i \cap U_v^{\succ 0} = S_+ \cup S_-$ holds. Since $S_v^i = S_0 \cup S_+ \cup S_-$ and $S_+ \cup S_- \subseteq U_v^{r,\succ 0} \cup L_v^{r,\succ 0} = U_v^{\succ 0}$ it holds that

$$S_v^i \cap U_v^{\succ 0} = (S_0 \cap U_v^{\succ 0}) \cup S_+ \cup S_- \ .$$

Thus, it is enough to show that $S_0 \cap U_v^{\succ 0} \subseteq S_+ \cup S_-$. Note that $S_0 = \mathrm{supp}(\mu_1) \cap \bar{A}^c$ where $\bar{A} = A \cup (U_v^{r,\succ 0} \setminus S_+) \cup (L_v^{r,\succ 0} \setminus S_-)$. Because of

$$U_v^{\succ 0} \setminus (S_+ \cup S_-) \subseteq (U_v^{r,\succ 0} \setminus S_+) \cup (L_v^{r,\succ 0} \setminus S_-) \subseteq \bar{A}$$

we obtain $S_0 \cap U_v^{\succ 0} \subseteq U_v^{\succ 0} \cap \bar{A}^c \subseteq S_+ \cup S_-$. This shows $S_v^i \cap U_v^{\succ 0} = S_+ \cup S_-$.

From the above discussion, we obtain $S_v^i \cap U_v^{\succ 0} = S_+ \cup S_-$. Both S_+ and S_- are countable unions of pairwise disjoint intervals, so $S_v^i \cap U_v^{\succ 0}$ is as well. As $S_v = \bigcup_{i\in\mathbb{N}} S_v^i$ is a countable union of the sets S_v^i, also $\Gamma_v = S_v \cap U_v^{\succ 0}$ is a countable union of pairwise disjoint intervals. □

Lemma 10 *Let x be a maximum s-t Borel flow. Then there exists an s-t Borel cut whose capacity equals the value of x.*

Proof Let $S = (S_v)_{v\in V}$ be the s-t Borel cut computed by the Reachability Procedure on the residual network with respect to x. In particular, we have $S_s = \mathbb{R}$. Moreover, the hypothesis of the lemma implies that $S_t = \emptyset$ since otherwise x is not maximum by Corollary 1. Further, it follows from Lemma 9 that, for each node $v \in V \setminus \{s, t\}$, the set $\Gamma_v := S_v \cap U_v^{\succ 0}$ can be written as $\bigcup_{i\in J_v} I_{v,i}$, where J_v is a countable set and $I_{v,i}$, $i \in J_v$, are pairwise disjoint intervals. Notice that each interval $I_{v,i}$ is supposed to be inclusion-wise maximal. Hence, S is a well defined s-t Borel cut.

In the remainder of the proof we show that $\mathrm{val}(x) = \mathrm{cap}(S)$. Recall from the proof of Lemma 2 that the value of x can be written as follows:

$$\begin{aligned}\mathrm{val}(x) = &\sum_{e=(v,w)\in E} \Big(x_e\big(S_v \cap (S_w - \tau_e)^c\big) - x_e\big(S_v^c \cap (S_w - \tau_e)\big)\Big)\\ &+ \sum_{v\in V\setminus\{s,t\}} \left(\sum_{i\in J_v^1} \big(Y_v(\beta_{v,i}-) - Y_v(\alpha_{v,i})\big) + \sum_{i\in J_v^2} \big(Y_v(\beta_{v,i}-) - Y_v(\alpha_{v,i}-)\big)\right)\\ &+ \sum_{v\in V\setminus\{s,t\}} \left(\sum_{i\in J_v^3} \big(Y_v(\beta_{v,i}) - Y_v(\alpha_{v,i})\big) + \sum_{i\in J_v^4} \big(Y_v(\beta_{v,i}) - Y_v(\alpha_{v,i}-)\big)\right) ,\end{aligned}$$

where $\alpha_{v,i}$ and $\beta_{v,i}$ are the left and right boundaries of the interval $I_{v,i}$ for each $v \in V$ and $i \in \mathbb{N}$. Further, J_v^1, J_v^2, J_v^3, and J_v^4 are the sets of indices i for which $I_{v,i}$ is open, left-closed & right-open, right-closed & left-open, and closed, respectively. On the other hand, the capacity of S is given by

$$\begin{aligned}\mathrm{cap}(S) = &\sum_{e=(v,w)\in E} u_e\big(S_v \cap (S_w - \tau_e)^c\big) +\\ &\sum_{v\in V\setminus\{s,t\}} \Big(\sum_{i\in J_v^1\cup J_v^2} U_v(\beta_{v,i}-) + \sum_{i\in J_v^3\cup J_v^4} U_v(\beta_{v,i})\Big) .\end{aligned}$$

Given the value of x and the capacity of S as above, it suffices to show that the following hold:

(i) $x_e\big(S_v \cap (S_w - \tau_e)^c\big) = u_e\big(S_v \cap (S_w - \tau_e)^c\big)$ and $x_e\big(S_v^c \cap (S_w - \tau_e)\big) = 0$ for all $e = (v, w) \in E$,
(ii) $Y_v(\beta_{v,i}-) = U_v(\beta_{v,i}-)$ and $Y_v(\alpha_{v,i}) = 0$ for all $v \in V \setminus \{s,t\}$ and $i \in J_1$,
(iii) $Y_v(\beta_{v,i}-) = U_v(\beta_{v,i}-)$ and $Y_v(\alpha_{v,i}-) = 0$ for all $v \in V \setminus \{s,t\}$ and $i \in J_2$,
(iv) $Y_v(\beta_{v,i}) = U_v(\beta_{v,i})$ and $Y_v(\alpha_{v,i}) = 0$ for all $v \in V \setminus \{s,t\}$ and $i \in J_3$,
(v) $Y_v(\beta_{v,i}) = U_v(\beta_{v,i})$ and $Y_v(\alpha_{v,i}-) = 0$ for all $v \in V \setminus \{s,t\}$ and $i \in J_4$.

Proof of case (i). By the definition of the residual network, (i) is equivalent to show that $u_e^r\big(S_v \cap (S_w - \tau_e)^c\big) = 0$ for all arcs $e = (v, w) \in E^r$ in the residual network. Since $S_v := \bigcup_{j\in\mathbb{N}} S_v^j$ it is enough to show $u_e^r\big(S_v^j \cap (S_w - \tau_e)^c\big) = 0$ for each $j \in \mathbb{N}$. Fix an $j \in \mathbb{N}$ and consider the execution of Step (3d) for node w in iteration j. We have (using the notations of the procedure) $u_e^r|_{S_v^j} - \tau_e \le \mu_1$. Moreover, we know from the definition of S_0 that $S_0 = \mathrm{supp}(\mu_1) \setminus \bar{A}$ and from Lemma 5 that $\mu_1(\bar{A}) = 0$. This shows that $\mu_1(S_0^c) = 0$ and consequently $(u_e^r|_{S_v^j} - \tau_e)(S_0^c) = u_e^r(S_v^j \cap (S_0 - \tau_e)^c) = 0$. Since $(S_w - \tau_e)^c \subseteq (S_0 - \tau_e)^c$ as $S_0 \subseteq S_w$ we can conclude $u_e^r\big(S_v^j \cap (S_w - \tau_e)^c\big) = 0$.

