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EXISTENCE RESULTS FOR SINGULAR CONVECTIVE ELLIPTIC PROBLEMS

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"Parler de ce qu'on ignore
finit par vous l'apprendre."
(A. Camus, *Les carnets*)

Contents

1 A historical introduction

One of the most common problems in the area of Partial Differential Equations is the following Dirichlet problem:

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

being $\Omega \subseteq \mathbb{R}^N$ a bounded domain and $f : (0, +\infty) \to [0, +\infty)$ a continuous function. This problem is said to be *singular* if f blows up somewhere, for instance at the origin:

$$\lim_{s \to 0^+} f(s) = +\infty.$$

Placing the singularity at the origin is not a random choice: indeed, this hypothesis forces $f \circ u$ to blow up near $\partial\Omega$, for any u solution to the problem.

Singular problems arise in the study of a plethora of physical matters: non-Newtonian fluids, boundary-layer phenomena for viscous fluids, chemical heterogeneous catalysts, theory of heat conduction in electric conducting materials, biological pattern formation by auto- and cross-catalysis, morphogenesis, cellular differentiation, communication, etc.; see [24, 25, 142, 155, 156, 35, 48, 161, 80, 159, 130] and the references therein.

From the mathematical point of view, the presence of a singularity prevents to treat the problem with variational methods, since it is not possible to extend f to a continuous function on the whole $\mathbb{R}$. Over more than sixty years, many valuable mathematicians attacked various kinds of singular problems, obtaining several outstanding results which lead the topic to assume a central role in the field of Partial Differential Equations; a quick search on MathSciNet reveals that there are more than seven thousand research articles on this subject.

Since it is impossible to expose all the relevant contributions about singular problems, in this introduction we will sketch only some of the main questions analyzed in the thesis; in the meantime, we will try to give a sufficient number of references, in order to guide the interested reader who wants to trace back the most important historical steps. We will restrict to elliptic problems, both equations and systems, keeping our attention mainly on two topics:

- problems in unbounded domains (especially in the whole $\mathbb{R}^N$);
- presence of convection terms (i.e., terms depending on the gradient of solution).

1.1 Singular equations

A pioneer work on the subject, published in 1987, is due to Crandall, Rabinowitz, and Tartar [51]: they investigated a problem whose prototype is

$$\begin{cases} -\Delta u = u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $\gamma > 0$. A classical solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ has been obtained via shifting method. Also, a priori estimates and regularity results have been established, as $u \in C^{0,\frac{2}{\gamma+1}}(\overline{\Omega})$ provided $\gamma > 1$ (which is called *strongly singular* case). An antecedent is represented by [154]. Coclite and Palmieri [47] partially generalized some results of [51] in the case that also a p-linear term appears on the right-hand side. More recently, in 2007, Giacomoni, Schindler, and Takáč [79] investigated existence, multiplicity, and regularity of solutions for singular problems exhibiting reaction terms with critical growth.

In their work [58], Diaz, Morel, and Oswald obtained existence and non-existence results for

$$\begin{cases} -\Delta u + u^{-\gamma} = g(x) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u^{-\gamma} \in L^1(\Omega), \end{cases} \tag{1.1.1}$$

being $g \in L^1(\Omega)$, $g \geq 0$ in Ω. They emphasized the role of variational techniques in this context and also delineated the connection between the solutions of (1.1.1) and φ_1, the first eigenfunction of $(-\Delta, W_0^{1,2}(\Omega))$. The latter consideration will turn out to be very useful in the analysis of behavior of solutions to (1.1.1) near the boundary: indeed, few years later, Lazer and McKenna showed in [106] that any solution u of

$$\begin{cases} -\Delta u = a(x)u^{-\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1.2}$$

with $a \in C^{0,\alpha}(\overline{\Omega})$, $a > 0$ in $\overline{\Omega}$, and $\gamma > 1$, behaves like $\varphi_1^{\frac{2}{\gamma+1}}$. Moreover, existence of a solution $u \in C^{2,\alpha}(\Omega) \cap C^0(\overline{\Omega})$ in the general case $\gamma > 0$ has been established; additionally,

$$0 < \gamma < 3 \Leftrightarrow u \in W^{1,2}(\Omega);$$
$$\gamma > 1 \Rightarrow u \notin C^1(\overline{\Omega}).$$

Global continuity of gradient of solutions was treated by Gomes in [83]; via fixed-point theory, a solution to (1.1.2) lying in $C^2(\Omega) \cap C^1(\overline{\Omega})$ has been found, under the condition

$$d^{-\theta} a \in L^\infty(\Omega) \quad \text{for some } \ \theta \in [0,1), \tag{1.1.3}$$

where $d(x) := \operatorname{dist}(x, \partial\Omega)$ is the distance function. On the importance of the distance function in singular problems, see also [40, p.306]. More general gradient estimates for singular problems have been provided by Del Pino (vide [57]).
The most part of the tools used in the earliest papers, as sub-super-solution and shifting techniques or variational and topological methods, are sufficiently powerful to be applied in more general settings, such as the quasilinear framework. Thus, since 2000, several articles are devoted to p-Laplacian, as [140], or more general operators (see the very recent [137]). On the other hand, the empowerment of nonlinear analysis offers a number of versatile tools, allowing to handle, e.g., p-super-linear reaction terms [100, 136] or measure sources [11, 131, 132]. To conclude, we point out [28], which furnishes a uniqueness result.

Alongside the development of analysis of singular equations in bounded domains, also problems in unbounded domains attracted the attention of analysts. In 1984, Kusano and Swanson (see [102]) studied the problem

$$\begin{cases} -\Delta u = a(x) u^{-\gamma} & \text{in } \mathbb{R}^N, \\ \quad u > 0 & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to +\infty, \end{cases} \tag{1.1.4}$$

with $N \geq 3$, $a \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^N)$, $a > 0$, and $\gamma \in (0,1)$. They proved existence of a solution $u \in C^{2,\alpha}_{\text{loc}}(\mathbb{R}^N)$ under the conditions

$$\inf_{t>0} \frac{\varphi(t)}{\psi(t)} > 0, \qquad \int_1^{+\infty} t^{N-1+\gamma(N-2)} \psi(t) \, \mathrm{d}t < +\infty, \tag{1.1.5}$$

where

$$\varphi(t) := \min_{|x|=t} a(x) \quad \text{and} \quad \psi(t) := \max_{|x|=t} a(x). \tag{1.1.6}$$

They also proved that the found solution decays at infinity like $|x|^{2-N}$. The argument used in proving existence is based on the sub-super-solution technique; sub- and super-solutions are radial functions obtained by solving ordinary differential equations. The result has been extended to strongly singular problems independently by Dalmasso [53] and Shaker [149]. A similar result is provided in [64] via fixed point approach.

Optimality of conditions (1.1.5) has been investigated in 1996 by Lair and Shaker [104]. Indeed, a solution to (1.1.4) can be found, for any $\gamma > 0$, by supposing

$$\int_0^{+\infty} t\psi(t)\,\mathrm{d}t < +\infty. \tag{1.1.7}$$

Moreover, if a is radial and (1.1.7) is false, then no solutions to (1.1.4) exist. Lair and Shaker actually achieved a nearly optimal condition; indeed, later, it has been proved the following general result (cf. [77, Theorem 4.7.1]):

Theorem 1.1.1. *Let φ, ψ as in* (1.1.6)*. The following properties are valid:*

- *if $\int_0^{+\infty} t\psi(t)\,\mathrm{d}t < +\infty$ then there exists a unique classical solution to* (1.1.4)*;*
- *if $\int_0^{+\infty} t\varphi(t)\,\mathrm{d}t = +\infty$ then there are no classical solutions to* (1.1.4)*.*

The introduction of the nearly-optimal condition (1.1.7) stimulated the study of more general reaction terms. Hence, in 1997, Lair and Shaker [105] (see also [162]) studied the following problem:

$$\begin{cases} -\Delta u = a(x)f(u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to +\infty, \end{cases} \tag{1.1.8}$$

with $f \in C^1(0, +\infty)$ satisfying the following conditions:

- $$f(s) > 0 \quad \forall s \in (0, +\infty),$$
- $$f'(s) < 0 \quad \forall s \in (0, +\infty),$$
- $$\lim_{s \to 0^+} f(s) = +\infty.$$

A classical solution to (1.1.8) is obtained provided a is a non-trivial, non-negative, continuous function satisfying (1.1.7); so, a is not required to be neither locally Hölder continuous nor strictly positive in $\mathbb{R}^N$. It is worth noticing that Theorem 1.1.1 still holds for (1.1.8). Some generalizations, pertaining both f and a, can be read in [46, 86, 84].
Existence of weak solutions to

$$\begin{cases} -\Delta u = a(x)u^{-\gamma} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1.9}$$

is provided by Chabrowski and König [38]; the solution u they found belongs to $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^N)$ and is obtained as limit of solutions u_n to Dirichlet problems on bounded domains Ω_n exhausting $\mathbb{R}^N$, that is, $\Omega_n \nearrow \mathbb{R}^N$. Moreover, the following summability results hold true:

$$\begin{aligned} 0 < \gamma \leq 1 &\Rightarrow u \in \mathcal{D}_0^{1,2}(\mathbb{R}^N); \\ \gamma > 1 &\Rightarrow u^{\frac{\gamma+1}{2}} \in \mathcal{D}_0^{1,2}(\mathbb{R}^N), \end{aligned}$$

where $\mathcal{D}_0^{1,2}(\mathbb{R}^N)$ denotes a Beppo Levi space (see paragraph 2.1). More general right-hand sides are treated in [37]. For further existence, uniqueness, and regularity results to (1.1.9), see also [1].

In the context of quasi-linear equations, one of the first contributions is given by Gonçalves and Santos in 2004 (see [85]): they considered

$$\begin{cases} -\Delta_p u = a(x)f(u) & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \\ u(x) \to 0 & \text{as } |x| \to +\infty, \end{cases} \tag{1.1.10}$$

assuming the following conditions:

$$\frac{f(s)}{s^{p-1}} \text{ is decreasing in } (0,+\infty), \quad \lim_{s\to 0^+} f(s) > 0, \quad \lim_{s\to+\infty} \frac{f(s)}{s^{p-1}} = 0,$$

$$a \text{ is radial, and } \quad \begin{cases} 0 < \displaystyle\int_1^{+\infty} t^{\frac{1}{p-1}} \psi(t)^{\frac{1}{p-1}} \, \mathrm{d}t < +\infty & \text{if } 1 < p \leq 2, \\ 0 < \displaystyle\int_1^{+\infty} t^{\frac{(p-2)N+1}{p-1}} \psi(t) \, \mathrm{d}t < +\infty & \text{if } p \geq 2, \end{cases}$$

being ψ as in (1.1.6). They proved that, under these assumptions, a radial solution $u \in (C^2(\mathbb{R}^N) \setminus \{0\}) \cap C^1(\mathbb{R}^N)$ to (1.1.10) exists if and only if $p < N$. Under the same assumptions, Covei [50] studied the case of a non-radial $a \in C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$: if $p < N$, then there exists a solution $u \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ to (1.1.10). Few years later, requiring neither summability nor radiality assumptions on a, Carl and Perera [34] found a distributional solution $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ to

$$\begin{cases} -\Delta_p u = a(x)u^{-\gamma} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1.11}$$

with $\gamma \geq 1$. Actually, their result is more general, since some regular perturbations of the right-hand side are allowed. Several generalizations to (1.1.10), such as adding p-super-linear terms [146], considering concave-convex nonlinearities [144], or working in exterior domains [62, 39], are nowadays subject of study.

Now we briefly recall some results about convective problems in bounded domains. In [73], Ghergu and Rădulescu studied the following Dirichlet problem:

$$\begin{cases} -\Delta u = f(u) + \lambda |\nabla u|^2 + \mu & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1.12}$$

with Ω being a bounded domain, $\lambda, \mu \geq 0$, and $f \in C^{0,\alpha}(0, +\infty)$ satisfying the following hypotheses:

- $$f(s) > 0 \quad \forall s \in (0, +\infty),$$
- $$f \text{ is decreasing in } (0, +\infty),$$
- $$\lim_{s \to 0^+} f(s) = +\infty.$$

Notice that the quadratic growth on the gradient is a natural requirement in order to apply the maximum principle; see [74, p.62]. Set

$$a := \lim_{s \to +\infty} f(s), \quad b_\gamma := \limsup_{s \to 0^+} s^\gamma f(s), \quad \text{and} \quad \lambda^*(\mu) = \frac{\lambda_1}{a + \mu},$$

where λ_1 is the first eigenvalue of $(-\Delta, W_0^{1,2}(\Omega))$ and $\gamma \in (0, 1)$. It has been proved that problem (1.1.12) admits a classical solution u if and only if $\lambda < \lambda^*$. Moreover, if $b_\gamma < +\infty$ for some $\gamma \in (0, 1)$, then $u \in C^2(\Omega) \cap C^{1,1-\gamma}(\overline{\Omega})$ and behaves like the distance function. Properties of the sequence $\{u_\lambda\}$, for $\lambda \nearrow \lambda^*$, are also investigated. The same authors proved various existence and non-existence results for the more general problem

$$\begin{cases} -\Delta u = f(u) + \lambda |\nabla u|^2 + \mu g(x, u) & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1.13}$$

highlighting the influence of convection terms with respect to existence of solutions; see [74]. Concerning the interesting topic of convective problems in unbounded domains we address, for instance, to [76, 87] and the references therein.

Existence of weak solutions to semi-linear convective problems in bounded domains, possibly including source terms with low summability, has been investigated, e.g., in [9, 5]. Just to give an idea about the typology of results, we report [5, Theorem 1.5].

Theorem 1.1.2. *Let $f \in L^q(\Omega)$, $q > \frac{N}{2}$, such that* $\operatorname{ess\,inf}_K f > 0$ *for all $K \Subset \Omega$. Then problem*

$$\begin{cases} -\Delta u = \dfrac{|\nabla u|^2}{u^\gamma} + f & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a weak solution $u \in W_0^{1,2}(\Omega)$ if and only if $\gamma < 2$.

During the last years, convective problems have been analyzed also in the quasi-linear setting. Moving from [68], concerning a semi-linear convective equation, in 2019 Liu, Motreanu, and Zeng studied the following problem (vide [113]):

$$\begin{cases} -\Delta_p u = f(x,u) + g(x,u,\nabla u) & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1.14}$$

Here $f : \Omega\times(0,+\infty) \to (0,+\infty)$, $g : \Omega\times\mathbb{R}\times\mathbb{R}^N \to (0,+\infty)$ are Carathéodory functions and f is weakly singular. Exploiting variational, topological, and set-valued methods, besides the sub-super-solution technique, they proved existence of a weak solution to (1.1.14), provided the reaction term is p-sub-linear in both u and ∇u, as well as f satisfies suitable monotonicity and summability assumptions near the origin. Also p-linear growths are allowed, provided a certain geometric condition (involving the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$) is fulfilled.
Very recently, Papageorgiou and Zhang proved existence of a regular solution for a problem similar to (1.1.14), but with a non-homogeneous principal operator, patterned after the (p,q)-Laplacian (i.e., $\Delta_p + \Delta_q$ with $1 < q < p < +\infty$); see [138].

1.2 Singular systems

Since 2000, singular elliptic systems have been systematically studied, looking for steady-state solutions to the Gierer-Meinhardt system, which arises from a biological model built three decades earlier (see [80]). A comprehensive study of elliptic (generalized) Gierer-Meinhardt systems was initiated by Choi and McKenna. In [41] the authors consider the Dirichlet problem

$$\begin{cases} -\Delta u + u = \dfrac{u}{v} & \text{in } \Omega, \\ -\Delta v + \alpha v = \dfrac{u}{v} & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.1}$$

being $\Omega \subseteq \mathbb{R}^N$, $N \geq 1$, a domain with smooth boundary. Existence of a solution $(u, v) \in (C^2(\Omega) \cap C^1(\overline{\Omega}))^2$ is proved; also, uniqueness of solution is shown, provided $\alpha = 1$ or $N = 1$. The proof of existence basically relies on a priori estimates and Schauder's fixed point theorem, but it differs according to $\alpha > 1$, $\alpha = 1$, or $\alpha < 1$. Then the same authors analyzed a slightly different problem (see [42]), namely,

$$\begin{cases} -\Delta u + u = \dfrac{u^r}{v} & \text{in } \Omega, \\ -\Delta v + v = u^r & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.2}$$

with $r > 1$. They proved existence of classical solutions in the following cases: (i) $N = 1$ and $\Omega = (0, 1)$; (ii) $N = 2$ and Ω is the unit ball centered at the origin. In case (ii), also radial symmetry is guaranteed. In analogy to (1.2.2), we mention [12].

A list of open questions, appearing at the end of [42], led various scholars to a deeper investigation of the problem. In 2007, existence and non-existence theorems have been furnished by Ghergu and Rădulescu in [75], which concerns the following generalization of (1.2.1):

$$\begin{cases} -\Delta u + \alpha u = \dfrac{f(u)}{g(v)} + \rho(x) & \text{in } \Omega, \\ -\Delta v + \beta v = \dfrac{h(u)}{k(v)} & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.3}$$

with $\alpha, \beta > 0$, $\rho \in C^{0,\gamma}(\Omega)$ being non-negative and non-trivial, and $f, g, h, k \in C^{0,\gamma}(\Omega)$ being non-negative, non-decreasing, and such that $g(0) = k(0) = 0$. The next year, Hernández, Mancebo, and Vega published a sub-super-solution theorem; cf. [97]. Among its consequences, existence of Hölder continuous solutions for the system

$$\begin{cases} -\Delta u + \alpha u = \lambda u^p v^q & \text{in } \Omega, \\ -\Delta v + \beta v = \mu u^r v^s & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.4}$$

is provided under the conditions $\lambda, \mu > 0$ and $-1 < p+q, r+s < 1$. If, moreover, $q, r > 0$, then the solution is unique. Incidentally, the proof of uniqueness is based on a concavity argument. Actually, more general uniformly elliptic operators are encompassed in [97].
Monotonicity assumptions on f, g, h, k in (1.2.3), as well as the sign conditions on p, q, r, s in (1.2.4), suggested to distinguish the cases of *cooperative* and *competitive* systems. Generally speaking, a system of the form

$$\begin{cases} -\Delta_p u = f(u, v) & \text{in } \Omega, \\ -\Delta_q v = g(u, v) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.5}$$

is called cooperative (resp., competitive) if $f(r, \cdot)$ and $g(\cdot, s)$ are non-decreasing (resp., non-increasing). Quasi-linear systems of type (1.2.5) are treated by Motreanu and Moussaoui in [124, 125, 126]. More specifically, [124] deals with cooperative systems having pure power nonlinearities, i.e., $f(u, v) = u^{\alpha_1} v^{\beta_1}$ and $g(u, v) = u^{\alpha_2} v^{\beta_2}$; [125] is devoted to competitive systems; finally, [126] treats problems having neither cooperative nor competitive structure. All of these works exploit the sub-super-solution method; for a degree-theoretic approach to quasi-linear problems, we address to [98] and the references therein. We also point out [60], containing a multiplicity result.

