

# A Single Server Retrial Queue with Impatient Customers

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## Abstract

In the present paper, a single server retrial queue with impatient customers is studied. The primary arrivals and repeating calls follow Poisson distribution. The service time is exponentially distributed. Explicit time dependent probabilities of exact number of arrivals and departures from the orbit are obtained by solving the differential-difference equations recursively. Steady state solution of the number of busy servers is obtained. The numerical results are graphically displayed to illustrate the effect of arrival rate, retrial rate and service rate on different probabilities against time. Some special cases of interest are also deduced.

**Keywords:** Retrial, arrivals, departures, impatient.

## 1. INTRODUCTION

When an incoming call finds the server free at the time of its arrival, this call begins to be served immediately and leaves the system after service completion. However, if the server is busy at the time of an incoming call then with probability  $1-a_1$  the call leaves the system without service and with probability  $a_1 > 0$  joins the secondary queue, named as orbit and retries from there with intensity  $\theta$  to get service. If an incoming repeated call finds the server free, it is served and leaves the system after service. Otherwise, if the server is busy at the time of a repeated call arrival with probability  $1-a_2$  the customer leaves the system without service and with probability  $a_2$  retries for service again.

Falin and Templeton (1997) studied a single server retrial queueing system and obtained the steady state probabilities for the number of units in the system when the server is free and busy. Garg et al. (2008) obtained the time dependent probabilities for the exact number of arrivals & departures for the secondary queue of a single channel retrial queueing system. In this paper, the concept of impatience is applied to obtain the time dependent probabilities for the exact number of arrivals and departures from the orbit for a single server retrial queueing system by a given time recursively in different cases i.e. when server is busy and idle. Numerical results along with graphs are also found.

*The queueing system investigated in this paper is described by the following assumptions:*

1. The primary calls arrive in a Poisson distribution with parameter  $\lambda$ .
2. The service time follows an exponential distribution with parameter  $\mu$ .

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3. The repeated calls follow a Poisson distribution with parameter  $\theta$ .
4. The stochastic processes involved viz.
  - (a) arrival of calls
  - (b) service times
  - (c) retrial times are statistically independent.

## 2. DEFINITIONS

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$p_{i,j,0}(t)$  =Probability that there are exactly  $i$  arrivals in the orbit and  $j$  departures from the orbit by time  $t$  when server is idle.

$p_{i,j,1}(t)$  =Probability that there are exactly  $i$  arrivals in the orbit and  $j$  departures from the orbit by time  $t$  when server is busy.

Initially

$$p_{0,0,0}(0) = 1; p_{i,j,1}(0) = 0, i < j; p_{i,0,1}(0) = 0, i \geq 0.$$

The difference-differential equations governing the system are

$$\frac{d}{dt} p_{0,0,0}(t) = -\lambda p_{0,0,0}(t) + \mu p_{0,0,1}(t) \quad (2.1)$$

$$\frac{d}{dt} p_{i,j,0}(t) = -(\lambda + (i-j)\theta) p_{i,j,0}(t) + \mu p_{i,j,1}(t), \quad j \leq i, j \geq 0, i > 0 \quad (2.2)$$

$$\begin{aligned} \frac{d}{dt} p_{i,j,1}(t) = & -(\lambda a_1 + \mu + (i-j)\theta(1-a_2)) p_{i,j,1}(t) + \lambda p_{i,j,0}(t) + \lambda a_1 p_{i-1,j,1}(t) \\ & + (i-j+1)\theta p_{i,j-1,0}(t) + (i-j+1)\theta(1-a_2) p_{i,j-1,1}(t), \\ & 0 \leq j \leq i \end{aligned} \quad (2.3)$$

Using the Laplace transformation  $\bar{f}(s)$  of  $f(t)$  given by

$$\bar{f}(s) = \int_0^{\infty} e^{-st} f(t) dt, \operatorname{Re}(s) > 0$$

in the equations (2.1)-(2.3) along with the initial conditions. We have

$$(s + \lambda) \bar{p}_{0,0,0}(s) = \mu \bar{p}_{0,0,1}(s) + 1 \quad (2.4)$$

$$(s + \lambda + (i-j)\theta) \bar{p}_{i,j,0}(s) = \mu \bar{p}_{i,j,1}(s), \quad j \leq i, j \geq 0, i > 0 \quad (2.5)$$

$$\begin{aligned} & (s + \lambda a_1 + \mu + (i-j)\theta(1-a_2)) \bar{p}_{i,j,1}(s) \\ & = \lambda \bar{p}_{i,j,0}(s) + \lambda a_1 \bar{p}_{i-1,j,1}(s) + (i-j+1)\theta \bar{p}_{i,j-1,0}(s) \\ & + (i-j+1)\theta(1-a_2) \bar{p}_{i,j-1,1}(s), \quad 0 \leq j \leq i \end{aligned} \quad (2.6)$$

### 3. MATHEMATICAL ANALYSIS

$$\bar{p}_{0,0,0}(s) = \frac{s + \lambda a_1 + \mu}{(s + \lambda a_1 + \mu)(s + \lambda) - \lambda \mu} \quad (3.1)$$

$$\bar{p}_{0,0,0}(s) = \frac{\lambda}{(s + \lambda a_1 + \mu)(s + \lambda) - \lambda \mu} \quad (3.2)$$

$$\bar{p}_{0,0,0}(s) = \sum_{z=0}^{\infty} \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \cdots \sum_{b_i=0}^{\infty} a_1 \lambda^{i+1+z} \mu^{1+z} \left( \prod_{x=1}^i (\lambda \mu)^{b_x} \right) \prod_{m=1}^{2i+2} (s + \eta_{m,0,i})^{-\xi_{m,0,i}} \times (s + \lambda + i\theta)^{-1} \quad i > 0 \quad (3.3)$$

