

Graphical Solution and Busy Period Analysis of a Queueing Model with Feedback and Reneging

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Abstract In the present paper, a busy period of a feedback queueing model is studied. The busy period is to begin with the arrival of customer to an idle system and to an end when the system next becomes idle. The customers arrive according to poisson process and are served by a single server according to an exponential distribution. Sometimes the customers get impatience and leave the queue without getting service with fixed probability. The probability generating function of busy period by using Laplace transformation and the graphical solution of the problem are obtained. Few interesting cases are also derived to match our results with earlier published work.

Keywords: Feedback, busy period, reneging, laplace transformation, generating functions, Infinite capacity, graphical results.

2010 Mathematics Subject Classification: 90Bxx, 90B20, 60525

1. INTRODUCTION

The research work in queueing theory has been extensively developed due to its significance in many areas like health sector, telecommunication system, manufacturing and production processes. A number of extensions in basic queueing systems have been made. There is now growing interest in the analysis of queueing system with feedback. In feedback queueing systems, the customers get additional service if they are not satisfied from their previous service. In Such Cases, a customer arrives and waits for his service. After getting his first service, if he is satisfied from the service then he leaves the system and if he is not satisfied from the service then he re-joined the queue again for another service, this is called in literature queues with feedback. Many

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researchers paid their attention toward this direction. Ayyappan & Shyamala [1], Choudhury & Paul [4], Thangaraj & Vanitha [11] and Kumari & Garg [5] studied a number of queueing systems with feedback.

The renegeing queueing systems study the characteristic of impatience of customers. In such cases, the arriving customer joins the waiting line if his waiting time is exceeded from his expected time then he leaves the system without getting service which is known as renegeing. Sharma & Kumar [8] studied an M/M/1/N queueing system with retention of renegeed customers & balking and also obtained the steady-state solution. Singh & Singh [9] studied a feedback queueing system with renegeing and also obtained the steady-state solution of multi-channel queueing system.

Busy period analysis plays a vital role in the study of queueing problems for forecasting the behaviour of the queueing systems. The busy period is the time measured between the instant a customer arrives to an empty system and the instant a last customer departs and leaving the system empty. Avignon & Disney [1], Thangaraj & Vanitha [11], Perry & Asmussen [6], Perry et.al [7], Bertsimas et al. [3] and Kumari and Garg [5] analysed the busy period of the various queueing systems.

In this paper, we have generalised our earlier research paper entitled "A feedback Queueing Model with Impatient Customers" and obtained the probability generating function of the busy period of M/M/1 feedback queueing problem with renegeing by using laplace transformation. The graphical solution of the problem has also been obtained.

The practical situation which corresponds to the above model can be that of a bank, wherein at cash counter, the customers arrived for withdraw their money. After being served once, some customers are not satisfied due to some ill-mannered services (e.g. the date and withdrawing amount is not noted on the passbook) and so they re-joined the counter again. Some customers have not enough time for waiting so they renege the queue. The manager of a bank can know the various probabilities of the number of customers to be serviced by any time.

The queueing system studied in this paper is described by the following assumptions

- (i) Arrivals follow the Poisson distribution with parameter λ and the service time distribution of every unit is exponential with parameter μ .
- (ii) The probability of re-joining the system is 'p' and that of leaving the system is 'q' for the units getting first service, so that $p + q = 1$.

However the units will have to leave the system after getting second service.

- (iii) The unit standing at the head of the line join the service channel for the first time with probability c_1 and for the second time with probability c_2 , so that $c_1 + c_2 = 1$.
- (iv) The probability of a unit reneging during Δt when there are n units in the queue is $r(n) \Delta t$ and also assumed that reneging follow the exponential distribution with density function $d(t) = \alpha e^{-\alpha t}$ where it is assumed that customer can renege at any time independent of no. of units in the queue i.e. $r(n) = \alpha, n > 1$.
- (v) The waiting space is infinite.
- (vi) The stochastic processes involved, viz
 - a) Arrival of units
 - b) Departure of units are statistically independent.