Proof of the first part of (ii) *and* (iii). We equivalently prove that $U_v^r(\beta_{v,i}-) = 0$ for all $v \in V \setminus \{s,t\}$ and $i \in J_1 \cup J_2$ in the residual network. Let $\beta := \beta_{v,i}$ be the right boundary of the right open interval $I_{v,i}$ for some node $v \in V\setminus\{s,t\}$ and some $i \in J_1\cup J_2$. Note that because of the definition of β we have $\beta \notin \Gamma_v$. We assume by contradiction that $U_v^r(\beta-) > 0$ and proceed to show that $\beta \in \Gamma_v$ or that β is not the right boundary of some inclusion-wise maximal interval of Γ_v.

Because of $U_v^r(\beta-) > 0$ we have $\beta \in U_v^{r,\succ 0}$. Since $U_v^{r,\succ 0}$ is the countable union of positive intervals there exists an inclusion-wise maximal positive interval $I \subseteq U_v^{r,\succ 0}$ with $\beta \in I$. Note that β is not the left boundary of I (if I is left closed) since this would imply $U_v^r(\beta-) = 0$. Hence, the set $\bar{I} := I \cap (-\infty, \beta]$ is a right closed interval with nonempty interior, i.e., $\bar{I} \setminus \{\beta\} \neq \emptyset$. We first consider the case that flow can be sent into v until time β over I at some iteration of the Reachability Procedure, i.e., $\mu_1|_{\bar{I}} > 0$. Recalling Step (3b), this shows that $\beta \in S_+$, and therefore $\beta \in \Gamma_v$.

Next we consider the case that $\mu_1|_{\bar{I}} = 0$ in each iteration which implies $\bar{I} \cap S_+ = \emptyset$. Recalling Step (3c) we know that $L_v^{r,\succ 0} = \bigcup_{k \in J} I_k$ is the countable union of disjoint positive intervals. We fix a $k \in J$ and show that $\bar{I} \cap I_k \cap H_k^{<|h_k|} = \emptyset$. Since $h_k = \mu_1|_{I_k}$ and $\mu_1|_{\bar{I}} = 0$ we have either $\bar{I} \subseteq H_k^{<|h_k|}$ or $\bar{I} \cap H_k^{<|h_k|} = \emptyset$. Let us assume $\bar{I} \subseteq H_k^{<|h_k|}$, as in the other case the assertion is trivial. Hence, if $\beta \notin I_k$, then β is on the left of I_k since otherwise $\beta \notin H_k^{<|h_k|}$ which contradicts our assumption $\bar{I} \subseteq H_k^{<|h_k|}$ due to $\beta \in \bar{I}$. Moreover, we know that β is the right boundary of $\bar{I}$ which implies $\bar{I} \cap I_k \cap H_k^{<|h_k|} = \emptyset$. On the other hand, if $\beta \in I_k$ we obtain either $\beta \in I_k \cap H_k^{<|h_k|}$ or $\bar{I} \cap I_k \cap H_k^{<|h_k|} = \emptyset$. Since $I_k \cap H_k^{<|h_k|} \subseteq S_- \subseteq \Gamma_v$ holds, $\beta \in I_k \cap H_k^{<|h_k|}$ would imply $\beta \in \Gamma_v$ contradicting $\beta \notin \Gamma_v$.

From the above discussion, we can deduce that $\bar{I} \cap I_k \cap H_k^{<|h_k|} = \emptyset$ for all $k \in J$. This shows $\bar{I} \cap S_- = \emptyset$. Recall that $\bar{I} \cap S_0 = \emptyset$ due to $\bar{I} \cap S_+ = \emptyset$ and $\bar{I} \subseteq U_v^{r,\succ 0}$. So we can conclude $\bar{I} \cap \Gamma_v = \emptyset$. This shows that β is not the right boundary of an inclusion-wise maximal interval of Γ_v, which contradicts the definition of β. Hence, we must have $U_v^r(\beta-) = 0$, which establishes the first part of (ii) and (iii).

Proof of the first part of (iv) *and* (v). Equivalently we show that $U_v^r(\beta_{v,i}) = 0$ for all $v \in V \setminus \{s,t\}$ and $i \in J_3 \cup J_4$ in the residual network. Let $\beta := \beta_{v,i}$ be the right boundary of the right closed interval $I_{v,i}$ for some node $v \in V \setminus \{s,t\}$ and some $i \in J_3 \cup J_4$. Note that because of the definition of β we have $\beta \in \Gamma_v$. We assume $U_v^r(\beta) > 0$ and proceed to derive a contradiction.

Because of $U_v^r(\beta) > 0$ we have $\beta \in U_v^{r,\succ 0}$. Since $U_v^{r,\succ 0}$ is the countable union of positive intervals there exists an inclusion-wise maximal positive interval $I \subseteq U_v^{r,\succ 0}$ with $\beta \in I$. Note that β is not the right boundary of I, since otherwise we must have $U_v^r(\beta) = 0$. We define the set $\bar{I} := I \cap (-\infty, \beta]$. Note that $\bar{I} = \{\beta\}$ if $U_v^{r,\succ 0}(\beta-) = 0$. As above, we first consider the case that $\mu_1|_{\bar{I}} > 0$. Recalling Step (3b), this shows that $I \cap [\beta, \infty) \subset S_+$ and as a consequence $I \cap [\beta, \infty) \subset \Gamma_v$. This implies that β is not the right boundary of some inclusion-wise maximal interval of Γ_v. Next we consider the case that $\mu_1|_{\bar{I}} = 0$ in each iteration which implies $\bar{I} \cap S_+ = \emptyset$. Here it follows along the same line as above that β is not the right boundary of an inclusion-wise maximal interval of Γ_v. This contradicts the definition of β. Hence, we must have $U_v^r(\beta) = 0$.

Proof of the second part of (ii) *and* (iv). It is equivalent to show that $L_v^r(\alpha_{v,i}) = 0$ for all $v \in V \setminus \{s,t\}$ and $i \in J_1 \cup J_3$ in the residual network. Let $\alpha := \alpha_{v,i}$ be the left boundary of the left open interval $I_{v,i}$ for some node $v \in V \setminus \{s,t\}$ and some $i \in J_1 \cup J_3$. By the definition of α we have $\alpha \notin \Gamma_v$. We assume by contradiction that $L_v^r(\alpha) > 0$ and proceed to show that $\alpha \in \Gamma_v$ or that α is not the left boundary of some inclusion-wise maximal interval of Γ_v.

We have $\alpha \in L_v^{r,\succ 0}$ as $L_v^r(\alpha) > 0$. Since $L_v^{r,\succ 0}$ is the countable union of positive intervals there exists an inclusion-wise maximal positive interval $I \subseteq L_v^{r,\succ 0}$ with $\alpha \in I$. Note that α is not the right boundary of I since otherwise we must have $L_v^r(\alpha) = 0$. Hence, the set $\bar{I} := I \cap [\alpha, \infty)$ is a left closed interval with $\bar{I} \setminus \{\alpha\} \neq \emptyset$. Let us first consider the case that flow can be sent into v on or after time α over I at some iteration of the REACHABILITY PROCEDURE, i.e., $\mu_1|_{\bar{I}} > 0$. Recalling Step (3c), this implies $\alpha \in S_-$, and therefore $\alpha \in \Gamma_v$.