$$\bar{p}_{0,0,0}(s) = \sum_{z=0}^{\infty} \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \cdots \sum_{b_i=0}^{\infty} a_1 \lambda^{i+1+z} \mu \left( \prod_{x=1}^i (\lambda \mu)^{b_x} \right) \prod_{m=1}^{2i+2} (s + \eta_{m,0,i})^{-\xi_{m,0,i}} \quad i > 0 \quad (3.4)$$

$$\bar{p}_{i,j,0}(s) = \frac{\mu}{(s + \lambda + (i-j)\theta)} \bar{p}_{i,j,1}(s), \quad 1 \leq j \leq i \quad (3.5)$$

$$\begin{aligned} \bar{p}_{i,j,0}(s) = & \sum_{z=0}^{\infty} \sum_{\alpha_j=1}^{i-j+1} \sum_{\alpha_{j-1}=1}^{\alpha_j+1} \cdots \sum_{\alpha_1=1}^{\alpha_z+1} \sum_{x_1=0}^1 \sum_{z_1=0}^{\infty} \cdots \sum_{x_j=0}^{\infty} \sum_{z_j=0}^{\infty} \sum_{w_{\alpha_1}}^{w_{\alpha_2}-w_{\alpha_1}+1} \sum_{w_{\alpha_2}}^{w_{\alpha_3}-w_{\alpha_2}+1} \cdots \\ & \cdots \sum_{w_{\alpha_j}}^{w_{\alpha_{j+1}}-w_{\alpha_j}+1} \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \cdots \sum_{b_{\alpha_2}=0}^{\infty} (\alpha_1 \alpha_2 \cdots \alpha_j) \theta^j a_1 \lambda^{1+z+\alpha_1 z} \\ & \left( \prod_{\theta=1}^j (\lambda \mu)^{z\theta} \lambda^{\beta_{\alpha_e}} \mu^{y_{\alpha_e} + \delta_{1,\alpha_e}} (1 - a_2)^{\delta_{0,\alpha_e}} (\lambda \mu)^{b_1 + b_2 + \cdots + b_{\alpha_1}} \right) \times \\ & \left( \prod_{q=1}^j \prod_{k=1}^{2\alpha_{q+1} - 4\alpha_q + 4} (s + r_{k,q})^{-u_{k,q}} \right) \left( \prod_{m=1}^{2b_{\alpha_1} + 3j + 2} (s + \eta_{m,j,b_{\alpha_1}})^{-\xi_{m,j,b_{\alpha_1}}} \right) \quad 1 \leq j \leq i \quad (3.6) \end{aligned}$$

Where

$$\sum_{w_{\alpha_v}}^{w_{\alpha_{v+1}} - \alpha_v + 1} = \sum_{w_{\alpha_v}=0}^{\infty} \sum_{w_{\alpha_{v+1}}=0}^{\infty} \cdots \sum_{w_{\alpha_{v+1}} - \alpha_{v+1} = 0}^{\infty}, \delta_{x,y} = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

$$\Upsilon_{k,q} = \begin{cases} \lambda + (\alpha_q + k - 1)\theta & \alpha_{q+1} - 2\alpha_q + 3 \leq k \leq 2\alpha_{q+1} - 4\alpha_q + 4 \\ \lambda a_1 + \mu + (k - \alpha_{q+1} + 3\alpha_q - 3)\theta(1 - \alpha_2) & 1 \leq k \leq \alpha_{q+1} - 2\alpha_q + 2 \end{cases}$$

$$u_{k,q} = \begin{cases} w_{\alpha_q+k-1} & 1 \leq k \leq \alpha_{q+1} - 2\alpha_q + 2 \\ w_{k-\alpha_{q+1}+3\alpha_{q-3}} & \alpha_{q+1} - 2\alpha_q + 3 \leq k \leq 2\alpha_{q+1} - 4\alpha_q + 4 \end{cases}$$

$$\eta_{m,j,b_{\alpha_1}} = \begin{cases} \lambda + (\alpha_m - 1)\theta & 1 \leq m \leq j \\ \lambda a_1 + \mu + (\alpha_{m-j} - 1)\theta(1 - a_2) & j + 1 \leq m \leq 2j \\ \lambda + \alpha_{m-2j}\theta & 2j + 1 \leq m \leq 3j \\ \lambda a_1 + \mu + (m - 3j)\theta(1 - a_2) & 3j + 1 \leq m \leq b_{\alpha_1} + 3j \\ \lambda + (m - b_{\alpha_1} - 3j)\theta & b_{\alpha_1} + 3j + 1 \leq m \leq 2b_{\alpha_1} + 3j \\ \lambda & m = 2b_{\alpha_1} + 3j + 1 \\ \lambda a_1 + \mu & m = 2b_{\alpha_1} + 3j + 2 \end{cases}$$

$$\xi_{m,j,b_{\alpha_1}} = \begin{cases} Z_m & 1 \leq m \leq j \\ Z_{m-j+\delta_{0,x_{m-j}}} & j + 1 \leq m \leq 2j \\ \delta_{1,x_{m-2j}} & 2j + 1 \leq m \leq 3j \\ b_{m-3j} + 1 & 3j + 1 \leq m \leq b_{\alpha_1} + 3j \\ b_{m-b_{\alpha_1}-3j} & b_{\alpha_1} + 3j + 1 \leq m \leq 2b_{\alpha_1} + 3j \\ Z + 1 & m = 2b_{\alpha_1} + 3j + 1 \\ Z + 1 & m = 2b_{\alpha_1} + 3j + 2 \end{cases}$$

$$w_{\alpha_v} + w_{\alpha_v+1} + \dots + w_{\alpha_{v+1}-\alpha_v+1} = y_{\alpha_v}$$

$$w_{\alpha_v} + w_{\alpha_v+1} + \dots + w_{\alpha_{v+1}-\alpha_v+1} + \alpha_{q+1} - 2\alpha_q + 2 = \beta_{\alpha_v}, \alpha_{j+1} = i - j$$