Definitions

$P_n^{(0)}(t)$ = Probability that there are n units in the system at any time t and next unit is to depart for the first time.

$P_n^{(1)}(t)$ = Probability that there are n units in the system at any time t and next unit is to depart for the second time.

$P_n(t)$ = Probability that there are n units in the system at any time t .

$$P_n(t) = P_n^{(0)}(t) + P_n^{(1)}(t) \quad , n \geq 0 \quad (1)$$

Initially

$$P_1^{(0)}(0) = 1, \quad P_0^{(0)}(t) = 0 \text{ and } P_0^{(1)}(t) = 0 \quad t \geq 0 \quad r(1) = 0$$

The difference – differential equations describing the system are

$$\frac{d}{dt} P_0^{(0)}(t) = \mu q P_1^{(0)}(t) + \mu P_1^{(1)}(t) \quad (2)$$

$$\begin{aligned} \frac{d}{dt} P_n^{(0)}(t) = & -\{\lambda + \mu + (1 - \delta_{n,1})\alpha\} P_n^{(0)}(t) + \lambda(1 - \delta_{n,1}) P_{n-1}^{(0)}(t) \\ & + \mu c_1 p (1 - \delta_{n,1}) P_n^{(0)}(t) + (\mu c_1 q + \alpha) P_{n+1}^{(0)}(t) + \mu c_1 P_{n+1}^{(1)}(t) \quad n \geq 1 \end{aligned} \quad (3)$$

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$$\begin{aligned} \frac{d}{dt} P_n^{(1)}(t) &= \{\lambda + \mu + (1 - \delta_{n,1})\alpha\} P_n^{(1)}(t) + \lambda(1 - \delta_{n,1}) P_{n-1}^{(1)}(t) \\ &+ (\mu c_2 + \alpha) P_{n+1}^{(1)}(t) + \mu p(c_1 \delta_{n,1} + c_2) P_n^{(0)}(t) \\ &+ \mu c_2 q P_{n+1}^{(0)}(t) \quad n \geq 1 \end{aligned} \quad (4)$$

Where $\delta_{n,1}$ is Kronecker delta.

Taking the Laplace transformation $\bar{P}_n(s) = \int_0^\infty e^{-st} P_n(t) dt$; **Res > 0 of (3) – (4)**

$$\begin{aligned} \{s + \lambda + \mu + (1 - \delta_{n,1})\alpha\} \bar{P}_n^{(0)}(s) &= 1\delta_{n,1} + \lambda(1 - \delta_{n,1}) \bar{P}_{n-1}^{(0)}(s) \\ &+ \mu c_1 (1 - \delta_{n,1}) \bar{P}_n^{(0)}(s) \\ &+ (\mu c_1 q + \alpha) \bar{P}_{n+1}^{(0)}(s) \\ &+ \mu c_1 \bar{P}_{n+1}^{(1)}(s) \quad n \geq 1 \end{aligned} \quad (5)$$

$$\begin{aligned} \{s + \lambda + \mu + \alpha(1 - \delta_{n,1})\} \bar{P}_n^{(1)}(s) &= \lambda(1 - \delta_{n,1}) \bar{P}_{n-1}^{(0)}(s) \\ &+ (\mu c_2 + \alpha) \bar{P}_{n+1}^{(1)}(s) + \mu c_2 q \bar{P}_{n+1}^{(0)}(s) \\ &+ \mu p(c_1 \delta_{n,1} + c_2) \bar{P}_n^{(0)}(s) \quad n \geq 1 \end{aligned} \quad (6)$$

Define

$$\begin{aligned} P^{(0)}(z, t) &= \sum_{n=1}^{\infty} P_n^{(0)}(t) z^n & \bar{P}^{(0)}(z, s) &= \int_0^\infty e^{-st} P^{(0)}(z, t) dt \\ P^{(1)}(z, t) &= \sum_{n=1}^{\infty} P_n^{(1)}(t) z^n & \bar{P}^{(1)}(z, s) &= \int_0^\infty e^{-st} P^{(1)}(z, t) dt \\ P(z, t) &= P^{(0)}(z, t) + P^{(1)}(z, t) & \bar{P}(z, s) &= \int_0^\infty e^{-st} P(z, t) dt \end{aligned}$$