We now assume that $\mu_1|_{\bar{I}} = 0$ in each iteration which implies $\bar{I} \cap S_- = \emptyset$. Recalling Step (3b) we know that $U_v^{r,\succ 0} = \bigcup_{k\in J} I_k$ is the countable union of disjoint positive intervals. We fix an arbitrary $k \in J$ and show that $\bar{I} \cap I_k \cap H_k^{>0} = \emptyset$. Since $h_k = \mu_1|_{I_k}$ and $\mu_1|_{\bar{I}} = 0$ we have either $\bar{I} \subseteq H_k^{>0}$ or $\bar{I} \cap H_k^{>0} = \emptyset$. We assume $\bar{I} \subseteq H_k^{>0}$, as in the other case the assertion is trivial. Hence, if $\alpha \notin I_k$, then α is on the right of I_k since otherwise $\alpha \notin H_k^{>0}$ which contradicts our assumption $\bar{I} \subseteq H_k^{>0}$ due to $\alpha \in \bar{I}$. Moreover, we know that α is the right boundary of $\bar{I}$ which implies $\bar{I} \cap I_k \cap H_k^{>0} = \emptyset$. On the other hand, if $\alpha \in I_k$ we get either $\alpha \in I_k \cap H_k^{>0}$ or $\bar{I} \cap I_k \cap H_k^{>0} = \emptyset$. Since $I_k \cap H_k^{>0} \subseteq S_+ \subseteq \Gamma_v$ holds, $\alpha \in I_k \cap H_k^{>0}$ would imply $\alpha \in \Gamma_v$ contradicting the fact that $\alpha \notin \Gamma_v$.

Now we can deduce that $\bar{I} \cap I_k \cap H_k^{>0} = \emptyset$ for all $k \in J$. This implies $\bar{I} \cap S_+ = \emptyset$. Recall that $\bar{I} \cap S_0 = \emptyset$ due to $\bar{I} \cap S_- = \emptyset$ and $\bar{I} \subseteq L_v^{r,\succ 0}$. So we can conclude $\bar{I} \cap \Gamma_v = \emptyset$. This shows that α is not the left boundary of an inclusion-wise maximal interval of Γ_v, which contradicts the definition of α. Hence, we must have $L_v^r(\alpha) = 0$, which establishes the second part of (ii) and (iv).

Proof of the second part of (iii) *and* (v). It is equivalent to show that $L_v^r(\alpha_{v,i}-) = 0$ for all $v \in V \setminus \{s,t\}$ and $i \in J_1 \cup J_3$ in the residual network. Let $\alpha := \alpha_{v,i}$ be the left boundary of the closed interval $I_{v,i}$ for some node $v \in V \setminus \{s,t\}$ and some $i \in J_2 \cup J_4$. By the definition of α we have $\alpha \in \Gamma_v$. We assume by contradiction that $L_v^r(\alpha-) > 0$ and seek a contradiction.

We have $\alpha \in L_v^{r,\succ 0}$ as $L_v^r(\alpha-) > 0$. Then there exists an inclusion-wise maximal positive interval $I \subseteq L_v^{r,\succ 0}$ with $\alpha \in I$. Note that α is not the left boundary of I. We consider the set $\bar{I} := I \cap [\alpha, \infty)$. We may have $\bar{I} = \{\alpha\}$ if $L_v^r(\alpha) = 0$. Let us first consider the case that flow can be sent into v on or after time α over I at some iteration of the REACHABILITY PROCEDURE, i.e., $\mu_1|_{\bar{I}} > 0$. Recalling Step (3c), this implies $(-\infty, \alpha] \cap I \subset S_-$, and therefore $v \subset \Gamma_v$. We now consider the case that $\mu_1|_{\bar{I}} = 0$ in each iteration which implies $\bar{I} \cap S_- = \emptyset$. In this case, in a similar way as in the proof of the second part of (ii) and (iv) to show that α is not the left boundary of an inclusion-wise maximal interval of Γ_v, which contradicts the definition of α. Hence, we must have $L_v^r(\alpha-) = 0$. □

Theorem 2 *For an s-t Borel flow x the following statements are equivalent:*

(i) The s-t Borel flow x is maximal.
(ii) There is no flow-carrying s-t path in the residual network with respect to x.
(iii) The sink t is not reachable in the residual network with respect to x.

Proof Following the proof of Corollary 1 we obtain the two implications (i) $\Longrightarrow$(ii) and (ii) $\Longrightarrow$ (iii). In particular, the implication (i) $\Longrightarrow$(ii) follows form Lemma 3.

To see that (iii) $\Longrightarrow$(i) holds let $S = (S_v)_{v\in V}$ be the Borel cut computed by the REACHABILITY PROCEDURE. Since t is not reachable, S is an *s-t* Borel cut. Moreover, by Lemma 10, we have $\text{cap}(S) = \text{val}(x)$. It then follows from Lemma 2 that x is maximum. □

Combining Theorem 1 and Lemma 10 we get the main result of this paper.

Theorem 3 (MaxFlow-MinCut Theorem) *There exists an s-t Borel flow x and an s-t Borel cut S for which val(x) = cap(S).*

Throughout the paper, we have considered the entire real line $\mathbb{R}$ as the time interval. However, all results remain valid if a time horizon $T > 0$ is given and the initial time is supposed to be zero, that is, flow originates at the source on or after time zero and must reach the sink strictly before time T. In this case, a Borel cut $S = (S_v)_{v \in V}$ is called an s-t cut if $S_s = [0, \infty)$ and $S_t := [T, \infty)$. To deal with this case, we introduce a source s_0 connected to s with an arc (s_0, s) and a sink t_0 connected to t with an arc (t, t_0). We assign a transit time of zero to both arcs (s_0, s) and (t, t_0), a capacity $u_{(s_0,s)} := \sum_{e \in \delta^+(s)} u_e|_{[0,\infty)}$ to arc (s_0, s) and capacity $u_{(t,t_0)} := \sum_{e \in \delta^-(t)} u_e|_{(-\infty,T)}$ to arc (t, t_0). Further, we let $U_s = U_t = \infty$ and $L_s = L_t = 0$. Then any s-t Borel flow obeying the additional departure and arrival time restrictions corresponds one-to-one to an s-t Borel flow on the constructed instance. Hence, an instance of (MBFP) with time horizon T and initial time 0 can be converted to an equivalent problem without time restrictions on the extended network. Therefore, all results can be translated to this situation as follows:

Theorem 4 *Consider an instance of (MBFP) with initial time 0 and time horizon T and let x be an s-t Borel flow on this instance. Then following statements are equivalent:*

(i) *The s-t Borel flow x is maximal.*
(ii) *There exists no flow-carrying s-t path in the residual network with respect to x along which flow is able to arrive at t strictly before time T.*
(iii) *The sink t is not reachable in the residual network with respect to x before time T.*
(iv) *There exists an s-t Borel S cut with $cap(S) = val(x)$.*

8 Conclusion and future work

We introduced the notion of *Borel flows* to unify discrete and continuous network flows over time into a single model. We focused on the *Maximum Borel Flow Problem* (MBFP) and gave a theoretical analysis of this problem, leading to a MaxFlow-MinCut Theorem. Our approach is based on a so-called REACHABILITY PROCEDURE, which is used to verify wether or not a given Borel flow x is maximal. Further, if x is not maximal, we can derive an augmenting s-t path. Sending flow along this path leads to a new s-t Borel flow with strictly larger value. Thus, the REACHABILITY PROCEDURE lays the ground for an algorithmic approach. Like the augmenting path algorithm for the static maximum flow problem, the algorithm maintains a feasible solution at each iteration and successively improves the solution towards optimality. More specifically, the algorithm starts with the zero flow x. Then, by calling the REACHABILITY PROCEDURE as a subroutine, it identifies augmenting s-t paths and sends flow along these paths, while preserving feasibility. The algorithm terminates when the sink t is not reachable any more. Corollary 1 implies that upon the termination of the algorithm it has found a maximum s-t Borel flow.

