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Bayesian Estimation of Augmented Exponential Strength Reliability Models Under Non-informative Priors

N. CHANDRA* AND V. K. RATHAUR

Department of Statistics, Ramanujan School of Mathematical Sciences, Pondicherry University, Puducherry-605 014, India,

***Email: nc.stat@gmail.com**

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Abstract In this article, the augmented strength reliability models are derived by assuming that the Inverse Gaussian stress(Y) is subjected to equipment having exponential strength(X) and are independent to each other.In a real life situations many manufactured new equipments/ products are being failed completely or partially at very early stage of its use, due to lack of its strength. Hence, ASP is proposed to protect such types of failures. The maximum likelihood (ML) and Bayes estimation of augmented strength reliability are considered. In Bayesian paradigm the non-informative types (uniform and Jeffrey's) priors are chosen under symmetric and asymmetric loss functions for better comprehension. A comparison between the ML and Bayes estimators of augmented strength reliability is carried out on the basis of their mean square errors (mse) by simulating Monte-Carlo samples from posterior distribution by using Metropolis-Hasting approximation.

Keywords: Stress-Strength reliability, Augmentation, Exponential Distribution, Inverse Gaussian distribution, Metropolis-Hasting Algorithm.

1. INTRODUCTION

The history of stress-strength reliability is old enough to the researchers. The problem of estimating $R = P(X > Y)$, where X and Y are considered to be independent random variables, has attracted the researchers because of its applicability in various directions of real life.It is considered as the reliability parameter that one random variable exceeds another. The strength reliability is defined as the probability that the system will survives its usual life by performing its intended function, provided that the random strength(X) should

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be higher than the random stress(Y) and vice versa for its failure processes. In reliability theory, the era of stress-strength reliability was introduced by [4] and after that the procedure for obtaining distribution free confidence interval for stress-strength reliability have discussed by [5] and [15]. A plenty of literature is available towards the stress-strength reliability problems for both single component as well as for multi component designed system. Initially, [12], [23] and [14] have attempted single component reliability under stress-strength set up, where stress and strength of inbuilt component are independent. Even, a number of work related to multi-component stress-strength reliability have discussed by well known researchers, like, [3], [13] and [24] for various life time distributions with the assumption that random stresses are independent and identically distributed with known distribution and also independent of strength. More reference works on stress-strength reliability for last four decades can be found out in the classical monograph of Kotz et. al. [22]. For recent developments related to system reliability, one may refer to [25] and [26] and references therein.

In the present real life scenario of competitive global market, many manufacturing industries are lunching several types of newly designed products/ equipments viz., electronic devices, automobiles etc., after conducting their strength breaking tests in order to meet quality standards and the customers' expectations. In fact, these products are highly sophisticated and costly. In some occasions, these products may have the impression of early failure in its first or subsequent use. Such types of failures are known as irrelevant failures and hence the strength of existing equipment becomes weaker. One can reuse such products by repair and may enhance its strength to dominate the imposed random stress by adopting the three possible cases of Augmentation Strategy Plan (ASP)(see., [7]).

The problem of augmenting strength reliability was firstly initiated by [1]. They derived augmenting strength reliability models for exponential distribution for three different possible cases of augmentation. After one decade, [7] extended the work of [1] and pointed out the applicability of ASP for augmenting the gamma strength reliability of equipment for same set up. In similar manner, [8] studied for augmenting Inverse Gaussian stress strength reliability under ASP. [9,10] have also attempted the problems of augmented strength reliability models under a coherent system set up for exponential and gamma life time models respectively. [2] have also studied strength reliability problem by assuming that the form of the distribution of strength (X) and stress(Y) follow power function distribution and exponential distribution respectively, a desired level of strength-reliability is achieved for the possible variations in model parameters, without using ASP.

Some of the authors also focused on strength reliability of equipment to face the array of stresses. [19, 20] discussed with the exponential strength reliability problems for Rayleigh and generalized Weibull Stresses respectively. But [17] and [21] have attempted the strength reliability problems for multi-component stresses having exponential and Half-normal Distributions respectively with common strength follows power function distribution for usual stress-strength set up.