Different forms of semi-linear Gierer-Meinhardt systems have been studied in the last two decades: here we mention [128], whose setting is $\mathbb{R}^N$, and [2], in which a Gierer-Meinhardt-type system perturbed with a convection term is investigated. Accordingly, in the quasi-linear framework, two paths are delineated: the analysis of singular systems in $\mathbb{R}^N$, and the study of convective singular systems in bounded domains. To the best of our knowledge, very few works have been written on these topics.

Regarding the first subject, the article [114], published in 2019, concerns the cooperative system

$$\begin{cases} -\Delta_p u = a(x)f(u,v) & \text{in } \mathbb{R}^N, \\ -\Delta_q v = b(x)g(u,v) & \text{in } \mathbb{R}^N, \\ \quad u,v > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2.6}$$

being $f,g : (0,+\infty)^2 \to (0,+\infty)$ two continuous functions satisfying the growth conditions

$$m_1 s^{\alpha_1} \leq f(s,t) \leq M_1 s^{\alpha_1}(1+t^{\beta_1}), \quad -1 < \alpha_1 < 0 < \beta_1, \quad m_1, M_1 > 0,$$
$$m_2 t^{\beta_2} \leq g(s,t) \leq M_2(1+s^{\alpha_2})t^{\beta_2}, \quad -1 < \beta_2 < 0 < \alpha_2, \quad m_2, M_2 > 0.$$

Moreover, the following *mixed conditions* (i.e., regarding both equations of (1.2.6) at the same time) are imposed:

$$\alpha_1 + \alpha_2 < p - 1 \quad \text{and} \quad \beta_1 + \beta_2 < q - 1.$$

Existence of a solution to (1.2.6) is guaranteed provided suitable summability assumptions on the weights a, b are fulfilled. We highlight that weights play a fundamental role for problems in unbounded domains, since they ensure a good summability of reaction terms, which implies a gain of regularity. Another investigation of singular systems in unbounded domains is represented by [147].

On the other hand, singular systems exhibiting convection terms have been studied in the recent paper [27], which is devoted to the following cooperative system:

$$\begin{cases} -\Delta_p u = f(x,u,v,\nabla u,\nabla v) & \text{in } \Omega, \\ -\Delta_q v = g(x,u,v,\nabla u,\nabla v) & \text{in } \Omega, \\ \quad u,v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.7}$$

where $f,g : \Omega \times (0,+\infty)^2 \times \mathbb{R}^{2N} \to (0,+\infty)$ are two Carathéodory functions subjected to the growth conditions

$$m_1 s^{\alpha_1} t^{\beta_1} \leq f(x,s,t,\xi_1,\xi_2) \leq M_1 s^{\alpha_1} t^{\beta_1} + |\nabla \xi_1|^{\gamma_1} + |\nabla \xi_2|^{\delta_1}, \quad m_1, M_1 > 0,$$
$$m_2 s^{\alpha_2} t^{\beta_2} \leq g(x,s,t,\xi_1,\xi_2) \leq M_2 s^{\alpha_2} t^{\beta_2} + |\nabla \xi_1|^{\gamma_2} + |\nabla \xi_2|^{\delta_2}, \quad m_2, M_2 > 0,$$

with

$$-1 < \alpha_1 < 0 < \beta_1, \gamma_1, \delta_1, \quad 0 \leq \alpha_1 + \beta_1 < \beta_1 - \alpha_1 < p - 1,$$
$$-1 < \beta_2 < 0 < \alpha_2, \gamma_2, \delta_2, \quad 0 \leq \alpha_2 + \beta_2 < \alpha_2 - \beta_2 < q - 1,$$
$$\gamma_1, \delta_1 < p - 1, \quad \gamma_2, \delta_2 < q - 1.$$

It is shown that there exists a solution to (1.2.7) that belongs to $C^1(\overline{\Omega})^2$ and whose components behave like the distance function. To complete the picture of singular convective systems in bounded domains, we mention the interesting work [127], published in 2017. It possesses two peculiarities: (i) its structure is, in general, neither cooperative nor competitive; (ii) it allows singularities on the gradient terms. More specifically, the problem under consideration is

$$\begin{cases} -\Delta_p u = f(u,v,\nabla u,\nabla v) & \text{in } \Omega, \\ -\Delta_q v = g(u,v,\nabla u,\nabla v) & \text{in } \Omega, \\ \quad u,v > 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.8}$$

with $f,g : (0,+\infty)^2 \times (\mathbb{R}^N \setminus \{0\})^2 \to (0,+\infty)$ continuous functions obeying the growth conditions

$$\begin{aligned} m_1(s+|\xi_1|)^{\alpha_1}(t+|\xi_2|)^{\beta_1} \le f(x,s,t,\xi_1,\xi_2) \le M_1(s+|\xi_1|)^{\alpha_1}(t+|\xi_2|)^{\beta_1}, \\ m_2(s+|\xi_1|)^{\alpha_2}(t+|\xi_2|)^{\beta_2} \le g(x,s,t,\xi_1,\xi_2) \le M_2(s+|\xi_1|)^{\alpha_2}(t+|\xi_2|)^{\beta_2}, \end{aligned}$$

being $m_1, M_1, m_2, M_2 > 0$ and $\alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that

$$\alpha_2, \beta_1 < 0, \quad \alpha_1\beta_2 > 0, \quad |\alpha_1| - \beta_1 < p-1, \quad |\beta_2| - \alpha_2 < q-1.$$

As observed above, all proofs of the aforementioned articles are based on sub- and super-solutions; the degree-theoretic counterpart is developed in [8], which deals with singular convective systems on bounded domains subjected to functional boundary conditions.

To the best of our knowledge, no articles about singular convective systems in the whole space are available in literature; in other words, singular problems exhibiting all the three features presented in this introduction seem to be not studied. We offer a result of this type in paragraph 4.2.2.

1.3 A brief guide for the reader

The aim of the present thesis is fourfold:

- furnishing a historical introduction to singular problems, highlighting their striking impact in Analysis, as well as focusing the mathematical issues and deals arising in this context. The 'state of art' presented in this chapter is clearly far from being complete, but it offers a large bibliography which could be used as a starting point for a deeper investigation.

- collecting, in a self-contained way, a wide class of tools commonly used for studying singular problems. Statements are written in their basic form, trying to retain a sufficiently high level of generality without entering into too technical details; the reader is addressed to authoritative references in order to find generalized versions, finer analyses, and further comments. Chapter 2 is devoted to this aspect.

- showing simple procedures to approach some basic singular problems, in order to show very important techniques, as sub-super-solution and shifting methods. Despite the arguments are simple, they allow to perform a complete analysis of particular classes of problems. This point of view is given in Chapter 3.

- offering some very recent research results about more sophisticated problems as, for instance, singular convective systems in the whole space. Proofs are given in symbiosis with both the tools shown in Chapter 2 and the techniques developed in Chapter 3, granting a harmonized aspect to the thesis. These results are contained in Chapter 4.

The thesis is divided in chapters, sections, and paragraphs; also enumeration of formulas and sub-paragraphs (including definitions, theorems, corollaries, etc.) follows this scheme. Notation is introduced in Section 2.1.
We highlight that elements of originality are contained in the proofs of Theorems 2.2.2, 2.2.11, 2.2.12, 2.2.19, 2.3.10, 2.5.10, as well as in the arguments employed in Remarks 2.2.57 and 2.4.1. Chapters 3 and 4 are entirely original.
Before going on, we warn the reader that, dealing with estimates, generic constants (often indicated by c) may change their value at each passage. The dependencies of such constants are explicitly indicated when any ambiguity could arise; some of these dependencies are marked as subscripts of c, in order to stress them.

2 Tools

2.1 The functional framework

The present section is devoted to briefly introduce the functional setting within we work, as well as to fix the notation used in the sequel. First, different function spaces are described; then some classical embedding theorems are stated, together with a useful compactness criterion. After recalling some information about dual spaces, two theorems (very useful in the context of singular problems) are presented, and various definitions of solution to an elliptic partial differential equation in divergence form are given, together with some related remarks.

2.1.1 Function spaces

For an organic exposition of the topics treated in this paragraph, we address to [101].

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a domain (i.e., an open, connected set). For any domain $\Omega' \subseteq \mathbb{R}^N$, we abbreviate $\overline{\Omega'} \subseteq \Omega$ with the symbol $\Omega' \Subset \Omega$. For any set E, we denote with χ_E the characteristic function of E.
We indicate with $C^k(\Omega)$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the set of functions possessing continuous derivatives up to k-th order in Ω, while $C^{k,\alpha}(\Omega)$, $k \in \mathbb{N}_0$ and $\alpha \in (0,1]$, contains the functions having Hölder-continuous derivatives (up to k-th order) with Hölder exponent α; when these spaces are endowed with the Fréchet topology of locally uniform convergence (up to derivatives of order k), we denote them with $C^k_{\rm loc}(\Omega)$ and $C^{k,\alpha}_{\rm loc}(\Omega)$, respectively.
We define $C^\infty(\Omega)$ as the set of the functions that are continuously differentiable infinitely many times. The k-th derivative of a function u will be indicated as $D^k u$; clearly, if u is a scalar function, then $D^k u$ can be represented at each point by a tensor of order k. If $k = 1$, we also use the symbol ∇ in place of D^1. If $U = U(x_1, \dots, x_N, y_1, \dots, y_M)$ is a differentiable function from $\mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^m$, we denote with $\nabla_x U : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^m \times \mathbb{R}^N$ (resp., $\nabla_y U : \mathbb{R}^N \times \mathbb{R}^M \to \mathbb{R}^m \times \mathbb{R}^M$) the gradient with respect to the x-variables (resp., y-variables).
With the notation

$$\|u\|_\infty := \sup_\Omega |u|, \quad [u]_\alpha := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha},$$

we also define, when Ω is bounded, the following spaces:

$$C^k(\overline{\Omega}) := \left\{ u \in C^k(\Omega) : \|u\|_{C^k} := \sum_{h=0}^{k} \|D^h u\|_\infty < +\infty \right\},$$
$$C^{k,\alpha}(\overline{\Omega}) := \left\{ u \in C^{k,\alpha}(\Omega) : \|u\|_{C^{k,\alpha}} := \sum_{h=0}^{k} \|D^h u\|_\infty + [D^k u]_\alpha < +\infty \right\},$$

with k, α as above.
A space of continuous functions with subscript c denotes its subspace consisting in the functions having compact support within the considered domain: for example,

$$C_c^\infty(\Omega) := \{u \in C^\infty(\Omega) : \operatorname{supp} u \subseteq \Omega, \operatorname{supp} u \text{ is bounded}\},$$

where $\operatorname{supp} u := \overline{\{x \in \Omega : u(x) \neq 0\}}$. A space of continuous functions with subscript 0 denotes that its functions vanish on the boundary: for instance,

$$C_0^{k,\alpha}(\overline{\Omega}) := \{u \in C^{k,\alpha}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}.$$

We will also make use of abstract function spaces: in this case, the image will be separated by the domain with a semi-colon; e.g.,

$$C^0([0,1]; X) := \{u : [0,1] \to X : u \text{ is continuous}\}.$$

Let us consider the following sets:

$$B := \{x = (x', x_N) \in \mathbb{R}^N : |x| < 1\},$$
$$B_+ := \{x = (x', x_N) \in \mathbb{R}^N : |x| < 1, \, 0 < x_N < 1\},$$
$$B_0 := \{x = (x', x_N) \in \mathbb{R}^N : |x| < 1, \, x_N = 0\}.$$

Ω is of class $C^{k,\alpha}$, $k \in \mathbb{N}_0$ and $\alpha \in (0,1]$, if for any $x \in \partial\Omega$ there exists a neighborhood U of x in $\mathbb{R}^N$ and a bijective map $H : B \to U$ (called *local chart*) such that

$$H \in C^{k,\alpha}(B), \quad H^{-1} \in C^{k,\alpha}(U), \quad H(B_+) = U \cap \Omega, \quad H(B_0) = U \cap \partial\Omega.$$

Accordingly, we adopt the following definition: if Ω is of class $C^{k,\alpha}$, a function $\psi : \partial\Omega \to \mathbb{R}$ belongs to $C^{k,\alpha}(\partial\Omega)$ if for any $x \in \partial\Omega$ the function $\psi_{|_{U\cap\partial\Omega}} \circ H_{|_{B_0}} : B_0 \to \mathbb{R}$ belongs to $C^{k,\alpha}(B_0)$, understanding B_0 as an open set of $\mathbb{R}^{N-1}$; cf. [81, p.94].

With $\mathcal{M}(\Omega)$ we mean the set of Lebesgue-measurable functions in Ω.
Given $m \in \mathbb{N}$, we say that $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a *Carathéodory function* if

$f(\cdot, s)$ is measurable for all $s \in \mathbb{R}^m$ and $f(x, \cdot)$ is continuous for almost every $x \in \Omega$. Dealing with Carathéodory functions, we use 'for all x' instead of 'for almost all x' to avoid unpleasant notation.

We define the Lebesgue space $L^p(\Omega)$, $p \in [1, +\infty)$, as

$$L^p(\Omega) := \left\{ u \in \mathcal{M}(\Omega) : \|u\|_p := \left(\int_\Omega |u(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} < +\infty \right\},$$

whose functions are considered modulo sets of measure zero (this identification holds true also for all the subsequent spaces). Using the notation of essential supremum $\|\cdot\|_\infty$, which is coherent with the one given for continuous functions, we introduce the space

$$L^\infty(\Omega) := \{u \in \mathcal{M}(\Omega) : \|u\|_\infty := \operatorname*{ess\,sup}_\Omega |u| < +\infty\}.$$

The Sobolev spaces $W^{k,p}(\Omega)$, $k \in \mathbb{N}_0$ and $p \in [1, +\infty)$ are defined as

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) : \|u\|_{k,p} := \sum_{h=0}^{k} \|D^h u\|_p < +\infty\}.$$

We also define the subspace $W^{k,p}_0(\Omega)$ as the completion of $C^\infty_c(\Omega)$ under the Sobolev norm $\|\cdot\|_{k,p}$. It can be proved that, if Ω is of class $C^{0,1}$, then $W^{k,p}_0(\Omega)$ consists of the functions whose derivatives up to order $k-1$ vanish on the boundary (in the sense of traces). When Ω is bounded, we endow $W^{k,p}_0(\Omega)$ with the equivalent norm $\|D^k u\|_p$ for any $u \in W^{k,p}_0(\Omega)$, stemming from Poincaré's inequality.
We make use of $L^p_{\text{loc}}(\Omega)$ and $W^{k,p}_{\text{loc}}(\Omega)$, being respectively the Lebesgue and Sobolev local spaces. A function $u : \Omega \to \mathbb{R}$ belongs to $L^p_{\text{loc}}(\Omega)$ (resp., $W^{k,p}_{\text{loc}}(\Omega)$) if $u_{|_K} \in L^p(K)$ (resp., $u_{|_K} \in W^{k,p}(K)$) for any $K \Subset \Omega$. These local spaces are endowed with the Fréchet topology induced by the semi-norms $\|\cdot\|_{L^p(K)}$ and $\|\cdot\|_{W^{k,p}(K)}$, with K varying among the sets such that $K \Subset \Omega$.
We also introduce a pivotal threshold, especially for Sobolev embeddings: the critical Sobolev exponent p^*_k, defined for any $p \in [1, +\infty)$ as

$$p^*_k := \begin{cases} \frac{Np}{N-kp}, & \text{if } kp < N, \\ \infty, & \text{if } kp \geq N. \end{cases}$$

When $k = 1$ we simply write p^*. In the sequel we will concentrate on the case $k = 1$ and $p < N$.

Solutions of elliptic equations in unbounded domains and their derivatives can have different behaviors at infinity (see, e.g., [71, p.80]), so it could occur that the summability of u differs from the one of ∇u (or with respect to derivatives of higher order). For this reason, in the case $kp < N$, we introduce the Beppo Levi spaces (also called homogeneous Sobolev spaces)

$$\mathcal{D}_0^{k,p}(\Omega) := \{u \in L^{p_k^*}(\Omega) : |D^k u| \in L^p(\Omega)\},$$

We equip $\mathcal{D}_0^{k,p}(\Omega)$ with the norm $\|D^k u\|_p$. The space $C_c^\infty(\Omega)$ is dense in $\mathcal{D}_0^{k,p}(\Omega)$ with respect to the norm just defined. Incidentally notice that, without any more stringent conditions, functions of $\mathcal{D}_0^{k,p}(\Omega)$ do not decay at infinity pointwise; however, see [71, Lemma II.6.3 and Theorem II.9.1] for integral and pointwise decay estimates, respectively.
An exhaustive introduction on Beppo Levi spaces may be found in [71, 151, 108].

2.1.2 Embedding theorems and dual spaces

We start by recalling some definitions regarding operators between Banach spaces (except monotonicity, which will be treated separately in Section 2.3; vide Definition 2.3.1). Given a sequence $\{u_n\}$ and a point u, we indicate with $u_n \to u$ the strong convergence of $\{u_n\}$ to u, while $u_n \rightharpoonup u$ stands for the weak convergence.