Following Bateman (1954), we note that the Laplace inverse transform of

$$\frac{Q(p)}{p(p)} = \sum_{k=1}^n \sum_{l=1}^{m_k} \frac{t^{m_k-l} e^{\alpha_k t}}{(m_k - l)!(l - 1)!} \frac{d^{l-1}}{dp^{l-1}} \left. \frac{Q(p)(p - \alpha_k^{m_k})}{p(p)} \right|_{p=\alpha_k}, \alpha_i \neq \alpha_k \text{ for } i \neq k$$

Where  $p(p) = (p - \alpha_1)^{m_1} (p - \alpha_2)^{m_2} \dots \dots \dots (p - \alpha_n)^{m_n}$  &  $Q(p)$  is a polynomial of degree  $< m_1 + m_2 + \dots \dots \dots m_n - 1$ , we obtain inverse Laplace transform of

$$\left( \prod_{Q=1}^j 2\alpha_{Q+1}^{-4\alpha_Q+4} \prod_{k=1}^{uk,q} (s+r_{k,q}) \right)^{-uk,q} \left( \prod_{m=1}^{2b_{\alpha_1}+3j+2} (s+\eta_{m,j,b_{\alpha_1}}) \right)^{-\xi_{m,j,b_{\alpha_1}}}$$

$$\prod_{m=3j+1}^{2i+3j+2} (s+\eta_{m,j,i})^{-\xi_{m,j,i}} \text{ are}$$

$$f_{u_{1,1}, u_{2,2}, \dots, u_{2\alpha_2-4\alpha_1+4,1}, u_{1,2}, u_{2,2}, \dots, u_{2\alpha_3-4\alpha_2+4,2}, \dots, u_{2\alpha_{j+1}-4\alpha_j+4,j}, \xi_{1,j,b_{\alpha_1}}, \dots, \xi_{2b_{\alpha_1}+3j+2,j,b_{\alpha_1}}}(t)$$

$$\& g_{\xi_{3j+1,j,i}, \xi_{3j+2,j,i}, \dots, \xi_{2i+3j+2,j,i}}(t)$$

Denoting the convolution as usual by\* and the inverse transform of equations (3.1)-(3.6) are

$$p_{0,0,0}(t) = e^{-\frac{\lambda(a_1+1)+\mu}{2}t} \cosh \frac{\sqrt{\lambda^2(1+a_1^2)+\mu^2+2\lambda\mu(a_1+1)}}{2}t + e^{-\frac{\lambda(a_1+1)\mu}{2}t} \frac{\lambda(a_1-1)+\mu}{\sqrt{\lambda^2(1+a_1^2)+2\lambda\mu(a_1+1)}} \sinh \frac{\sqrt{\lambda^2(1+a_1^2)+\mu^2+2\lambda\mu(a_1+1)}}{2}t \quad (3.7)$$

$$p_{0,0,0}(t) = \frac{2\lambda}{\sqrt{\lambda^2(1+a_1^2)+\mu^2+2\lambda\mu(a_1+1)}} e^{-\frac{\lambda(a_1+1)+\mu}{2}t} \sinh \frac{\sqrt{\lambda^2(1+a_1^2)+\mu^2+2\lambda\mu(a_1+1)}}{2}t \quad (3.8)$$

$$p_{i,0,0}(t) = \sum_{z=0}^{\infty} \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \dots \sum_{b_i=0}^{\infty} a_1 \lambda^{i+1+z} \mu^{1+z} \left( \prod_{x=1}^i (\lambda\mu)^{b_x} \right) \left( e^{-(\lambda+i\theta)t} * g_{\xi_{1,0,i}, \xi_{2,0,i}, \dots, \xi_{2i+2,0,i}}(t) \right) \quad i > 0 \quad (3.9)$$

$$p_{i,0,1}(t) = \sum_{z=0}^{\infty} \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \dots \sum_{b_i=0}^{\infty} a_1 \lambda^{i+1+z} \mu \left( \prod_{x=1}^i (\lambda\mu)^{b_x} \right) g_{\xi_{1,0,i}, \xi_{2,0,i}, \dots, \xi_{2i+2,0,i}}(t) \quad i > 0 \quad (3.10)$$

$$P_{i,j,0}(t) = \mu \left( e^{-(\lambda+(i-j)\theta)t} * P_{i,j,1}(t) \right), \quad 1 \leq j \leq i \quad (3.11)$$

$$P_{i,j,1}(t) = \sum_{z=0}^{\infty} \sum_{\alpha_j=1}^{i-j+1} \sum_{\alpha_{j-1}=1}^{\alpha_j+1} \dots \sum_{\alpha_1=1}^{\alpha_2+1} \sum_{x_1=0}^1 \sum_{z_1=0}^{\infty} \dots \sum_{x_j=0}^{\infty} \sum_{z_j=0}^{\infty} \sum_{w_{\alpha_1}}^{w_{\alpha_2}-w_{\alpha_1}+1} \sum_{w_{\alpha_2}}^{w_{\alpha_3}-w_{\alpha_2}+1} \dots \dots \sum_{w_{\alpha_i}}^{w_{\alpha_{j+1}}-w_{\alpha_j}+1} \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \dots \sum_{b_{\alpha_1}=0}^{\infty} (\alpha_1 \alpha_2 \dots \alpha_j) \theta^j \alpha_1 \lambda^{1+z+\alpha_1} \mu^z$$

$$\left( \prod_{e=1}^j (\lambda \mu)^{z^e} \lambda^{\beta_{ae}} \mu^{y_{ae} + \delta_{1,xe}} (1 - a_2)^{\delta_{0,xe}} (\lambda \mu)^{b_1 + b_2 + \dots + b_{a_1}} \right) f_{u_{1,1}, u_{2,1}, \dots, u_{2\alpha_2 - 4\alpha_1 + 4, 1}, u_{1,2}, u_{2,2}, \dots, u_{2\alpha_3 - 4\alpha_2}, \dots, u_{2\alpha_{j+1} - 4\alpha_j + 4, j}, \xi_1, j, b_{a_1}, \xi_2, j, b_{a_1}, \dots, \xi_{2j+2}, j, b_{a_1}^{(i)}} \quad 1 \leq j \leq i \quad (3.12)$$