We consider the problem of augmented strength reliability models of any equipment which have minimum possibility of survival in nature i.e., facing early failure due to weak in potential under the generalized case of ASP. It is assumed that initially the strength of equipment follows exponential distribution with scale parameter (θ) and stress follows Inverse Gaussian distribution with parameters (μ, λ).The detailed descriptions and usefulness of the Inverse Gaussian distribution (also known as Wald Distribution) in the study of life testing and reliability problem is reported by [11]. In this article, a comparison between the maximum likelihood (ML) and Bayes estimators of augmented strength reliability for the generalized case of ASP has been carried out on the basis of their mean square errors. In Bayesian context, the non-informative (uniform and Jeffrey's) types of priors are considered under squared error loss function (SELF) and LINEX loss function (LLF).

The rest of the paper is organized as follows. Section 2 presents the development of the augmented strength-reliability expression for generalized case of ASP. In section 3, the maximum Likelihood estimators of parameters of generalized augmented strength reliability have presented. Bayes estimators of augmented strength reliability for uniform and Jeffrey's prior under symmetric and asymmetric loss functions have been presented in section 4. Section 5 presents the simulation study and discussion. In section 6, the concluding remarks and further scope of proposed research are given.

2. AUGMENTED STRENGTH RELIABILITY MODELS

Let X and Y represent the strength and stress of the equipment respectively which are independently distributed. The strength (X) follows exponential distribution with parameter θ and the stress (Y) follow an Inverse Gaussian (IG) distribution with $\mu > 0$ (mean) and $\lambda > 0$ (scale) parameters. The probability density functions (pdf) of random variables X and Y are given as

$$f_X(x)=\frac{1}{\theta}\ e^{\frac{-x}{\theta}} \quad ;x>0,\theta>0 \tag{1}$$

and

$$g_Y(y)=\sqrt{\frac{\lambda}{2\pi y^3}}\ \exp\left(\frac{-\lambda(y-\mu)^2}{2\mu^2 y}\right) \qquad ; y>0,\ \mu,\ \lambda>0 \tag{2}$$

To enhance the strength of a weaker strength / failed equipment, we implement the Augmentation Strategy Plan (ASP), which comprises three possible cases to enhance the strength reliability of an equipment to face the common stress proposed by [7]. It is noticed that case-I and case-II of ASP were special cases of case-III, which we call it as generalized case of ASP. Under the generalized case of ASP the enhanced strength $Z_k=\sum_{i=1}^{n} S_i$ follow Gamma distribution with parameters $(m\theta, n)$, where the subscript 'k' denotes the k^{th} (1, 2, 3) case of ASP and each $S_i\,(i=1,2,3,...,n)$ are defined as m times of the initial strength $(S_i=mX_i)$ of the equipment with distribution as $\exp(m\theta)$. The probability density function (pdf) of augmented strength (Z_k) under generalized case of ASP can be given as

$$f_{Z_k}(z_k)=\frac{(m\theta)^n}{\Gamma(n)}\exp(-m\theta\, z_k)\, z_k^{\,n-1} \quad ;\ z_k>0, \mathrm{m},\ \theta>0 \tag{3}$$

Where, 'm' is a positive real number and 'n' is positive integer. The probability density of two special cases of ASP can also be obtained from equation (3) as, case-I can be found by substituting $k=1, n=1$ and case-II can be obtained by substituting $k=2, m=1$.
Thus, the strength reliability of the equipment under generalized case of ASP is given by

$$\begin{aligned} R_k &= \Pr(Z_k>Y) \\ &= \frac{(m\theta)^n}{2\Gamma(n)}\left\{\begin{aligned} &\int_0^\infty erfc\left[\frac{\sqrt{\lambda/z_k}\,(-z_k+\mu)}{\sqrt{2}\,\mu}\right] e^{-m\theta z_k}\, z_k^{\,n-1}\, dz_k \\ &+\exp(2\lambda/\mu)\int_0^\infty erfc\left\{\frac{\sqrt{\lambda/z_k}\,(z_k+\mu)}{\sqrt{2}\,\mu}\right\} e^{-m\theta z_k}\, z_k^{\,n-1}\, dz_k \end{aligned}\right\} \end{aligned} \tag{4}$$

Where $erfc(.)$ is complementary error function and it can also be defined in terms of error function. The error function equals the twice of the integral of a normalized Gaussian function between 0 *and* $x/\sigma\sqrt{2}$. Thus the error and complementary error functions are respectively defined by

$$erf(x)=\frac{2}{\sqrt{\pi}}\int_{0}^{x/\sigma\sqrt{2}} e^{-t^2}\,dt \text{ and } erfc(x)=1-erf(x) \tag{5}$$

Here, one may notice that the augmented strength reliability for other two special cases of ASP can be found as, case-I can be found by substituting $k=1, n=1$ and case-II can be obtained by substituting $k=2, m=1$ in equation (4).

