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Approximate Analytical Solution of Advection-Dispersion Equation By Means of OHAM

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ABSTRACT

This work deals with the analytical solution of advection dispersion equation arising in solute transport along unsteady groundwater flow in finite aquifer. A time dependent input source concentration is considered at the origin of the aquifer and it is assumed that the concentration gradient is zero at the other end of the aquifer. The optimal homotopy analysis method (OHAM) is used to obtain numerical and graphical representation.

1. Introduction

Solutions of advection-dispersion equation (ADE) may be used to predict the concentration of solutes in unsteady groundwater flow. Advection causes the contaminant plum to flow in the direction of groundwater water without any change in the shape. Dispersion of plum arises due to the variation in groundwater velocity. The heterogeneity of the porous medium is responsible for dispersion. The solute transport in heterogeneous aquifer is thus the combined process of advection and dispersion. The dispersion in the direction of groundwater flow is called longitudinal dispersion and the transverse dispersion is perpendicular to the groundwater flow direction. The ADE can be derived using Fick's law and low of conservation of mass.

Analytical solutions in one-dimensional problems through semi-infinite or finite porous media have been presented by several researchers: (Mazaheri *et. al.* 2013, Kumar *et. al.* 2010, Marino *et al.* 1974, Singh *et al.* 2008) etc. The objective of this work is to derive an approximate analytical solution of ADE with the help of Optimal Homotopy Analysis Method (OHAM). In this work, an approximate analytical solution of one-dimensional ADE in heterogeneous finite aquifer is derived for continuous time dependent input source concentration of increasing nature.

2. Problem Formulation

We assume that the solute transport is primarily one-dimensional. The solutes are assumed to be conservative in the unsteady field. Consider an isotopic heterogeneous finite aquifer of length $L=1\ km$.

Let $C(x,t)$ be the contaminant concentration in the aquifer, u the groundwater velocity and D_L the longitudinal dispersion coefficient. Then the partial differential equation describing the one dimensional advection and dispersion is[ii]

$$\frac{\partial C}{\partial t}+\frac{\partial}{\partial x}(uC)=\frac{\partial}{\partial x}\left(D_L\frac{\partial C}{\partial x}\right) \tag{1}$$

Expanding (1) we obtain

$$\frac{\partial C}{\partial t}+\left(u-\frac{\partial D_L}{\partial x}\right)\frac{\partial C}{\partial x}+C\frac{\partial u}{\partial x}=D_L\frac{\partial^2 C}{\partial x^2} \tag{2}$$

In heterogeneous medium, porosity changes with position. The velocity is non uniform as it depends on porosity. Hence the velocity of the flow field transporting the contaminants is considered spatially dependent. Let velocity at the origin

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$x = 0$ of the domain be u_0 which increases to $u_0(1+b)$ at $x = L$ where a real constant $b<1$ ensures that the change in velocity is of small order i.e. the laminar condition of the flow is not affected (Kumar *et al.*, 2010) . The expression for velocity at any position x may be linearly interpolated as

$$u(x) = \frac{x-L}{0-L}u_0 + \frac{x-0}{L-0}u_0(1+b) \tag{3}$$

Simplifying we have

$$u(x) = u_0(1+ax) \tag{4}$$

where $a = \frac{b}{L}$ is a constant less than 1 and serves as a heterogeneity parameter. Its different values represent media of varying heterogeneity.

Since mechanical dispersion depends on the flow, it is expected to increase with increasing flow speed. The dispersion parameter is considered to be proportional to the square of the velocity.

$$D_L = D_0(1+ax)^2 \tag{5}$$

where $D_0 = \alpha u_0^2$ where α is known as longitudinal dispersivity.

Using equation (4) and equation (5) in equation (2), we obtain

$$\begin{aligned} &\frac{\partial C}{\partial t} + (u_0 - 2aD_0)(1+ax)\frac{\partial C}{\partial x} \\ &\quad + au_0 C = D_0(1+ax)^2\frac{\partial^2 C}{\partial x^2} \end{aligned} \tag{6}$$

which is linear advection-dispersion equation with variable coefficients. The solution $C(x,t)$ of this equation represents the concentration of contaminants at any position and at any time.