The problem arising in the implementation of the algorithm for computing a maximum Borel flow is that, in general, the procedure never terminates and even requires infinite memory if the node and arc capacities have pathological structure. This makes the procedure problematic for computing a maximum Borel flow. Hence the question under which circumstances the procedure is a finite-time algorithm is of great interest and certainly deserves attention. For example, if the arc capacities are concentrated on a finite set and no restrictions on storage at nodes are made, then the REACHABILITY PROCEDURE terminates in finite time. This remains also true if we forbid storage at nodes, i.e., if we set all node capacities to zero, but in general we need an oracle deciding wether flow can be stored between two given points in time or not. In general, one has to examine the following aspects in developing a finite algorithm for computing a maximum flow: the decomposition of the sets $U_v^{\succ 0}$ and $L_v^{\succ 0}$ in Steps (3b) and (3c), respectively, into a countable union of disjoint positive intervals, Lebesgue decomposition of measure μ_1 in Step (3d) (note that there is a constructive proof), the number of iterations within the Reachability Procedure, and the number of calls of the Reachability Procedure. Further details are beyond the scope of the paper and are left for future work.

We conclude the paper by considering a possible extension of a MaxFlow-MinCut Theorem to the more general setting of *time/inflow/Load-dependent transit times*. In (MBFP), although arc and node capacities are subject to fluctuations over time, the transit times are constant. A natural generalization of (MBFP) is the case where transit times are time-dependent (that is, the transit time of an arc depends on the time a flow enters the arc) or inflow-dependent (that is, the transit time of an arc depends on the amount of flow entering the arc). However, in many real-world applications, such as road traffic control, production systems, and communication networks, a difficult but more realistic feature is that the amount of time needed to traverse an arc increases as the arc becomes more congested. Introducing this into (MBFP) leads to the case of load-dependent transit times (that is, transit time of an arc is not necessarily constant but depends on the amount of flow currently sent on the arc). These generalizations of (MBFP) make the problem much harder to analyze and require a more complicated formulation. The basic problem arising here is whether or not the MaxFlow-MinCut Theorem holds in these more general settings. This problem is theoretically interesting and certainly deserves further study.

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References

1. E. J. Anderson and P. Nash. *Linear Programming in Infinite-Dimensional Spaces.* Wiley, New York, 1987.
2. E. J. Anderson, P. Nash, and A. B. Philpott. A class of continuous network flow problems. *Mathematics of Operations Research*, 7:501–514, 1982.
3. T. M. Apostol. *Mathematical Analysis.* 2th Edition, Addison-Wesley, 1974.
4. J. E. Aronson. A survey of dynamic network flows. *Annals of Operations Research*, 20:1–66, 1989.
5. J. B. Conway. *A Course in Functional Analysis.* 2nd Edition, New York, Springer-Verlag, 1990.
6. L. K. Fleischer and É. Tardos. Efficient continuous-time dynamic network flow algorithms. *Operations Research Letters*, 23:71–80, 1998.
7. L. R. Ford and D. R. Fulkerson. Constructing maximal dynamic flows from static flows. *Operations Research*, 6:419–433, 1958.
8. L. R. Ford and D. R. Fulkerson. *Flows in Networks.* Princeton University Press, 1962.
9. J. F. C. Kingman and S. J. Taylor. *Introduction to Measure and Probability.* Cambridge University Press, Cambridge, 1966.
10. B. Kotnyek. An annotated overview of dynamic network flows. Rapport de recherche 4936, INRIA Sophia Antipolis, 2003.

11. S. E. Lovetskii and I. I. Melamed. Dynamic network flows. *Automation and Remote Control*, 48:1417–1434, 1987. Translated from Avtomatika i Telemekhanika, 11 (1987), pp. 7–29.
12. E. Nasrabadi. *Dynamic Flows in Time-varying Networks*. PhD thesis, Technische Universität Berlin, Berlin, 2009.
13. K. R. Parthasarathy. *Probability Measures On Metric Spaces*. Academic Press, New York, 1967.
14. A. B. Philpott. *Algorithms For Continuous Network Flow Problems*. PhD thesis, University of Cambridge, UK, 1982.
15. A. B. Philpott. Continuous-time flows in networks. *Mathematics of Operations Research*, 15:640–661, 1990.
16. A. B. Philpott. Continuous-time shortest path problems and linear programming. *SIAM Journal on Control and Optimization*, 32:538–552, 1994.
17. W. B. Powell, P. Jaillet, and A. Odoni. Stochastic and dynamic networks and routing. In M. O. Ball, T. L. Magnanti, C. L. Monma, and G. L. Nemhauser, editors, *Network Routing*, volume 8 of *Handbooks in Operations Research and Management Science*, chapter 3, pages 141–295. North–Holland, Amsterdam, The Netherlands, 1995.
18. M. Skutella. An introduction to network flows over time. In W. Cook, L. Lovász, and J. Vygen, editors, *Research Trends in Combinatorial Optimization*, pages 451–482. Springer, Berlin, 2009.

A Preliminaries on measure theory

In this appendix we present some definitions and notations that are frequently used throughout the paper. For a detailed treatment we refer to, e.g., [9, 13].

A *σ-algebra* on the real line $\mathbb{R}$ is a nonempty collection of subsets of $\mathbb{R}$ that is closed under countable unions and complements. The smallest σ-algebra on $\mathbb{R}$ containing all open sets (or, equivalently, closed sets) is called the *Borel σ-algebra*. The elements of the Borel algebra are called *measurable sets* or *Borel sets*. Let $\mathcal{B}$ denote the collection of all Borel sets on $\mathbb{R}$. A function $\mu : \mathcal{B} \to \mathbb{R}_{\geq 0}$ is called a *Borel measure* on $\mathbb{R}$ if

(i) $\mu(\emptyset) = 0$,
(ii) $\mu(B) \geq 0$ for any $B \in \mathcal{B}$,
(iii) let $\{B_i\}_{i\in\mathbb{N}}$ be a countable collection of pairwise disjoint sets in $\mathcal{B}$, then

$$\mu\Big(\bigcup_{i\in\mathbb{N}} B_i\Big) = \sum_{i\in\mathbb{N}} \mu(B_i) \ .$$

Measures are by definition nonnegative, i.e., a nonnegative real number is assigned to each measurable set. However, it is sometimes convenient to allow that a measure also takes negative values. A measure which can take negative and positive values is called a *signed* measure. The space of finite signed measures becomes a vector space under the standard addition and scalar multiplication operations. In particular, for any two finite signed Borel measures μ_1 and μ_2 and any real value λ, the addition $\mu_1 + \mu_2$ and scalar multiplication $\lambda \cdot \mu_1$ are defined as

$$(\mu_1 + \mu_2)(B) = \mu_1(B) + \mu_2(B) \qquad \forall B \in \mathcal{B} \ ,$$
$$(\lambda \cdot \mu_1)(B) = \lambda \cdot \mu_1(B) \qquad \forall B \in \mathcal{B} \ .$$

We use also 0 to denote the null element of this vector space, i.e., the measure which assigns 0 to each $B \in \mathcal{B}$. For a signed Borel measure μ, a Borel set B is called a μ-null set if $\mu(B) = 0$ and a strict μ-null set if $\mu(A) = 0$ for all $A \subseteq B$. Note that if μ is not a signed measure then both definitions coincide.