Definition 2.1.1. Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be two Banach spaces and $T : X \to Y$ a (possibly nonlinear) operator. T is said to be

- *bounded* if T maps bounded sets into bounded sets,
- *compact* if T maps bounded sets into relatively compact sets,
- *continuous* if $u_n \to u$ implies $T(u_n) \to T(u)$,
- *weakly continuous* if $u_n \rightharpoonup u$ implies $T(u_n) \rightharpoonup T(u)$,
- *strongly continuous* if $u_n \rightharpoonup u$ implies $T(u_n) \to T(u)$,
- *demi-continuous* if $u_n \to u$ implies $T(u_n) \rightharpoonup T(u)$.
- *completely continuous* if T is continuous and compact.

Remark 2.1.2. It is readily seen that strong continuity implies both continuity and weak continuity, and each of them in turn imply demi-continuity; moreover, compactness implies boundedness. Nevertheless, in more stringent hypotheses for X, Y, and T, we can say more:

- if X is reflexive then strong continuity implies complete continuity;

- if X is finite-dimensional then continuity is equivalent to strong continuity, and demi-continuity is equivalent to weak continuity;
- if T has finite rank then its weak continuity is equivalent to the strong one, and its demi-continuity is equivalent to continuity;
- if T is a linear operator then its boundedness is equivalent to continuity, and its complete continuity is equivalent to the strong one.

For further implications, we address to the survey [69]; see also [150, Chapter II].

Now let us discuss some basic facts regarding embeddings between the function spaces considered in paragraph 2.1.1. Retaining the notation of Definition 2.1.1, if $X \subseteq Y$ and $T : X \to Y$ is the identity operator then T is said to be an *embedding*. According to Remark 2.1.2, any compact embedding is a continuous embedding. In the sequel, we will write $X \hookrightarrow Y$ (resp., $X \overset{c}{\hookrightarrow} Y$) to signify that X is continuously (resp., compactly) embedded in Y.

Theorem 2.1.3 (Ascoli-Arzelà). *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain. Then, for any $k \in \mathbb{N}_0$ and $0 < \beta < \alpha \leq 1$,*

$$C^{k,\alpha}(\overline{\Omega}) \overset{c}{\hookrightarrow} C^{k,\beta}(\overline{\Omega}) \overset{c}{\hookrightarrow} C^{k}(\overline{\Omega}).$$

It is not true that in general $C^{k+1}(\overline{\Omega}) \hookrightarrow C^{k,\alpha}(\overline{\Omega})$, but this embedding holds true provided Ω satisfies the following geometric condition:

<u>(S)</u>: There exists $M > 0$ such that, for any $x, y \in \Omega$, there exists $\{z_i\}_{i=0}^{n}$ such that $z_0 = x$, $z_n = y$, as well as

$$[z_i, z_{i+1}] \subseteq \Omega \quad \forall\, i = 0, \dots, n-1, \quad \sum_{i=0}^{n-1} |z_{i+1} - z_i| \leq M\, |x - y|,$$

where $[a, b]$ denotes the segment joining a with b.

In other words, condition (S) requires that for any pair of points (x, y) there exists a polygonal whose length does not exceed a multiple, independent of x, y, of the distance between x and y. This condition is satisfied, for instance, by star-shaped domains; on the other hand, a plane domain lying between two infinite, disjoint spirals converging to the same point does not satisfy the condition; see [101, pp.23-25].

Theorem 2.1.4 (Rellich-Kondrachov)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain of class $C^{0,1}$, $k \in \mathbb{N}_0$, and $p \in [1, +\infty)$. Then:*

$$\begin{array}{lll} W^{k,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega) & \forall\, q \in [1, p_k^*), & \text{if } kp < N, \\ W^{k,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega) & \forall\, q \in [1, +\infty), & \text{if } kp = N, \\ W^{k,p}(\Omega) \overset{c}{\hookrightarrow} C^{m,\alpha}(\overline{\Omega}), & m = [k - N/p],\ \alpha = k - \frac{N}{p} - m, & \text{if } kp > N, \end{array}$$

being $[a]$ the integer part of $a \in (0, +\infty)$.

Further properties related to Theorem 2.1.4 and, more generally, to Sobolev spaces can be found, e.g., in [123, Chapter 1].
If Ω is unbounded, Theorem 2.1.4 is no longer valid: this phenomenon is usually called *lack of compactness for Sobolev embeddings*. The best we can expect in the case $kp < N$ is a Sobolev inequality, which furnishes a continuous embedding.

Theorem 2.1.5 (Sobolev)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be a domain of class $C^{0,1}$. Then, for any $k \in \mathbb{N}_0$ and $p \in [1, +\infty)$ such that $kp < N$,*

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [p, p_k^*].$$

Unlike Sobolev spaces, which compactly embed in Lebesgue spaces via Theorem 2.1.4, in Lebesgue spaces compactness of subsets is hard to deduce. Then the following characterization of relatively compact sets of $L^p(\Omega)$ can be a helpful tool. For a proof we refer to [101, Section 2.13]; see also [23, Corollary 4.27 and Remark 13] for an extension in domains having infinite measure. Given $h \in \mathbb{R}^N$ and $f : \Omega \to \mathbb{R}$, we define

$$\delta_h f(x) := f(x + h) - f(x) \quad \forall\, x \in \mathbb{R}^N,$$

with the convention $f(x + h) = 0$ whenever $x + h \notin \Omega$.

Theorem 2.1.6 (Fréchet-Riesz-Kolmogorov)**.** *Let $p \in [1, +\infty)$, $\Omega \subseteq \mathbb{R}^N$ be a domain with finite measure, and $F \subseteq L^p(\Omega)$. Then F is relatively compact if and only if it is bounded in $L^p(\Omega)$ and*

$$\lim_{|h| \to 0^+} \|\delta_h f\|_p = 0 \quad \textit{uniformly with respect to } \ f \in F.$$

We conclude this paragraph by recalling some duality properties. Given $p \in (1, +\infty)$, we define the conjugate exponent $p' := \frac{p}{p-1}$. The spaces $L^p(\Omega)$, $W^{k,p}(\Omega)$, $\mathcal{D}_0^{k,p}(\Omega)$ are reflexive for $p \in (1, +\infty)$. The dual spaces of $L^p(\Omega)$, $W_0^{k,p}(\Omega)$, and $\mathcal{D}_0^{k,p}(\Omega)$ are denoted (coherently to the notation introduced above) by $L^{p'}(\Omega)$, $W^{-1,p'}(\Omega)$, and $\mathcal{D}^{-k,p'}(\Omega)$, respectively.
Hereafter, with X^* we mean the topological dual of the space X; $\langle \cdot, \cdot \rangle$ represent the duality brackets. We also recall (cf. [99]) that if $X \hookrightarrow Y$ (resp., $X \overset{c}{\hookrightarrow} Y$) then $Y^* \hookrightarrow X^*$ (resp., $Y^* \overset{c}{\hookrightarrow} X^*$).

2.1.3 Auxiliary results

We begin this paragraph by recalling few definitions about multi-functions. We address to [31] for further details.
Given a set A, we define

$$2^A := \{B : B \subseteq A\}.$$

Given two sets A and B, a *multi-function* (or *set-valued map*) is a function $\mathscr{S} : A \to 2^B$. The *domain* of $\mathscr{S}$ is defined as

$$\operatorname{dom} \mathscr{S} := \{a \in A : \mathscr{S}(a) \neq \emptyset\}.$$

A function $T : A \to B$ is called *selection* of $\mathscr{S}$ if $T(a) \in \mathscr{S}(a)$ for all $a \in A$. Let (X, d_X) and (Y, d_Y) be two metric spaces. A multi-function $\mathscr{S} : X \to 2^Y$ is said to be *compact* if it maps bounded sets into relatively compact sets; it is *lower semi-continuous* if, for every $x_n \to x$ in X and $y \in \mathscr{S}(x)$, there exists $\{y_n\} \subseteq Y$ such that $y_n \to y$ in Y and $y_n \in \mathscr{S}(x_n)$ for all $n \in \mathbb{N}$.

Dealing with problems subjected to homogeneous Dirichlet boundary conditions and possessing a reaction term which is singular at the origin, it is crucial to understand the behavior of $W^{1,p}_0(\Omega)$ functions near the boundary. Given a domain $\Omega \neq \mathbb{R}^N$ of class $C^{0,1}$, we can define ($\mathscr{H}^{N-1}$-almost everywhere on $\partial\Omega$, being $\mathscr{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure) the outer normal $\nu : \partial\Omega \to \mathbb{S}^{N-1}$, where $\mathbb{S}^{N-1}$ is the $(N-1)$-dimensional sphere, according to Rademacher's theorem (see [67, Theorem 3.2]). If Ω is bounded, we define the distance function d as

$$d(x) := \operatorname{dist}(x, \partial\Omega) := \inf_{y \in \partial\Omega} |x - y| \quad \forall\, x \in \Omega.$$

For all $u \in C^1(\overline{\Omega})$, we consider the normal derivative $\partial_\nu u := \nabla u \cdot \nu$.

The following theorem (vide [133, Theorem 21.3]) furnishes an information, of integral type, for functions in $W^{1,p}_0(\Omega)$.

Theorem 2.1.7 (Hardy-Sobolev)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be a domain of class $C^{0,1}$, $p \in (1, N)$, $\theta \in [0, 1]$. Set $\frac{1}{r} = \frac{1}{p} - \frac{1-\theta}{N}$. Then there exists $C = C(\Omega, N, p, \theta) > 0$ such that*

$$\|d^{-\theta} u\|_r \leq C\, \|\nabla u\|_p \quad \forall\, u \in W^{1,p}_0(\Omega).$$

Regarding the pointwise behavior near the boundary, we state and prove the result below.

Theorem 2.1.8. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain of class C^2, and let $u \in C^{1,\alpha}_0(\overline{\Omega})$ with $\alpha \in (0, 1]$. Then $d^{-1}u \in C^{0,\beta}(\overline{\Omega})$ for $\beta = \frac{\alpha}{1+\alpha}$. More precisely, there exists $C = C(\Omega) > 0$ such that*

$$\|d^{-1}u\|_{C^{0,\beta}(\overline{\Omega})} \leq C\, \|u\|_{C^{1,\alpha}(\overline{\Omega})}.$$

Proof. Set $\Omega_\delta := \{x \in \Omega : d(x) < \delta\}$. Since Ω is of class C^2, there exists $\delta \in (0,1)$ and $\Pi \in C^1(\Omega_\delta; \partial\Omega)$ such that, for any $x \in \Omega_\delta$,

$$d(x) = |x - \Pi(x)|, \quad \frac{x - \Pi(x)}{|x - \Pi(x)|} = -\nu(\Pi(x)), \quad]\Pi(x), x[\subseteq \Omega. \tag{2.1.1}$$

The theorem is proved once we show that there exists $C = C(\Omega) > 0$ such that

$$\sup \left\{ \frac{\left| \frac{u(x)}{d(x)} - \frac{u(y)}{d(y)} \right|}{|x-y|^\beta} : x, y \in \Omega,\, 0 < |x-y| < \frac{\delta}{2} \right\} \le C \, \|u\|_{C^{1,\alpha}}. \tag{2.1.2}$$

So pick any $x, y \in \Omega$ such that $0 < |x-y| < \frac{\delta}{2}$. Without loss of generality, we can suppose $d(x) \ge d(y)$. Three cases can occur.
Case 1: $d(x) \ge \delta$. By the mean value theorem one has

$$\frac{\left| \frac{u(x)}{d(x)} - \frac{u(y)}{d(y)} \right|}{|x-y|^\beta} = \nabla \left(\frac{u(z)}{d(z)} \right) |x-y|^{1-\beta} \quad \text{for some } \; z \in]x, y[. \tag{2.1.3}$$

Observe that

$$d(z) \ge d(x) - |z - x| > d(x) - |x - y| > \delta - \frac{\delta}{2} = \frac{\delta}{2}.$$

Then, recalling that d is 1-Lipschitz continuous,

$$\left| \nabla \left(\frac{u(z)}{d(z)} \right) \right| \le \frac{d(z)|\nabla u(z)| + |u(z)||\nabla d(z)|}{d(z)^2} \le \frac{2}{\delta} \, \|\nabla u\|_\infty + \frac{4}{\delta^2} \, \|u\|_\infty. \tag{2.1.4}$$

By (2.1.3)–(2.1.4), besides $|x-y| < \frac{\delta}{2} < 1$, we get

$$\frac{\left| \frac{u(x)}{d(x)} - \frac{u(y)}{d(y)} \right|}{|x-y|^\beta} \le \left(\frac{2}{\delta} + \frac{4}{\delta^2} \right) \|u\|_{C^1} \quad \text{if } \; d(x) \ge \delta. \tag{2.1.5}$$

Case 2: $|x-y|^{1-\beta} \le d(x) < \delta$. Lipschitz continuity of u produces

$$|u(x)| \le \|\nabla u\|_\infty \, d(x),$$

so the mean value theorem and the 1-Lipschitz continuity of d entail

$$\begin{aligned}
\left| \frac{u(x)}{d(x)} - \frac{u(y)}{d(y)} \right| &\le \frac{|u(x) - u(y)|}{d(x)} + |u(y)| \frac{|d(x) - d(y)|}{d(x)d(y)} \\
&\le \|\nabla u\|_\infty \frac{|x-y|}{d(x)} + \|\nabla u\|_\infty \, d(y) \, \frac{|x-y|}{d(x)d(y)} \\
&= 2\|\nabla u\|_\infty \frac{|x-y|}{d(x)} \le 2\|\nabla u\|_\infty |x-y|^\beta.
\end{aligned}$$

We obtain

$$\frac{\left|\frac{u(x)}{d(x)} - \frac{u(y)}{d(y)}\right|}{|x-y|^\beta} \le 2\,\|\nabla u\|_\infty \quad \text{if } \ |x-y|^{1-\beta} \le d(x) < \delta. \tag{2.1.6}$$

Case 3: $d(x) < \min\{|x-y|^{1-\beta}, \delta\}$**.** According to (2.1.1) and the fact that $u = 0$ on $\partial\Omega$, the mean value theorem furnishes $\hat{x} \in]\Pi(x), x[$ and $\hat{y} \in]\Pi(y), y[$ such that

$$\begin{aligned} \frac{u(x)}{d(x)} &= \frac{u(x) - u(\Pi(x))}{|x - \Pi(x)|} = -\nabla u(\hat{x})\,\nu(\Pi(x)), \\ \frac{u(y)}{d(y)} &= \frac{u(y) - u(\Pi(y))}{|y - \Pi(y)|} = -\nabla u(\hat{y})\,\nu(\Pi(y)). \end{aligned} \tag{2.1.7}$$

Hence, the inequalities $d(y) \le d(x) \le |x-y|^{1-\beta} < 1$ yield

$$\begin{aligned} &|\hat{x} - \hat{y}| \le |\hat{x} - x| + |x - y| + |y - \hat{y}| \\ &\le d(x) + |x-y| + d(y) \le 3\,|x-y|^{1-\beta}. \end{aligned} \tag{2.1.8}$$

By (2.1.7) and the mean value theorem we get

$$\begin{aligned} &\left|\frac{u(x)}{d(x)} - \frac{u(y)}{d(y)}\right| \le |\nabla u(\hat{x}) - \nabla u(\hat{y})| + |\nabla u(\hat{y})||\nu(\Pi(x)) - \nu(\Pi(y))| \\ &\le \|\nabla u\|_{C^{0,\alpha}}\,|\hat{x} - \hat{y}|^\alpha + \|\nabla u\|_\infty \|\nabla \nu\|_\infty \|\nabla \Pi\|_\infty\,|x-y| \\ &\le c\,|x-y|^\beta \|u\|_{C^{1,\alpha}}\,, \end{aligned} \tag{2.1.9}$$

since $\alpha(1-\beta) = \beta$. Thus, we infer that

$$\frac{\left|\frac{u(x)}{d(x)} - \frac{u(y)}{d(y)}\right|}{|x-y|^\beta} \le c\,\|u\|_{C^{1,\alpha}} \quad \text{if } \ d(x) < \min\{|x-y|^{1-\beta}, \delta\}. \tag{2.1.10}$$

Estimates (2.1.5), (2.1.6), and (2.1.10) yield (2.1.2), completing the proof. $\square$

Remark 2.1.9.

- There is another version of Theorem 2.1.8; if $u \in C^1_0(\overline{\Omega})$ then $d^{-1}u \in C^0(\overline{\Omega})$, and in particular there exists $C = C(\Omega) > 0$ such that

$$\|d^{-1}u\|_{C^0(\overline{\Omega})} \le C\,\|u\|_{C^1(\overline{\Omega})}.$$

  Indeed, according to (2.1.1), we have

$$|u(x)| = |u(x) - u(\Pi(x))| \le \|\nabla u\|_\infty\,|x - \Pi(x)| = \|\nabla u\|_\infty\,d(x) \quad \forall\, x \in \Omega_\delta,$$

while

$$|u(x)| < \frac{d(x)}{\delta}|u(x)| \leq \delta^{-1}\|u\|_\infty \, d(x) \quad \forall\, x \in \Omega \setminus \overline{\Omega}_\delta,$$

providing the estimate

$$\|d^{-1}u\|_{L^\infty(\Omega)} \leq C\|\nabla u\|_{C^1(\overline{\Omega})}.$$

It remains to prove the uniform continuity of $d^{-1}u$. We can reason as in the proof of Theorem 2.1.8, but here (2.1.10) does not hold. Hence the uniform continuity follows from (2.1.5), (2.1.6), and the first inequality of (2.1.9), besides observing that

$$\lim_{|x-y|\to 0^+} |\nabla u(\hat{x}) - \nabla u(\hat{y})| = 0,$$

because of (2.1.8), the regularity of u, and the Lipschitz continuity of $\nu \circ \Pi$.