The boundary condition at the origin is of uniform nature as defined by many researchers. It means that the input concentration at the origin of the medium (water bodies on the surface, groundwater level) and hence its source of pollution on the earth surface remains uniform at all times. But this is not the real situation at all times. Actually, the pollution at the source and so the input will increase with time due to increasing human activities on the surface. So we have considered the inlet boundary condition as

$$C(0,t) = t \;,\; t > 0 \tag{7}$$

The concentration change is very negligible at the other end L=1 *km* of the aquifer. So the solute transport may not be affected at the other end of the aquifer. Hence we prescribe the outlet boundary condition as no flux boundary condition. i.e.

$$\frac{\partial C}{\partial x}(1,t) = 0 \;,\; t > 0 \tag{8}$$

We solve equation (6) with boundary conditions equation (7) and equation (8) using optimal homotopy analysis method.

3. Solution of the Problem Using OHAM

To solve the problem by OHAM, we choose the initial guess of the solution $C(x,t)$ as

$$C_0(x,t) = t(e^{-x} + xe^{-1}) \tag{9}$$

which satisfies boundary conditions equation (7) and equation (8). The linear operator is selected as

$$\mathrm{L}[\phi(x,t;q)] = \frac{\partial^2\phi(x,t;q)}{\partial x^2} \tag{10}$$

satisfying the property

$$\mathrm{L}[f] = 0 \quad when \quad f = 0. \tag{11}$$

where $0 \le q \le 1$ is the embedding parameter.

Further according to equation (6), the other operator is defined as

$$\begin{aligned} \mathrm{N}[\phi(x,t;q)] = {} & D_0(1+ax)^2\frac{\partial^2\phi(x,t;q)}{\partial x^2} - au_0\phi(x,t;q) \\ & -(u_0 - 2aD_0)(1+ax)\frac{\partial\phi(x,t;q)}{\partial x} \\ & -\frac{\partial\phi(x,t;q)}{\partial t} \end{aligned} \tag{12}$$

Let c_0 denote a nonzero auxiliary parameter. According to Liao[iii], the zeroth order deformation equation is

$$(1-q)\mathrm{L}[\phi(x,t;q) - C_0(x,t)] = c_0 qH(x,t)\mathrm{N}[\phi(x,t;q)] \tag{13}$$

where $H(x,t)$ is non-zero auxiliary function and $\phi(x,t;q)$ is an unknown function.

Clearly, when $q = 0$ and $q = 1$, we have from equation (11) and equation (13),

$$\phi(x,t;0) = C_0(x,t) \tag{14}$$

and $\phi(x,t;1) = C(x,t)$ (15)

Hence as q increases from *0* to *1*, the solution $\phi(x,t;q)$ deforms from the initial choice $C_0(x,t)$ to the exact solution $C(x,t)$ of equation (6). The determination of $\phi(x,t;q)$ depends on the proper choices of the operator ***L***, the initial choice $C_0(x,t)$ and the parameter c_0. We assume that all of them are properly chosen, the Maclaurin series

$$\phi(x,t;q) = C_0(x,t) + \sum_{m=1}^{\infty} C_m(x,t)q^m \tag{16}$$

exists and converges at $q = 1$. Hence we have the homotopy series solution

$$C(x,t) = C_0(x,t) + \sum_{m=1}^{\infty} C_m(x,t) \tag{17}$$

where $C_m(x,t) = \dfrac{1}{m!}\dfrac{\partial^m \phi(x,t;q)}{\partial q^m}\Big|_{q=0}$ (18)