Let M be a real-valued function on $\mathbb{R}$. The *total variation* of M within the interval $[a, b]$ is defined by

$$V(M; [a, b]) := \sup\Big\{\sum_{i=2}^{n} M(a_i) - M(a_{i-1}) \mid \{a_1, \dots, a_n\} \text{ is a partition of } [a, b]\Big\} \ .$$

The function M is said to be of *bounded variation* on $\mathbb{R}$ if there exists a constant $K < \infty$ such that $V(M; [a, b]) < K$ for any (finite) interval $[a, b] \subset \mathbb{R}$. It is a well-known result that a function is of bounded variation if and only if it is the difference between two monotonic increasing functions (see, e.g., Chapter 6 in [3]).

A function $M : \mathbb{R} \to \mathbb{R}$ is called a *distribution function* if it is of bounded variation, continuous from right, and $M(-\infty) = 0$. A Borel measure μ on $\mathbb{R}$ is called finite if the norm $|\mu| := \mu(\mathbb{R})$ of μ is finite, i.e., $|\mu| < \infty$. It is well known that the formula $\mu\big((-\infty, b]\big) = M(b)$ sets up a one-to-one correspondence between finite signed Borel measures and distribution functions. In particular, if μ is a nonnegative measure, then its corresponding distribution function M is monotonic increasing. Throughout the paper, we denote the measures by small letters (such as μ, ν, f, y, z, u, h) and their corresponding distribution functions by capital letters (such as M, N, F, Y, Z, U, H).

Let μ_1 and μ_2 be two signed Borel measures, respectively, with corresponding distribution functions M_1 and M_2. We write $\mu_1 = \mu_2$ ($\mu_1 \leq \mu_2$) if $\mu_1(B) = \mu_2(B)$ ($\mu_1(B) \leq \mu_2(B)$) for each $B \in \mathcal{B}$. We also write $M_1 \leq M_2$ if $M_1(\theta) \leq M_2(\theta)$ for each $\theta \in \mathbb{R}$. Note that $\mu_1 \leq \mu_2$ implies $M_1 \leq M_2$ but the other direction does not hold in general.

Suppose that μ is a Borel measure with corresponding distribution function M. For a measurable set A, the restriction $\mu|_A$ of μ to A is a measure defined by $\mu|_A(B) := \mu(B \cap A)$ for each $B \in \mathcal{B}$. Hence A is a strict μ-null set if $\mu|_A = 0$. Moreover, the restriction $M|_A : A \to \mathbb{R}$ of M is defined by $\theta \mapsto M(\theta)$ for all $\theta \in A$. Note that $M|_A$ is not a distribution function since it is not defined on the whole real line $\mathbb{R}$ and, in particular, it is not the distribution function of $\mu|_A$. In addition, we write $M|_A > \epsilon$ for some ϵ if $M(\theta) > \epsilon$ for all $\theta \in A$.

Moreover, for a real value τ we define the *shifted measure* $\mu - \tau$ by $(\mu - \tau)(B) = \mu(B - \tau)$ for each $B \in \mathcal{B}$, where $B - \tau := \{\theta - \tau \mid \theta \in B\}$. Similarly, the *shifted function* $M - \tau : \mathbb{R} \to \mathbb{R}$ of M is defined by $\theta \mapsto M(\theta - \tau)$. Note that $M - \tau$ is the distribution function of $\mu - \tau$.

For the distribution function M, we define $M^{\succ 0}$ ($M^{>0}$ and $M^{<|\mu|}$) to denote the set of all points $\theta \in \mathbb{R}$ such that M or its left limit is positive at θ ($M(\theta) > 0$ and $M(\theta-) < |\mu|$, respectively). More precisely,

$$\begin{aligned} M^{\succ 0} &:= \big\{\theta \in \mathbb{R} \mid M(\theta-) > 0 \text{ or } M(\theta) > 0\big\} \ , \\ M^{>0} &:= \big\{\theta \in \mathbb{R} \mid M(\theta) > 0\big\} \ , \\ M^{<|\mu|} &:= \big\{\theta \in \mathbb{R} \mid M(\theta-) < |\mu|\big\} \ . \end{aligned}$$

Note that if M is a distribution function of a nonnegative measure, then $M^{\succ 0} = M^{>0}$. Since M is right continuous $M^{\succ 0}$ is the countable union of pairwise disjoint intervals. Moreover, we can assume that each interval I of the countable union is positive, i.e., for all $a, b \in I$ with $a < b$ there exists an $\epsilon > 0$ such that $M|_{[a,b)} > \epsilon$ (see Lemma 4). Throughout the paper we implicitly assume that each (positive) interval is inclusion-wise maximal.

Given a Borel measure μ, the *support* of μ is defined to be the set of all points in $\mathbb{R}$ with a neighborhood of positive measure, that is,

$$\operatorname{supp}(\mu) := \big\{\theta \in \mathbb{R} \mid \mu(I) > 0 \ \text{ for every open neighborhood } I \text{ of } \theta\big\} \ .$$

A point $\theta \in \mathbb{R}$ is called an *atom* of μ if $\mu(\{\theta\}) > 0$. Obviously, if μ is finite, the set of atoms of μ is countable. In this case, we define the *discrete part* μ^d and *continuous part* μ^c of μ by

$$\mu^d(B) := \sum_{\text{atoms } \theta \in B} \mu\big(\{\theta\}\big) \qquad \text{and} \qquad \mu^c(B) := \mu(B) - \mu^d(B)$$

for every measurable set B.

A measure μ is called *discrete* (*continuous*[1]) if its continuous (discrete) part is zero. It can be shown that a finite Borel measure is continuous (discrete) if and only if its corresponding distribution function is a continuous function (a step function) (see, e.g., [9, Section 9.3]). Hence, there is a decomposition of a finite Borel measure into a sum of a discrete and a continuous measure. This decomposition is unique.

A measure μ is said to be *concentrated* on a measurable set A if $\mu(B) = 0$ whenever $A \cap B = \emptyset$ for each measurable set B. We can easily see that a finite measure is concentrated on a countable set if and only if it is discrete.

Two Borel measures μ_1 and μ_2 are called *mutually singular* if there exist two disjoint measurable sets A and B whose union is $\mathbb{R}$ such that μ_1 is zero on all measurable subsets of B

[1] A continuous measure is also called *nonatomic* measure.

while μ_2 is zero on all measurable subsets of A, i.e., $\mu_1(B) = 0$ and $\mu_2(A) = 0$. Moreover, μ_1 is *absolutely continuous* with respect to μ_2 if $\mu_2(A) = 0$ implies $\mu_1(A) = 0$ for every measurable set A.

The following theorem shows that any signed measure can be expresses as the difference of two mutually singular measures (see, e.g., [9] for a proof).

Theorem 5 (Jordan Decomposition) *Every signed measure μ can be expressed as the difference of two (nonnegative) measures μ^+ and μ^- such that μ^+ and μ^- are mutually singular and at least one of them is finite. If μ is finite, then both μ_1 and μ_2 are finite. Moreover, if $\mu = \mu_1 - \mu_2$, then $\mu^+ \leq \mu_1$ and $\mu^- \leq \mu_2$. The measures μ^+ and μ^- are called the* positive *and* negative part of μ*, respectively. The pair (μ^+, μ^-) is called the* Jordan decomposition *of μ.*

Theorem 5 helps us to define the minimum of two measures. Let μ_1 and μ_2 be two nonnegative measures on $\mathbb{R}$. The minimum of μ_1 and μ_2 is a nonnegative measure defined by $\min\{\mu_1, \mu_2\} := \mu_1 - \mu^+ = \mu_2 - \mu^-$, where (μ^+, μ^-) is the Jordan decomposition of the signed measure $\mu_1 - \mu_2$. It is not hard to see that $\min\{\mu_1, \mu_2\}$ is positive if μ_1 and μ_2 are positive and not mutually singular. In particular, if μ_2 is positive and μ_2 is absolutely continuous with respect to μ_1, then $\min\{\mu_1, \mu_2\}$ is positive.