#### 4. STEADY STATE SOLUTION OF THE NUMBER OF BUSY SERVERS

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Define  $\bar{P}_{n,k}(s) = \sum_{j=0}^{\infty} \bar{P}_{j+n,j,k}(s), k = 0, 1; s\bar{P}_{n,0}(s) \rightarrow P_{n,0}$  as  $s \rightarrow 0$  and substituting in equations (2.4) to (2.6), we get

$$(\lambda + n\theta)P_{n,0} = \mu P_{n,1} \quad (4.1)$$

$$\begin{aligned} & (\lambda a_1 + \mu + n\theta(1 - a_2))P_{n,1} \\ & = \lambda P_{n,0} + \lambda a_1 P_{n-1,1} + (n+1)\theta P_{n+1,0} + (n+1)\theta(1 - a_2)P_{n+1,1} \end{aligned} \quad (4.2)$$

Define  $P_m(z) = \sum_{n=0}^{\infty} z^n P_{n,m}, m = 0, 1$

$$\lambda P_0(z) + \theta P'(z)_0 = \mu P_1(z) \quad (4.3)$$

$$\begin{aligned} & (\lambda a_1 + \mu)P_1(z) + z\theta(1 - a_2)P'(z)_1 \\ & = \lambda P_0(z) + \lambda a_1 z P_1(z) + \theta P'_0(z) + \theta(1 - a_2)P'_1(z) \end{aligned} \quad (4.4)$$

To get the steady state distribution of the number of busy servers

$P_m = P$  (Number of busy servers =  $m$ ),  $m = 0, 1$  put  $z=1$  in above equations, we have

$$\lambda P_0 + \theta N_0 = \mu P_1 \quad (4.5)$$

Where  $N_0$  is the number of repeated calls when no server is busy. Denote the ratio  $\frac{\theta N_0}{P_0} = r_0$  which equals the rate of flow of repeated calls given that the number of busy servers is zero. Equation (4.5) can be written as

$$P_1 = \frac{\lambda + r_0}{\mu} P_0 \quad (4.6)$$

And from the normalizing condition  $P_0 + P_1 = 1$ , we have

$$P_0 = \frac{\mu}{\mu + \lambda + r_0} \quad (4.7)$$

Thus the steady state distribution of the number of busy servers in this model is identical to the steady distribution of the number of busy servers in the Erlang loss Model.

### 5. NUMERICAL EXAMPLE

The numerical results are generated by using BASIC program following the work of Bunday (1986). Assume that the mean service time is 1. In table 1(a) for retrial rate  $\theta = .09, a_1 = .03$  &  $a_2 = .04$ , values of probabilities of no arrival and no departures are found for two different cases i.e. when the server is free & busy for two different values of  $\rho \left( = \frac{\lambda}{\mu} \right) = 0.1, 0.2$ .

Table 1(a)

t	$P_{0,0,0}(t)$		$P_{0,0,1}(t)$	
	$\rho = .1$	$\rho = .2$	$\rho = .1$	$\rho = .2$
0	1	1	0	0
1	0.939322	0.883419	6.06E-02	0.116164
2	0.919014	0.847899	0.080661	0.150862
3	0.912107	0.836674	0.087314	0.161145
4	0.90965	0.83273	0.089506	0.164112
5	0.908668	0.830968	0.090218	0.164885
6	0.908178	0.829863	0.090438	0.165004
7	0.907849	0.828953	9.05E-02	0.164924
8	0.90757	0.828113	9.05E-02	0.164775
9	0.907357	0.827178	9.05E-02	0.164627

Table 1(b)

t	$P_{3,0,0}(t)$	$P_{3,0,1}(t)$	$P_{2,1,0}(t)$	$P_{2,1,1}(t)$	$P_{3,1,0}(t)$	$P_{3,1,1}(t)$	$P_{3,2,0}(t)$	$P_{3,2,1}(t)$
0	0	0	0	0	0	0	0	0
1	4.95E-11	1.11E-12	2.14E-10	1.17E-09	1.96E-13	1.11E-12	1.13E-15	5.07E-15
2	3.70E-10	3.15E-11	6.84E-09	1.75E-08	1.06E-11	3.15E-11	1.85E-13	5.97E-13
3	9.35E-10	1.77E-10	4.28E-08	6.89E-08	8.79E-11	1.77E-10	2.72E-12	8.07E-12
4	1.57E-09	5.08E-10	1.39E-07	1.61E-07	3.54E-10	5.08E-10	2.03E-11	3.87E-11
5	2.17E-09	1.04E-09	3.18E-07	2.88E-07	9.48E-10	1.04E-09	8.72E-11	1.15E-10
6	2.74E-09	1.78E-09	5.95E-07	4.37E-07	1.92E-09	1.78E-09	2.39E-10	2.90E-10
7	3.20E-09	2.60E-09	9.67E-07	6.06E-07	3.43E-09	2.60E-09	5.96E-10	5.24E-10
8	3.68E-09	3.69E-09	1.44E-06	7.82E-07	5.23E-09	3.69E-09	1.11E-09	9.92E-10
9	3.99E-09	4.55E-09	1.98E-06	9.72E-07	7.81E-09	4.55E-09	2.13E-09	1.35E-09

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In table 1(b) for retrial rate  $\theta = .09, \rho = 0.1, a_1 = .03$  &  $a_2 = .04$ , various probabilities of number of arrivals and departures are calculated when the server is free and busy.

