Common Fixed Point Theorem for Weakly Compatible Maps in Intuitionistic Fuzzy Metric Spaces using Implicit Relation

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Abstract In this paper, we use the notion of property *E.A.* in an intuitionistic fuzzy metric space to prove a common fixed point theorem which generalizes Theorem-2 of Turkoglu *et al.* (2006).

Key words: Intuitionistic fuzzy metric space, property *E.A*, implicit relation. **Subject classification**: 2001 AMS: 47H10, 54H25

1. INTRODUCTION

in 1986, Jungck introduced the notion of compatible maps for a pair of self mappings. Several papers involving compatible maps proved the existence of common fixed points in the classical and fuzzy metric spaces (Grorge and Veeramani, 1994, Kramosil and Michalek, 1975). Aamri and Moutawakil (2002) generalized the concept of non compatibility by defining the notion of property E.A. and proved common fixed point theorems under strict contractive conditions. Atanassove (1986) introduced and studied the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets and later there has been much progress in the study of intuitionistic fuzzy sets by many authors (Alaca, 2006; Atanassov, 1986; Coker, 1997; Manro et al., 2010, 2012; Park, 2004; Park et al. 2005; Saadati and Park, 2006). In 2004, Park defined the notion of intuitionistic fuzzy metric space with the help of continuous *t*-norms and continuous *t*-conorms as a generalization of fuzzy metric space due to George and Veeramani (1994). Fixed point theory has important applications in diverse disciplines of mathematics, statistics, engineering, and economics in dealing with problems arising in: Approximation theory, potential theory, game theory, mathematical economics, etc. Several authors (George and Veeramani 1994; Kramosil and Michalek, 1975) proved some fixed point theorems for various generalizations of contraction mappings in probabilistic and fuzzy metric space. Turkoglu et al. (2006) gave a generalization of Jungck's common fixed point theorem (Jungck, 1976) to intuitionistic fuzzy metric spaces. In this paper, we

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©2014 by Chitkara University. All Rights Reserved. Manro, S. use the notion of property *E.A.* in an intuitionistic fuzzy metric space to prove a common fixed point theorem for a quadruplet of self mappings in intuitionistic fuzzy metric space.

2. PRELIMINARIES.

The concepts of triangular norms (t - norm) and triangular conorms (t - conorm) are were originally introduced by Schweizer and Sklar (1960) in the study of statistical metric spaces.

Definition 2.1. (Schweizer and Sklar, 1960) A binary operation *: $[0,1] \times [0,1] \rightarrow [0,1]$ is continuous *t*-norm if * satisfies the following conditions:

- (i) * is commutative and associative;
- (ii) * is continuous;
- (iii) a * 1 = a for all $a \in [0, 1]$;
- (iv) $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Definition 2.2. (Schweizer and Sklar, 1960) A binary operation \Diamond : $[0,1] \times [0,1] \rightarrow [0,1]$ is continuous *t*-conorm if \Diamond satisfies the following conditions:

- (i) \Diamond is commutative and associative;
- (ii) \Diamond is continuous;
- (iii) $a \diamond 0 = a$ for all $a \in [0, 1]$;
- (iv) $a \diamondsuit b \le c \diamondsuit d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0, 1]$.

Alaca *et al.* (2006) using the idea of Intuitionistic fuzzy sets, defined the notion of intuitionistic fuzzy metric space with the help of continuous *t*-norm and continuous *t*- conorms as a generalization of fuzzy metric space due to Kramosil and Michalek (1975) as :

Definition 2.3. (Alaca *et al.*, 2006) A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, * is a continuous *t*-norm, \Diamond is a continuous *t*-conorm and *M*, *N* are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions:

- (i) $M(x, y, t) + N(x, y, t) \le 1$ for all $x, y \in X$ and t > 0;
- (ii) M(x, y, 0) = 0 for all $x, y \in X$;
- (iii) M(x, y, t) = 1 for all $x, y \in X$ and t > 0 if and only if x = y;
- (iv) M(x, y, t) = M(y, x, t) for all $x, y \in X$ and t > 0;
- (v) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$ for all $x, y, z \in X$ and s, t > 0;

(vi) for all $x, y \in X$, $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous;

- (vii) $\lim_{n\to\infty} M(x, y, t) = 1$ for all $x, y \in X$ and t > 0;
- (viii) N(x, y, 0) = 1 for all $x, y \in X$;
- (ix) N(x, y, t) = 0 for all $x, y \in X$ and t > 0 if and only if x = y;
- (x) N(x, y, t) = N(y, x, t) for all $x, y \in X$ and t > 0;
- (xi) $N(x, y, t) \Diamond N(y, z, s) \ge N(x, z, t + s)$ for all $x, y, z \in X$ and s, t > 0;
- (xii) for all $x, y \in X$, $N(x, y, .) : [0, \infty) \rightarrow [0, 1]$ is right continuous;

(xiii) $\lim_{n\to\infty} N(x, y, t) = 0$ for all $x, y \in X$.