We also need the following basic theorem of measure theory (see, e.g., [9] for a proof).

Theorem 6 (Lebesgue Decomposition) *Suppose that μ_1 and μ_2 are two finite Borel measures. There exist two finite Borel measures ν^{ac} and ν^s such that*

- $\mu_2 = \nu^{ac} + \nu^s$*;*
- ν^{ac} *is absolutely continuous with respect to* μ_1*;*
- ν^s *and* μ_1 *are mutually singular.*

The proof of the Lebesgue Decomposition Theorem is constructive and the measures ν^{ac} and ν^{s} are constructed through the proof. The construction also gives a set A such that $\mu_1(A) = 0$ and $\nu^{\mathrm{s}}(A^c) = 0$.

B Proof of technical lemmas

In this Appendix, we provide the proofs of Lemmas 1, 4 and, 7 that were omitted from the main text. The proof of Lemma 1 is based on the next lemma together with the two subsequently corollaries.

Lemma 11 *Suppose that μ_1 and μ_2 are two finite continuous Borel measures on $\mathbb{R}$ with distribution functions M_1 and M_2, respectively. Let $M_1 \geq M_2$ on some interval $I := (-\infty, \theta]$, $\theta \in \mathbb{R}$, and $A := \{\vartheta \in I \mid M_1(\vartheta) = M_2(\vartheta)\}$ be the set of points in I where the two distribution functions are equal. Then $\mu_1(A) = \mu_2(A)$.*

Proof For a given $\epsilon > 0$, let $A_\epsilon := \{\vartheta \in (-\infty, \theta) \mid M_1(\vartheta) - M_2(\vartheta) < \epsilon\}$ be the set of points in $(-\infty, \theta)$ where the two distribution functions differ by less than ϵ. We know that the distribution functions M_1 and M_2 are continuous since μ_1 and μ_2 are continuous measures. Hence, A_ϵ is an open set, so we can express it as a countable union of pairwise disjoint open intervals: $A_\epsilon = \bigcup_{i \in J} (a_i, b_i)$, where J is a countable set of indices and $a_i = -\infty$ for one $i \in J$. Note that, for each $i \in J$, the interval (a_i, b_i) is maximal in the following sense. There exists no open interval $(a', b') \subseteq A_\epsilon$ strictly containing (a_i, b_i). Since the distribution functions M_1 and M_2 are continuous we can conclude that

$$M_1(a_i) - M_2(a_i) = \begin{cases} \epsilon & \text{if } a_i > -\infty \\ 0 & \text{if } a_i = -\infty \end{cases} \qquad \text{and} \qquad M_1(b_i) - M_2(b_i) \begin{cases} = \epsilon & \text{if } b_i < \theta \\ \leq \epsilon & \text{if } b_i = \theta \end{cases} .$$

It follows that

$$\mu_1(A_\epsilon) - \mu_2(A_\epsilon) = \sum_{i \in J} \mu_1\big((a_i, b_i)\big) - \mu_2\big((a_i, b_i)\big) \leq \epsilon .$$

Now we let ϵ tend to 0 and get $\mu_1(A \setminus \{\theta\}) = \mu_2(A \setminus \{\theta\})$. Since μ_1 and μ_2 are continuous this shows $\mu_1(A) = \mu_2(A)$. □

The next corollary generalizes Lemma 11 from $\mu_1(A) = \mu_2(A)$ to $\mu_1|_A = \mu_2|_A$, even for the more general case when the assumption of $M_1 \geq M_2$ is not met.

Corollary 2 *Let μ_1 and μ_2 be two finite continuous Borel measures on $\mathbb{R}$ with distribution functions M_1 and M_2, respectively. Further, let $A := \{\theta \in \mathbb{R} \mid M_1(\theta) = M_2(\theta)\}$ be the set of points where the two distribution functions are equal. Then, $\mu_1|_A = \mu_2|_A$.*

Proof We first assume that $M_1 \geq M_2$. Then, Lemma 11 implies

$$\mu_1|_A\big((-\infty,\theta]\big) = \mu_1|_{(-\infty,\theta]}(A) = \mu_2|_{(-\infty,\theta]}(A) = \mu_2|_A\big((-\infty,\theta]\big), \qquad \text{for all } \theta \in \mathbb{R}.$$

It follows from this relation that the distribution functions with respect to $\mu_1|_A$ and $\mu_2|_A$ coincide on $\mathbb{R}$. This implies $\mu_1|_A = \mu_2|_A$.

For the general case, we define $M_{\max} : \mathbb{R} \to \mathbb{R}$ by $M_{\max}(\theta) := \max\{M_1(\theta), M_2(\theta)\}$. It is clear that $M_{\max}$ is monotonically increasing and continuous. So it is the distribution function of some finite continuous measure $\mu_{\max}$. Applying the previous result for $M_{\max}$ and M_1, and also for $M_{\max}$ and M_2, we get $\mu_1|_A = \mu_{\max}|_A = \mu_2|_A$. □

Corollary 3 *Let μ be a finite signed Borel measure on $\mathbb{R}$ with distribution function M and let $Q \subset \mathbb{R}$ be a countable set of real numbers. If μ is continuous, then $A := \{\theta \mid M(\theta) \in Q\}$ is a strict μ-null set, i.e., $\mu|_A = 0$.*

Proof For each $q \in Q$ define $A_q := \{\theta \mid M(\theta) = q\}$. Since A is the disjoint countable union of the sets A_q, we have $\mu|_A = \sum_{q\in Q} \mu|_{A_q}$. Hence, in order to establish the lemma it is enough to show that $\mu|_{A_q} = 0$ for each $q \in Q$.

Let $q \in Q$ be fixed and assume, without loss of generality, that $q \geq 0$. Further, let μ^+ and μ^- be the positive and negative part of μ with distribution functions M^+ and M^-, respectively. Since μ is continuous, $a := \min\{\theta \mid M^+(\theta) \geq q\} \in \mathbb{R} \cup \{\infty\}$ is well-defined and $M^+(a) = q$. We define $\bar{M} : \mathbb{R} \to \mathbb{R}_+$ by

$$\bar{M}(\theta) := \begin{cases} 0 & \text{if } \theta < a \ , \\ M^+(\theta) - q & \text{if } \theta \geq a \ . \end{cases}$$

Then, $\bar{M}$ is the distribution function of the measure $\bar{\mu} := \mu^+|_{[a,\infty)}$. Further, defining the set $\bar{A}_q$ by $\bar{A}_q := \{\theta \mid \bar{M}(\theta) = M^-(\theta)\}$, Corollary 2 shows $\bar{\mu}|_{\bar{A}_q} = \mu^-|_{\bar{A}_q}$. Since $M(\theta) = q$ implies $M^+(\theta) - M^-(\theta) = q$ it holds that $A_q \subseteq \bar{A}_q$. Together with $A_q \cap (-\infty, a) = \emptyset$ it follows that $\mu^+|_{A_q} = \bar{\mu}|_{A_q} = \mu^-|_{A_q}$ and, as a direct consequence, $\mu|_{A_q} = 0$. □

We can now give a proof of Lemma 1.