3. MAXIMUM LIKELIHOOD ESTIMATION OF GENERALIZED AUGMENTED STRENGTH RELIABILITY

Suppose $Z_k=\{Z_{k1}, Z_{k2}, \ldots, Z_{kn_1}\}$ and $Y=\{Y_1, Y_2, \ldots, Y_{n_2}\}$ be the two independent random samples of sizes n_1 and n_2 drawn from the augmented exponential strength and inverse Gaussian stress distributions respectively. Then the generalized form of likelihood function based on the observed random samples is defined as follows

$$\begin{aligned} L_k(z_k, y/\theta,\mu,\lambda) &= \frac{(m\theta)^{nn_1}}{(\Gamma n)^{n_1}}\left(\frac{1}{\sqrt{2\pi}}\right)^{n_2}\left(\prod_{i=1}^{n_1} z_{ki}^{(n-1)}\right)\left(\prod_{j=1}^{n_2} y_j^{(-3/2)}\right)\lambda^{n_2/2} \\ &\exp\{-mn_1\theta\,\overline{z}_k\}\exp\left[-\lambda\sum_{j=1}^{n_2}\left(\frac{-y_j}{2\mu^2}-\frac{1}{2y_j}+\frac{1}{\mu}\right)\right] \end{aligned} \tag{6}$$

The log-likelihood function is given by

$$\begin{aligned} \log L_k(z_k, y/\theta,\mu,\lambda) &= nn_1(\log m+\log\theta) \\ &-n_1\log(\Gamma n)-n_2\log\left(\sqrt{2\pi}\right) \\ &+(n-1)\sum_{i=1}^{n_1}\log z_{ki}-\frac{3}{2}\sum_{i=1}^{n_2}\log y_j-n_1 m\theta\,\overline{z}_k \\ &+\frac{n_2}{2}\log\lambda-\lambda\sum_{j=1}^{n_2}\left(\frac{-y_j}{2\mu^2}-\frac{1}{2y_j}+\frac{1}{\mu}\right) \end{aligned} \tag{7}$$

Thus the maximum likelihood estimators of θ, μ and λ can be obtained by solving the following likelihood equations with respect to θ, μ and λ

$$\frac{\partial \log L_k(z_k, y/\theta,\mu,\lambda)}{\partial\theta} = 0; \frac{\partial \log L_k(z_k, y/\theta,\mu,\lambda)}{\partial\mu} = 0 \text{ and } \frac{\partial \log L_k(z_k, y/\theta,\mu,\lambda)}{\partial\lambda} = 0 \tag{8}$$

The maximum likelihood estimators $\hat{\theta}_k, \hat{\mu}_k$ and $\hat{\lambda}_k$ are respectively given as

$$\hat{\theta}_k = \frac{n}{m\,\overline{z}_k}, \hat{\mu}_k = \overline{y} \text{ and } \hat{\lambda}_k = \left[\frac{1}{n_2}\sum_{j=1}^{n_2}\left(\frac{1}{y_j} - \frac{1}{\overline{y}}\right)\right]^{-1} \tag{9}$$

ML estimator $(\hat{R}_k)$ of augmented strength reliability (R_k) can be obtained through invariance property of ML estimators by substituting $\hat{\theta}_k, \hat{\mu}_k$ and $\hat{\lambda}_k$ in place of θ, μ and λ respectively in the equation (4).

4. BAYES ESTIMATE OF AUGMENTED STRENGTH RELIABILITY MODEL

In this section, we considered the Bayes estimation of $R_k\left(k=1,2,3\right)$ for the generalized case of ASP by assuming the model parameters θ, μ and λ as independently distributed random variables. In Bayesian paradigm, the choice of a prior distribution is a challenging task and there is no any hard and fast rule available in the literature. Choosing prior distribution totally depends on the subjectivity and personal belief of the concern experimenter. However, if one has adequate information about the parameter(s) one should use informative prior(s); otherwise it is preferable to use non informative prior(s).The notion of a non-informative prior has attracted much attention in recent years. There are different notions of non-informative priors. In this study we consider non-informative (uniform and Jeffrey’s) types of priors by assuming that no information is known priori about the underlying parameters of interest. For a comprehensive comparison of the proposed Bayes estimators, we consider two different loss functions known as symmetric (SELF) and asymmetric (LLF). In fact, the squared error loss function considers the overestimation and underestimation as equally penalized, whereas, in LINEX loss function the overestimation is more serious than the underestimation or vice-versa (see; [27] and [28]).