- If the function u appearing in Theorem 2.1.8 is a solution to a differential equation, sometimes it is possible to apply both the strong maximum principle and the boundary point lemma (see Theorems 2.3.7 and 2.3.8 below) in order to guarantee that the extension of $d^{-1}u$ in $\overline{\Omega}$ is a strictly positive function, which in turn implies that there exists $l > 0$ (depending on u) such that $u \geq ld$ in $\overline{\Omega}$, thanks to the continuity of $d^{-1}u$ ensured by Theorem 2.1.8. Indeed, the strong maximum principle yields $u > 0$ in Ω, so the same holds for $d^{-1}u$. Since $d^{-1}u$ can be extended to a continuous function in $\overline{\Omega}$, we can evaluate it on a generic $x_0 \in \partial\Omega$ by computing, through (2.1.1) and the boundary point lemma,

$$\begin{aligned} \frac{u(x_0)}{d(x_0)} &= \lim_{t\to 0^+} \frac{u(x_0 - t\nu(x_0))}{d(x_0 - t\nu(x_0))} = \lim_{t\to 0^+} \frac{u(x_0 - t\nu(x_0)) - u(x_0)}{t} \\ &= -\partial_\nu u(x_0) > 0, \end{aligned}$$

after recalling that $u(x_0) = 0$ and $\Pi(x_0 - t\nu(x_0)) = x_0$ whenever t is small (see (2.1.1)).

In this final part of the section we provide different notions of solution for elliptic equations in divergence form. Before going on, it is worth noticing that any definition of solution has to be contextualized with respect to the features of the problem one deals with: some authors usually perform slight modifications to the meaning of ‘solution’ to adapt it to the problem they would like to discuss. Here we limit ourselves to the problems we will treat

in the sequel.
Let $\Omega \subseteq \mathbb{R}^N$ be a domain of class C^1, $N \geq 2$, with outer normal ν to $\partial\Omega$, and let $p \in (1, +\infty)$. We consider the differential problem

$$-\mathrm{div}\,\mathcal{A}(x, u, \nabla u) = \mathcal{B}(x, u, \nabla u) \quad \text{in } \Omega, \tag{P}$$

possibly coupled with *Dirichlet boundary conditions*

$$u = \psi_1(x) \quad \text{on } \partial\Omega, \tag{D}$$

or *Robin boundary conditions* (called *Neumann conditions* if $\psi_2 \equiv 0$)

$$\partial_\nu u = \psi_2(x, u) \quad \text{on } \partial\Omega, \tag{R}$$

where $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$, $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, and $\psi_2 : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, while $\psi_1 \in L^p(\partial\Omega)$. We suppose that $\mathcal{A}$ satisfies

$$|\mathcal{A}(x, s, \xi)| \leq a_1 |\xi|^{p-1} + a_2 s^{\frac{p^*}{p'}} + \psi(x), \tag{A}$$

for all $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, being $a_1, a_2, a_3 > 0$ and $\psi \in L^{p'}(\Omega)$.

Definition 2.1.10.

- $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is said to be a *distributional solution* to (P) if $\mathcal{B}(x, u, \nabla u) \in L^1_{\mathrm{loc}}(\Omega)$ and, for all $\varphi \in C^\infty_c(\Omega)$,

$$\int_\Omega \mathcal{A}(x, u, \nabla u) \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega \mathcal{B}(x, u, \nabla u) \, \varphi \, \mathrm{d}x. \tag{2.1.11}$$

- $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is said to be a classical solution if $\mathcal{A}(\cdot, u, \nabla u)$ is of class C^1 and (P) holds true pointwise.

Suppose that Ω is bounded. Then:

- $u \in W^{1,p}(\Omega)$ is said to be a *weak solution* to (P) + (D) if, for all $\varphi \in W^{1,p}_0(\Omega)$, it holds

$$\mathcal{B}(\cdot, u, \nabla u) \, \varphi \in L^1(\Omega), \tag{2.1.12}$$

$$\int_\Omega \mathcal{A}(x, u, \nabla u) \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega \mathcal{B}(x, u, \nabla u) \, \varphi \, \mathrm{d}x, \tag{2.1.13}$$

  and (D) is satisfied in the sense of traces.

- $u \in W^{1,p}(\Omega)$ is said to be a *weak solution* to (P) + (R) if, for all $\varphi \in W^{1,p}(\Omega)$,

$$\mathcal{B}(\cdot, u, \nabla u)\, \varphi \in L^1(\Omega), \quad \psi_2(\cdot, u)\varphi \in L^1(\partial\Omega) \tag{2.1.14}$$

hold true (identifying $\varphi_{|\partial\Omega}$ with the trace of φ on $\partial\Omega$), besides

$$\begin{aligned}&\int_\Omega \mathcal{A}(x, u, \nabla u) \cdot \nabla\varphi \, \mathrm{d}x - \int_{\partial\Omega} \psi_2(x, u)\varphi \, \mathrm{d}\sigma \\ &= \int_\Omega \mathcal{B}(x, u, \nabla u)\, \varphi \, \mathrm{d}x.\end{aligned} \tag{2.1.15}$$

If Ω is unbounded, then:

- $u \in \mathcal{D}^{1,p}_0(\Omega)$ is said to be a weak solution to (P) if (2.1.12)–(2.1.13) hold true for all $\varphi \in \mathcal{D}^{1,p}_0(\Omega)$.

u is said to be a *classical sub-solution* to (P) if (P) holds true (pointwise) with the sign of '$\leq$' in place of '$=$'. Moreover, u is said to be a *distributional* (resp., *weak*) *sub-solution* to (P) if (2.1.11) (resp., (2.1.13) or (2.1.15)) holds true with '$\leq$' instead of '$=$' for any $\varphi \geq 0$. In the case of Dirichlet problems, sub-solutions must satisfy also $u \leq \psi_1$ on $\partial\Omega$ in the sense of traces; for Robin problems sub-solutions must satisfy also $\psi_2(x, u) \geq 0$ on $\partial\Omega$ in the sense of traces. Super-solutions are defined as sub-solutions, substituting '$\leq$' with '$\geq$'.

To be more precise, we highlight some important considerations in the following remark.

Remark 2.1.11.

- We observe that the number p in Definition 2.1.10 is related to the growth condition of the principal part $\mathcal{A}$: indeed, assumption (A) is sufficient to ensure that the integrals involving $\mathcal{A}$ in (2.1.11), (2.1.13), and (2.1.15) are finite for any u. On the other hand, growth conditions for $\mathcal{B}$ and ψ_2 ensure (2.1.12) and (2.1.14); for instance, thanks to Sobolev's embedding theorem (see Theorem 2.1.5), we can suppose

$$\begin{aligned}|\mathcal{B}(x, s, \xi)| &\leq c_1|\xi|^{\frac{p}{(p^*)'}} + c_2|s|^{p^*-1} + f(x), \quad && f \in L^{(p^*)'}(\Omega), \\ |\psi_2(x, s)| &\leq c_3|s|^{\frac{p_*}{p'}} + g(x), && g \in L^{p'}(\partial\Omega),\end{aligned}$$

with $p_* := \frac{(N-1)p}{N-1-p}$. Anyway, we chose not to make assumptions of this type, because they are not satisfied for singular problems without imposing further restrictions on the class of solutions.

- Different definitions of '$u \geq 0$ on $\partial\Omega$' are available also for distributional solutions: see, e.g., [143, p.52] and [28, Definition 1.2]. Following [143], we say that $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ satisfies $u \geq 0$ on $\partial\Omega$ if for any $\delta > 0$ there exists a neighborhood Γ_δ of $\partial\Omega$ such that $u \geq -\delta$ in Γ_δ.

- We will not treat classical solutions, because even for the p-Poisson equation the best regularity of solutions is $C^{1,\alpha}$ (see, e.g., [148, p.1]). On the other hand, in the linear setting, solutions can achieve $C^{2,\alpha}$ regularity; this is essentially due to the Calderón-Zygmund theory (see paragraph 2.2.4 below). In the context of quasilinear problems, the concept of strong solution is introduced (vide [81, Section 9]) to recover a sort of 'pointwise' definition of solution; actually, strong solutions are weak solutions with a further differentiability property, as the one ensured by Theorems 2.2.17 and 2.2.18 below.

- Definition 2.1.10 can be generalized to systems. Another generalization consists in defining particular couples of sub-super-solutions: this aspect will be treated in paragraph 2.3.3 (in particular, see Definition 2.3.11).

2.2 Nonlinear regularity theory

The present section is devoted to recall some basic facts about regularity of weak solutions to elliptic problems in divergence form. Both local and global regularity will be discussed, with particular emphasis about a priori estimates for entire solutions, i.e., solutions defined on the whole space $\mathbb{R}^N$. Anyway, we will collect the results depending on regularity, not on domain; a list ordered according to the typology of domain will be given in Remark 2.2.21 below.
Regularity theory for semilinear problems can be found, e.g., in [96] (cf. [81] for a classical reference). There is a few literature about nonlinear problems, even for the p-Laplacian; we refer to [123] for some brief sketches about some of the results exposed here, while for further details we address to the articles cited in itinere.

In order to expose the main ideas about nonlinear regularity theory, let $N \geq 2$, $p \in (1, +\infty)$, and $\Omega \subseteq \mathbb{R}^N$ be a bounded domain of class C^2. Consider the following problem:

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2.1}$$

If f is a continuous function with p-sub-linear growth, i.e.,

$$|f(s)| \leq c_1|s|^{q-1} + c_2 \quad \forall s \in \mathbb{R},$$

with $q \in [1, p)$ and $c_1, c_2 > 0$, then (2.2.1) admits a weak solution $u \in W_0^{1,p}(\Omega)$ (see Theorem 2.4.3 and Remark 2.4.4). Anyway, from the regularity point of view, we can assume *a priori* that (2.2.1) has a solution u such that $f \circ u$ is measurable; in particular, we do not require any continuity or growth hypotheses on f. A simple way to deduce regularity of solutions is to *freeze* the right-hand side (that is, to keep it fixed), so that a generic solution u solves also

$$\begin{cases} -\Delta_p u = g(x) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2.2}$$

with $g(x) = f(u(x))$ for all $x \in \Omega$. Standard results (see Corollary 2.2.3 and Theorem 2.2.12 below) say

1. $g \in L^q(\Omega)$, $q > \frac{N}{p} \Rightarrow u \in L^\infty(\Omega)$;

2. $g \in L^q(\Omega)$, $q > N \Rightarrow u \in C^{1,\alpha}(\overline{\Omega})$.

In this way the regularity can be deduced by the properties of the Nemitskii operator $N_f(u) := f \circ u$. But there is a loss of information: in reducing to problem (2.2.2) we forgot that u is a solution to a more 'balanced' problem, in which the function appears in both members of the equation. Hence we can gain regularity with the aim of more sophisticated theorems concerning directly problem (2.2.1). For instance (see [123, Section 8.1]),

1. $|f(s)| \leq c_1|s|^{r-1} + c_2$, $r \in [p, p^*) \Rightarrow u \in L^\infty(\Omega)$;
2. $f \in L^\infty_{\mathrm{loc}}(\mathbb{R})$, $u \in L^\infty(\Omega) \Rightarrow u \in C^{1,\alpha}(\overline{\Omega})$.

The requirement $u \in L^\infty(\Omega)$ in (2) is usually a consequence of (1). We explicitly notice that the hypothesis in (1), called sub-critical growth condition, implies that the Nemitskii operator N_f maps $W^{1,p}_0(\Omega)$ in $L^{\frac{p^*}{r-1}}(\Omega)$, which could be very close to $L^1(\Omega)$, yielding a very low summability; hence, in the context of (2.2.1), we obtained a better regularity result.
In more general cases, a good combination of the two methods presented above (for instance, freezing some terms on the right-hand side and leaving the other ones as unfrozen) allows to achieve different types of regularity of solutions (boundedness, Hölder continuity, Sobolev differentiability, etc.), furnishing a priori estimates which are very useful in the context of existence theory, especially for singular problems, to gain compactness.

2.2.1 Moser's technique

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a domain of class $C^{0,1}$ and let $p \in (1, N)$. We would like to investigate boundedness of weak solutions to (P) (see paragraph 2.1.3). Results of this type are usually achieved via Moser's technique: here we are going to present this argument in the context of positive solutions to singular problems, addressing to some research articles for different settings.
We assume the following structure conditions: $\mathcal{A} : \Omega \times (0, +\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ and $\mathcal{B} : \Omega \times (0, +\infty) \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions satisfying

$$(\mathrm{A}_1) \qquad |\mathcal{A}(x, s, \xi)| \leq a_1|\xi|^{p-1} + a_2 s^{\frac{p^*}{p'}} + \psi(x),$$

$$(\mathrm{A}_2) \qquad \mathcal{A}(x, s, \xi) \cdot \xi \geq a_3|\xi|^p,$$

$$(\mathrm{B}) \qquad |\mathcal{B}(x, s, \xi)| \leq b_1|\xi|^{p-1} + b_2 s^{p-1} + b_3 s^{-\delta} + f(x),$$

for all $(x, s, \xi) \in \Omega \times (0, +\infty) \times \mathbb{R}^N$, being $\delta \in (0, 1)$, $f \in L^r(\Omega)$ with $r > \frac{N}{p}$, $\psi \in L^{p'}(\Omega)$, and $a_i, b_i > 0$, $i = 1, 2, 3$.

We briefly justify the hypotheses considered here. Assumptions (A_1) and $\delta < 1$ are used only to guarantee that (P) has a weak solution (cf. [118, p.2]), and they have no role in the proof of Theorem 2.2.2 below, while (A_2) is a standard growth condition that seems not to be weakened in the context of singular problems (cf. [118, Remark 2.2]). Finally, hypothesis (B) incorporates the totality of problems we will treat in the sequel: they possess reaction terms which are convective, p-linear, and singular; moreover, f usually represents some fixed terms that allow us to handle, amongst several problems, systems of equations like (P) or singularities with sufficiently high summability.

Before reporting a proof of a Moser-type theorem for the illustrated problem, we mention some recent results obtained by Marino and Winkert about convective problems in bounded domains: [116] concerns critical problems with critical boundary conditions, while in [118] the singular setting is investigated; moreover, [117] is devoted to systems. Proofs of Moser-type results for entire solutions of (P), which are particular cases of Theorem 2.2.2 below, can be found in [114, 91].

For any $u \in \mathcal{M}(\Omega)$ and $a, b \in \mathbb{R}$, we introduce the sets

$$\Omega_a := \{x : \, u(x) \geq a\}, \quad \Omega_a^b := \{x : \, a \leq u(x) \leq b\}.$$

We also set $u_c := \min\{u, c\}$ for any $c \in \mathbb{R}$. For convenience of exposition we will consider $u \in X$, $X := W_0^{1,p}(\Omega)$ (being $\Omega \subseteq \mathbb{R}^N$ bounded) or $X := \mathcal{D}_0^{1,p}(\mathbb{R}^N)$ (setting $\Omega := \mathbb{R}^N$), weak solution to (P).

Lemma 2.2.1. *Suppose* (A_1)–(A_2) *and* (B). *Let* $u \in X$, $u > 0$ *in* Ω, *be a weak solution to* (P). *Then, for any* $\varphi \in X$, $\varphi \geq 0$,

$$\int_{\Omega_1} \mathcal{A}(x, u, \nabla u) \cdot \nabla \varphi \, \mathrm{d}x \leq \int_{\Omega_1} (b_1 |\nabla u|^{p-1} + b_2 u^{p-1} + b_3 u^{-\delta} + f) \, \varphi \, \mathrm{d}x. \tag{2.2.3}$$

Proof. Pick a cut-off function $\eta \in C^\infty(\mathbb{R})$ satisfying

$$\eta(t) = \begin{cases} 0 & \text{if } \; t \leq 0, \\ \text{increasing} & \text{if } \; 0 < t < 1, \\ 1 & \text{if } \; t \geq 1, \end{cases} \tag{2.2.4}$$

and let $\eta_\varepsilon(t) := \eta(\frac{t-1}{\varepsilon})$ for all $t \in \mathbb{R}$. Take any $\varphi \in C_c^1(\Omega)$ with $\varphi \geq 0$. Test (P) with $(\eta_\varepsilon \circ u) \, \varphi$ and exploit (B) to get

$$\begin{aligned} &\int_\Omega (\mathcal{A}(x, u, \nabla u) \cdot \nabla \varphi)(\eta_\varepsilon \circ u) \, \mathrm{d}x \\ &+ \int_\Omega (\mathcal{A}(x, u, \nabla u) \cdot \nabla u)(\eta_\varepsilon' \circ u) \, \varphi \, \mathrm{d}x \\ &\leq \int_\Omega (b_1 |\nabla u|^{p-1} + b_2 u^{p-1} + b_3 u^{-\delta} + f)(\eta_\varepsilon \circ u) \, \varphi \, \mathrm{d}x. \end{aligned} \tag{2.2.5}$$

By (A_2) and $\eta'_\varepsilon \circ u \geq 0$ we have

$$\int_\Omega (\mathcal{A}(x,u,\nabla u)\cdot\nabla u)(\eta'_\varepsilon \circ u)\,\varphi\,\mathrm{d}x \geq a_3 \int_\Omega |\nabla u|^p (\eta'_\varepsilon \circ u)\,\varphi\,\mathrm{d}x \geq 0. \tag{2.2.6}$$

From (2.2.5)–(2.2.6), besides observing that $\eta_\varepsilon \circ u = 0$ outside Ω_1, we obtain

$$\begin{aligned} &\int_{\Omega_1} (\mathcal{A}(x,u,\nabla u)\cdot\nabla\varphi)(\eta_\varepsilon \circ u)\,\mathrm{d}x \\ &\leq \int_{\Omega_1} (b_1|\nabla u|^{p-1} + b_2 u^{p-1} + b_3 u^{-\delta} + f)(\eta_\varepsilon \circ u)\,\varphi\,\mathrm{d}x. \end{aligned} \tag{2.2.7}$$

Letting $\varepsilon \to 0^+$ in (2.2.7) and applying Lebesgue's dominated convergence theorem proves (2.2.3) for any $\varphi \in C^1_c(\Omega)$; then a density argument permits to conclude. □

Now we are ready for the main result of the paragraph.