With $|z| \leq 1$

Laplace transformation of probability generating function of transient – state queue length probabilities

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$$\bar{P}^{(0)}(z,s) = \frac{z[-A\{\mu c_1(q+pz) + \alpha(1-z)\} + \mu c_2\alpha(1-z) + \mu^2 c_1 p z \bar{P}_1^{(0)}(s) - [A\mu c_1 + \mu c_1\alpha(1-z)]\bar{P}_1^{(1)}(s) + z(A - \mu c_2)]}{\{A - \mu c_1(q+pz)\}\{A - \mu c_2\} - \mu^2 c_1 c_2(q+pz)}, \quad (7)$$

$|z| \leq 1$

$$\bar{P}^{(1)}(z,s) = \frac{z[-A\{\mu c_2(q+pz) - \mu p z\} + \mu c_2(q+pz)\alpha(1-z) - \mu^2 c_1 p z(q+pz)]\bar{P}_1^{(0)}(s) + [-A\{\mu c_2 + \alpha(1-z)\} + \mu c_1(q+pz)\alpha(1-z)]\bar{P}_1^{(1)}(s) + z\mu c_2(q+pz)}{\{A - \mu c_1(q+pz)\}\{A - \mu c_2\} - \mu^2 c_1 c_2(q+pz)}, \quad (8)$$

$|z| \leq 1$

$$z(1-z)\left\{-A\alpha + \mu^2 c_1 p^2 z + \mu c_2 p \alpha(1-z)\right\}\bar{P}_1^{(0)}(s) + \left\{-A\alpha + \mu c_1 p \alpha(1-z)\right\}\bar{P}_1^{(1)}(s) - z\mu c_2 p - z[A\mu q \bar{P}_1^{(0)}(s) + A\mu \bar{P}_1^{(1)}(s) - zA]$$

$$\bar{P}(z,s) = \frac{z(1-z)\left\{-A\alpha + \mu^2 c_1 p^2 z + \mu c_2 p \alpha(1-z)\right\}\bar{P}_1^{(0)}(s) + \left\{-A\alpha + \mu c_1 p \alpha(1-z)\right\}\bar{P}_1^{(1)}(s) - z\mu c_2 p - z[A\mu q \bar{P}_1^{(0)}(s) + A\mu \bar{P}_1^{(1)}(s) - zA]}{\{A - \mu c_1(q+pz)\}\{A - \mu c_2\} - \mu^2 c_1 c_2(q+pz)}, \quad (9)$$

$|z| \leq 1$

Where $A = -\lambda z^2 + (s + \lambda + \mu + \alpha)z - \alpha$

Let $D = K_1(z)K_2(z) - \mu^2 c_1 c_2(q+pz)$

Where $K_1(z) = (-\lambda z^2 + (s + \lambda + \mu + \alpha - \mu c_1 p)z - (\mu c_1 q + \alpha))$

$K_2(z) = (-\lambda z^2 + (s + \lambda + \mu + \alpha)z - (\mu c_2 + \alpha))$

Obviously $K_1(z)$ and $K_2(z)$ have two zeros inside the unit circle.

Let $f(z) = K_1(z)K_2(z)$ and $g(z) = \mu^2 c_1 c_2(q+pz)$

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$$\begin{aligned}
|f(z)| &= |K_1(z)K_2(z)| \\
&= |(-\lambda z^2 + (s + \lambda + \mu + \alpha - \mu c_1 p)z - (\mu c_1 q + \alpha)) \\
&\quad |(-\lambda z^2 + (s + \lambda + \mu + \alpha)z - (\mu c_2 + \alpha))| \\
&\geq (\xi + \mu c_2)(\xi + \mu c_2) \text{ For } s = \xi + i\eta, |z| = 1 \\
&\geq \mu^2 c_1 c_2 \geq |g(z)|
\end{aligned}$$