4.1 Assuming Uniform prior

In this study, we consider θ, μ and λ are independent random variables having non-informative uniform prior with joint prior probability density function given as follows

$$g_1(\theta,\mu,\lambda) = \frac{1}{\theta\,\mu\,\lambda}\ ;\ 0<\theta,\mu,\lambda<\infty \tag{10}$$

The joint posterior probability distribution of random variables θ, μ and λ is obtained by combining both likelihood function $L_k\left(\theta,\mu,\lambda / z_k, y\right)$ and joint prior probability density function $g_1(\theta,\mu,\lambda)$ given by

$$\Pi_{k1}\left(\theta,\mu,\lambda / z_k, y\right) \propto K_1\,\mu^{-1}\,\theta^{n n_1-1}$$
$$\exp\{-mn_1\theta\,\overline{z}_k\}\,\lambda^{\frac{n_2-2}{2}}\exp\left[\lambda\sum_{j=1}^{n_2}\left(\frac{-y_j}{2\mu^2}-\frac{1}{2y_j}+\frac{1}{\mu}\right)\right] \tag{11}$$

Where; $K_1 = \left(\frac{m^n}{\Gamma n}\right)^{n_1}\left(\frac{1}{\sqrt{2\pi}}\right)^{n_2}\left(\prod_{i=1}^{n_1} z_{ki}^{(n-1)}\right)\left(\prod_{j=1}^{n_2} y_j^{\left(-3/2\right)}\right)$

Thus, the proposed Bayes estimators of augmenting strength reliability $\{R_k; k=1,2,3\}$ under squared error loss function for general case of ASP, is given by

$$\hat{R}_k^{\,self} = \int_{(\theta,\mu,\lambda)} R_k\,\Pi_{k1}\left(\theta,\mu,\lambda / z_k, y\right) d\theta\, d\mu\, d\lambda \tag{12}$$

Under Linex loss function (LLF), the Bayes estimate ($\hat{R}_k^{\,llf}$) of augmented strength reliability model is given by

$$\hat{R}_k^{\,llf} = -\frac{1}{p}\ln\left\{\int_{(\theta,\mu,\lambda)}\exp\left(-pR_k\right)\Pi_{k1}\left(\theta,\mu,\lambda / z_k, y\right) d\theta\, d\mu\, d\lambda\right\} \tag{13}$$

The expressions for Bayes estimates of augmented strength reliability obtained under uniform prior are not in explicit form and cannot be evaluated manually, thus the numerical methods (MCMC) can be applied to evaluate the expressions.

4.2. Assuming Jeffrey's prior

Considering the parameters θ, μ and λ as independent random variables having non-informative Jeffrey's prior. [18] proposed a type of non-informative prior, which is defined as

$$J\left(\theta,\mu,\lambda\right) \propto \sqrt{\det I\left(\theta,\mu,\lambda\right)}$$

Where $I(\theta,\mu,\lambda)$ is Fisher information matrix. The joint Jeffrey prior of θ, μ and λ for general case of ASP is given as

$$J(\theta,\mu,\lambda) \propto \theta^{-1}\,\mu^{-3/2}\left(\frac{1}{2\lambda}+\frac{2}{\mu^2}\right)^{1/2} \tag{14}$$

The joint posterior distribution of θ, μ and λ is defined by

$$\Pi_{k2}(\theta,\mu,\lambda / z_k, y) \propto K_2\,\theta^{n n_1 -1} \exp\{-mn_1\theta\,\bar{z}_k\}\,\lambda^{\frac{n_2}{2}}\,\mu^{\frac{-3}{2}}\left(\frac{1}{2\lambda}+\frac{2}{\mu^2}\right)^{1/2} \exp\left[\lambda\sum_{j=1}^{n_2}\left(\frac{-y_j}{2\mu^2}-\frac{1}{2y_j}+\frac{1}{\mu}\right)\right] \tag{15}$$