Theorem 2.2.2. *Let* (A_1)–(A_2) *and* (B) *be satisfied, and let* $u \in X$, $u > 0$ *in* Ω, *be a weak solution to* (P) *such that, for a suitable* $C > 0$,

$$\|\nabla u\|_p \leq C.$$

Then there exists $M = M(C, \|f\|_r, \Omega, N, p, a_3, b_1, b_2, b_3) > 0$, *such that*

$$\|u\|_\infty \leq M.$$

Proof. Let $k > 0$ and $h > 1$. Lemma 2.2.1, applied with $\varphi = u u_h^{kp}$, gives

$$\begin{aligned} &\int_{\Omega_1} (\mathcal{A}(x,u,\nabla u)\cdot\nabla u)\, u_h^{kp}\,\mathrm{d}x \\ &+ kp \int_{\Omega_1} (\mathcal{A}(x,u,\nabla u)\cdot\nabla u_h)\, u u_h^{kp-1}\,\mathrm{d}x \\ &\leq \int_{\Omega_1} (b_1|\nabla u|^{p-1} + b_2 u^{p-1} + b_3 u^{-\delta} + f)\, u u_h^{kp}\,\mathrm{d}x \end{aligned} \tag{2.2.8}$$

Now we estimate both sides of (2.2.8) separately. Regarding the left-hand side, using (A_2) we obtain

$$\int_{\Omega_1} (\mathcal{A}(x,u,\nabla u)\cdot\nabla u)\, u_h^{kp}\,\mathrm{d}x \geq a_3 \int_{\Omega_1} |\nabla u|^p u_h^{kp}\,\mathrm{d}x \tag{2.2.9}$$

and

$$\begin{aligned}
&kp\int_{\Omega_1}(\mathcal{A}(x,u,\nabla u)\cdot\nabla u_h)\,uu_h^{kp-1}\,\mathrm{d}x\\
&=kp\int_{\Omega_1^h}(\mathcal{A}(x,u_h,\nabla u_h)\cdot\nabla u_h)\,u_h^{kp}\,\mathrm{d}x\\
&\geq a_3kp\int_{\Omega_1}|\nabla u_h|^p u_h^{kp}\,\mathrm{d}x.
\end{aligned}\tag{2.2.10}$$

On the other side, according to Young's and Hölder's inequalities, besides recalling that we are working in Ω_1, we have

$$\begin{aligned}
\int_{\Omega_1}|\nabla u|^{p-1}uu_h^{kp}\,\mathrm{d}x&=\int_{\Omega_1}|\nabla u|^{p-1}u_h^{k(p-1)}uu_h^k\,\mathrm{d}x\\
&\leq\varepsilon\int_{\Omega_1}|\nabla u|^p u_h^{kp}\,\mathrm{d}x+c_\varepsilon\int_{\Omega_1}(uu_h^k)^p\,\mathrm{d}x,
\end{aligned}\tag{2.2.11}$$

$$\int_{\Omega_1}u^p u_h^{kp}\,\mathrm{d}x=\int_{\Omega_1}(uu_h^k)^p\,\mathrm{d}x,\tag{2.2.12}$$

$$\int_{\Omega_1}u^{1-\delta}u_h^{kp}\,\mathrm{d}x\leq\int_{\Omega_1}(uu_h^k)^p\,\mathrm{d}x,\tag{2.2.13}$$

and

$$\int_{\Omega_1}fuu_h^{kp}\,\mathrm{d}x\leq\int_{\Omega_1}|f|(uu_h^k)^p\,\mathrm{d}x\leq\|f\|_r\left(\int_{\Omega_1}(uu_h^k)^{pr'}\,\mathrm{d}x\right)^{\frac{1}{r'}}.\tag{2.2.14}$$

Putting (2.2.9)–(2.2.14) into (2.2.8) gives

$$\begin{aligned}
&a_3\left(\int_{\Omega_1}|\nabla u|^p u_h^{kp}\,\mathrm{d}x+kp\int_{\Omega_1}|\nabla u_h|^p u_h^{kp}\,\mathrm{d}x\right)\\
&\leq b_1\varepsilon\int_{\Omega_1}|\nabla u|^p u_h^{kp}\,\mathrm{d}x+(b_1c_\varepsilon+b_2+b_3)\|uu_h^k\|_p^p+\|f\|_r\|uu_h^k\|_{pr'}^p.
\end{aligned}$$

Choosing $\varepsilon=\frac{a_3}{2b_1}$ allows us to re-absorb on the left the first integral of the right-hand side, yielding

$$\begin{aligned}
&\frac{a_3}{2}\left(\int_{\Omega_1}|\nabla u|^p u_h^{kp}\,\mathrm{d}x+kp\int_{\Omega_1}|\nabla u_h|^p u_h^{kp}\,\mathrm{d}x\right)\,\mathrm{d}x\\
&\leq(b_1c_\varepsilon+b_2+b_3)\|uu_h^k\|_p^p+\|f\|_r\|uu_h^k\|_{pr'}^p.
\end{aligned}\tag{2.2.15}$$

Notice that

$$\begin{aligned}&\frac{kp+1}{(k+1)^p}|\nabla(uu_h^k)|^p \le \frac{kp+1}{(k+1)^p}(u_h^{kp}|\nabla u|^p + k^p u^p u_h^{(k-1)p}|\nabla u_h|^p)\\&= \frac{kp+1}{(k+1)^p}(u_h^{kp}|\nabla u|^p + k^p u_h^{kp}|\nabla u_h|^p) \le u_h^{kp}|\nabla u|^p + kp u_h^{kp}|\nabla u_h|^p.\end{aligned}$$

Hence the left-hand side of (2.2.15) can be estimated from below by Sobolev's inequality (Theorem 2.1.5) producing, for a suitable $c = c(\Omega, N, p) > 0$,

$$\begin{aligned}&\int_{\Omega_1} |\nabla u|^p u_h^{kp}\,\mathrm{d}x + kp\int_{\Omega_1} |\nabla u_h|^p u_h^{kp}\,\mathrm{d}x\\&\ge \frac{kp+1}{(k+1)^p}\,\|\nabla(uu_h^k)\|_p^p \ge c\,\frac{kp+1}{(k+1)^p}\,\|uu_h^k\|_{p^*}^p\,.\end{aligned}\tag{2.2.16}$$

Hence, summarizing the constants (we observe that c_ε is independent of k), by (2.2.15) we get

$$\frac{kp+1}{(k+1)^p}\,\|uu_h^k\|_{p^*}^p \le c\,(\|uu_h^k\|_p^p + \|uu_h^k\|_{pr'}^p),$$

for an opportune $c = c(\|f\|_r, \Omega, N, p, a_3, b_1, b_2, b_3) > 0$. Re-arranging the terms gives

$$\|uu_h^k\|_{p^*} \le c\,\frac{k+1}{(kp+1)^{\frac{1}{p}}}\,(\|uu_h^k\|_p + \|uu_h^k\|_{pr'}).\tag{2.2.17}$$

The condition $r > \frac{N}{p}$ produces $p < pr' < p^*$. Thus, by interpolation and Young's inequality, we obtain

$$\|uu_h^k\|_{pr'} \le \|uu_h^k\|_p^{1-\theta}\|uu_h^k\|_{p^*}^{\theta} \le \sigma\|uu_h^k\|_{p^*}^{\theta s} + c_\sigma\|uu_h^k\|_p^{(1-\theta)s'},\tag{2.2.18}$$

with

$$\frac{1}{pr'} = \frac{1-\theta}{p} + \frac{\theta}{p^*} \quad \text{and} \quad c_\sigma = \sigma^{\frac{1}{1-s}}.$$

Choosing $\theta s = 1$ we get

$$\theta = \frac{N}{pr}, \quad s = \frac{pr}{N}, \quad c_\sigma = \sigma^{\frac{N}{N-pr}}, \quad \text{and} \quad (1-\theta)s' = 1.$$

Hence (2.2.18) becomes

$$\|uu_h^k\|_{pr'} \le \sigma\|uu_h^k\|_{p^*} + \sigma^{\frac{N}{N-pr}}\|uu_h^k\|_p$$

that, inserted into (2.2.17), furnishes

$$\begin{aligned}\|uu_h^k\|_{p^*} &\leq c\,\frac{k+1}{(kp+1)^{\frac{1}{p}}}\,\left[\sigma\|uu_h^k\|_{p^*}+(\sigma^{\frac{N}{N-pr}}+1)\|uu_h^k\|_p\right]\\ &\leq c\,\frac{k+1}{(kp+1)^{\frac{1}{p}}}\,\left[\sigma\|uu_h^k\|_{p^*}+2\sigma^{\frac{N}{N-pr}}\|uu_h^k\|_p\right]\end{aligned}$$

for any $\sigma \leq 1$ (recall that $N - pr < 0$ by hypothesis). We can choose $\sigma = \frac{1}{2c}\frac{(kp+1)^{1/p}}{k+1}$, which is less than 1 for any $k > 0$, assuming (without loss of generality) that $c > \frac{1}{2}$. Reabsorbing to the left the first term on the right we get

$$\|uu_h^k\|_{p^*} \leq 2^{\frac{N}{pr-N}+2}\left(c\,\frac{k+1}{(kp+1)^{\frac{1}{p}}}\right)^{\frac{pr}{pr-N}}\|uu_h^k\|_p.$$

Applying Fatou's lemma on the left-hand side and Beppo Levi's monotone convergence theorem on the right we get, up to constants,

$$\|u\|_{(k+1)p^*} \leq c^{\mu_k}\lambda_k^{\gamma}\,\|u\|_{(k+1)p} \quad \forall k > 0, \tag{2.2.19}$$

being

$$\mu_k := \frac{1}{k+1}, \quad \lambda_k := \left(\frac{k+1}{(kp+1)^{\frac{1}{p}}}\right)^{\frac{1}{k+1}}, \quad \gamma := \frac{pr}{pr-N}.$$

Observe that (2.2.19) is a *reverse-Hölder inequality* (referring to the fact that higher norms are controlled in terms of lower ones), so we can perform a bootstrap argument. Pick the sequence $\{k_n\}$ defined as

$$\begin{cases}(k_{n+1}+1)\,p = (k_n+1)\,p^* & \forall\, n \in \mathbb{N},\\ \quad (k_1+1)\,p = p^*.\end{cases} \tag{2.2.20}$$

Inequality (2.2.19) for $k = k_1$ implies

$$\|u\|_{(k_1+1)p^*} \leq c^{\mu_{k_1}}\lambda_{k_1}^{\gamma}\|u\|_{p^*} < +\infty.$$

Reasoning inductively, by (2.2.19) we get

$$\|u\|_{(k_n+1)p^*} \leq c^{\mu_{k_n}}\lambda_{k_n}^{\gamma}\|u\|_{(k_n+1)p} \quad \forall\, n \in \mathbb{N},$$

with all norms being finite. Proceeding by induction again, (2.2.20) yields

$$\|u\|_{(k_n+1)p^*} \leq c^{\sum_{i=1}^n \mu_{k_i}}\left[\prod_{i=1}^n \lambda_{k_i}\right]^{\gamma}\|u\|_{p^*} \quad \forall\, n \in \mathbb{N}. \tag{2.2.21}$$

It remains to prove that both the series and the infinite product stemming from (2.2.21) are convergent. Observe that, since $k_n \nearrow +\infty$ as $n \to \infty$,

$$\frac{k_n}{k_{n-1}} \simeq \frac{k_n + 1}{k_{n-1} + 1} = \frac{p^*}{p}, \quad \text{whence} \quad k_n \simeq \left(\frac{p^*}{p}\right)^n.$$

Thus

$$\sum_{n=1}^{\infty} \mu_{k_n} \le \sum_{n=1}^{\infty} \frac{1}{k_n} \simeq \sum_{n=1}^{\infty} \left(\frac{p}{p^*}\right)^n < +\infty.$$

Moreover, since

$$\left(\frac{k+1}{(kp+1)^{\frac{1}{p}}}\right)^{\frac{1}{\sqrt{k+1}}} \ge 1 \;\; \forall\, k > 0 \quad \text{and} \quad \left(\frac{k+1}{(kp+1)^{\frac{1}{p}}}\right)^{\frac{1}{\sqrt{k+1}}} \xrightarrow{k \to \infty} 1,$$

there exists $K = K(p) > 1$ such that

$$\begin{aligned}\prod_{i=1}^{n} \lambda_{k_i} &= \prod_{i=1}^{n} \left[\left(\frac{k_i+1}{(k_i p+1)^{\frac{1}{p}}}\right)^{\frac{1}{\sqrt{k_i+1}}}\right]^{\frac{1}{\sqrt{k_i+1}}} \le \prod_{i=1}^{n} K^{\frac{1}{\sqrt{k_i+1}}} \\ &\le K^{\sum_{n=1}^{\infty} \frac{1}{\sqrt{k_n}}} \simeq K^{\sum_{n=1}^{\infty}\left(\sqrt{\frac{p}{p^*}}\right)^n} < +\infty.\end{aligned}$$

Letting $k \to \infty$ in (2.2.21) we then get $u \le M$ in Ω_1 for some $M > 1$. Since $u \le 1$ outside Ω_1, we obtain $u \le M$ in Ω. $\square$

Corollary 2.2.3. *Let $u \in W_0^{1,p}(\Omega)$ be a weak solution to*

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2.22}$$

being $f \in L^r(\Omega)$ with $r > \frac{N}{p}$. Then $u \in L^\infty(\Omega)$.

Remark 2.2.4.

- Although we have proved Theorem 2.2.2 for Dirichlet problems, the same conclusion can be achieved also in the setting of Neumann problems, provided hypothesis $\|\nabla u\|_p \le C$ is replaced by $\|u\|_{1,p} \le C$: indeed, it suffices to add the term $\frac{a_3}{2}\|uu_h^k\|_p^p$ in both sides of (2.2.15) and estimate it from below through

$$\frac{a_3}{2}\|uu_h^k\|_p^p \ge \frac{a_3}{2}\,\frac{kp+1}{(k+1)^p}\,\|uu_h^k\|_p^p \quad \forall\, k > 0,$$

  in order to couple it with $\|\nabla(uu_h^k)\|_p^p$ in (2.2.16), when Sobolev's inequality is used. Other generalizations, related for instance to Robin boundary conditions or critical exponents (vide [116]), can be possibly implemented for singular problems.

- We highlight the fact that (B) allows to treat systems of differential equations like (P), because one can freeze the mixed terms; this point of view will be adopted in paragraph 4.2.2. Let us clarify this fact with an example. Consider [117]; hypotheses (E_4)–(E_5) and (E_7)–(E_8) force the mixed terms appearing in (H_5) to fall into $L^r(\Omega)$ with $r > \frac{N}{p}$: indeed, for instance,

$$\frac{b_3}{p^*} + \frac{b_4}{q^*} < \frac{p^* - p}{p^*} \quad \Rightarrow \quad |u(x)|^{b_3}|v(x)|^{b_4} \in L^r(\Omega), \quad r > \left(\frac{p^*}{p}\right)' = \frac{N}{p},$$

  for all $(u, v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$. The same thing occurs in [91] with hypothesis H_1(a). This fact also reveals the importance of the exponent $\frac{N}{p}$ for L^∞ bounds in the context of p-coercive elliptic operators.

2.2.2 Hölder regularity

The development of Hölder regularity theory is one of the bedrock results of Mathematical Analysis in the twentieth century: it suffices to say that this theory led to a complete solution to Hilbert's XIX problem. It is clearly impossible to summarize, even in a partial way, the literature concerning this topic, so we limit ourselves to address the interested reader to the classical book by Gilbarg and Trudinger [81]. An introduction to the topic can be found, e.g., in [96] for the linear setting; cf. also [4].
Since we are interested in nonlinear equations in divergence form, we only mention that, moving from the works by Ladyzhenskaya and Ural'tseva [103] and Uhlenbeck [160], many authors investigated local regularity of weak solutions: Evans [65], Lewis [107], DiBenedetto [59], Tolksdorf [157], and Lieberman [110]. We deem that regularity up to the boundary has been less studied: up to our knowledge, we can cite only the well-known Lieberman's paper [109] and, in view of singular problems, [79, 95]. Very recent results about p-Laplacian systems have been obtained in [21] and [22], regarding local and global estimates respectively.

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a domain and let $p \in (1, +\infty)$. In this paragraph we deal again with problem (P) of paragraph 2.1.3 but, in order to gain regularity, we have to strengthen the structural conditions considered in paragraph 2.2.1. Hence we suppose that $\mathcal{A}$ is a C^0 function, whose restriction to $\Omega \times \mathbb{R} \times \mathbb{R}^N \setminus \{0\}$ is C^1, and $\mathcal{B}$ is a Carathéodory function; moreover, the following structure hypotheses are assumed:

(A_1) $$\mu^T \partial_\xi \mathcal{A}(x, s, \xi)\mu \geq \gamma_1 |\xi|^{p-2}|\mu|^2,$$

(A_2) $$|\partial_\xi \mathcal{A}(x, s, \xi)| \leq \gamma_2 |\xi|^{p-2},$$

$$(\mathrm{A}_3) \qquad |\mathcal{A}(x,s,\xi)-\mathcal{A}(y,t,\xi)| \leq \gamma_2(1+|\xi|^{p-1})[|x-y|^\beta+|s-t|^\beta],$$

$$(\mathrm{B}) \qquad |\mathcal{B}(x,s,\xi)| \leq \gamma_2|\xi|^p+f(x),$$

for all $(x,s,\xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, $(y,t) \in \Omega \times \mathbb{R}$, and $\mu \in \mathbb{R}^N$, being $\beta \in (0,1]$, $f \in L^r_{\mathrm{loc}}(\Omega)$, $r > N$, and $\gamma_1, \gamma_2 > 0$.

The following local result is due to Lieberman; for a more general statement, see Remark 2.2.9 below.

Theorem 2.2.5. *Suppose* (A_1)–(A_3) *and* (B). *Let* $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ *be a distributional solution to* (P) *and* $\Omega' \Subset \Omega$ *such that, for a suitable* $C_{\Omega'} > 0$,

$$\|u\|_{L^\infty(\Omega')} \leq C_{\Omega'}.$$

Then there exist two constants $M_{\Omega'} = M_{\Omega'}(\Omega', C_{\Omega'}, \beta, f, N, p, \gamma_1, \gamma_2) > 0$ *and* $\alpha = \alpha(\beta, N, p, \gamma_2/\gamma_1) \in (0,1]$ *such that*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega'})} \leq M_{\Omega'}.$$

Corollary 2.2.6. *Let* $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ *be a distributional solution to*

$$-\Delta_p u = f(x) \in L^q_{\mathrm{loc}}(\Omega), \quad q > N. \tag{2.2.23}$$

Then $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$.