Differentiating equation (13) m times with respect to q and then dividing them by m! and then taking $q = 0$, we obtain high order deformation equations

$$\mathrm{L}[C_m(x,t) - \chi_m C_{m-1}(x,t)] = c_0 H(x,t) R_m(x,t) \tag{19}$$

subject to the boundary conditions

$$C_m(0,t) = 0, \quad \frac{\partial C_m}{\partial x}(1,t) = 0, \quad m \geq 1 \tag{20}$$

where

$$\begin{aligned} R_m(x,t) &= \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathrm{N}[\phi(x,t;q)]}{\partial q^{m-1}}\Big|_{q=0} \\ &= D_0(1+ax)^2 \frac{\partial^2 C_{m-1}}{\partial x^2} \\ &-(u_0 - 2aD_0)(1+ax)\frac{\partial C_{m-1}}{\partial x} \\ &-au_0 C_{m-1} - \frac{\partial C_{m-1}}{\partial t} \end{aligned} \tag{21}$$

and $\chi_m = \begin{cases} 0 & \text{if } m \leq 1, \\ 1 & \text{if } m > 1. \end{cases}$ (22)

We assume $H(x,t) = 1$ for simplicity. Equations (19) are nonhomogeneous linear ordinary differential equations with constant coefficients for all $m \geq 1$. Solving equations (19)-(20) for $m = 1$, we obtain the first order homotopy approximation as

$$C_1(x,t) = c_0 \left[\begin{array}{l} a^2 D_0 t x^2 e^{-x} + \left(2a^2 D_0 + 2aD_0 + au_0\right) txe^{-x} + \left(u_0 + 2a^2 D_0 + 2aD_0 + D_0 + au_0\right) te^{-x} \\ -e^{-x} - \dfrac{x^3 e^{-1}}{6} + \left(aD_0 e^{-1} - \dfrac{u_0 e^{-1}}{2}\right) tx^2 + \left(\dfrac{a^2 D_0 e^{-1}}{3} - \dfrac{au_0 e^{-1}}{3}\right) tx^3 - \dfrac{xe^{-1}}{2} + 1 \\ + \left(D_0 e^{-1} + 2u_0 e^{-1} + 2au_0 e^{-1}\right) tx - \left(2a^2 D_0 + 2aD_0 + D_0 + u_0 + au_0\right) t \end{array} \right] \tag{23}$$

Similarly the *mth* order homotopy approximation $C_m(x,t)$ can be obtained for successive values of *m*. The convergence of the *mth* order homotopy approximation depends on the proper choice of c_0. The homotopy series solution can then be written as

$$C(x,t) = C_0(x,t) + C_1(x,t) + \ldots \tag{24}$$

The solution represents solute or contaminant concentration at distance *x* for any time *t* whose convergence depends on the proper choice of the convergence-control parameter c_0. To select proper value of c_0., we use the discrete squared residual at the *mth* order homotopy approximation denoted by E_m and defined by (Liao *et al.* 2010) as

$$E_m = \frac{1}{(M+1)(N+1)} \sum_{i=0}^{M}\sum_{j=0}^{N} \left\{ \mathrm{N}\left[\sum_{n=0}^{m} C_m\left(\frac{i}{M}, \frac{j}{N}\right) \right] \right\}^2 \tag{25}$$

As fast the residual E_m decreases, the accuracy of the corresponding homotopy approximation increases. At the *m*th order approximation, the corresponding optimal value of the convergence-control parameter c_0 is given by the minimum of E_m corresponding to a nonlinear algebraic equation to be solved from

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$$\frac{dE_m(c_0)}{dc_0} = 0 \tag{26}$$

This approach for obtaining the optimal value of c_0 has been used for solving a number of problems for nonlinear ordinary and partial differential equations by many researchers (Liao, 2010, Prajapati *et al.* 2015, 2016, 2017, Liao *et al.* 2012, 2013, vejravelu *et al.*, 2012). We have obtained the solution up to 10th order approximation. The optimal value of c_0 is found by the minimum of E_{10} using Mathematica. Here E_{10} attains its minimum value 5.23828×10^{-7} at $c_0 = -2.42025$ which we can observe in Figure 1 also. We take $M = N = 50$ for finding E_{10}.

