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A Note on Two Diophantine Equations $17^x + 19^y = z^2$ and $71^x + 73^y = z^2$

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Abstract In this short note we study some Diophantine equations of the form $p^x+q^y = z^2$, where x,y and z are non-negative integers and, p and q are both primes, $p < q$, with distance two.

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1 INTRODUCTION

In 2004, Mihailescu (2004) proved the Catalan's conjecture: (3, 2, 2, 3) is a unique solution (a, b, x, y) for the Diophantine equation a^x-$b^y = 1$ where a, b, x and y are integers with min $\{a, b, x, y\} > 1$. This result plays an important role in the study of exponential Diophantine equations. In 2007, Acu (2007) proved that the Diophantine equation $2^x + 5^y = z^2$ has exactly two solutions (x, y, z) in non-negative integers. The solutions are (3, 0, 3) and (2, 1, 3).

In 2011, Suvarnamani, Singta, and Chotchaisthit (2011) showed that the two Diophantine equations $4^x + 7^y = z^2$ and $4^x + 11^y = z^2$ have no solutions in non-negative integers. On the otherhand, in 2012, Sroysang used Catalan's conjecture to study Diophantine equations of the form $a^x + b^y = c^z$. In particular, Sroysang (2012) showed that the Diophantine equation $8^x + 19^y = z^2$ has a unique non-negative integer solution. The solution (x, y, z) is (1, 0, 3). In Sroysang (2012a), Sroysang showed that $(x, y, z) = (1, 0, 2)$ is the only solution to the Diophantine equation $3^x + 5^y = z^2$ in non-negative integers. Contrariwise, Sroysang (2012b) proved the impossibility of the Diophantine equation $31^x + 32^y = z^2$ in non-negative integers. Furthermore, in Sroysang (2012, 2012a) Sroysang posed some open problems in relation to the Diophantine equation $a^x + b^y = c^z$. Rabago (2013, 2013a) gave all solutions to these open problems. Particularly, in Rabago (2013), Rabago found all solutions to the Diophantine equation $8^x + 17^y = z^2$ in non-negative integers. The solutions (x, y, z) are (1, 0, 3),(1, 1, 5), (2, 1, 9), and (3, 1, 23). On the other hand, in

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Rabago (2013a), Rabago showed that the Diophantine equation $2^x + 31^y = z^2$ has exactly two solutions in non-negative integers. The solutions (x, y, z) are (3, 0, 3) and (7, 2, 33). Moreover, in Sroysang (2012b), Sroysang asked for the set of all solutions (x, y, z) for the Diophantine equation $p^x + q^y = z^2$ where x, y and z are non-negative integers for positive odd primes p and q such that $q-p = 2$.

In this note, we consider two particular cases of the Diophantine equation $p^x + q^y = z^2$. To be exact, we show that the two Diophantine equations $17^x + 19^y = z^2$ and $71^x + 73^y = z^2$ has unique solutions in non-negative integers.

2 MAIN RESULTS

We begin this section by proving an important theorem.

Theorem 2.1 *Let p be a prime. Then, the Diophantine equation* $p^y + 1 = z^2$ *has exactly two solutions* (p, y, z) *in non-negative integers. The solutions are* (2, 3, 3) *and* (3, 1, 2).

Proof. For cases $y = 0$ and $z = 0$, the result is obvious since $2 \neq z^2$ and $p^y \neq -1$ for all natural numbers p, y, z. We let $y, z > 0$. So, $z^2 - 1 = (z + 1)(z - 1) = p^y$. Then $2 = (z + 1) - (z - 1) = p^\beta - p^\alpha$, where $\alpha + \beta = y$ and $\alpha < \beta$. We have two possibilities. If $p^\alpha = 1$ and $p^{\beta - \alpha} - 1 = 2$ then this implies that $\alpha = 0$ and $p^y - 1 = p^\beta - 1 = 2$. Thus, $p^y = 3^1$, giving us the solution $(p, y, z) = (3, 1, 2)$. On the other hand, if $p^\alpha = 2$ and $p^{\beta - \alpha} - 1 = 1$, it follows that $p = 2$ and $\alpha = 1$. Hence, $p^{\beta - 1} = 2$ or $\beta = 2$. Here we obtain the solution $(p, y, z) = (2, 3, 3)$. This proves the theorem.

Theorem 2.2 *The Diophantine equation* $p^y + 1 = z^2$ *has no positive integer solution for prime* $p > 3$.

The above theorem follows directly from Theorem 2.1.

Corollary 2.3 *The Diophantine equation* $1 + 19^y = z^2$ *has no solution in non-negative integers.*

Corollary 2.4 *The Diophantine equation* $17^x + 1 = z^2$ *has no solution in non-negative integers.*

Theorem 2.5 *The only solution to the Diophantine equation* $17^x + 19^y = z^2$ *in non-negative integers is* $(x, y, z) = (1, 1, 6)$.