Now we come to regularity up to the boundary, reporting another result by Lieberman [109]; it investigates the problems

$$\begin{cases} -\operatorname{div} \mathcal{A}(x,u,\nabla u) = \mathcal{B}(x,u,\nabla u) & \text{in } \Omega, \\ u = \psi_1(x) & \text{on } \partial\Omega, \end{cases} \tag{2.2.24}$$

and

$$\begin{cases} -\operatorname{div} \mathcal{A}(x,u,\nabla u) = \mathcal{B}(x,u,\nabla u) & \text{in } \Omega, \\ \mathcal{A}(x,u,\nabla u) \cdot \nu = \psi_2(x,u) & \text{on } \partial\Omega, \end{cases} \tag{2.2.25}$$

where Ω is bounded domain of class $C^{1,\beta}$, $\beta \in (0,1]$, with outer normal ν, and

$$\begin{gathered} \psi_1 \in C^{1,\beta}(\partial\Omega), \quad |\psi_2(x,s)| \leq c, \\ |\psi_2(x,s)-\psi_2(y,t)| \leq c[|x-y|^\beta+|s-t|^\beta], \end{gathered} \tag{2.2.26}$$

for all $(x,s),(y,t) \in \Omega \times \mathbb{R}$.

Theorem 2.2.7. *Let* (A_1)–(A_3)*,* (B)*, and* (2.2.26) *be satisfied. Let* $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a weak solution of* (2.2.24) *or* (2.2.25) *such that, for a suitable* $C > 0$*,*

$$\|u\|_{L^\infty(\Omega)} \le C.$$

Then there exist $\alpha \in (0,1]$ *and* $M > 0$*, depending on* $C, \Omega, \beta, N, p, \gamma_1, \gamma_2$*, such that*

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le M.$$

Corollary 2.2.8. *Let* $u \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be a weak solution to*

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad \textit{or} \quad \begin{cases} -\Delta_p u = f(x) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f \in L^q(\Omega)$*,* $q > N$*. Then* $u \in C^{1,\alpha}(\overline{\Omega})$*.*

Remark 2.2.9. It is possible to prove regularity, both local and up to the boundary, also for problems with non-standard growth, as the (p,q)-Laplacian, i.e., $\Delta_p + \Delta_q$ with $p, q \in (1, +\infty)$, $p > q$. Indeed, a theorem by Lieberman (cf. [110, p.320]) asserts that the conclusions of Theorems 2.2.5 and 2.2.7 remain valid if (A_1)–(A_3) and (B) are replaced by the following assumptions:

(A_1') $$\mu^T \partial_\xi \mathcal{A}(x,s,\xi)\mu \ge \gamma_1 \tfrac{\omega(|\xi|)}{|\xi|}|\mu|^2,$$

(A_2') $$|\partial_\xi \mathcal{A}(x,s,\xi)| \le \gamma_2 \tfrac{\omega(|\xi|)}{|\xi|},$$

(A_3') $$|\mathcal{A}(x,s,\xi) - \mathcal{A}(y,t,\xi)| \le \gamma_2(1+\omega(|\xi|))[|x-y|^\beta + |s-t|^\beta],$$

(B') $$|\mathcal{B}(x,s,\xi)| \le \gamma_2 \omega(|\xi|)|\xi| + f(x),$$

for a suitable $\omega \in C^1(0,+\infty)$ such that $\omega > 0$ in $(0,+\infty)$ and

$$C_1 \le \frac{s\omega'(s)}{\omega(s)} \le C_2 \quad \forall s \in (0,+\infty), \tag{2.2.27}$$

for some $C_1, C_2 > 0$.

As we will see in paragraph 3.1.1, Hölder regularity for solutions to singular problems is often deduced by the following theorem (see [95, Lemma 3.1]), which is patterned after [79]. We recall that the distance function d was defined in paragraph 2.1.3.

Theorem 2.2.10. *Let $u \in W_0^{1,p}(\Omega)$ be the weak solution to the Dirichlet problem* (2.2.22)*, with $f \in L^\infty_{\rm loc}(\Omega)$ obeying*

$$|f(x)| \le Cd^{-\gamma}$$

for some $\gamma \in (0,1)$ and $C > 0$. Then there exists $M = M(C, \gamma, \Omega) > 0$ such that

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le M.$$

Flattening $\partial\Omega$ and using a partition of unity argument, with a procedure similar to [106, pp.726-727]), it can be proved that

$$d^{-\gamma} \in L^r(\Omega) \quad \forall r \in \left[1, \frac{1}{\gamma}\right). \tag{2.2.28}$$

Hence, Theorem 2.2.10 cannot be deduced by Theorem 2.2.7. In other words, the importance of Theorem 2.2.10 relies on the possibility of handling right-hand sides possessing very low summability.

Concluding this paragraph, we would like to give another proof of Corollaries 2.2.6 and 2.2.8 (in the Dirichlet case), based on coupling linear regularity theory with regularity results for equations with right-hand side in divergence form (these type of results are common in literature, since they produce 'natural' estimates). Local estimates are a consequence of [21, Corollary 5.2], while the global ones can be deduced from [22, Corollary 2.7].

Theorem 2.2.11. *Let $\Omega \subseteq \mathbb{R}^N$ be a domain, $N \ge 2$, and $p \in (1, +\infty)$. Let $u \in W^{1,p}_{\rm loc}(\Omega)$ be a distributional solution to* (2.2.23)*. Then, for any $\alpha \in (0,1)$ sufficiently small, one has $u \in C^{1,\alpha}_{\rm loc}(\Omega)$.*

Proof. Fix any $B_{4R} \Subset \Omega$ and consider the linear problem

$$\begin{cases} -\Delta v = f(x) & \text{in } B_{4R}, \\ v = 0 & \text{on } \partial B_{4R}, \end{cases}$$

which admits a unique solution $v \in W^{2,q}(B_{4R})$, according to Minty-Browder's theorem (see Theorem 2.3.4 below) and the Calderón-Zygmund theory (vide paragraph 2.2.4). By Rellich-Kondrachov's theorem (Theorem 2.1.4) one has $W^{2,q}(B_{4R}) \hookrightarrow C^{1,\beta}(\overline{B}_{4R})$ for a suitable $\beta \in (0,1)$, so $\nabla v \in C^{0,\beta}(\overline{B}_{4R})$. In particular, the Calderón-Zygmund estimate (see (2.2.45) below) entails

$$\|\nabla v\|_{C^{0,\beta}(\overline{B}_{4R})} \le c\, \|f\|_{L^q(B_{4R})}. \tag{2.2.29}$$

Taking a smaller β if necessary, we apply [21, Corollary 5.2] to

$$-\Delta_p u = -\mathrm{div}\,(\nabla v) \quad \text{in } B_{4R},$$

obtaining

$$\begin{aligned}\||\nabla u|^{p-2}\nabla u\|_{C^{0,\beta}(\overline{B}_R)} &\le c_1\,\|\nabla v\|_{C^{0,\beta}(\overline{B}_{2R})} + c_2(R)\,\|\nabla u\|^{p-1}_{L^p(B_{2R})} \\ &\le c_1\,\|f\|_{L^q(B_{4R})} + c_2(R)\,\|\nabla u\|^{p-1}_{L^p(B_{2R})} < +\infty,\end{aligned} \tag{2.2.30}$$

because of (2.2.29) and the fact that $u \in W^{1,p}(B_{2R})$. We deduce

$$|\nabla u|^{p-2}\nabla u \in C^{0,\beta}(\overline{B}_R). \tag{2.2.31}$$

In particular there exists $M > 0$ such that

$$\|\nabla u\|_{L^\infty(B_R)} \le M. \tag{2.2.32}$$

Let us consider the bijective map $\Psi : \mathbb{R}^N \to \mathbb{R}^N$ defined as

$$\Psi(y) = \begin{cases} |y|^{\frac{2-p}{p-1}}\,y & \text{if } y \neq 0, \\ 0 & \text{if } y = 0, \end{cases}$$

as well as its inverse $\Psi^{-1}(y) = |y|^{p-2}y$. Inequalities (I) and (VII) of [111, Chapter 12] ensure that, for any $y_1, y_2 \in \mathbb{R}^N$,

$$\begin{aligned}&(\Psi^{-1}(y_1) - \Psi^{-1}(y_2)) \cdot (y_1 - y_2) \\ &\ge \begin{cases} 2^{2-p}|y_1 - y_2|^p & \text{if } p \ge 2, \\ (p-1)(1 + |y_1|^2 + |y_2|^2)^{\frac{p-2}{2}}|y_1 - y_2|^2 & \text{if } 1 < p < 2. \end{cases}\end{aligned} \tag{2.2.33}$$

In particular, if $|y_1| + |y_2| \le K$ for some $K > 0$, then the Schwartz inequality yields

$$|\Psi^{-1}(y_1) - \Psi^{-1}(y_2)| \ge \begin{cases} c(p)\,|y_1 - y_2|^{p-1} & \text{if } p \ge 2, \\ c(p,K)\,|y_1 - y_2| & \text{if } 1 < p < 2. \end{cases}$$

Hence $\Psi \in C^{0,\frac{1}{p-1}}(\mathbb{R}^N)$ for $p \ge 2$ and $\Psi \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^N)$ for $1 < p < 2$. Then (2.2.31)–(2.2.32) produce

$$\nabla u = \Psi(|\nabla u|^{p-2}\nabla u) \in C^{0,\alpha}(\overline{B}_R), \quad \alpha = \begin{cases} \frac{\beta}{p-1} & \text{if } p \ge 2, \\ \beta & \text{if } 1 < p < 2. \end{cases} \tag{2.2.34}$$

In particular,

$$\begin{aligned}&\|\nabla u\|_{C^{0,\alpha}(\overline{B}_R)} \\ &= \sup_{\substack{x,y \in B_R \\ x \neq y}} \frac{|\Psi(|\nabla u(x)|^{p-2}\nabla u(x)) - \Psi(|\nabla u(y)|^{p-2}\nabla u(y))|}{|x-y|^\alpha} \\ &\le c(p,u) \sup_{\substack{x,y \in B_R \\ x \neq y}} \left(\frac{||\nabla u(x)|^{p-2}\nabla u(x) - |\nabla u(y)|^{p-2}\nabla u(y)|}{|x-y|^\beta}\right)^{\frac{\alpha}{\beta}} \\ &\le c(p,u)\||\nabla u|^{p-2}\nabla u\|^{\frac{\alpha}{\beta}}_{C^{0,\beta}(\overline{B}_R)},\end{aligned} \tag{2.2.35}$$

with (cf. (2.2.33))

$$c(p,u) = \begin{cases} 2^{p-2} & \text{if } p \geq 2, \\ \frac{1}{p-1}(1+2\|\nabla u\|^2_{L^\infty(B_R)})^{\frac{2-p}{2}} & \text{if } 1 < p < 2. \end{cases} \tag{2.2.36}$$

□

Theorem 2.2.12. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, $N \geq 2$, with $\partial\Omega$ of class $C^{1,\gamma}$, $\gamma \in (0,1)$, and $p \in (1,+\infty)$. Let $u \in W^{1,p}_0(\Omega)$ be a weak solution to the Dirichlet problem (2.2.22) with $f \in L^q(\Omega)$, $q > N$. Then one has $u \in C^{1,\alpha}(\overline{\Omega})$, for a sufficiently small $\alpha \in (0,1)$. More precisely,*

$$\|\nabla u\|_{C^{0,\alpha}(\overline{\Omega})} \leq c\, \|f\|^{\frac{1}{p-1}}_{L^q(\Omega)}. \tag{2.2.37}$$

Proof. The proof of the first part follows by adapting the argument used for Theorem 2.2.11. Regarding estimate (2.2.37), notice that inequality (2.2.30) is replaced by

$$\||\nabla u|^{p-2}\nabla u\|_{C^{0,\beta}(\overline{\Omega})} \leq c\, \|\nabla v\|_{C^{0,\beta}(\overline{\Omega})} \leq c\, \|f\|_{L^q(\Omega)}, \tag{2.2.38}$$

according to [22, Corollary 2.7]. If $p \geq 2$, then a computation analogous to (2.2.35), besides (2.2.34), (2.2.36), and (2.2.38), entails

$$\|\nabla u\|_{C^{0,\alpha}(\overline{\Omega})} \leq 2^{p-2} \||\nabla u|^{p-2}\nabla u\|^{\frac{1}{p-1}}_{C^{0,\beta}(\overline{\Omega})} \leq c\, \|f\|^{\frac{1}{p-1}}_{L^q(\Omega)}. \tag{2.2.39}$$

Let us suppose $p \in (1,2)$. It is not restrictive to assume $\|f\|_{L^q(\Omega)} = 1$: indeed, setting $\lambda := \|f\|^{\frac{1}{1-p}}_{L^q(\Omega)}$, we have

$$-\Delta_p(\lambda u) = \lambda^{p-1}(-\Delta_p u) = \lambda^{p-1} f \quad \text{and} \quad \|\nabla(\lambda u)\|_{C^{0,\alpha}(\overline{\Omega})} = \lambda \|\nabla u\|_{C^{0,\alpha}(\overline{\Omega})}.$$

Since $\|\lambda^{p-1} f\|_{L^q(\Omega)} = 1$, then (2.2.37) would give $\|\nabla u\|_{C^{0,\alpha}(\Omega)} \leq c\lambda^{-1}$, which is exactly (2.2.37) in the general case.
Reasoning as above, besides exploiting (2.2.38)–(2.2.39) and $\|f\|_{L^q(\Omega)} = 1$, we get

$$\begin{aligned} \|\nabla u\|_{C^{0,\alpha}(\overline{\Omega})} &\leq \frac{1}{p-1}(1+2\|\nabla u\|^2_{L^\infty(\Omega)})^{\frac{2-p}{2}} \||\nabla u|^{p-2}\nabla u\|_{C^{0,\beta}(\overline{\Omega})} \\ &\leq \frac{1}{p-1}\left(1 + c\|f\|^{\frac{2}{p-1}}_{L^q(\Omega)}\right)^{\frac{2-p}{2}} \|f\|_{L^q(\Omega)} \leq c\, \|f\|^{\frac{1}{p-1}}_{L^q(\Omega)}. \end{aligned}$$

□

2.2.3 Lipschitz regularity

Let $\Omega \subseteq \mathbb{R}^N$ be a domain, $N \geq 2$, and $p \in (1, +\infty)$. Whenever we are not in the position to apply Theorem 2.2.7, for instance treating reaction terms with low summability or dealing with unbounded domains, we have no information about boundedness of the gradients of solutions. Anyway, in some particular situations it is possible to produce an a priori estimate on $\|\nabla u\|_\infty$, where $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ is a distributional solution to the problem

$$-\mathrm{div}\, a(\nabla u) = f(x) \in L^{(p^*)'}_{\mathrm{loc}}(\Omega). \tag{2.2.40}$$

We suppose $a : \mathbb{R}^N \to \mathbb{R}^N$ to be a C^0 function, whose restriction to $\mathbb{R}^N \setminus \{0\}$ is C^1, satisfying the following structure conditions:

$$(\mathrm{a}_1) \qquad \mu^T \, \nabla a(\xi) \, \mu \geq \gamma_1 |\xi|^{p-2} |\mu|^2,$$

$$(\mathrm{a}_2) \qquad |\nabla a(\xi)| \leq \gamma_2 |\xi|^{p-2},$$

$$(\mathrm{a}_3) \qquad |a(\xi)| \leq \gamma_2 |\xi|^{p-1},$$

for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and $\mu \in \mathbb{R}^N$, being $0 < \gamma_1 \leq \gamma_2$. The a priori estimate we are going to present is provided by [63], and its proof is based on nonlinear potential theory. Accordingly, for any $f \in L^2_{\mathrm{loc}}(\Omega)$ we define the nonlinear potential

$$P_f(x,R) := \int_0^R \left(\frac{|f|^2(B(x,\rho))}{\rho^{N-2}} \right)^{\frac{1}{2}} \frac{\mathrm{d}\rho}{\rho}, \quad \text{with } \; |f|^2(B(x,\rho)) := \|f\|^2_{L^2(B(x,\rho))}.$$

Theorem 2.2.13. *Suppose* (a_1)–(a_3). *Let* $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ *be a distributional solution to* (2.2.40) *with* $f \in L^r_{\mathrm{loc}}(\Omega)$, $r := \max\{2, (p^*)'\}$. *Then there exists* $c = c(N, p, \gamma_1, \gamma_2) > 0$ *such that*

$$\|\nabla u\|_{L^\infty(B_R)} \leq c \left(\frac{1}{|B_{2R}|} \int_{B_{2R}} |\nabla u|^p \, \mathrm{d}x \right)^{\frac{1}{p}} + c \|P_f(\cdot, 2R)\|^{\frac{1}{p-1}}_{L^\infty(B_{2R})}$$

for any $B_{2R} \Subset \Omega$.