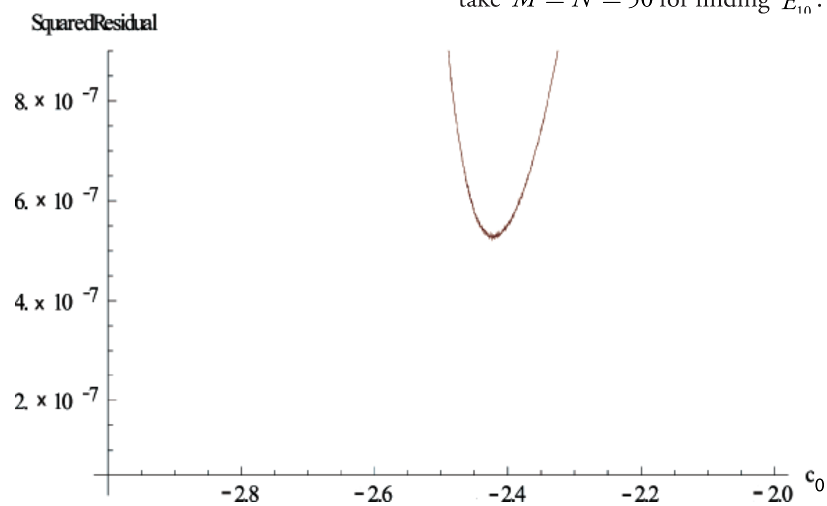

Fig 1 The discrete squared residual at 10th order homotopy approximation

Table 1. The discrete squared residual of governing equation (6) by means of $c_0 = -2.42025$.

Order of approximation *m*	Discrete Squared Residual E_m
2	1.24554E-1
4	3.53453E-3
6	2.38709E-4
8	9.68075E-6
10	4.7776E-7

From the Table 1, we can see that the squared residual decreases as we increase the order of approximation. So the 10th order homotopy approximation is accurate.

4. Numerical and Graphical Representation

BVPh, a Mathematica package, is used to obtain numerical values of concentration. Table 2 indicates the numerical values of the solute concentration

$$C(x,t)\left[\frac{kg}{(km)^3}\right]$$

for different distance *x (km)* and time *t (year)* up to 10th order approximation using $c_0 = -2.42025$.

We have considered

$$L = 1(km),\ a = 0.1(km)^{-1},\ u_0 = 0.11\left(\frac{km}{yr}\right) \text{ and}$$
$$D_0 = 0.21\left[\frac{(km)^2}{yr}\right]$$

to obtain numerical values of the solution.

Figure 2 shows the concentration profiles of the contaminant for fixed values of times . Numerical values of Table 2 are used for Figure 2. Figure 3 shows the graph of concentration for fixed positions. We use numerical values of Table 2 for Figure 3 also.

Table 2. Numerical values of the solute concentration *C*(*x, t*).

x	t=1	t=1.1	t =1.2	t =1.3	t =1.4	t=1.5	t =1.6	t =1.7
0.0	1.00000	1.10000	1.20000	1.30000	1.40000	1.50000	1.60000	1.70000
0.01	0.96338	1.06334	1.16330	1.26326	1.36322	1.46318	1.56314	1.66310
0.02	0.92713	1.02705	1.12697	1.22689	1.32681	1.42673	1.52665	1.62657
0.03	0.89124	0.99112	1.09100	1.19088	1.29076	1.39064	1.49052	1.59040

0.04	0.85571	0.95555	1.05539	1.15523	1.25507	1.35491	1.45475	1.55459
0.05	0.82054	0.92034	1.02014	1.11995	1.21975	1.31955	1.41935	1.51915
0.06	0.78574	0.88550	0.98526	1.08503	1.18479	1.28455	1.38432	1.48408
0.07	0.75129	0.85102	0.95074	1.05047	1.15019	1.24992	1.34965	1.44937
0.08	0.71721	0.81690	0.91659	1.01628	1.11596	1.21565	1.31534	1.41503
0.09	0.68349	0.78314	0.88280	0.98244	1.08210	1.18175	1.28140	1.38105
0.10	0.65014	0.74975	0.84936	0.94898	1.04859	1.14821	1.24782	1.34743