Then (M, N) is called an intuitionistic fuzzy metric space on X. The functions M(x, y, t) and N(x, y, t) denote the degree of nearness and the degree of non-nearness between x and y w.r.t. t respectively.

Remark 2.1.(Alaca *et al.*, **2006**) Every fuzzy metric space (X, M, *) is an intuitionistic fuzzy metric space of the form (X, M, 1-M, *, \Diamond) such that *t*-norm * and *t*-conorm \Diamond are associated as

 $x \diamondsuit y = 1 - ((1 - x)^* (1 - y))$ for all $x, y \in X$

Remark 2.2.(Alaca *et al.*, 2006) In intuitionistic fuzzy metric space (*X*, *M*, *N*, *, \Diamond), *M*(*x*, *y*, .) is non-decreasing and *N*(*x*, *y*, .) is non-increasing for all *x*, *y* \in *X*.

Alaca et al. (2006) introduced the following notions:

Definition 2.4. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to be Cauchy sequence if, for all t > 0 and p > 0,

 $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1 \text{ and } \lim_{n\to\infty} N(x_{n+p}, x_n, t) = 0$

(b) a sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$ if, for all t > 0,

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \text{ and } \lim_{n\to\infty} N(x_n, x, t) = 0$$

Definition 2.5.(Alaca *et al.*, 2006) An intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to be complete if and only if every Cauchy sequence in *X* is convergent.

Example 2.1.(Alaca *et al.*, **2006**) Let $X = \left\{\frac{1}{n} : n = 1, 2, 3, ...\right\} \cup \{0\}$ and let * be the continuous *t*-norm and \Diamond be the continuous *t*-conorm defined by a * b = ab

Manro, S.

and $a \diamond b = \min\{1, a + b\}$ respectively, for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$ and $x, y \in X$, define (M, N) by

$$M(x,y,t) = \begin{cases} \frac{t}{t+|x-y|}, t > 0, \\ 0, t=0 \end{cases} \text{ and } N(x,y,t) = \begin{cases} \frac{|x-y|}{t+|x-y|}, t > 0, \\ 1, t=0. \end{cases}$$

Clearly, $(X, M, N, *, \Diamond)$ is complete intuitionistic fuzzy metric space.

Definition 2.6. A pair of self mappings (f, g) of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to be commuting if

$$M(fgx, gfx, t) = 1$$
 and $N(fgx, gfx, t) = 0$ for all $x \in X$

Definition 2.7. (Aamri and Moutawakil, 2002) A pair of self mappings (f, g) of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to satisfy the property *E*.*A* if there exist a sequence $\{x_n\}$ in *X* such that $\lim_{n\to\infty} M(fx_n, gx_n, t) = 1$ and $\lim_{n\to\infty} N(fx_n, gx_n, t) = 0$.

Example 2.2.(Aamri and Moutawakil, 2002) Let $X = [0,\infty)$. Consider (*X*, *M*, *N*, ^{*}, \Diamond) be an intuitionistic fuzzy metric space as in Example 2.1. Define

 $f, g: X \to X$ by $fx = \frac{x}{5}$ and $gx = \frac{2x}{5}$ for all $x \in X$. Then for sequence $\{x_n\} = \{\frac{1}{n}\}$, $\lim_{n\to\infty} M(fx_n, gx_n, t) = 1$ and $\lim_{n\to\infty} N(fx_n, gx_n, t) = 0$. Then f and g satisfy property E.A.

Definition 2.8.(Alaca et al., 2006) A pair of self mappings (f, g) of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ is said to be weakly compatible if they commute at coincidence points i.e. if fu = gu for some $u \in X$, then fgu = gfu.

It is easy to see that two compatible maps are weakly compatible.

Lemma 2.1.(Alaca *et al.*, 2006) Let $(X, M, N, *, \Diamond)$ be intuitionistic fuzzy metric space and for all $x, y \in X, t > 0$ and if for a number $k \in (0, 1)$,

$$M(x, y, kt) \ge M(x, y, t)$$

and

$$N(x, y, kt) \leq N(x, y, t),$$

then x = y.

3. MAIN RESULTS

Turkoglu et al. (2006) proved the following Theorem:

Theorem 3.1. Let $(X, M, N, *, \Diamond)$ be a complete intuitionistic fuzzy metric space. Let *f* and *g* be self mappings of *X* satisfying the following conditions:

(a)
$$g(X) \subseteq f(X);$$

(b) there exist 0 < k < 1 such that

$$M(gx, gy, kt) \ge M(fx, fy, t)$$

and

$$N(gx, gy, kt) \le N(fx, fy, t);$$

(c) f is continuous.