Lemma 1 *Let μ be a finite signed Borel measure on $\mathbb{R}$ with a nonnegative distribution function M. Let $A := \mathbb{R} \setminus M^{\succ 0}$ be the set of points $\theta \in \mathbb{R}$ for which M is continuous and zero at θ. Then $\mu|_A = 0$, i.e., the set A is a strict μ-null set.*

Proof Let μ^d be the discrete part of μ and M^d be its distribution function. As μ is finite, the support of μ^d is countable, and thus the set $Q = \{M^d(\theta) \mid \theta \in \mathbb{R}\}$ is countable.

Let M^c be the distribution function of the continuous part μ^c and define the set $\bar{A}$ by $\bar{A} := \{\theta \mid -M^c(\theta) \in Q\}$. It now follows from Corollary 3 that $\mu^c|_{\bar{A}} = 0$ since Q is countable. On the other hand, we know that $A \subseteq \bar{A}$ and $A \cap \operatorname{supp}(\mu^d) = \emptyset$ implying $\mu|_A = 0$. This concludes the proof. □

Next we prove Lemma 4 which shows that $M^{\succ 0}$ is a countable union of pairwise disjoint positive intervals. Recall that an interval I is called positive if for all $a, b \in I$ with $a < b$ there exists an $\epsilon > 0$ such that $M|_{[a,b)} > \epsilon$.

Lemma 4 *Let M be a distribution function. Then the set $M^{\succ 0}$, defined by (5), can be written as a countable union of pairwise disjoint positive intervals.*

Proof The set $M^{\succ 0}$ can be written as the union of two disjoint sets $M^{\succ 0}_{\text{con}}$ and $M^{\succ 0}_{\text{dis}}$, where

$$\begin{aligned} M^{\succ 0}_{\text{con}} &:= \{\theta \in M^{\succ 0} \mid M(\theta-) = M(\theta) > 0\} \ , \\ M^{\succ 0}_{\text{dis}} &:= \{\theta \in M^{\succ 0} \mid M(\theta-) \neq M(\theta)\} \ . \end{aligned}$$

The set $M_{\text{con}}^{\succ 0}$ is an open set and hence can be expressed as the countable union of pairwise disjoint intervals. We thus let $M_{\text{con}}^{\succ 0} = \bigcup_{i \in J_{\text{con}}} I_i$ where J_{con} is a countable set of indices and I_i is an open interval for each $i \in J_{\text{con}}$. For each $i \in J$, we have $M|_{I_i} > 0$ and hence, I_i is positive.

We next consider the set $M_{\text{dis}}^{\succ 0}$. Each member of this set is a discontinuous point of M. On the other hand, since M is right-continuous, it has a countable number of discontinuous points (see, e.g., [3]). Thus, $M_{\text{dis}}^{\succ 0}$ is a countable set. So let $M_{\text{dis}}^{\succ 0} = \bigcup_{i \in J_{\text{dis}}} \{a_i\}$ where J_{dis} is a countable set of indices and a_i is a real number for each $i \in J_{\text{dis}}$. For each a_i there exists an interval I_j for some $j \in J_{\text{con}}$ such that either a_i is the right boundary of some I_j and $M(a_i-) > 0$ or otherwise it must be $M(a_i-) = 0$ and a_i is the left boundary of I_j. We then extent I_j as $I_j := I_j \cup \{a_i\}$. Note that I_j remains positive.

The above construction gives a decomposition of $M^{\succ 0}$ into a countable union of pairwise disjoint positive intervals I_i, $i \in J_{\text{con}}$. □

For the proof of Lemma 7, we need the concept of regularity for measures and a simple result. A Borel measure μ is called *regular* if for every Borel set B

$$\mu(B) \;=\; \sup\{\mu(C) \mid C \subseteq B, C \text{ closed}\} \;=\; \inf\{\mu(O) \mid B \subseteq O, O \text{ open}\} \;.$$

It is well known that any finite Borel measure on $\mathbb{R}$ is regular (see, e.g., [13]). Using the result, the following lemma can be established.

Lemma 12 *Let μ and ν be two finite Borel measures on $\mathbb{R}$ with distribution functions M and N, respectively. Then $\mu \leq \nu$ if and only if $M(b) - M(a) \leq N(b) - N(a)$ for all $a, b \in \mathbb{R}$ with $a \leq b$.*

Proof Assume $\mu \leq \nu$. Then, for all $a, b \in \mathbb{R}$ with $a \leq b$, it holds that $M(b) - M(a) = \mu((a, b]) \leq \nu((a, b]) = N(b) - N(a)$. Hence, it remains to prove the other direction.

Let $B \in \mathcal{B}$ be any Borel set. Since every finite Borel measure on $\mathbb{R}$ is regular (see, e.g., [13]) we know $\nu(B) = \inf\{\nu(O) \mid B \subseteq O, O \text{ open}\}$. Hence, for every $\epsilon > 0$ there exists an open set O containing B such that $\nu(O) \leq \nu(B) + \epsilon$. Since O is open it is the countable union of disjoint open intervals, i.e., $O = \bigcup_{i \in J}(a_i, b_i)$ for some countable set J and $a_i, b_i \in \mathbb{R} \cup \{-\infty, \infty\}$. Assuming $M(b) - M(a) \leq N(b) - N(a)$ for all $a, b \in \mathbb{R}$ with $a \leq b$ we have:

$$\mu((a_i, b_i)) = \lim_{b \nearrow b_i} M(b) - M(a_i) \leq \lim_{b \nearrow b_i} N(b) - N(a_i) = \mu((a_i, b_i)) \qquad \forall\, i \in J \;.$$

This shows $\mu(O) \leq \nu(O)$. Since $B \subseteq O$ this implies $\mu(B) \leq \mu(O) \leq \nu(O) \leq \nu(B) + \epsilon$. Now we let tend ϵ to zero and obtain $\mu(B) \leq \nu(B)$. □

Using the last lemma we are able to prove Lemma 7.

Lemma 7 *Let μ_1, μ_2, and ν_1 be finite Borel measures on $\mathbb{R}$ with corresponding distribution functions M_1, M_2, and N_1, respectively. Further, assume that $|\mu_1| \geq |\mu_2|$ and $\nu_1 \leq \mu_1$. Then there exists a (finite) Borel measure $\nu_2 \leq \mu_2$ with distribution function N_2 such that $|N_1(\theta) - N_2(\theta)| \leq |M_1(\theta) - M_2(\theta)|$ for each $\theta \in \mathbb{R}$, i.e, the vertical distance between the distribution functions does not increase when replacing μ_1 and μ_2 with ν_1 and ν_2, respectively.*

Proof We define ν_2 via its distribution function N_2. The key idea is to construct N_2 such that the horizontal distance between the distribution functions is preserved. That is, for each $\theta_1, \theta_2 \in \mathbb{R}$ with $M_1(\theta_1) = M_2(\theta_2)$ it holds that $N_1(\theta_1) = N_2(\theta_2)$. Since a measure can be interpreted as the slope of its distribution function, $\nu_1 \leq \mu_1$ ensures that the vertical distance between the distribution functions does not increase. Unfortunately, N_2 is not well-defined in this manner because the distribution functions have jumps in general.