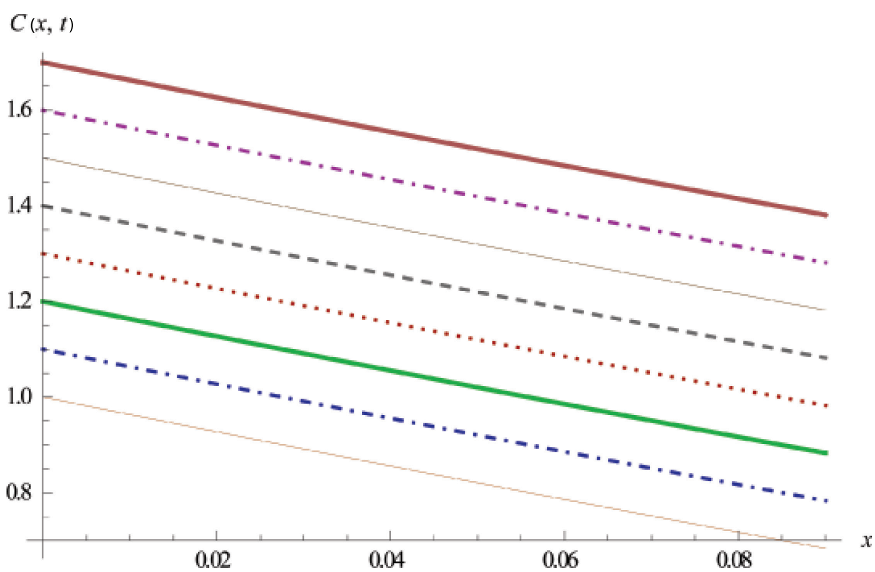

Fig 2 The spatial distribution of solute concentration for fixed values of times

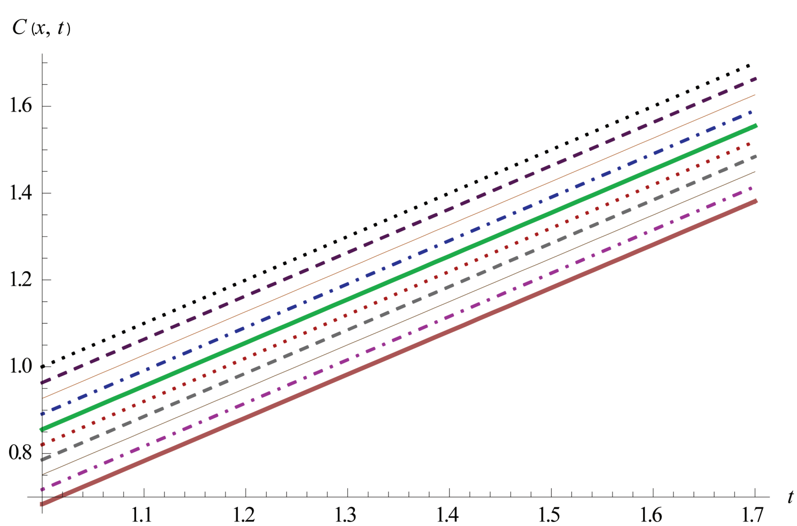

Fig 3 The temporal distribution of solute concentration for fixed values of distances

5. Conclusion

The approximate analytical solution of advection-dispersion equation with variable coefficients is obtained by optimal homotopy analysis method with time-dependent input source concentration. The contaminant transport behaves as expected i.e. the concentration at a fixed time decreases as the distance increases and the concentration at a fixed position advances with time. The solution is useful as a preliminary predictive tool for simulating the solute migration in aquifer due to the release of a time-dependent source.

References

[1] Baxter M., Van Gorder R. A., Vajravelu K (2014). On the choice of auxiliary linear operator in the optimal homotopy analysis of the Cahn-Hilliard initial value problem, Numer Algor, **66**(2) 269–298.