Where; $K_2 = n_2\sqrt{n n_1}\left(\frac{m^n}{\Gamma n}\right)^{n_1}\left(\frac{1}{\sqrt{2\pi}}\right)^{n_2}\left(\prod_{i=1}^{n_1} z_{ki}^{(n-1)}\right)\left(\prod_{j=1}^{n_2} y_j^{(-3/2)}\right)$

Under squared error loss function, the Bayes estimator of augmented strength reliability for Jeffrey's prior of augmented strength reliability R_k $(k=1,2,3)$ for general case of ASP is given as

$$\hat{R}_k^{self} = \int_{(\theta,\mu,\lambda)} R_k\,\Pi_{k2}(\theta,\mu,\lambda / z_k, y)\,d\theta\,d\mu\,d\lambda \tag{16}$$

The Bayes estimate ($\hat{R}_k^{llf}$) of augmented strength reliability model $(R_k; k=1,2,3)$ for Jeffrey's prior under Linex loss function (LLF), is given by

$$\hat{R}_k^{llf} = -\frac{1}{p}\ln\left\{\int_{(\theta,\mu,\lambda)} \exp(-pR_k)\,\Pi_{k2}(\theta,\mu,\lambda / z_k, y)\,d\theta\,d\mu\,d\lambda\right\} \tag{17}$$

Here one can notice that in each of the cases the joint posterior densities have not in any distributional form and it is difficult to get analytical solution. In this situation the Markov Chain Monte Carlo (MCMC) sampling method can be used to approximate the integrals (see; [6] and [16]) numerically. To approximate the above integrals for finding Bayes estimates, we used the Metropolis-Hastings algorithm.

5. SIMULATION STUDY AND DISCUSSIONS

In this section, we studied the behavior of ML and Bayesian estimators of augmented strength reliability parameters under the augmentation strategy plan through simulated samples with different combinations of sample sizes and stress-strength reliability parameters. The comparison between the ML and Bayes estimators has carried out on the basis of their mean square errors. The Bayesian estimators have calculated for uniform and Jeffery's types of priors under two different loss function (SELF and LLF). Performances of proposed maximum likelihood estimates and Bayes estimates of augmented strength reliability for uniform prior under two different loss functions (i.e. self and llf) were compared through its mean square errors (MSE). The whole procedure was replicated randomly 1000 times in order to evaluate its MSEs. In order to observe the effect of sample sizes to the proposed estimation procedures, we drawn the random samples from the stress and augmented strength distributions with different combinations sample sizes (s,t).

It may be noticed from the expressions of Bayes estimates of augmented strength reliability that finding the posterior expectations are difficult because of the form of joint posterior density are not in any standard distributional form, which cannot be solved analytically. In such a situation, the well-known MCMC technique viz. Metropolis-Hastings algorithm is used for drawing the samples from any arbitrary posterior distribution. For generating a random sample of size N (say) from a posterior distribution $\pi\left(\boldsymbol{\theta}/data\right); \boldsymbol{\theta}=(\theta,\mu,\lambda)'$, the basic Metropolis-Hastings algorithm consist the following steps:

1. Choose initial value for the parameter $\boldsymbol{\theta}^{(0)}=\left(\theta^{(0)},\mu^{(0)},\lambda^{(0)}\right)$ such that $\pi\left(\boldsymbol{\theta}^{(0)}\right)>0$.
2. For $j=1,2,...,N$ repeat the following steps
   i. Set $\boldsymbol{\theta}=\boldsymbol{\theta}^{(j-1)}$
   ii. Draw a 'candidate' value $\boldsymbol{\theta}^{c}$ from a proposal density say $q(\boldsymbol{\theta}^{c}/\boldsymbol{\theta})$.
   iii. Generate 'U' uniform variate on range 0 and 1 i.e., $u\sim U(0,1)$.
   iv. If $u \,''\, \min(1,R)$, accept the candidate point $\left(\boldsymbol{\theta}^{c}\right)$ with probability $\min\left\{1, R=\frac{\pi\left(\boldsymbol{\theta}^{c}/\mathbf{data}\right)q\left(\boldsymbol{\theta}/\boldsymbol{\theta}^{c}\right)}{\pi\left(\boldsymbol{\theta}/\mathbf{data}\right)q\left(\boldsymbol{\theta}^{c}/\boldsymbol{\theta}\right)}\right\}$, otherwise set $\boldsymbol{\theta}^{(j)}=\boldsymbol{\theta}$.