Corollary 2.2.14. *Let* $u \in \mathcal{D}^{1,p}_0(\mathbb{R}^N)$ *be a distributional solution to*

$$-\Delta_p u = f(x) \in L^r(\mathbb{R}^N), \quad r > N.$$

Then $\nabla u \in L^\infty(\mathbb{R}^N)$. *More precisely, there exists* $c = c(N,p) > 0$ *such that*

$$\|\nabla u\|^{p-1}_{L^\infty(\mathbb{R}^N)} \leq c(\|\nabla u\|^{p-1}_{L^p(\mathbb{R}^N)} + \|f\|_{L^r(\mathbb{R}^N)}).$$

Proof. Pick any $x \in \mathbb{R}^N$. Set $B_R := B(x, R)$ for all $R > 0$. By Theorem 2.2.13 and Hölder's inequality, after observing that $r > N \geq \max\{2, (p^*)'\}$, we get

$$\begin{aligned}
|\nabla u(x)|^{p-1} &\leq \|\nabla u\|_{L^\infty(B_1)}^{p-1} \\
&\leq c \left(\frac{1}{|B_2|} \int_{B_2} |\nabla u|^p \, \mathrm{d}x \right)^{\frac{1}{p'}} + c \, \|P_f(\cdot, 2)\|_{L^\infty(B_2)} \\
&\leq c \, \|\nabla u\|_{L^p(\mathbb{R}^N)}^{p-1} + c \sup_{y \in B_2} \int_0^2 \rho^{-\frac{N}{2}} \|f\|_{L^2(B(y,\rho))} \, \mathrm{d}\rho \\
&\leq c \, \|\nabla u\|_{L^p(\mathbb{R}^N)}^{p-1} + c \, \|f\|_{L^r(\mathbb{R}^N)} \int_0^2 \rho^{-\frac{N}{r}} \, \mathrm{d}\rho \\
&\leq c \, (\|\nabla u\|_{L^p(\mathbb{R}^N)}^{p-1} + \|f\|_{L^r(\mathbb{R}^N)}).
\end{aligned}$$

Taking the supremum in $x \in \mathbb{R}^N$ on the left yields the conclusion. □

Remark 2.2.15. As observed in [63, p.1363], assumption $r > (p^*)'$ in Theorem 2.2.13 is irrelevant in the proof, but it guarantees that u is a weak solution, and not merely a *very weak solution*; in the latter case, an approximation procedure leads to the existence of a very weak solution $u \in W^{1,p-1}(\Omega)$ to (2.2.40). For a thorough treatment on approximable solutions, see [44].

Now we turn to regularity up to the boundary. We mention the following counterpart of Corollary 2.2.14, due to Cianchi and Maz'ya (see [43, Theorem 3.1 and Remark 3]), which investigates the Dirichlet problem

$$\begin{cases} -\operatorname{div} a(\nabla u) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2.41}$$

and the Neumann problem

$$\begin{cases} -\operatorname{div} a(\nabla u) = f(x) & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2.42}$$

where Ω is bounded domain of class $C^{1,\gamma}$, $\gamma \in (0, 1]$, with outer normal ν, and

$$f \in L^r(\Omega), \quad r > N. \tag{2.2.43}$$

In this case a is required to possess Uhlenbeck structure, i.e.,

$$a(\xi) = a_0(|\xi|) \, \xi, \tag{2.2.44}$$

being $a_0 : [0, +\infty) \to [0, +\infty)$ whose restriction to $(0, +\infty)$ is C^1, and the following structural hypotheses:

$$(\mathrm{a}_1') \qquad -1 < i_a := \inf_{t>0} \frac{t a_0'(t)}{a_0(t)} \le \sup_{t>0} \frac{t a_0'(t)}{a_0(t)} =: s_a < +\infty,$$

$$(\mathrm{a}_2') \qquad c_1 t^{p-1} \le t a_0(t) \le c_2(t^{p-1}+1) \quad \forall\, t > 0,$$

where $0 < c_1 \le c_2$.

Theorem 2.2.16. *Let* (2.2.43)–(2.2.44) *and* (a$_1'$)–(a$_2'$) *be satisfied. Let* $u \in W_0^{1,p}(\Omega)$ *(resp.,* $u \in W^{1,p}(\Omega)$*) be a weak solution to* (2.2.41) *(resp.,* (2.2.42)*). Then there exists* $c = c(\Omega, p) > 0$ *such that*

$$\|\nabla u\|_\infty \le c\, \|f\|_r^{\frac{1}{p-1}}.$$

2.2.4 Differentiability and compactness results

Let $\Omega \subseteq \mathbb{R}^N$ be a domain, $N \ge 2$, and $p \in (1, +\infty)$. Another branch of regularity theory, often called L^p-theory, investigates the summability of higher order derivatives of weak solutions under a summability assumption on the datum. The theory moves from the Calderón-Zygmund theorem, which asserts that any $u \in W_0^{1,2}(\Omega)$ weak solution to

$$\begin{cases} -\Delta u = f(x) \in L^r(\Omega) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

belongs to $W^{2,r}(\Omega)$ and satisfies the estimate

$$\|u\|_{2,r} \le C\, \|f\|_r. \tag{2.2.45}$$

A proof, based on Harmonic Analysis, can be found in [81, Theorem 9.9]. In the nonlinear context, much less can be said. To the best of our knowledge, the only positive result available il literature concerns the p-Laplacian in the singular case $p \in (1,2)$: in [55, 152] the authors prove that the solutions of

$$-\Delta_p u = f(x) \tag{2.2.46}$$

belong to $W^{2,p}(\mathbb{R}^N)$, provided $f \in L^{p'}(\mathbb{R}^N)$.
In the degenerate case $p > 2$, some differentiability results for the field

$$\mathcal{V} := |\nabla u|^{\frac{p-2}{2}} \nabla u$$

are provided, e.g., by Mingione [121, 122], and they lead to fractional differentiability of ∇u. Another, maybe more natural, way to get differentiability results is to study the so-called stress field

$$V := |\nabla u|^{p-2} \nabla u.$$

Nowadays, a well-known conjecture for solutions to (2.2.46) is

$$f \in L^r_{\mathrm{loc}}(\Omega) \overset{?}{\Leftrightarrow} V \in W^{1,r}_{\mathrm{loc}}(\Omega;\mathbb{R}^N). \tag{2.2.47}$$

It is readily seen that all of the differentiability results discussed above allow to gain compactness, because of the Rellich-Kondrachov theorem (see Theorem 2.1.4). Here we state two results about conjecture (2.2.47) and a compactness argument based on Theorem 2.1.6.

We start by reporting a theorem (see [45]) which confirms (2.2.47) for $r = 2$. Let us consider the problem

$$-\mathrm{div}\, a(\nabla u) = f(x), \tag{2.2.48}$$

with a possessing Uhlenbeck structure (2.2.44) and obeying (a_1')–(a_2') in paragraph 2.2.3.

Theorem 2.2.17. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a distributional solution to* (2.2.48) *with $f \in L^r_{\mathrm{loc}}(\Omega)$, $r := \max\{2,(p^*)'\}$. Then $a(\nabla u) \in W^{1,2}_{\mathrm{loc}}(\Omega)$. More precisely, there exists $c = c(N, i_a, s_a) > 0$ such that*

$$\|a(\nabla u)\|_{W^{1,2}(B_R)} \le c\,(R^{-\frac{N}{2}} + R^{-\frac{N}{2}-1})\|a(\nabla u)\|_{L^1(B_{2R})} + c\,\|f\|_{L^2(B_{2R})}$$

for any $B_{2R} \Subset \Omega$.

Remark 2.2.15 holds also for Theorem 2.2.17.

Theorem 2.2.17 can be improved (see [92]), showing that conjecture (2.2.47) holds true whenever p is sufficiently near 2, in the spirit of [119].

Theorem 2.2.18. *Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ be a distributional solution to* (2.2.48) *with $f \in L^r_{\mathrm{loc}}(\Omega)$. Then $a(\nabla u) \in W^{1,r}_{\mathrm{loc}}(\Omega)$, provided $|p-2| < \delta$, with $\delta = \delta(r,N,p)$ small enough. More precisely, there exists $c = c(N,r,R) > 0$ such that*

$$\|a(\nabla u)\|_{W^{1,r}(B_R)} \le c\,(\|a(\nabla u)\|_{L^r(B_{2R})} + \|f\|_{L^r(B_{2R})})$$

for any $B_{4R} \Subset \Omega$.

We conclude the paragraph by proving an L^p compactness result for gradients of solutions, a very useful tool to pass to the limit convection terms. We set, for any $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$, $x \in B_R \Subset \Omega$, and $h \in \mathbb{R}^N$ such that $|h| < d(B_R, \partial\Omega)$,

$$u_h(x) := u(x+h), \quad \delta_h u := u_h - u.$$

Theorem 2.2.19. *Let $\{u_n\} \subseteq W^{1,p}_{\mathrm{loc}}(\Omega)$ and $\{f_n\} \subseteq L^{r'}_{\mathrm{loc}}(\Omega)$ such that u_n is a distributional solution to*

$$-\Delta_p u_n = f_n \quad \text{in } \Omega \tag{2.2.49}$$

for all $n \in \mathbb{N}$. Suppose that, for some $M > 0$ and $r \in (1, +\infty)$,

(H$_1$) $\{\nabla u_n\}$ *is bounded in* $L^p_{\mathrm{loc}}(\Omega)$,

(H$_2$) $\{f_n\}$ *is bounded in* $L^{r'}_{\mathrm{loc}}(\Omega)$,

(H$_3$) $u_n \to u$ *in* $L^p_{\mathrm{loc}}(\Omega) \cap L^r_{\mathrm{loc}}(\Omega)$.

Then $\{\nabla u_n\}$ admits a strongly convergent subsequence in $L^p_{\mathrm{loc}}(\Omega)$.

Proof. Fix $R > 0$ such that $B_R \Subset \Omega$. A density argument proves that

$$\int_{B_R} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi \, \mathrm{d}x = \int_{B_R} f_n \varphi \, \mathrm{d}x \tag{2.2.50}$$

for any $n \in \mathbb{N}$ and $\varphi \in W^{1,p}_0(B_R)$. Now pick $t, s > 0$ such that $B_t \Subset B_s \Subset B_R$ and $\eta \in C^\infty_c(B_s)$, $\eta \geq 0$, such that $\eta \equiv 1$ in B_t and $|\nabla \eta| \leq \frac{c}{s-t}$. For all $n \in \mathbb{N}$ set $V_n := |\nabla u_n|^{p-2} \nabla u_n$. Using (2.2.50) with $\varphi := \eta^2 \delta_h u_n$, with $|h| < R - s$, gives

$$\int_{B_R} \eta^2 \, V_n \cdot \delta_h(\nabla u_n) \, \mathrm{d}x + 2 \int_{B_R} \eta \, \delta_h u_n \, V_n \cdot \nabla \eta \, \mathrm{d}x = \int_{B_R} f_n \varphi \, \mathrm{d}x, \tag{2.2.51}$$

Analogously, exploiting (2.2.50) with φ_{-h} and performing the change of variable $x \mapsto x + h$ on the left-hand side, besides recalling that $B_{s+|h|} \subseteq B_R$, furnish

$$\int_{B_R} \eta^2 \, (V_n)_h \cdot \delta_h(\nabla u_n) \, \mathrm{d}x + 2 \int_{B_R} \eta \, \delta_h u_n \, (V_n)_h \cdot \nabla \eta \, \mathrm{d}x = \int_{B_R} f_n \varphi_{-h} \, \mathrm{d}x. \tag{2.2.52}$$

Subtracting (2.2.51) from (2.2.52) yields

$$\int_{B_R} \eta^2 \, \delta_h V_n \cdot \delta_h(\nabla u_n) \, \mathrm{d}x + 2 \int_{B_R} \eta \, \delta_h u_n \, \delta_h V_n \cdot \nabla \eta \, \mathrm{d}x = \int_{B_R} f_n \delta_{-h} \varphi \, \mathrm{d}x.$$

Using the monotonicity of the p-Laplacian (cf. (2.2.33)) and rearranging the terms produces the estimate

$$
\begin{aligned}
&\int_{B_t} \delta_h V_n \cdot \delta_h(\nabla u_n)\,\mathrm{d}x \le \int_{B_R} \eta^2\,\delta_h V_n \cdot \delta_h(\nabla u_n)\,\mathrm{d}x \\
&\le 2\int_{B_R} |\delta_h u_n||\delta_h V_n||\nabla\eta|\,\mathrm{d}x + \int_{B_R} |f_n||\delta_{-h}\varphi|\,\mathrm{d}x \\
&\le \frac{c}{s-t}\,\|\delta_h u_n\|_{L^p(B_R)}\|\delta_h V_n\|_{L^{p'}(B_R)} + \|f_n\|_{L^{r'}(B_R)}\|\delta_{-h}\varphi\|_{L^r(B_R)} \\
&\le \frac{2c}{s-t}\,\|\delta_h u_n\|_{L^p(B_R)}\|V_n\|_{L^{p'}(B_R)} + 2\,\|f_n\|_{L^{r'}(B_R)}\|\delta_h u_n\|_{L^r(B_R)} \\
&\le c\,(\|\delta_h u_n\|_{L^p(B_R)}\|\nabla u_n\|^{p-1}_{L^p(B_R)} + \|f_n\|_{L^{r'}(B_R)}\|\delta_h u_n\|_{L^r(B_R)}),
\end{aligned}
\tag{2.2.53}
$$

being $c = c(N,t,s) > 0$. Notice that the last member of (2.2.53) vanishes uniformly in n when $h \to 0^+$ because of (H_1)–(H_3) and the Fréchet-Riesz-Kolmogorov theorem (Theorem 2.1.6). The first member of (2.2.53) can be estimated from below, but we have to distinguish two cases: $p \ge 2$ and $p \in (1,2)$.

Case 1. For $p \ge 2$ we get (see (2.2.33))

$$
\begin{aligned}
&\int_{B_t} \delta_h V_n \cdot \delta_h(\nabla u_n)\,\mathrm{d}x \\
&= \int_{B_t} (|\nabla(u_n)_h|^{p-2}\nabla(u_n)_h - |\nabla u_n|^{p-2}\nabla u_n)\cdot(\nabla(u_n)_h - \nabla u_n)\,\mathrm{d}x \\
&\ge c\,\|(\nabla u_n)_h - \nabla u_n\|^p_{L^p(B_t)} = c\,\|\delta_h(\nabla u_n)\|^p_{L^p(B_t)}.
\end{aligned}
\tag{2.2.54}
$$

By (2.2.53)–(2.2.54) we obtain $\delta_h(\nabla u_n) \to 0$ in $L^p(B_t)$ uniformly in n when $h \to 0^+$, so Theorem 2.1.6 completes the proof in this case.

Case 2. For $p \in (1,2)$ we obtain (see (2.2.33))

$$
\begin{aligned}
&\int_{B_t} \delta_h V_n \cdot \delta_h(\nabla u_n)\,\mathrm{d}x \\
&= \int_{B_t} (|\nabla(u_n)_h|^{p-2}\nabla(u_n)_h - |\nabla u_n|^{p-2}\nabla u_n)\cdot(\nabla(u_n)_h - \nabla u_n)\,\mathrm{d}x \\
&\ge c\int_{B_t} (1 + |\nabla(u_n)_h|^2 + |\nabla u_n|^2)^{\frac{p-2}{2}}|\nabla(u_n)_h - \nabla u_n|^2\,\mathrm{d}x \\
&= c\int_{B_t} W_{nh}\,|\delta_h(\nabla u_n)|^2\,\mathrm{d}x,
\end{aligned}
\tag{2.2.55}
$$

with $W_{nh}(x) := (1 + |\nabla(u_n)_h|^2 + |\nabla u_n|^2)^{\frac{p-2}{2}}$. Hölder's inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$, as well as (H$_1$), yields

$$\begin{aligned}
\|\delta_h(\nabla u_n)\|^p_{L^p(B_t)} &= \int_{B_t} W_{nh}^{\frac{p}{2}} \, |\delta_h(\nabla u_n)|^p \, W_{nh}^{-\frac{p}{2}} \, \mathrm{d}x \\
&\leq \left(\int_{B_t} W_{nh} \, |\delta_h(\nabla u_n)|^2 \, \mathrm{d}x \right)^{\frac{p}{2}} \left(\int_{B_t} W_{nh}^{\frac{p}{p-2}} \, \mathrm{d}x \right)^{\frac{2-p}{2}} \\
&\leq \left(\int_{B_t} W_{nh} \, |\delta_h(\nabla u_n)|^2 \, \mathrm{d}x \right)^{\frac{p}{2}} \left(|B_t| + 2 \, \|\nabla u_n\|^p_{L^p(B_R)} \right)^{\frac{2-p}{2}} \\
&\leq c \left(\int_{B_t} W_{nh} \, |\delta_h(\nabla u_n)|^2 \, \mathrm{d}x \right)^{\frac{p}{2}} .
\end{aligned} \tag{2.2.56}$$

Now the conclusion follows by (2.2.53) and (2.2.55)–(2.2.56), reasoning as in the first case. □

Remark 2.2.20. If $r < p^*$, then Theorem 2.2.19 can be proved also (in a less direct way) by exploiting a result by Boccardo and Murat [10] which ensures, under the hypotheses of Theorem 2.2.19 with $r < p^*$, that

$$\nabla u_n \to \nabla u \quad \text{in} \ \ L^q_{\mathrm{loc}}(\Omega), \quad \forall \, q \in (1,p). \tag{2.2.57}$$

In particular, (2.2.57) implies $\nabla u_n \to \nabla u$ a.e. in Ω. Now we report this alternative proof of Theorem 2.2.19.
Hypothesis (H$_2$) furnishes $f \in L^{r'}(E)$ such that $f_n \rightharpoonup f$ in $L^{r'}(E)$, by reflexivity. Then (2.2.57), together with (H$_1$) and Lebesgue's dominated convergence theorem, allows to pass to the limit in (2.2.50), producing

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega f \varphi \, \mathrm{d}x \tag{2.2.58}$$

for any $E \Subset \Omega$ and $\varphi \in W^{1,p}_0(E)$. Fix any $\varphi \in C^\infty_c(\Omega)$ and let $E := \operatorname{supp} \varphi$. Using (2.2.50) with $u_n \varphi$ in place of φ we get

$$\int_\Omega |\nabla u_n|^p \varphi \, \mathrm{d}x + \int_\Omega u_n \, |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega f_n u_n \varphi \, \mathrm{d}x. \tag{2.2.59}$$

Similarly, exploiting (2.2.58) with $u\varphi$ instead of φ gives

$$\int_\Omega |\nabla u|^p \varphi \, \mathrm{d}x + \int_\Omega u \, |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega f u \varphi \, \mathrm{d}x. \tag{2.2.60}$$

Observe that

$$\| |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi \|_{L^{p'}(\Omega)} \leq \|\nabla \varphi\|_{L^\infty(\Omega)} \|\nabla u_n\|^{p-1}_{L^p(\Omega)},$$

as well as $|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi \to |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi$ a.e. in Ω by (2.2.57). Thus, [23, Exercise 4.16] and (H_1) ensure that $|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi \rightharpoonup |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi$ in $L^{p'}(\Omega)$. Through (H_3) we get

$$\lim_{n\to\infty} \int_\Omega u_n \, |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega u \, |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \,\mathrm{d}x. \tag{2.2.61}$$

On the other hand, (H_3) produces

$$\int_E |u_n\varphi - u\varphi|^r \,\mathrm{d}x \le \|\varphi\|_\infty^r \int_E |u_n - u|^r \,\mathrm{d}x \to 0,$$

proving that $u_n\varphi \to u\varphi$ in $L^r(E)$. Then

$$\lim_{n\to\infty} \int_\Omega f_n u_n \varphi \,\mathrm{d}x = \int_\Omega f u \varphi \,\mathrm{d}x. \tag{2.2.62}$$

Passing to the limit (2.2.59) through (2.2.61)–(2.2.62) and subtracting (2.2.60) we get

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^p \varphi \,\mathrm{d}x = \int_\Omega |\nabla u|^p \varphi \,\mathrm{d}x. \tag{2.2.63}$$

Now take any $K \Subset \Omega$ and $\varphi \in \mathcal{C}_K := \{\psi \in C_c^\infty(\Omega) : \chi_K \le \psi\}$. By (2.2.63) we have

$$\limsup_{n\to\infty} \int_K |\nabla u_n|^p \,\mathrm{d}x \le \limsup_{n\to\infty} \int_\Omega |\nabla u_n|^p \varphi \,\mathrm{d}x = \int_\Omega |\nabla u|^p \varphi \,\mathrm{d}x.$$

Taking the infimum in $\varphi \in \mathcal{C}_K$, we get

$$\limsup_{n\to\infty} \int_K |\nabla u_n|^p \,\mathrm{d}x \le \int_K |\nabla u|^p \,\mathrm{d}x.$$

Hence, according to (H_1) and [23, Proposition 3.32], up to subsequences we get $\nabla u_n \to \nabla u$ in $L^p(K)$. Since K was arbitrary, we conclude $\nabla u_n \to \nabla u$ in $L^p_{\mathrm{loc}}(\Omega)$.