Hence $|f(z)| \geq |g(z)|$ on $|z| = 1$

Since all the condition of Rouché's theorem are satisfied, so D has two zeroes inside the unit circle. Let these zeroes be z_m ($m = 0, 1$). Numerator must also vanish for these two zeroes since $\bar{P}(z, s)$ is analytical function of z . These two equations will determine the two unknown's $P_1^{(0)}(s)$ and $P_1^{(1)}(s)$. By taking the inverse laplace of $P_1^{(0)}(s)$ and $P_1^{(1)}(s)$, the probabilities $P_1^{(0)}(t)$ and $P_1^{(1)}(t)$ can be determined. Substituting the values of $P_1^{(0)}(t)$ and $P_1^{(1)}(t)$ in equation (2), $\frac{d}{dt} P_0^{(0)}(t)$ can be obtained.

Special case

When there is no feedback and no reneging

putting $q = 1, p = 0, \alpha = 0, \bar{P}^{(1)}(z, s) = 0, \bar{P}^{(0)}(z, s) = \bar{P}(z, s), \bar{P}_0^{(0)}(s) = \bar{P}_0(s),$

$\bar{P}_n^{(0)}(s) = \bar{P}_n(s), \quad \bar{P}_n^{(1)}(s) = 0$ in equation (9) we get

$$\bar{P}(z, s) = \frac{z - \mu \bar{P}_1(s)}{-\lambda z + (s + \lambda + \mu) - \frac{\mu}{z}} \quad (10)$$

This coincides with generating function of busy period of M/M/1 model.

2. GRAPHICAL SOLUTION OF THE PROBLEM

The difference – differential equations describing the problem are

$$\frac{d}{dt} P_0^{(0)}(t) = -\lambda P_0^{(0)}(t) + \mu q P_1^{(0)}(t) + \mu P_1^{(1)}(t) \quad (11)$$

$$\begin{aligned} \frac{d}{dt} P_n^{(0)}(t) = & -\{\lambda + \mu + \alpha(1 - \delta_{n,1})\} P_n^{(0)}(t) \\ & + \lambda P_{n-1}^{(0)}(t) + \mu c_1 p(1 - \delta_{n,1}) P_n^{(0)}(t) \\ & + (\mu c_1 q + \alpha) P_{n+1}^{(0)}(t) + \mu c_1 P_{n+1}^{(1)}(t) \quad n \geq 1 \end{aligned} \quad (12)$$

$$\begin{aligned} \frac{d}{dt} P_n^{(1)}(t) = & -\{\lambda + \mu + \alpha(1 - \delta_{n,1})\} P_n^{(1)}(t) \\ & + \lambda P_{n-1}^{(1)}(t) + (\mu c_2 + \alpha) P_{n+1}^{(1)}(t) \\ & + \mu p(c_1 \delta_{n,1} + c_2) P_n^{(0)}(t) + \mu c_2 q P_{n+1}^{(0)}(t) \quad n \geq 1 \end{aligned} \quad (13)$$

Various probabilities are plotted vs. time for the data obtained by using Matlab programming. Fig 1.1 shows plot of probability $P_0^{(0)}$ with respect to time t (average service time). It is clear from the graph that probability $P_0^{(0)}$ decreasing rapidly in the starting and then become almost steady from the initial value (at time t=0).

Probability $P_0^{(0)}$ Vs. time

Figs 1.2 and 1.3 show relative change in probabilities $P_1^{(0)}, P_1^{(1)}, P_2^{(0)}$ and $P_2^{(1)}, P_3^{(0)}, P_3^{(1)}, P_4^{(0)}, P_4^{(1)}, P_5^{(0)}, P_5^{(1)}, P_6^{(0)}, P_6^{(1)}, P_7^{(0)}$ and $P_7^{(1)}$ with respect to time (average service time). Probability $P_1^{(0)}$ increasing rapidly in the starting and attain some steady values for higher values of t. $P_2^{(0)}$ & $P_3^{(0)}$ (shown in fig1.2) and $P_4^{(0)}, P_5^{(0)}, P_6^{(0)}, P_7^{(0)}$ (shown in fig 1.3) also increasing in starting and then attain some steady values. Though the probabilities $P_1^{(1)}, P_2^{(1)}, P_3^{(1)}$ (shown in fig 1.2) and, $P_4^{(1)}, P_5^{(1)}, P_6^{(1)}, P_7^{(1)}$ (shown in fig 1.3) increasing in the starting but the increase comparatively less than their corresponding counterparts $P_1^{(0)}, P_2^{(0)}, P_3^{(0)}, P_5^{(0)}, P_6^{(0)}$ and $P_7^{(0)}$. These also attain some steady values for higher values of t. From the above interpretations we can say that the probability of joining the server for the first time is more than that of joining server for the second time.