Table 1(c)

t	$P_{3,2,0}(t)$			
	$\theta = .1$	$\theta = .2$	$\theta = .3$	$\theta = .4$
0	0	0	0	0
1	2.35E-09	8.82E-09	1.86E-08	5.57E-08
2	1.84E-07	6.46E-07	1.28E-06	3.65E-06
3	1.95E-06	6.35E-06	1.17E-05	3.23E-05
4	9.24E-06	2.79E-05	4.79E-05	1.28E-04
5	2.85E-05	7.97E-05	1.28E-04	3.34E-04
6	6.76E-05	1.75E-04	2.62E-04	6.76E-04
7	1.34E-04	3.23E-04	4.55E-04	1.16E-03
8	2.37E-04	5.30E-04	7.12E-04	1.77E-03
9	3.80E-04	8.01E-04	9.38E-04	2.65E-03

Table 1(d)

t	$P_{3,2,1}(t)$			
	$\theta = .1$	$\theta = .2$	$\theta = .3$	$\theta = .4$
0	0	0	0	0
1	1.63E-08	6.14E-08	1.31E-07	3.94E-07
2	6.29E-07	2.22E-06	4.43E-06	1.29E-05
3	4.42E-06	1.45E-05	2.71E-05	7.69E-05
4	1.57E-05	4.81E-05	8.41E-05	2.35E-04
5	3.92E-05	1.11E-04	1.83E-04	5.06E-04
6	7.87E-05	2.08E-04	3.22E-04	8.86E-04
7	1.37E-04	3.38E-04	4.93E-04	1.36E-03
8	2.17E-04	5.01E-04	7.27E-04	1.95E-03
9	3.18E-04	7.13E-04	6.66E-04	2.20E-03



Table 1(e)

t	$P_{3,2,0}(t)$			
	$\rho = .4$	$\rho = .5$	$\rho = .6$	$\rho = .7$
0	0	0	0	0
1	2.64E-08	1.85E-07	6.11E-08	4.30E-07
2	1.84E-06	6.45E-06	4.07E-06	1.45E-05
3	1.75E-05	4.12E-05	3.72E-05	9.01E-05
4	7.45E-05	1.34E-04	1.54E-04	2.88E-04
5	2.08E-04	3.08E-04	4.19E-04	6.52E-04
6	4.49E-04	5.71E-04	8.85E-04	1.20E-03
7	8.16E-04	9.24E-04	1.58E-03	1.93E-03
8	1.33E-03	1.39E-03	2.57E-03	2.95E-03
9	1.85E-03	1.71E-03	3.17E-03	3.08E-03

In tables 1(c) & 1(d) for  $\rho = 0.3, a_1 = .4$  &  $a_2 = .5$ , probabilities of 3 arrivals and 2 departures is calculated for two different cases when the server is free & busy at different retrial rates  $\theta = 0.1, 0.2, 0.3, 0.4$ . In tables 1(e) & 1(f) for  $\theta = 0.2, a_1 = .4$  &  $a_2 = .5$ , probabilities of 3 arrivals and 2 departures is calculated for two different cases when the server is free & busy at different values of  $\rho = 0.4, 0.5, 0.6, 0.7$ .

Table 1(f)

t	$P_{3,2,1}(t)$			
	$\rho = .4$	$\rho = .5$	$\rho = .6$	$\rho = .7$
0	0	0	0	0
1	1.20E-07	8.49E-07	2.10E-07	1.50E-06
2	7.63E-06	2.76E-05	1.28E-05	4.70E-05
3	6.71E-05	1.67E-04	1.08E-04	2.77E-04
4	2.69E-04	5.24E-04	4.21E-04	8.49E-04
5	7.15E-04	1.17E-03	1.09E-03	1.86E-03
6	1.48E-03	2.11E-03	2.19E-03	3.30E-03
7	2.57E-03	3.31E-03	3.75E-03	5.17E-03
8	4.11E-03	5.54E-03	5.96E-03	8.23E-03
9	6.74E-03	1.20E-02	8.11E-03	2.77E-02

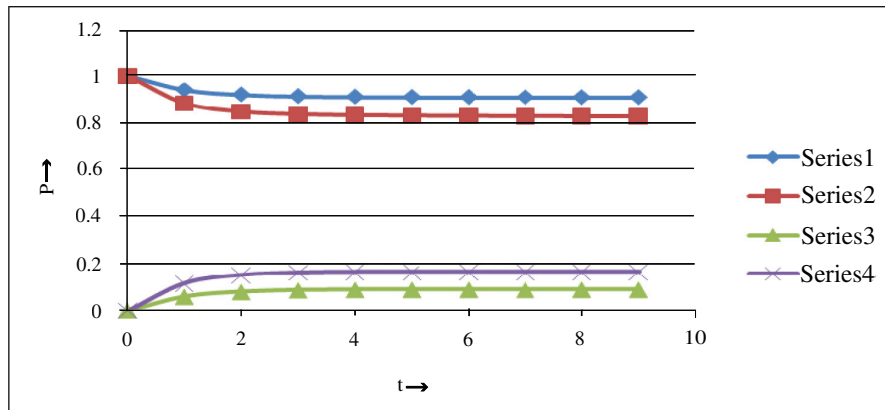


Figure 2(a)

In figure 2(a) for retrial rate  $\theta = .09, a_1 = .03$  &  $a_2 = .04$ , values of probabilities of no arrival and no departures are plotted versus time for two different cases i.e. when the server is free & busy for two different values of  $\rho \left( = \frac{\lambda}{\mu} \right) = 0.1, 0.2$ . When there is no arrival, no departure & server is free, probabilities decrease with time. Also when there is no arrival, no departure & server is busy, probabilities increase with time in starting moments and then decrease with time.