Proof. For the case $x = 0$, we use Corollary 2.3 and for the case $y = 0$ we use Corollary 2.4. The case $z = 0$ is obvious so we only consider the following remaining cases.

Case 1. $x = 1$. If $x = 1$ then we have $17 + 19^y = z^2$. Note that $17 \equiv 1 \pmod 4$ and $19^y \equiv 1 \pmod 4$ for even integer y and $19^y \equiv 3 \pmod 4$ for even ineger y. Since $z^2 \equiv 0, 1 \pmod 4$ then y must be odd and z is even. Let $y = 2k + 1$, $k = 0$ or a natural number and $z = 2m$, m a positive integer. Then $17 + 19^{2k+1} = 4m^2$. So, $1 + 19^{2k+1} = 4m^2 - 16 = 4(m + 2)(m - 2)$. This follows that $(m + 2)(m - 2) = (19^{2k+1} + 1)/4$. Hence, $m + 2 = (19^{2k+1} + 1)/4$ and $m - 2 = 1$, giving us the values $m = 3$ and $19^{2k+1} = 19$ or $k = 0$. Here we obtain the solution $y = 2k + 1 = 1$ and $z = 2m = 6$. That is, we have $(x, y, z) = (1, 1, 6)$, a solution to $17^x + 19^y = z^2$.

Case 2. $y = 1$. This case is equivalent to the first case. That is, for $y = 1$, we'll also have the solution $(x, y, z) = (1, 1, 6)$.

Case 3. $x, y, z > 1$. We note that $17^x + 19^y = z^2$ is possible only when y is odd because $z^2 \equiv 0, 1 \pmod 4$. Then, we have $17^x + 19^{2k+1} = z^2$, k a natural number. We divide x into two cases.

If x is even, say $x = 2n$ for some natural number n, and suppose that $17^x + 19^y = z^2$ is true for positive integers $x, y, z > 1$, then $z^2 - (17^n)^2 = 19^{2k+1}$. Hence, $2.\,17^n = (z + 17^n) - (z - 17^n) = 19^\beta - 19^\alpha$, where $\alpha + \beta = 2k + 1$ and $\alpha < \beta$. This implies that $2.17^n = 19^\alpha(19^{\beta - \alpha} - 1)$. Thus, $\alpha = 0$ and $19^{2k+1} - 1 = 19^\beta - 1 = 2.17^n$. But $19^{2k+1} - 1 \equiv 0 \pmod 3$ and on the other side, $2.17^n \equiv 2 \pmod 3$ for even n and $2.17^n \equiv 1 \pmod 3$ for odd n. This is a contradiction.

Now, suppose that $17^x + 19^y = z^2$ is true for positive integers $x, y, z > 1$ where x is odd, then we have $17^{2n+1} + 19^{2k+1} = z^2$. Take note that $z^2 \equiv 0, 1, 4 \pmod 5$. But, $19^{2k+1} \equiv 4 \pmod 5$ and $17^x \equiv 2 \pmod 5$ for even integer x and $17^x \equiv 3 \pmod 5$ for odd integer x. So, $17^{2n+1} + 19^{2k+1} = z^2$ is possible only when $2n + 1$ is even, this is clearly a contradiction. Thus, for $x, y, z > 1$, the Diophantine equation $17^x + 19^y = z^2$ is not solvable. This completes the proof of the theorem.

Corollary 2.6. *Let $n \geq 2$ be a natural number. Then, the Diophantine equation $17^x + 19^y = w^{2n}$ has no solution in non-negative integers.*

Proof. Let $2 \leq n \in \mathbb{N}$ and $w, x, y \in \mathbb{N}$. Then, $17^x + 19^y = w^{2n} = z^2$ where $z = w^n \in \mathbb{N}$. By Theorem 2.5, we see that $z = 6$. So, $w^n = 6$ or equivalently, $w = 6$ and $n = 1$. This impossible since, by assumption, $n \geq 2$. Thus, the Diophantine equation $17^x + 19^y = w^{2n}$ has no solution in non-negative integers.

Theorem 2.7. *The only solution to the Diophantine equation $71^x + 73^y = z^2$ in non-negative integers is $(x, y, z) = (1, 1, 12)$.*

Proof. The case when $z = 0$ is obviously impossible. Likewise, for $x = 0$ and $y = 0$ we use Theorem 2.2 in which implies that $1 + 73^y = z^2$ and $71^x + 1 = z^2$ has no solutions. We consider the following remaining cases.

Case 1. $x = 1$. If $x = 1$ then we have $71 + 73^y = z^2$. Taking *modulo* 4 both sides, we see that $71 + 73^y \equiv z^2 \equiv 0 \pmod 4$. That is, $z = 4m$ for some natural number m. Then, $71 + 73^y = 16m^2$. It follows that

$$73(1+73^{y-1}) = 16m^2 + 2 = 2(8m^2+1).$$

So, $73^{y-1} + 1 = 2$ and $8m^2 + 1 = 73$. Thus, $y = 1$ and $m = 3$ in which follows that $z = 4m = 12$. Therefore, $(x, y, z) = (1, 1, 12)$ is a solution to $71^x + 73^y = z^2$.