Comparison among probabilities when unit at the head of the queue is to join the server for the first time i.e. among $P_1^{(0)}, P_1^{(0)}, P_2^{(0)}, P_3^{(0)}, P_4^{(0)}, P_5^{(0)}, P_6^{(0)}$ and $P_7^{(0)}$ is done through Fig.1.4. It is interpreted that probability decreasing as n (number of the units in the system) increase for the case under study. It is also seen that all the probabilities finally approach some steady values. From Fig 1.5 it is interpreted that the probabilities $P_1^{(1)}, P_2^{(1)}, P_3^{(1)}, P_4^{(1)}, P_5^{(1)}, P_6^{(1)}$ and $P_7^{(1)}$ also show same relationship among themselves as shown by corresponding probabilities $P_0^{(0)}, P_1^{(0)}, P_2^{(0)}, P_3^{(0)}, P_4^{(0)}, P_5^{(0)}, P_6^{(0)}, P_7^{(0)}$ (Fig. 1.4).

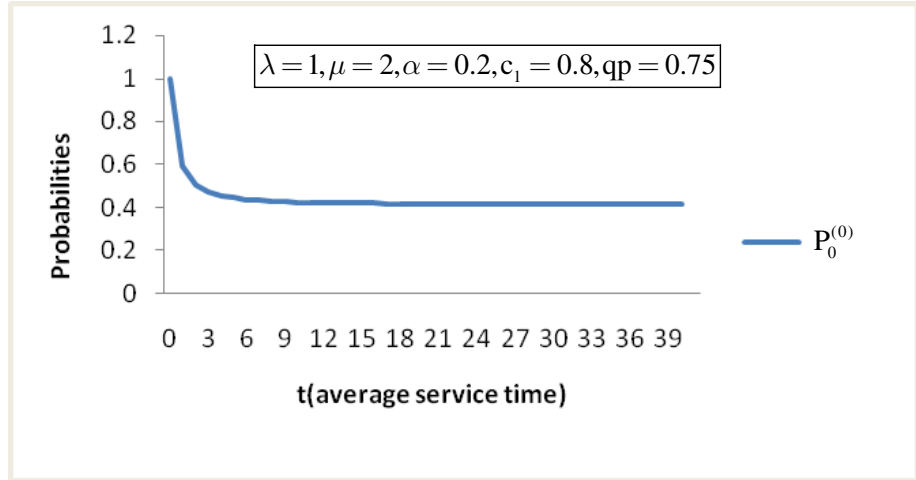


Figure 1.1

Probabilities $P_1^{(0)}, P_1^{(1)}, P_2^{(0)}, P_2^{(1)}, P_3^{(0)}, P_3^{(1)}$ Vs. time