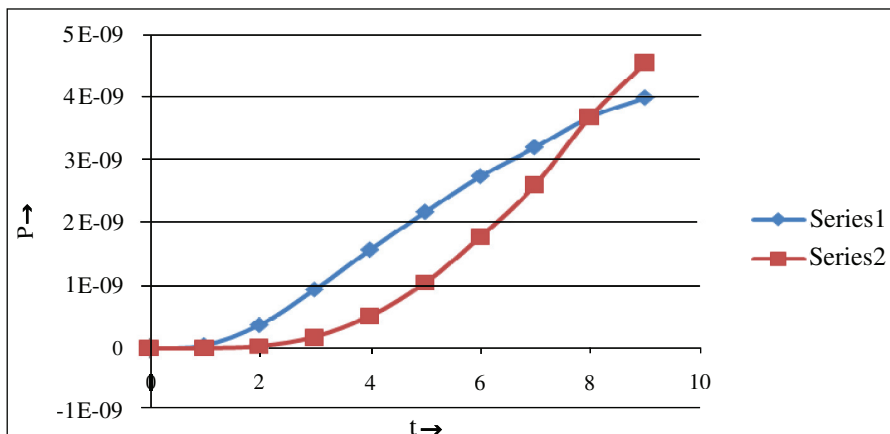


Figure 2.1(b)

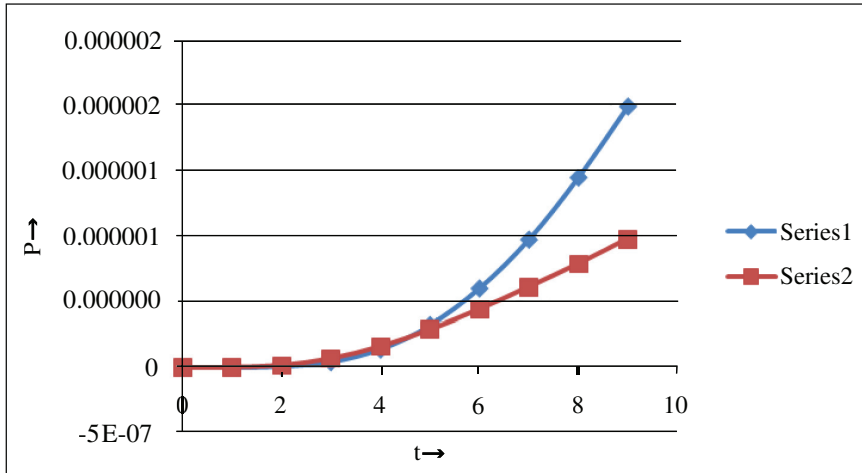


Figure 2.2(b)

In figure 2.1(b)-2.4(b) for retrial rate  $\theta = .09$ ,  $\rho = 0.1$ ,  $a_1 = .03$  &  $a_2 = .04$ , various probabilities of number of arrivals and departures are plotted versus time when the server is free and busy. In the initial moments, probabilities of exact number of arrivals and departures, when the server is busy, are greater than those probabilities of exact number of arrivals and departures, when the server is free. With the passage of time, probabilities of exact number of arrivals and departures, when the server is free, become greater than those probabilities of exact number of arrivals and departures, when the server is busy.

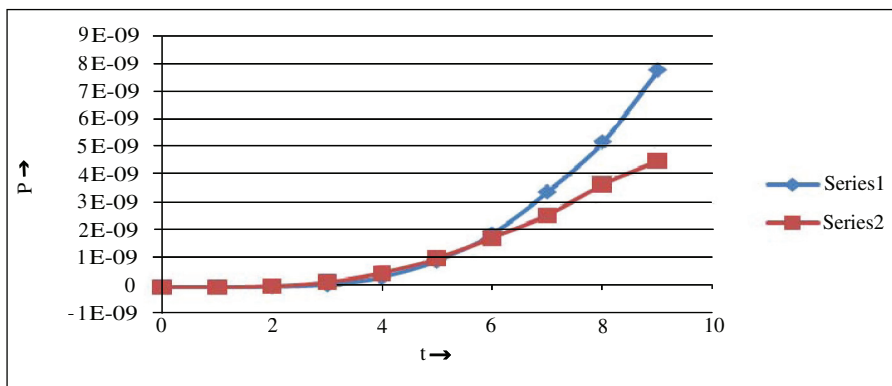


Figure 2.3(b)

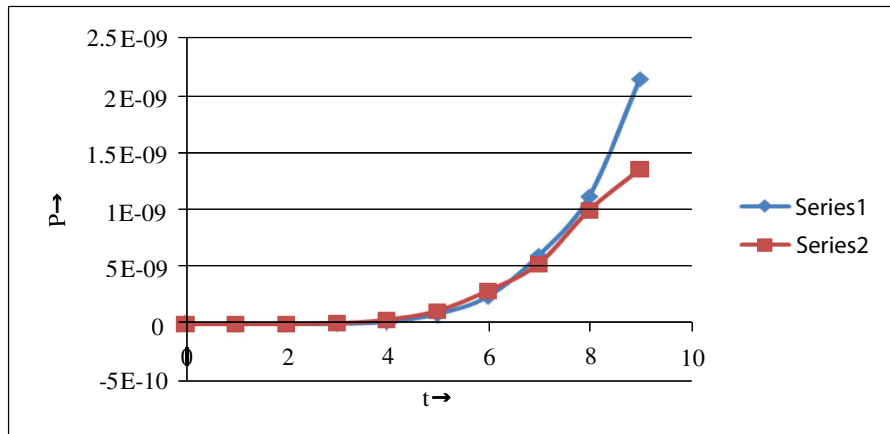


Figure 2.4(b)

In figures 2(c) & 2(d) for  $\rho = 0.3, a_1 = .4$  &  $a_2 = .5$ , probabilities of 3 arrivals and 2 departures versus time for two different cases when the server is free & busy at different retrial rates  $\theta = 0.1, 0.2, 0.3, 0.4$  are shown and these probabilities increase with the increasing value of  $\theta$ .