Table 1: AVG and MSE for estimates of Rk for variation of μ when $\lambda = 2.5; \theta = 1.5; n = m = 2$.

$\mu = 1.5$, R = 0.891697						
(n_1, n_2)	Statistic	MLE	Uniform prior		Jeffrey's prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg.	0.891084	0.8298	0.828725	0.828546	0.827459
	Mse	0.001463	0.003926	0.004224	0.004085	0.004224
(20,30)	Avg.	0.8916	0.910775	0.910664	0.910631	0.910522
	Mse	0.000505	0.000376	0.000366	0.00037	0.000366
(30,20)	Avg.	0.891846	0.881349	0.881088	0.881224	0.880961
	Mse	0.000786	0.000123	0.000132	0.000126	0.000132
(50,50)	Avg.	0.891135	0.882973	0.882844	0.882841	0.882712
	Mse	0.000288	0.000092	0.000097	0.000095	0.000097

$\mu = 3.5$, R = 0.724949						
(n_1, n_2)	Statistic	MLE	Uniform prior		Jeffrey's prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg.	0.724734	0.74529	0.743276	0.743257	0.741267
	Mse	0.003053	0.000309	0.000249	0.000281	0.000249
(20,30)	Avg.	0.727523	0.772715	0.772185	0.772093	0.771569
	Mse	0.002058	0.002347	0.00224	0.002289	0.00224
(30,20)	Avg.	0.730443	0.715866	0.714668	0.715096	0.713907
	Mse	0.002864	0.000203	0.000241	0.00022	0.000241
(50,50)	Avg.	0.725943	0.727525	0.727089	0.727171	0.726734
	Mse	0.001237	0.000068	0.000061	0.000063	0.000061

Here q is the transition probability matrix of the Markov chain with same support as that of likelihood function and $q(\theta^c/\theta)$ is the transition probability from θ to θ^c. we assume asymptotic normal distribution as proposal distribution. The initial values $\theta^{(0)} = \hat{\theta}, \mu^{(0)} = \hat{\mu}, \lambda^{(0)} = \lambda$ are fixed as the ML estimates along with asymptotic variance covariance matrix to draw the initial random samples from proposal density. The Bayes estimates were calculated by assuming non-informative types (uniform and Jeffrey's) of priors under squared error loss function and Linex loss functions. To evaluate the Bayes estimate of augmented strength reliability we generated 10000 MCMC random samples from posterior density by using Metropolis-Hastings algorithm. The

Table 2: AVG and MSE for estimates of R_k for variation of λ when $\mu = 2.5; \theta = 1.5; n = m = 2$.

$\lambda = 2.5$, R = 0.795637						
(n_1, n_2)	Statistic	MLE	Uniform prior		Jeffrey's prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg.	0.797896	0.770767	0.76896	0.769128	0.767329
	Mse	0.004208	0.000832	0.001026	0.000927	0.001026
(20,30)	Avg.	0.796616	0.83334	0.833037	0.833079	0.832777
	Mse	0.001416	0.001455	0.001413	0.001435	0.001413
(30,20)	Avg.	0.798803	0.785671	0.784901	0.785169	0.784395
	Mse	0.00225	0.000178	0.000206	0.000189	0.000206
(50,50)	Avg.	0.795286	0.799179	0.798903	0.799145	0.798867
	Mse	0.000861	0.000048	0.000044	0.000046	0.000044

$\lambda = 4.5$, R = 0.78848						
(n_1, n_2)	Statistic	MLE	Uniform prior		Jeffrey's prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg.	0.787564	0.733922	0.731773	0.730101	0.727994
	Mse	0.00323	0.003213	0.003904	0.00365	0.003904
(20,30)	Avg.	0.788923	0.822565	0.822278	0.822522	0.822238
	Mse	0.001143	0.001198	0.001174	0.001193	0.001174
(30,20)	Avg.	0.789566	0.797431	0.79683	0.79662	0.796027
	Mse	0.001797	0.000124	0.000101	0.00011	0.000101
(50,50)	Avg.	0.787752	0.783514	0.78323	0.783299	0.783019
	Mse	0.000653	0.000063	0.000071	0.000068	0.000071

first thousand samples have been discarded as burn-in period of Markov chain. We also tested the autocorrelation and it is noticed that the chains are highly auto correlated. For reducing the autocorrelation within the chain, we thinned the chain equally spaced at every second simulation.