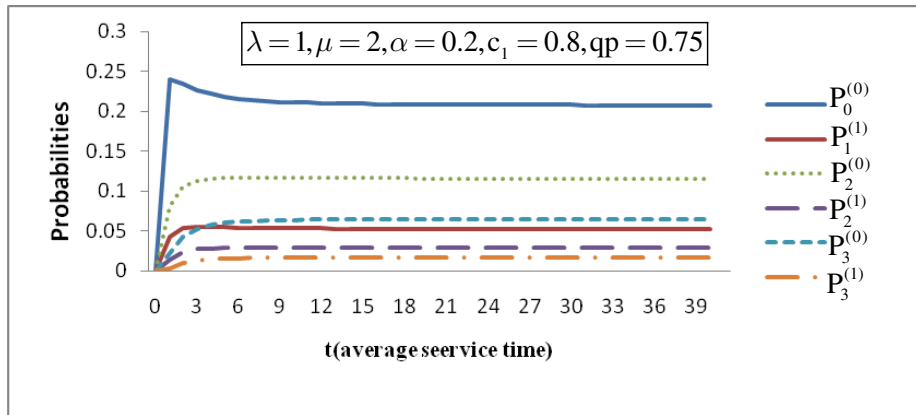


Figure 1.2

Probabilities $P_4^{(0)}, P_4^{(1)}, P_5^{(0)}, P_5^{(1)}, P_6^{(0)}, P_6^{(1)}, P_7^{(0)}, P_7^{(1)}$ Vs. time

To study the effect of reneging on different probabilities of the model, the data of various probabilities is generated for different values of α keeping other parameters constant. The values that α took are $\{0.2 \text{ \& } 0.5\}$. The other parameters are fixed at $\lambda = 1, \mu = 2, c_1 = 0.8, q = 0.75$. Fig. 1.6 concluded that as α increase probabilities $P_2^{(0)}, P_3^{(0)}, P_4^{(0)}$ & $P_5^{(0)}$ are decreasing.

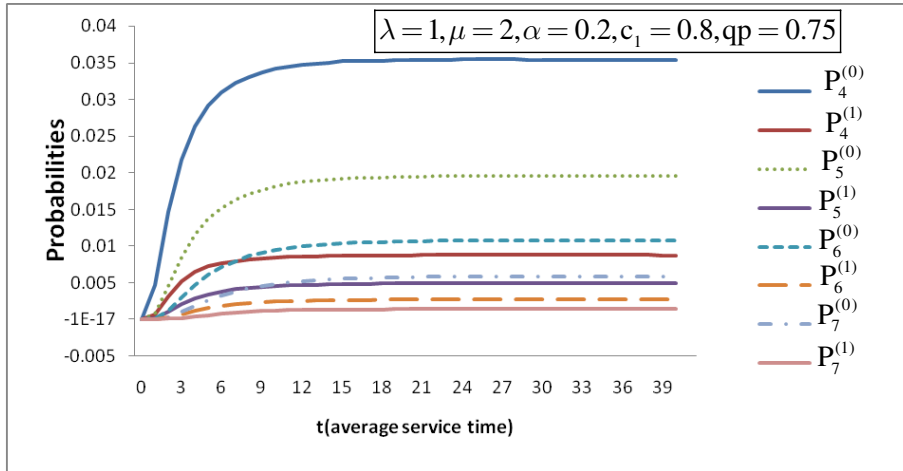


Figure 1.3

Probabilities $P_1^{(0)}, P_2^{(0)}, P_3^{(0)}, P_4^{(0)}, P_5^{(0)}, P_6^{(0)}, P_7^{(0)}$ Vs. time