Then f and g have a unique common fixed point provided f and g commute.

Now, we prove a common fixed point theorem using property *E.A.* in an intuitionistic fuzzy metric space, which is a generalization of Theorem 3.1 in the following way:

(i) to relax the continuity requirement of maps completely,

(ii) property *E*.*A* buys containment of ranges.

Theorem 3.2. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space with continuous *t*-norm and continuous *t*- conorm defined by $a * a \ge a$ and $(1 - a)\Diamond(1 - a) \le (1 - a)$ where $a \in [0, 1]$. Let *f* and *g* be two weakly compatible self mappings of *X* satisfying the following conditions:

(3.1) f and g satisfy the property E.A.;

(3.2) for each $x, y \in X, t > 0$, there exist 0 < k < 1 such that

$$M(gx, gy, kt) \ge M(fx, fy, t)$$

and

 $N(gx, gy, kt) \leq N(fx, fy, t);$

(3.3) f(X) or g(X) is complete subspace of X.

Then f and g have a unique common fixed point.

Proof: In view of (3.1), there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} fx_n = u$ for some $u \in X$. Suppose that f(X) is complete subspace of X,

Manro, S. therefore, every convergent sequence of points of f(X) has a limit point in f(X) implies $\lim_{n\to\infty} fx_n = fa = u = \lim_{n\to\infty} gx_n$, for some $a \in X$, which implies that $u = fa \in f(X)$.

Now, we prove that ga = fa. From (3.2) take $x = x_n$, y = a, we get

 $M(gx_n, ga, kt) \ge M(fx_n, fa, t)$

and

 $N(gx_n, ga, kt) \leq N(fx_n, fa, t).$

Taking limit $n \to \infty$ on both sides, we get

$$M(fa, ga, kt) \ge M(fa, fa, t)$$

and

 $N(fa, ga, kt) \leq N(fa, fa, t).$

This implies by Lemma 2.1, fa = ga.

Therefore, u = fa = ga.

This shows that 'a' is coincident point of g and f. As g and f are weakly compatible, therefore, gfa = fga = ffa = gga. Now, we show that ga is the common fixed point of g and f. From (3.2) take x = a, y = ga,

 $M(ga, gga, kt) \ge M(fa, fga, t)$

 $M(ga, gga, kt) \ge M(ga, gga, t)$

and

 $N(ga, gga, kt) \leq N(fa, fga, t),$

 $N(ga, gga, kt) \leq N(ga, gga, t),$

This implies by Lemma 2.1, gga = ga = fga.

This proves that ga is the common fixed point of g and f.

Now, we prove the uniqueness of common fixed point of g and f. If possible, let x_0 and y_0 be two common fixed points of f and g. Then by condition (3.2),

$$M(x_0, y_0, kt) = M(gx_0, gy_0, kt) \ge M(fx_0, fy_0, t) = M(x_0, y_0, t)$$

and

$$N(x_0, y_0, kt) = N(gx_0, gy_0, kt) \le N(fx_0, fy_0, t) = N(x_0, y_0, t).$$

Then by Lemma 2.1, we have $x_0 = y_0$.

Therefore, the mappings f and g have a unique common fixed point. This completes the proof.

Example 3.1. Let $X = \left\{\frac{1}{n} : n = 1, 2, 3, ...\right\} \cup \{0\}$. Consider $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space as in Example 2.1. Define $fx = \frac{x}{4}$ and $gx = \frac{x}{12}$ for all $x \in X$. Clearly, *f* and *g* are weakly compatible mappings on *X*, Also,

(i) *f* and *g* satisfy the property *E*.*A* for the sequence $\{x_n\} = \left\{\frac{1}{n}\right\}$,

(ii) for $k = \frac{1}{3}$, the condition (3.2) of above theorem is satisfied,

(iii) f(X) is complete subspace of X.

Thus all the conditions of Theorem 3.2 are satisfied and so *f* and *g* have the common fixed point x = 0.

Theorem 3.3. Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space with continuous *t*-norm and continuous *t*- conorm defined by $a * a \ge a$ and (1 - a) $\Diamond (1 - a) \le (1 - a)$, where *a* in [0, 1]. Let *f* and *g* be two weakly compatible self mappings of *X* satisfying (3.1) and the following conditions: (3.4) for each $x, y \in X, t > 0$, there exist 0 < k < 1 such that

$$\begin{split} M(fx, fy, kt) &\geq \varphi(M(gx, gy, t), M(fx, gx, t), M(fy, gy, t), \\ M(fx, gy, t), M(fy, gx, t)) \end{split}$$

and

 $N(fx, fy, kt) \le \psi (N(gx, gy, t), N(fx, gx, t), N(fy, gy, t),$ N(fx, gy, t), N(fy, gx, t))

where φ , ψ is a mapping from [0,1] to [0,1], which is upper semi-continuous, non-decreasing in each coordinate variable and such that

$$\phi(1,1,t,1,t) \ge t$$

$$\phi(t,1,1,t,t) \ge t$$

and

$$\psi(1,1,t,1,t) \le t$$

 $\psi(t, 1, 1, t, t) \le t$ where $t \in [0, 1]$,

(3.5) g(X) is a closed subspace of X.