The comparison among the proposed estimators of augmented strength reliability for third (viz. generalized) case of ASP was carried out for varying values of stress-strength parameters μ, λ, θ, m and n for different combinations of sample sizes (s, t). The findings of simulation study are presented in Tables 1-5 for variations in μ, λ, θ, n and m respectively. The

Table 3: AVG and MSE for estimates of R_k for variation of θ when $\mu = \lambda = 2.5; n = m = 2$.

θ = 2.5, R = 0.890163						
(n_1, n_2)	Statistic	MLE	Uniform prior		Jeffrey's prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg.	0.890856	0.857471	0.856563	0.85746	0.856539
	Mse	0.001852	0.001067	0.001132	0.001075	0.001132
(20,30)	Avg.	0.89044	0.907436	0.907302	0.907353	0.90722
	Mse	0.000694	0.000311	0.000304	0.000308	0.000304
(30,20)	Avg.	0.891592	0.870237	0.869862	0.870403	0.870022
	Mse	0.001079	0.000426	0.000435	0.00042	0.000435
(50,50)	Avg.	0.889715	0.883705	0.883564	0.883695	0.883554
	Mse	0.00042	0.000057	0.000059	0.000057	0.000059

θ = 4.5, R = 0.9587555						
(n_1, n_2)	Statistic	MLE	Uniform prior		Jeffrey's prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg.	0.952590	0.925414	0.925055	0.926187	0.92582
	Mse	0.000504	0.000613	0.000597	0.000582	0.000597
(20,30)	Avg.	0.953118	0.958349	0.958302	0.95837	0.958323
	Mse	0.000219	0.000029	0.000029	0.000029	0.000029
(30,20)	Avg.	0.953509	0.934705	0.934562	0.935141	0.934997
	Mse	0.000338	0.000355	0.000344	0.000339	0.000344
(50,50)	Avg.	0.952890	0.944992	0.944935	0.94500	0.944944
	Mse	0.000130	0.000075	0.000076	0.000075	0.000076

Table 4: AVG and MSE for estimates of R_k for variation of n when $\mu = \lambda = \theta = 2.5;\ m = 2$.

n = 3, R = 0.967818						
(n_1, n_2)	Statistic	MLE	Uniform prior		Jeffrey's prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg.	0.96674	0.962099	0.961990	0.962976	0.962866
	Mse	0.000595	0.000038	0.000030	0.000029	0.00003
(20,30)	Avg.	0.967674	0.958019	0.957954	0.958193	0.958130
	Mse	0.000206	0.000100	0.000098	0.000096	0.000098

(30,20)	Avg. Mse	0.967472 0.000307	0.960727 0.000054	0.960653 0.000050	0.961108 0.000049	0.961034 0.00005
(50,50)	Avg. Mse	0.967671 0.000118	0.954997 0.000171	0.954926 0.000176	0.954873 0.000174	0.954804 0.000176

n = 5, R = 0.99556

(n_1, n_2)	Statistic	MLE	Uniform prior		Jeffrey's prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg. Mse	0.994383 0.000054	0.997294 0.000003	0.99729 0.000004	0.997558 0.000004	0.997555 0.000004
(20,30)	Avg. Mse	0.995104 0.000018	0.9551 0.001639	0.954959 0.001591	0.955842 0.00158	0.9557 0.001591
(30,20)	Avg. Mse	0.994735 0.000028	0.978553 0.000291	0.978493 0.000273	0.979153 0.000271	0.979094 0.000273
(50,50)	Avg. Mse	0.995151 0.000009	0.973871 0.000318	0.973828 0.000309	0.974230 0.000308	0.974186 0.000309

following observations have been made based on the given tables, which are stated as follow.