In order to construct N_2, let $\theta_2 \in \mathbb{R}$ be some real number. Further, let $\theta_1 \in \mathbb{R}$ be such that

$$M_1(\theta_1-) \leq M_2(\theta_2) \leq M_1(\theta_1) \;. \tag{14}$$

Note that such a real number θ_1 exists since we know that $M_1(-\infty) = M_2(-\infty) = 0$ and that $M_1(\infty) = |\mu_1| \geq |\mu_2| = M_2(\infty)$. Then we define:

$$N_2(\theta_2) := \begin{cases} N_1(\theta_1) & \text{if } \mu_1(\{\theta_1\}) = 0 \\ N_1(\theta_1-) + \frac{\nu_1(\{\theta_1\})}{\mu_1(\{\theta_1\})}(M_2(\theta_2) - M_1(\theta_1-)) & \text{if } \mu_1(\{\theta_1\}) > 0 \end{cases} \;. \tag{15}$$

Note that $\mu_1(\{\theta_1\})$ and $\nu_1(\{\theta_1\})$ are the heights of a jump of M_1 and N_1 at θ_1, respectively. So if $\mu_1(\{\theta_1\}) > 0$ then $N_2(\theta_2)$ is defined in such a way that $N_2(\theta_2)$ divides the jump $\nu_1(\{\theta_1\})$ of N_1 with the same ratio as $M_2(\theta_2)$ divides the jump $\mu_1(\{\theta_1\})$ of M_1.

Furthermore, note that although the definition of $N_2(\theta_2)$ depends on some θ_1 satisfying (14), it is independent on a special choice of θ_1. This can be seen as follows: First recall that M_1 is monotonically increasing and right-continuous. Hence, if θ_1 is not unique then θ_1 must be taken out of some closed interval $I := [a, b]$ satisfying $M_1(\theta_1) = M_2(\theta_2)$ for all $\theta_1 \in [a, b)$. Since $\nu_1 \leq \mu_1$ by the hypothesis of the lemma, also N_1 must be constant to some value k over $[a, b)$ as desired. Thus using (15) we obtain $N_2(\theta_2) = k$ for all $\theta_1 \in [a, b]$. Hence, $N_2(\theta_2)$ is well-defined.

Alternatively, $N_2(\theta_2)$ can be defined in terms of $M_1(\theta_1)$ and $N_1(\theta_1)$, instead of $M_1(\theta_1-)$ and $N_1(\theta_1-)$, respectively. This can be achieved by substituting $M_1(\theta_1-) = M_1(\theta_1) - \mu_1(\{\theta_1\})$ and $N_1(\theta_1-) = N_1(\theta_1) - \nu_1(\{\theta_1\})$ in (15) leading to:

$$N_2(\theta_2) = N_1(\theta_1) + \frac{\nu_1(\{\theta_1\})}{\mu_1(\{\theta_1\})}(M_2(\theta_2) - M_1(\theta_1)) \ . \tag{16}$$

Next we show some relations which we use subsequently to show that ν_2 can defined via N_2. Since θ_1 was chosen such that (14) holds we have $0 \leq M_2(\theta_2) - M_1(\theta_1-) \leq \mu_1(\{\theta_1\})$. Then, the definition of N_2 in (15) implies that (14) holds also for N_1 and N_2 instead of M_1 and M_2, that is,

$$N_1(\theta_1-) \leq N_2(\theta_2) \leq N_1(\theta_1) \ . \tag{17}$$

Since we assume $\nu_1 \leq \mu_1$ we know $\nu_1(\{\theta_1\}) \leq \mu_1(\{\theta_1\})$. Further we get from (14) the inequalities $0 \leq M_2(\theta_2) - M_1(\theta_1-)$ and $0 \geq M_2(\theta_2) - M_1(\theta_1)$. Hence, (15) and (16) show

$$N_2(\theta_2) \ \leq \ N_1(\theta_1-) + M_2(\theta_2) - M_1(\theta_1-) \tag{18}$$

and

$$-N_2(\theta_2) \ \leq \ -N_1(\theta_1) - M_2(\theta_2) + M_1(\theta_1) \ , \tag{19}$$

respectively.

Now we show that N_2 is the distribution function of some Borel measure ν_2. First we show that N_2 is monotonically increasing. Let $\theta_2', \theta_2'' \in \mathbb{R}$ with $\theta_2' < \theta_2''$ and let θ_1' and θ_1'' be such that $M_1(\theta_1'-) \leq M_2(\theta_2') \leq M_1(\theta_1')$ and $M_1(\theta_1''-) \leq M_2(\theta_2'') \leq M_1(\theta_1'')$. Since M_1 and M_2 are monotonically increasing we can choose θ_1' and θ_1'' such that $\theta_1' \leq \theta_1''$. If $\theta_1' = \theta_1''$ then the inequality $N_2(\theta_2') \leq N_2(\theta_2'')$ follows from $M_2(\theta_2') \leq M_2(\theta_2'')$. On the other hand, if $\theta_1' < \theta_1''$ then $N_2(\theta_2') \leq N_2(\theta_2'')$ follows from (17) and the fact that N_1 is monotonically increasing. For proving the right-continuity we use $\nu_1 \leq \mu_1$, (18), and (19) to obtain:

$$\begin{aligned} N_2(\theta_2'') - N_2(\theta_2') \ &\leq \ N_1(\theta_1''-) + M_2(\theta_2'') - M_1(\theta_1''-) - N_1(\theta_1') - M_2(\theta_2') + M_1(\theta_1') \\ &= \ \nu_1([\theta_1', \theta_1'')) - \mu_1([\theta_1', \theta_1'')) + M_2(\theta_2'') - M_2(\theta_2') \\ &\leq \ M_2(\theta_2'') - M_2(\theta_2') \ . \end{aligned} \tag{20}$$

Since N_2 is monotonically increasing and M_2 is right-continuous, (20) shows that N_2 is right-continuous (let θ'' tend to θ'). Hence, N_2 is a distribution function and defines a Borel measure ν_2.

Lemma 12 and equation (20) show $\nu_2 \leq \mu_2$. Hence, it remains to show that the vertical distance does not increase, i.e., for all $\theta_2 \in \mathbb{R}$ it holds that $|N_1(\theta_2) - N_2(\theta_2)| \leq |M_1(\theta_2) - M_2(\theta_2)|$. First we assume that $0 \leq M_1(\theta_2) - M_2(\theta_2)$. From the definition of θ_1 we obtain the chain of inequalities $M_1(\theta_1-) \leq M_2(\theta_2) \leq M_1(\theta_2)$. Since M_1 is monotonically increasing this implies $\theta_1 \leq \theta_2$. Thus, we obtain with (19) and $\nu_1 \leq \mu_1$:

$$\begin{aligned} 0 \ \leq \ N_1(\theta_2) - N_2(\theta_2) \ &= \ N_1(\theta_2) - N_1(\theta_1) - M_2(\theta_2) + M_1(\theta_1-) \\ &\leq \ M_1(\theta_2) - M_1(\theta_1) - M_2(\theta_2) + M_1(\theta_1-) \ = \ M_1(\theta_2) - M_2(\theta_2) \end{aligned}$$

Hence, we have $|N_1(\theta_2) - N_2(\theta_2)| \leq |M_1(\theta_2) - M_2(\theta_2)|$ in the case of $0 \leq M_1(\theta_2) - M_2(\theta_2)$. If $M_1(\theta_2) - M_2(\theta_2) \leq 0$, we can follow the same line of arguments as above and we therefore omit the details. This completes the proof. □