In figures 2(e) & 2(f) for  $\theta = 0.2, a_1 = .4$  &  $a_2 = .5$ , probabilities of 3 arrivals and 2 departures versus time for two different cases when the server is free & busy at different retrial rates  $\rho = 0.4, 0.5, 0.6, 0.7$  are shown and these probabilities increase with increase in  $\rho$

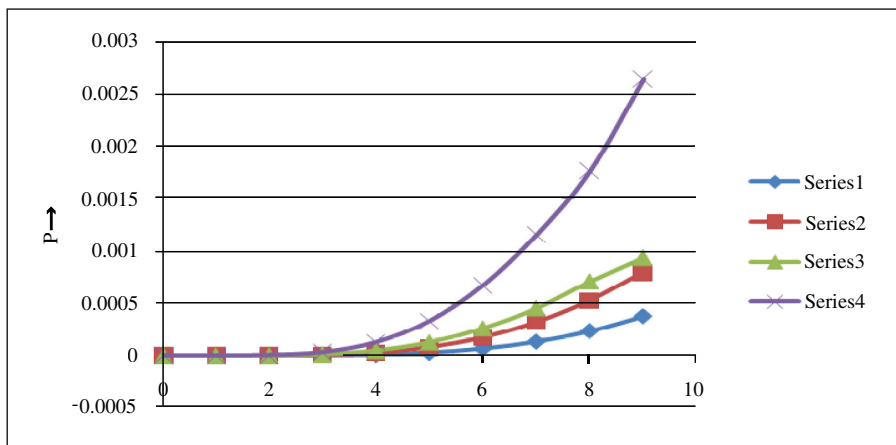


Figure 2(c)

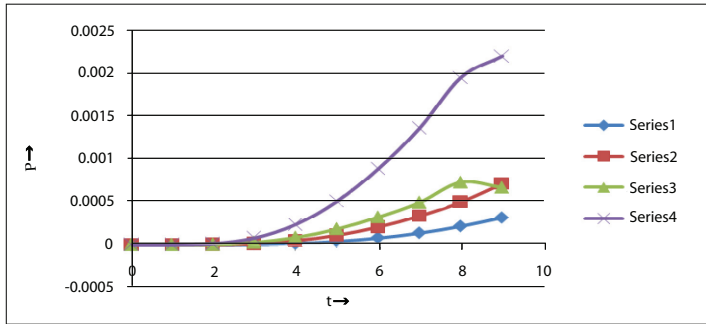


Figure 2(d)

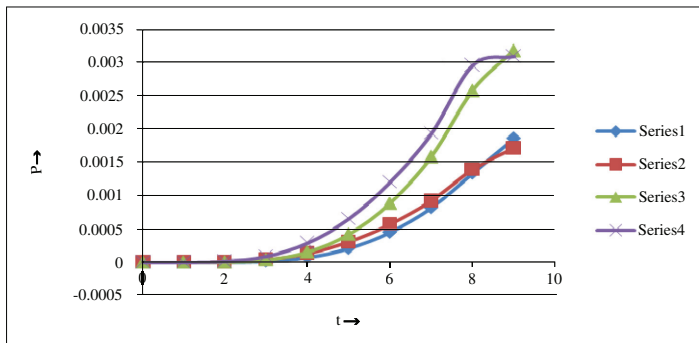


Figure 2(e)

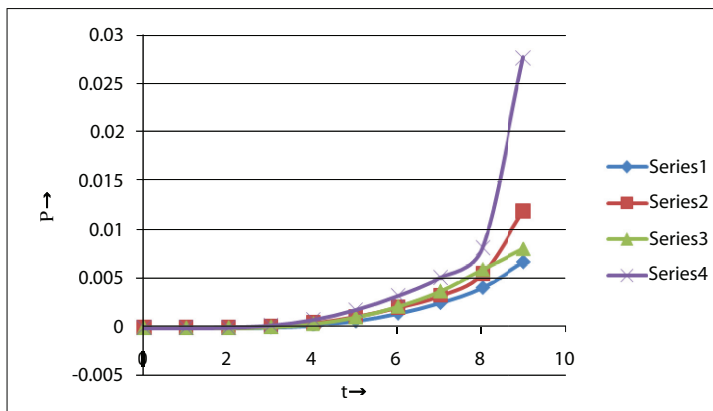


Figure 2(f)