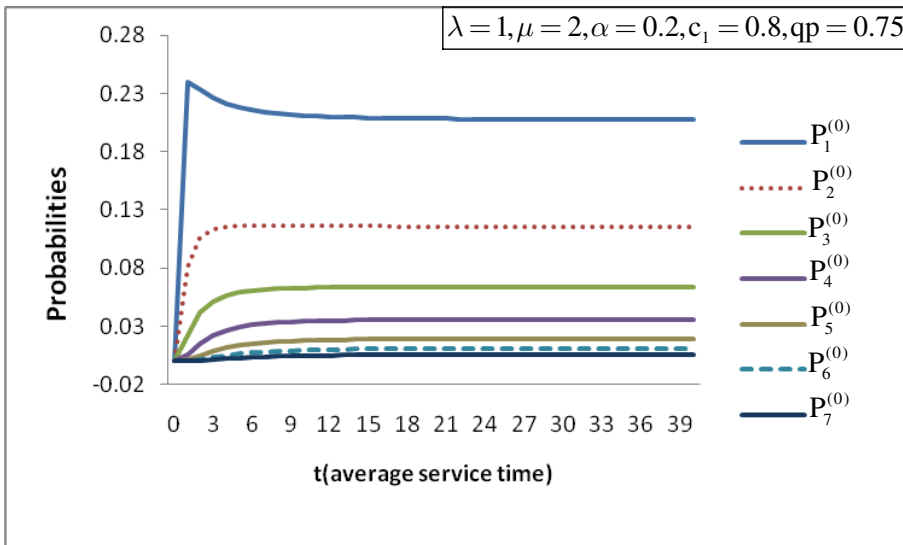


Figure 1.4

Probabilities $P_1^{(1)}, P_2^{(1)}, P_3^{(1)}, P_4^{(1)}, P_5^{(1)}, P_6^{(1)}, P_7^{(1)}$ Vs. time

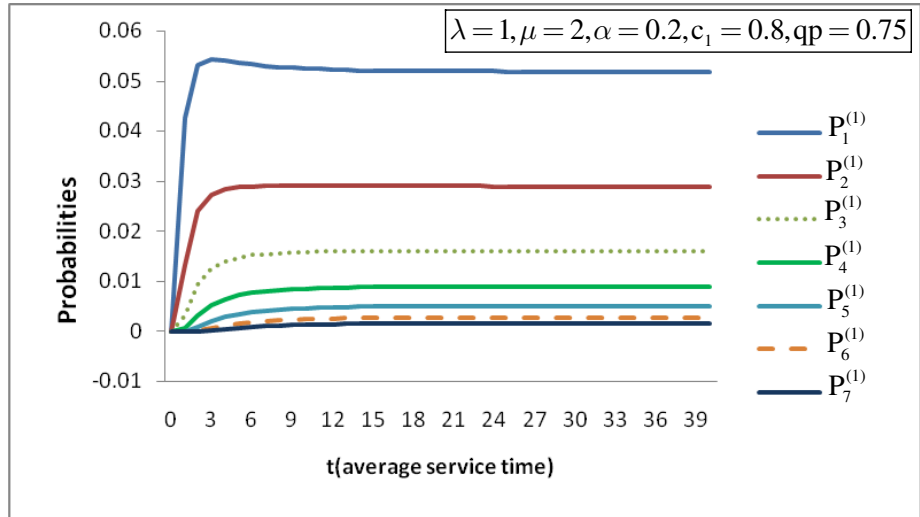


Figure 1.5

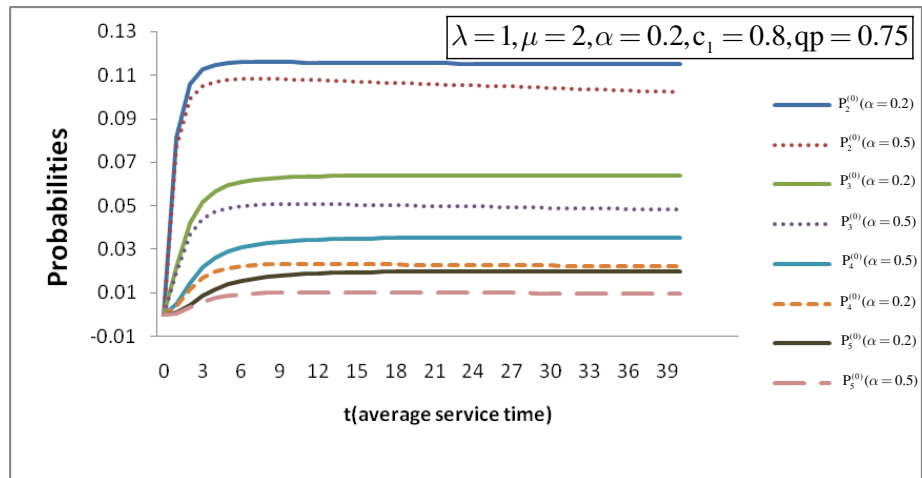


Figure 1.6

Probabilities $P_2^{(0)}, P_3^{(0)}, P_4^{(0)}, P_5^{(0)}$ for $\alpha = 0.2$ & 0.5 Resp.

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