[2] Das P., Begam S., Singh M. K. (2017). Mathematical modeling of groundwater contamination with varying velocity field, *Journal of Hydrology and Hydromechanics*, **65**(2) 192–204.

[3] Kumar A., Jaiswal D. K., Kumar A. (2010). Analytical solutions to one-dimensional advection diffusion equation with variable coefficients in semi-infinite media, *Journal of Hydrology*, **380**, 330–337.

[4] Liao S. J. (2010). An optimal homotopy analysis approach for strongly nonlinear differential equations, Commun. Nonlinear Sci. Numer. Simul., **15**, 2003–2016.

[5] Marino M. A. (1974). Distribution of contaminants in porous media flow, *Water Resources Research*, **10**(5).

[6] Mazaheri M., Samani J. M. V., Samani H. M. V. (2013). Analytical solution to one-dimensional advection-diffusion equation with several point sources through arbitrary time-dependent emission rate patterns, J. Agr. Sci. Tech., **15**, 1231–1245.

[7] Prajapati D. J., Desai N. B. (2015). The solution of immiscible fluid flow by means of optimal homotopy analysis method, *International Journal of Computer and Mathematical Sciences*, **4**(8).

[8] Prajapati D. J., Desai N. B. (2016). Application of the basic optimal homotopy analysis method to fingering phenomenon, *Global Journal of Pure and Applied Mathematics*, **12**(3), 2011–2022.

[9] Prajapati D. J., Desai N. B. (2017). Optimal homotopy analysis solution of fingero-imbibition phenomenon in homogeneous porous medium with magnetic fuid effect, *Kalpa Publications in Computing*, **2**, 85–94.

[10] Prajapati D. J., Desai N. B. A. (2017). pproximate analytical solution for the fingero-imbibition phenomenon by optimal homotopy analysis method, *International Journal of Computational and Applied Mathematics*, **12**(3), 751–761.

[11] Prajapati D. J., Desai N. B. (2017). Analytic analysis for oil Recovery during cocurrent imbibition in inclined homogeneous porous medium, *International Journal on Recent and Innovation Trends in Computing and Communication*, **5**(7), 189–194.

[12] Prajapati D. J., Desai N. B. (2017). Mathematical investigation of counter-current imbibition phenomenon in heterogeneous porous medium, PRAJÑĀ - *Journal of Pure and Applied Sciences*, 24-2**5** 38–46.

[13] Singh M. K., Singh P., Singh V. P. (2010). Analytical solution for solute transport along and against time dependent source concentration in homogeneous finite aquifers, *Advances in Theoretical and Applied Mechanics*, **3**(3), 99–119.

[14] Singh M. K., Mahto N. K., Singh P. (2008). Longitudinal dispersion with time-dependent source concentration in semi-infinite aquifer, *Journal of Earth System Science*, **117**(6) 945–949.

[15] Scheidegger A. E. (1961). General theory of dispersion in porous media, *Journal of Geophysical Research*, **66**(10) 3273–3278.

[16] Xu M., Eckstein Y. (1997). Statistical analysis of the relationships between dispersivity and other physical properties of porous media. *Hydrogeology Journal*, **5(4)** 4–20.

Books

i. Bear, J. (1972). Dynamics of fluids in porous media, American Elsevier Publishing Company, Inc., New York.

ii. Batu, V. (2006). Applied Flow and Solute Transport Modeling in Aquifers: Fundamental Principles and Analytical and Numerical Methods., CRC, Boca Raton, FL.

iii. Liao, S. J. (2012). Homotopy analysis method in nonlinear differential equations, Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg,

iv. Liao, S. J. (2013). Advances in the homotopy analysis method, World Scientific,

v. Vajravelu, K.; Van Gorder, R. A. (2012). Nonlinear flow phenomena and homotopy analysis: Fluid flow and Heat transfer, Higher Education Press, Beijing and Springer-Verlag Berlin Heidelberg.