The case when $y = 1$ is equivalent and follows the same argument applied on the case $x = 1$.

Case 2. $x, y > 1$. Suppose $71^x + 73^y = z^2$ is true for positive integers x, y and z different from one. We note that $71^x \equiv 1 \pmod 4$ for even integer x and $71^x \equiv 3 \pmod 4$ for odd integer x. So, $71^x + 73^y = z^2$ is only possible for odd integer x. We let $x = 2k + 1$ for some natural number k. We have two possibilities for y.

If y is even, say $y = 2n$, where n is a natural number. Then, $(z + 73^n)(z - 73^n) = z^2 - 73^{2n} = 71^{2k+1}$. Hence, $71^\beta - 71^\alpha = 2.73^n$ where , $\alpha + \beta = 2k+1 = x$, $\alpha < \beta$. It follows that, $71^\alpha(71^{\beta-\alpha} - 1) = 2.73^n$. So, $71^\alpha = 1$ and $71^{\beta-\alpha} - 1 = 2.73^n$.Thus, $\alpha = 0$ and $71^\beta - 1 = 71^{2k+1} - 1 = 2.73^n$. But, $2.73^n \equiv 1, 2, 3, 4 \pmod 5$ and $71^{2k+1} - 1 \equiv 0 \pmod 5$, a contradiction.

Oppositely, if y is odd then $71^{2k+1} + 73^{2n+1} = z^2$. One can check that $z^2 \equiv 0 \pmod{72}$ for all z a multiple of 12. On the other hand, it can also be verified that $71^{2k+1} \equiv 71 \pmod{72}$ and $73^{2n+1} \equiv 1 \pmod{72}$. So, $z^2 \equiv 71^{2k+1} + 73^{2n+1} \equiv 0 \pmod{72}$ for all z a multiple of 12. Let $z = 12m$. Hence, $71^{2k+1} + 73^{2n+1} = 144m^2$. This implies that,

$$71(71^k + m)(71^k - m) = 73(m + 73^n)(m - 73^n).$$

If $73 = 71^k - m$ then $m = 73^k - 71$. So,

$$71(71^k + 71^k - 73) = 71(2.71^k - 73) = (71^k - 73 + 73^n)(71^k - 73 - 73^n).$$

Hence $71 = 71^k - 73^k - 73$ then $71(71^{k-1} - 1) = 73(73^{n-1} + 1)$. Here, we see that $71^{k-1} = 74$ and $73^{n-1} = 70$, a contradiction.

If $73 = 71^k + m$ then $m = 73 - 71^k$. This implies that

$$71(71^k - 73 + 71^k) = 71(2.71^k - 73) = (73 - 71^k + 73^n)(73 - 71^k - 73^n).$$

Then, $73 - 71^k + 73^n = 71$. It follows that, $73(73^{n-1} + 1) = 71(71^{k-1} + 1)$. So $73^{n-1} = 70$ and $71^{k-1} = 72$ which is also a contradiction. Thus, $71^x + 73^y = z^2$ is impossible in positive integers x, y, z with min $\{x, y, z\} > 1$. This completes the proof of the theorem.

Corollary 2.8. *Let $n \geq 2$ be a natural number. Then, the Diophantine equation $71^x + 73^y = w^{2n}$ has no solution in non-negative integers.*

Proof. Let $2 \leq n \in \mathbb{N}$ and $w, x, y \in \mathbb{N}$. Then, $71^x + 73^y = w^{2n} = z^2$ where $z = w^n \in \mathbb{N}$. By Theorem 2.7, we see that $z = 12$. Hence, $w^n = 12$. This is only possible when $w = 12$ and $n = 1$. A contradiction since, by assumption, $n \geq 2$. Thus, the Diophantine equation $71^x + 73^y = w^{2n}$ has no solution in non-negative integers.

3 AN OPEN PROBLEM

From the above discussion, some may hypothesize immediately that if z is a multiple of 6 then there exist two odd primes p and q, where $q - p = 2$, such that the Diophantine equation $p^x + q^y = z^2$ is true. Unfortunately, this hypothesis is not always true since for $z = 18$, we have $18^2 = 161 + 163 = 324$. The number 163 is a prime but 161 is not because 161 = 7.23. Furthermore, for $z = 24$ we have $24^2 = 287 + 289 = 576$. The numbers 287 and 289 are both composites. But, for $z = 42$, we have $42^2 = 881 + 1764$. The numbers 881 and 883 are both primes. Thus, we may pose the question, "What is the set of all solutions to the Diophantine equation $p^x + q^x = z^2$ in non-negative integers, where p and q are odd primes such that $q - p = 2$ and $z \equiv 0 \pmod 6$?"

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