Then f and g have a unique common fixed point.

Manro, S. **Proof.** In view of (3.1), there exist a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = p$ for some $p \in X$. As g(X) is a closed subspace of X, there is $u \in X$ such that p = gu. Therefore,

$$\lim_{n\to\infty} fx_n = p = gu = \lim_{n\to\infty} gx_n.$$

Now, we prove that fu = gu.

From (3.4) take $x = x_{y}, y = u$,

$$M(fx_n, fu, kt) \ge \varphi \ (M(gx_n, gu, t), M(fx_n, gx_n, t), M(fu, gu, t),$$

 $M(fx_n, gu, t), M(fu, gx_n, t))$

$$N(fx_n, fu, kt) \leq \psi (N(gx_n, gu, t), N(fx_n, gx_n, t), N(fu, gu, t), N(fx_n, gu, t), N(fu, gx_n, t)).$$

Taking the limit as $n \to \infty$, we get

$$M(gu, fu, kt) \ge \varphi(M(gu, gu, t), M(gu, gu, t), M(fu, gu, t), M(gu, gu, t), M(fu, gu, t))$$

 $= \varphi(1, 1, M(gu, fu, t), 1, M(gu, fu, t)) \ge M(gu, fu, t)$

and

and

$$\begin{split} N(gu, fu, kt)) &\leq \psi \left(N(gu, gu, t), N(gu, gu, t), N(fu, gu, t), \right. \\ & N(gu, gu, t), N(fu, gu, t)) \\ &= \psi \left(0, 0, N(gu, fu, t), 0, N(gu, fu, t) \right) \leq N(gu, u, t). \end{split}$$

By using Lemma 2.1, we deduce that fu = gu. Denote fu by z.

Therefore, fu = gu = z.

This shows that 'u' is coincident point of f and g. From weak compatibility of the mappings f and g it follows that fgu = gfuThis implies, fz = gz. Now, we show that z is the common fixed point of f and g. From (3.4) take x = z, y = u,

M(fz, z, t) = M(fz, fu, t)

$$\geq \varphi(M(gz, gu, t), M(fz, gz, t), M(fu, gu, t), M(fz, gu, t), M(fu, gz, t)),$$

= $\varphi(M(fz, z, t), 1, 1, M(fz, z, t), M(z, fz, t)) \geq M(z, fz, t)$

and

$$\begin{split} N(fz, z, t) &= N(fz, fu, t) \\ &\leq \psi(N(gz, gu, t), \, N(fz, gz, t), \, N(fu, gu, t), \, N(fz, gu, t), \, N(fu, gz, t)), \\ &= \psi(N(fz, z, t), 0, 0, N(fz, z, t), N(z, fz, t)) \leq N(fz, z, t). \end{split}$$

By using Lemma 2.1, we deduce that, fz = z = gz and thus we obtain that z is a common fixed point of f and g.

Now, we prove the uniqueness of common fixed point of f and g. If possible, let 'a' and 'b' be two common fixed points of f and g.

Then by condition (3.4) take x = a, y = b we get,

$$\begin{split} M(fa, fb, kt) &\geq \varphi \; (M(ga, gb, t), M(fa, ga, t), M(fb, gb, t), M(fa, gb, t), M(fb, ga, t)) \\ M(a, b, kt) &\geq \varphi \; (M(a, b, t), M(a, a, t), M(b, b, t), M(a, b, t), M(b, a, t)) \\ &\geq \varphi(M(a, b, t), 1, 1, M(a, b, t), M(a, b, t)) \end{split}$$

 $M(a, b, kt) \ge M(a, b, t)$

and

 $N(fa, fb, kt) \le \psi (N(ga, gb, t), N(fa, ga, t), N(fy, gb, t), N(fa, gb, t), N(fb, ga, t))$ $N(a, b, kt) \le \psi (N(a, b, t), N(a, a, t), N(b, b, t), N(a, b, t), N(b, a, t))$ $\le \psi(N(a, b, t), 0, 0, N(a, b, t), N(a, b, t))$

 $N(a, b, kt) \leq \psi N(a, b, t).$

Then by Lemma 2.1, we have a = b.

Therefore, the mappings f and g have a unique common fixed point. This completes the proof.

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