* In Table 1 the effect of the variation in $\mu(1.5,3.5)$ has been seen by fixing rest of the parameters $(\lambda = 2.5,\ \theta = 1.5,\ n = m = 2)$ and it is noticed that the mean square errors decrease for increasing sample sizes. The Bayes

Table 5: AVG and MSE for estimates of R_k for variation of m when $\mu = \lambda = \theta = 2.5;\ n = 2$.

m = 2.5, R = 0.919202

(n_1, n_2)	Statistic	MLE	Uniform prior		Jeffrey's prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg. Mse	0.918349 0.000679	0.886958 0.000508	0.88631 0.000517	0.887386 0.000497	0.886725 0.000517
(20,30)	Avg. Mse	0.91922 0.000466	0.930568 0.000138	0.930476 0.000135	0.930537 0.000137	0.930446 0.000135
(30,20)	Avg. Mse	0.920021 0.000721	0.898595 0.000442	0.898329 0.000441	0.898917 0.00043	0.898648 0.000441

(Table 5: Continued)

(Table 5: Continued)

m = 2.5, R = 0.919202						
(n_1, n_2)	Statistic	MLE	Uniform prior		Jeffrey's prior	
			SELF	LLF	SELF	LLF
(50,50)	Avg. Mse	0.918723 0.000281	0.911116 0.000076	0.911013 0.000077	0.911115 0.000076	0.911012 0.000077

m = 4.5, R = 0.967641						
(n_1, n_2)	Statistic	MLE	Uniform prior		Jeffrey's prior	
			SELF	LLF	SELF	LLF
(10,10)	Avg. Mse	0.966201 0.000271	0.943231 0.000401	0.942989 0.000383	0.944067 0.000375	0.943821 0.000383
(20,30)	Avg. Mse	0.96736 0.00013	0.970308 0.00001	0.970278 0.00001	0.970343 0.00001	0.970314 0.00001
(30,20)	Avg. Mse	0.967587 0.00020	0.951277 0.000272	0.951183 0.000261	0.951717 0.000258	0.951622 0.000261
(50,50)	Avg. Mse	0.967236 0.000077	0.960015 0.000061	0.959978 0.000062	0.960027 0.000061	0.95999 0.000062

estimators under uniform and Jeffrey priors performs better with minimum MSEs than that of maximum likelihood estimates for increasing values of the sample sizes (s, t). Among the Bayes estimators, the Jeffery's prior under SELF performs better than others.

* The ML and Bayesian estimates of augmented strength reliability and their mean square errors are presented in Table 2 for variation in $\lambda(2.5,4.5)$ by fixing other parameters $(\mu = 2.5,\ \theta = 1.5,\ n = m = 2)$. It is observe that the Bayes estimators give more precise estimates with minimum mean square errors. The augmented strength reliability decreases for higher values of λ.
* Similarly, Table 3 presents the variations in $\theta(2.5,4.5)$ for fixed values of rest of the parameters $(\mu = \lambda = 2.5; n = m = 2)$ and it is observed that the Bayes estimators give precise estimates with minimum MSE than ML estimator. The MSEs decrease for increasing values of sample sizes.
* In Table 4, the effect of variation in augmentation parameter $n(3,5)$ is presented by keeping other model parameters $(\mu = \lambda = \theta = 2.5;\ m = 2)$

fixed and it is observed that the strength reliability of the equipment get enhanced by 99% by adding $n=5$ components to the system. The Bayes estimators dominate the ML estimator and give the minimum MSEs.

* Table 5 presents the effect of variation in augmentation parameter $m(2.5, 4.5)$ by fixing rest of the other parameters $(\mu=\lambda=\theta=2.5;\ n=2)$ and it is seen that strength reliability get enhanced by increasing the values of m. The Bayes estimators dominate the ML estimator with lesser MSE. It is also observed that the MSEs gradually decrease for increasing combinations of samples sizes.

6. CONCLUDING REMARKS

In this article, the augmented strategy plan is considered for enhancing the strength of an unreliable equipment/system. A system becomes unreliable due to its unwanted frequent failure occurs and hence assessing the life of such kind of equipments are very difficult to the experimenter. ASP may be a useful technique for boosting the system reliability and its durability. We have attempted the estimation of augmented strength reliability under generalized case of ASP through ML and Bayes methods. The Bayes estimation of augmented strength reliability for different types of non-informative (uniform and Jeffrey's) priors under both of squared error and LINEX loss functions separately for generalized case (case-III) of ASP are considered. Overall, it may be concluded from the given Tables that the Bayes estimators performs quite well than that of ML estimators. Thus, all three possible cases of ASP are useful to augment the strength of a system; even adding new components for some desired level to the existing system may be suggestive.

In further, one may think over for attempting the Bayes estimation of augmenting strength reliability for different censoring schemes.

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