## 6. SPECIAL CASES

1. When the customers are served singly according to a Poisson distribution with parameter  $\mu$ , the Laplace transform of the  $P_{i,j,0}(t)$  and  $P_{i,j,1}(t)$  probabilities and can be obtained by putting  $a_1 = a_2 = 1$  in equations (3.1)-(3.6), we get

$$\bar{P}_{0,0,0}(s) = \frac{s + \lambda + \mu}{(s + \lambda + \mu)(s + \lambda) - \lambda\mu} \quad (6.1)$$

$$\bar{P}_{0,0,1}(s) = \frac{\lambda}{(s + \lambda + \mu)(s + \lambda) - \lambda\mu} \quad (6.2)$$

$$\bar{P}_{i,j,0}(s) = \frac{\mu}{s + \lambda + (i - j)\theta} \bar{P}_{i,j,1}(s), \quad j \leq i, i > 0, j \geq 0 \quad (6.3)$$

$$\bar{P}_{i,0,1}(s) = \frac{\lambda^{i+1+z} \mu^z \prod_{x=1}^i (\lambda\mu)^{b_x}}{\sum_{z=0}^{\infty} \sum_{b_1=0}^{\infty} \sum_{b_2=0}^{\infty} \cdots \sum_{b_i=0}^{\infty} (s + \lambda)^{z+1} (s + \lambda + \mu)^{z+1+i+b_1+b_2+\cdots+b_i} (s + \lambda + \theta)^{b_1} (s + \lambda + 2\theta)^{b_2} \cdots (s + \lambda + i\theta)^{b_i}}, \quad i \geq 1 \quad (6.4)$$

$$\begin{aligned} \bar{P}_{i,i,1}(s) &= (\mu\lambda\theta)^i \sum_{\alpha_1=1}^1 \sum_{\alpha_{r-1}=1}^{\alpha_r+1} \cdots \sum_{\alpha_1=1}^{\alpha_2+1} \sum_{z=0}^{\infty} \sum_{y=0}^{\infty} \sum_{z_1=0}^{\infty} \sum_{z_2=0}^{\infty} \cdots \sum_{z_i=0}^{\infty} \sum_{y_1=0}^{\infty} \sum_{y_2=0}^{\infty} \cdots \sum_{y_i=0}^{\infty} (\alpha_1 \alpha_2 \cdots \alpha_i) \\ &\quad \lambda^{y+1+z} \mu^{z+y} \prod_{x=1}^i (\lambda\mu)^{z_x+y_x} (s + \lambda)^{-(z+1+y)} (s + \lambda + \mu)^{-(z+1+2i+y+\sum_{e=1}^i (z_e+y_e))} \\ &\quad \prod_{r=1}^i (s + \lambda + (\alpha_r - 1)\theta)^{-z_r} (s + \lambda + \alpha_r\theta)^{-(y_r+1)} \end{aligned} \quad (6.5)$$

$$\begin{aligned} \bar{P}_{i,j,1}(s) &= (\lambda)^i (\mu\theta)^j \\ &\quad \sum_{\alpha_j=1}^{i-j+1} \sum_{\alpha_{j-1}=1}^{\alpha_j+1} \cdots \sum_{\alpha_1=1}^{\alpha_2+1} \sum_{z=0}^{\infty} \sum_{r_1=0}^{\infty} \cdots \sum_{r_{i-j}=0}^{\infty} \sum_{z_1=0}^{\infty} \sum_{z_2=0}^{\infty} \cdots \sum_{z_j=0}^{\infty} \sum_{y_1=0}^{\infty} \sum_{y_2=0}^{\infty} \cdots \sum_{y_j=0}^{\infty} (\alpha_1 \alpha_2 \cdots \alpha_j) \\ &\quad \lambda^{1+z} \mu^z \prod_{x=1}^j (\lambda\mu)^{z_x+y_x} \prod_{c=1}^{i-j} (\lambda\mu)^{r_c} (s + \lambda)^{-(z+1+\sum_{u=1}^{i-j} r_u)} \\ &\quad (s + \lambda + \mu)^{-(z+1+i+j+\sum_{u=1}^{i-j} r_u + \sum_{e=1}^j (z_e+y_e))} \\ &\quad \prod_{r=1}^j (s + \lambda + (\alpha_r - 1)\theta)^{-z_r} (s + \lambda + \alpha_r\theta)^{-(y_r+1)}, \quad 1 \leq j < i \end{aligned} \quad (6.6)$$

The results (6.1) – (6.6) coincide with the results of Garg, Bansal & Srivastava (2009).

2. After getting the above case (i.e. when the customers are served singly according to a Poisson distribution with parameter  $\mu$  by putting  $a_1 = a_2 = 1$ ), when the concept of retrial is ended further by fixing  $z_0, z_1, z_2, \dots, z_{i-j}, x_1, \dots, x_j, y_1, \dots, y_j$  to zero and letting  $\bar{P}_{i,j,1}(s) = \bar{P}_{i,j}(s), \bar{P}_{0,0,1}(s) = \bar{P}_{0,0}(s) = \bar{P}_{0,0}(s)$

$$\begin{aligned} \frac{1}{s + \lambda + \mu} &= 1, \frac{(\alpha_1 \alpha_2 \alpha_3 \dots \alpha_j) \lambda \theta^j}{(s + \lambda + \mu)^{j+1}} = \left( \frac{1}{s + \lambda} \right)^j \frac{1}{s + \lambda + \alpha_2 \theta} = \frac{1}{s + \lambda + \alpha_3 \theta} \\ &= \dots = \frac{s + \lambda}{s + \lambda + \mu} \end{aligned}$$

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in equations (3.1)-(3.4), we obtain

$$\bar{P}_{0,0}(s) = \frac{1}{s + \lambda} \tag{6.7}$$

$$\bar{P}_{0,0}(s) = \frac{\lambda^i}{(s + \lambda)(s + \lambda + \mu^i)}, \quad i \geq 1 \tag{6.8}$$

$$\bar{P}_{i,j}(s) = \left( \frac{\lambda}{s + \lambda + \mu} \right)^i \left( \frac{\mu}{s + \lambda} \right)^j \sum_{k=0}^i \frac{(i-k)(i+k-1)!}{k!i!} \frac{(s + \lambda)^{k-1}}{(s + \lambda + \mu)^{kj}} \tag{6.9}$$

$1 \leq j \leq i$

Which coincide with equation (5) of Pegden and Rosenshine (1982).

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