Remark 2.2.21. For the sake of completeness, we list all the results contained in this section according to the typology of the domain Ω.

- Results holding for a possibly unbounded Ω: Theorem 2.2.5 and Corollary 2.2.6, Theorem 2.2.11, Theorem 2.2.13, Theorem 2.2.17, Theorem 2.2.18, Theorem 2.2.19.

- Results holding for a bounded Ω: Theorem 2.2.2 and Corollary 2.2.3, Theorem 2.2.7 and Corollary 2.2.8, Theorem 2.2.10, Theorem 2.2.12, Theorem 2.2.16.

- Results holding for $\Omega = \mathbb{R}^N$: Theorem 2.2.2, Corollary 2.2.14.

2.3 Monotonicity methods

2.3.1 Pseudo-monotone operators

Definition 2.3.1. Let $(X, \|\cdot\|)$ be a Banach space and $A : X \to X^*$ be an operator. A is said to be

- *coercive* if $\frac{\langle A(u),u\rangle}{\|u\|} \to +\infty$ whenever $\|u\| \to +\infty$,
- *monotone* if $\langle A(u) - A(v), u - v\rangle \geq 0$ for all $u, v \in X$,
- *strictly monotone* if $\langle A(u) - A(v), u - v\rangle > 0$ for all $u, v \in X$, $u \neq v$,
- *uniformly monotone* if $\langle A(u) - A(v), u - v\rangle \geq a(\|u - v\|)\|u - v\|$ for all $u, v \in X$, where $a : [0, +\infty) \to [0, +\infty)$ is a strictly increasing function satisfying $a(0) = 0$ and $a(t) \to +\infty$ for $t \to +\infty$,
- *pseudo-monotone* if
$$u_n \rightharpoonup u \quad \text{and} \quad \limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \leq 0 \tag{2.3.1}$$
imply, for all $v \in X$,
$$\langle A(u), u - v\rangle \leq \liminf_{n\to\infty}\langle A(u_n), u_n - v\rangle, \tag{2.3.2}$$
- a (S_+) *operator* if (2.3.1) implies $u_n \to u$.

If A is bounded, then (2.3.2) can be replaced with

$$A(u_n) \rightharpoonup A(u) \quad \text{and} \quad \langle A(u_n), u_n\rangle \to \langle A(u), u\rangle.$$

We observe that

$$\begin{aligned} &A \text{ uniformly monotone } \Rightarrow A \text{ strictly monotone} \\ &\Rightarrow A \text{ monotone } \Rightarrow A \text{ pseudo-monotone}\end{aligned}$$

and ([69, Lemma 6.5])

$$A \text{ uniformly monotone } \Rightarrow A \text{ coercive, } (S_+) \text{ operator.}$$

Another important result is the following.

Lemma 2.3.2. *Let $(X, \|\cdot\|)$ be a Banach space. Suppose that $A : X \to X^*$ is a (S_+) operator and $B : X \to X^*$ is a compact operator. Then $C := A + B$ is a (S_+) operator. If, in addition, C is demi-continuous, then C is pseudo-monotone.*

Proof. Let $\{u_n\} \subseteq X$ and $u \in X$ such that $u_n \rightharpoonup u$ in X and

$$\limsup_{n\to\infty} \langle C(u_n), u_n - u\rangle \leq 0. \tag{2.3.3}$$

Up to subsequences, compactness of B produces $B(u_n) \to \psi$ in X^* for some $\psi \in X^*$. So (2.3.3) reads as

$$0 \geq \limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle + \lim_{n\to\infty} \langle B(u_n), u_n - u\rangle = \limsup_{n\to\infty} \langle A(u_n), u_n - u\rangle.$$

Hence, the (S_+) property of A gives $u_n \to u$ in X, as desired.
Now assume that C is demi-continuous and suppose (2.3.3). Then $u_n \to u$ in X, since C is of type (S_+), and $C(u_n) \rightharpoonup C(u)$, by demi-continuity. Accordingly,

$$\begin{aligned}\liminf_{n\to\infty} \langle C(u_n), u_n - v\rangle &= \lim_{n\to\infty} \langle C(u_n), u_n - u\rangle + \lim_{n\to\infty} \langle C(u_n), u - v\rangle \\ &= \langle C(u), u - v\rangle.\end{aligned}$$

□

For further implications, we address to [69]. Here we also point out that strongly continuous operators are pseudo-monotone, and that the sum of two pseudo-monotone operators is pseudo-monotone.

Now we are ready to state the main theorem on pseudo-monotone operators; see [163, Theorem 27.A] and [112, Théorèmes 2.1-2.2].

Theorem 2.3.3. *Let $(X, \|\cdot\|)$ be a reflexive Banach space and let $A : X \to X^*$ be a bounded, continuous, coercive, pseudo-monotone operator. Then A is surjective.*

Theorem 2.3.3 has an important consequence.

Corollary 2.3.4 (Minty-Browder)**.** *Let $(X, \|\cdot\|)$ be a reflexive Banach space and let $A : X \to X^*$ be a bounded, continuous, coercive, strictly monotone operator. Then A is bijective.*

Some extensions are available also for closed convex subsets of X in place of X itself (see, e.g., [23, Problem 31]).

2.3.2 Maximum and comparison principles

The regularity results discussed in Section 2.2 can be viewed as a special investigation about a more general question: what can be said about the solutions of partial differential equations? Since, in general, it is impossible

to find solutions in explicit form, it should be good to have some information about their qualitative behavior.
In this section we will present some maximum principles, which furnish a sign information about super-solutions, as well as comparison theorems, which allow to ‘compare’ (in pointwise sense) sub- and super-solutions (say $\underline{u}$ and $\overline{u}$, respectively) of the same equation; also a boundary point lemma will be stated, giving a sign information on the normal derivative of a sub- or super-solution. Incidentally, we explicitly notice that comparison principles can be directly deduced by maximum principles only for linear operators.
The aforementioned principles will be given in two different forms: the weak form and the strong one. Weak forms produce non-strict inequalities (of the type $\overline{u} \geq 0$ for the maximum principle and $\underline{u} \leq \overline{v}$ for the comparison principle), while strong forms furnish strict inequalities (of type $\overline{u} > 0$ and $\underline{u} < \overline{v}$): this additional information is paid (essentially, but not only) by higher regularity of solutions.
The main reference about these topics is [143]: Chapter 3 treats the weak maximum principle and the weak comparison principle, while Chapter 5 concerns their strong forms.

Let $\Omega \subseteq \mathbb{R}^N$ be a domain, $N \geq 2$, and $p \in (1, +\infty)$. Throughout this paragraph we consider distributional sub- and super-solutions to problem (P) of paragraph 2.1.3, making the following structural assumptions on $\mathcal{A} : \Omega \times (0, +\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ and $\mathcal{B} : \Omega \times (0, +\infty) \times \mathbb{R}^N \to \mathbb{R}$:

$$(\mathrm{A}_1) \qquad |\mathcal{A}(x, s, \xi)| \leq a_1 |\xi|^{p-1} + a_2 s^{p-1} + \psi(x),$$

$$(\mathrm{A}_2) \qquad \mathcal{A}(x, s, \xi) \cdot \xi \geq a_3 |\xi|^p - a_4 s^p,$$

$$(\mathrm{B}) \qquad \mathcal{B}(x, s, \xi) \geq -(b_1 |\xi|^{p-1} + b_2 s^{p-1}),$$

for all $(x, s, \xi) \in \Omega \times (0, +\infty) \times \mathbb{R}^N$, being $\psi \in L^{p'}(\Omega)$.

Before stating the weak maximum principle, we recall that the definition of ‘$u \geq 0$ on $\partial\Omega$’ can be found in Remark 2.1.11.

Theorem 2.3.5 (Weak maximum principle)**.** *Suppose that Ω is bounded, as well as* (A$_1$)–(A$_2$) *and* (B) *are satisfied with $a_4 = b_2 = 0$ or $b_1 = b_2 = 0$. Let $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ a distributional super-solution to* (P)*. If $u \geq 0$ on $\partial\Omega$, then $u \geq 0$ in Ω. The same conclusion holds true if Ω is unbounded, provided*

$$\liminf_{\substack{x \in \Omega \\ |x| \to \infty}} u(x) \geq 0.$$

Theorem 2.3.5 is proved in [143, Theorems 3.2.1 and 3.2.2] (see also ibid., p.58, Remark 3). Hypothesis (B) can substituted by a dual one to obtain a weak maximum principle for sub-solutions.

In order to compare sub- and super-solutions of (P) we have to require, in particular, some monotonicity hypotheses on $\mathcal{A}$ and $\mathcal{B}$. We have the following result (cf. [143, Theorem 3.4.1]).

Theorem 2.3.6 (Weak comparison principle)**.** *Let $\mathcal{A} = \mathcal{A}(x,\xi)$ and $\mathcal{B} = \mathcal{B}(x,s)$, with $\mathcal{A}$ strictly monotone in ξ and $\mathcal{B}$ non-increasing in s. Moreover, suppose that Ω is bounded and* (A_1) *is satisfied. Let $u,v \in W^{1,p}_{\mathrm{loc}}(\Omega)$, being u a distributional sub-solution and v a distributional super-solution to* (P)*. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω. The same conclusion holds true if Ω is unbounded, provided*

$$\liminf_{\substack{x\in\Omega\\ |x|\to\infty}} (u(x) - v(x)) \geq 0.$$

Now we discuss the strong forms of maximum and comparison principles. To do this, we particularize our problem considering

$$-\operatorname{div} a(\nabla u) = b(x,u,\nabla u) \quad \text{in } \ \Omega, \tag{2.3.4}$$

where a possesses the Uhlenbeck structure (2.2.44), and $b \in L^\infty_{\mathrm{loc}}(\Omega\times(0,+\infty)\times\mathbb{R}^N)$. Setting $\omega(s) := sa_0(s)$, we further assume

(a) ω is strictly increasing in $(0,+\infty)$ and $\displaystyle\lim_{s\to 0^+} \omega(s) = 0$,

(b) $\mathcal{B}(x,s,\xi) \geq -c\,\omega(|\xi|) - f(s)$,

$$f \in C^0([0,+\infty)), \quad f(0) = 0, \quad f_{|(0,\delta)} \text{ non-decreasing for some } \delta > 0.$$

We also set

$$H(s) := s\omega(s) - \int_0^s \omega(t)\,\mathrm{d}t, \quad F(s) := \int_0^s f(t)\,\mathrm{d}t.$$

Theorem 2.3.7 (Strong maximum principle)**.** *Let* (a) *and* (b) *be satisfied, and let $u \in C^1(\Omega)$ be a distributional super-solution to* (2.3.4) *such that $u \geq 0$ in Ω and $u(x_0) = 0$ for some $x_0 \in \Omega$. Moreover, suppose there exists $\varepsilon > 0$ such that either*

$$f(s) = 0 \quad \forall\, s \in (0,\varepsilon) \tag{2.3.5}$$

or

$$\int_0^\varepsilon \frac{\mathrm{d}s}{H^{-1}(F(s))} = +\infty. \tag{2.3.6}$$

Then $u \equiv 0$ in Ω.

As a consequence of Theorem 2.3.7, under hypotheses (a) and (b), besides either (2.3.5) or (2.3.6), any $u \in C^1(\Omega)$ non-trivial, non-negative, distributional super-solution to (2.3.4) is strictly positive in Ω.
For a proof of Theorem 2.3.7, see [143, Theorem 5.3.1]; a generalization for x-dependent operators can be found in ibid., Theorem 5.4.1.
As a consequence of Theorem 2.3.7 we get the following crucial information about the behavior of super-solutions to (2.3.4) on the boundary.

Theorem 2.3.8 (Boundary point lemma)**.** *Assume* (a) *and* (b)*. Let $u \in C^1(\overline{\Omega})$ be a distributional super-solution to* (2.3.4) *such that $u \geq 0$ in Ω and $u(x_0) = 0$ for some $x_0 \in \partial\Omega$. Moreover, suppose either* (2.3.5) *or* (2.3.6)*. If Ω satisfies the interior sphere condition at x_0, then $\partial_\nu u(x_0) < 0$.*

The proof of Theorem 2.3.8, in a slightly wider context, is given in [143, Theorem 5.5.1].

A general strong comparison principle for nonlinear operators is not available in literature, even for p-harmonic functions, as pointed out in [111, p.14]; hence, different versions of this result appeared in the last forty years: we mention Tolksdorf [158], Guedda-Veron [94], Cuesta-Takáč [52] and, more recently, Sciunzi [148]. Here we present a version for Dirichlet p-Laplacian problems by Arcoya and Ruiz; see [6, Proposition 2.6].

Theorem 2.3.9 (Strong comparison principle)**.** *Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a bounded domain of class C^2 with outer normal ν to $\partial\Omega$, $p \in (1, +\infty)$, $\lambda \geq 0$, $f, g \in L^\infty(\Omega)$, and $u, v \in W_0^{1,p}(\Omega)$ solutions to*

$$\begin{cases} -\Delta_p u + \lambda |u|^{p-2} u = f & \text{in } \Omega, \\ \qquad\qquad\quad u = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta_p v + \lambda |v|^{p-2} v = g & \text{in } \Omega, \\ \qquad\qquad\quad v = 0 & \text{on } \partial\Omega. \end{cases}$$

If for any compact $K \Subset \Omega$ there exists $\varepsilon_K > 0$ such that $g - f \geq \varepsilon_K$ almost everywhere in K, as well as $v > 0$ in Ω and $\partial_\nu v < 0$ on $\partial\Omega$, then $u < v$ in Ω and $\partial_\nu u > \partial_\nu v$ on $\partial\Omega$.

Actually, according to Theorems 2.2.2 and 2.2.7, $u, v \in C^{1,\alpha}(\overline{\Omega})$. In addition, the hypotheses $v > 0$ in Ω and $\partial_\nu v < 0$ on $\partial\Omega$ can be verified through Theorems 2.3.7–2.3.8 in case that $g \geq 0$ in Ω; notice also that these hypotheses can be substituted by dual assumptions on u (cf. [6, Remark 2.8]).

2.3.3 Sub-super-solutions and trapping region

The notion of sub- and super-solution is strongly related to existence results: it suffices to think about the Perron method for solving the Dirichlet-Laplace problem. This notion gains importance especially in the context of singular problems; indeed, as we will see in Chapter 3, there are mainly two methods for solving singular problems: using sub- and super-solutions to 'avoid' the singularities of the reaction term, or 'shifting' the values of the reaction by a quantity ε, obtaining in this way a family of *regular* (i.e., not singular) problems, and letting $\varepsilon \to 0^+$ to recover a solution to the main problem. Theorems which provide a solution by assuming the existence of a sub-solution and a super-solution are usually called *sub-super-solution theorems.*

Before stating a basic result of this type, let us discuss the sub-super-solution point of view with an example, partially reported in [26, Lemma 2.1]. A generalization will be exposed in paragraph 3.1.1.
Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, be a bounded domain of class C^2 and $p \in (1, +\infty)$. Consider the problem

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3.7}$$

where $f : \Omega \times (0, +\infty) \to (0, +\infty)$ is a Carathéodory function satisfying

$$\text{(f}_1\text{)} \qquad \lim_{s \to 0^+} f(x,s) = +\infty \quad \text{uniformly in } x \in \Omega,$$

$$\text{(f}_2\text{)} \qquad f(x,s) \leq c_1 s^{-\gamma} + c_2 s^{q-1}, \quad \text{with } 0 < \gamma < 1 \leq q < p.$$

Claim 1: under the only hypothesis (f_1), we claim that there exists $\underline{u} \in C_0^{1,\alpha}(\overline{\Omega})$ sub-solution to (2.3.7). To this aim, consider the *torsion problem*

$$\begin{cases} -\Delta_p e = 1 & \text{in } \Omega, \\ e > 0 & \text{in } \Omega, \\ e = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3.8}$$

Problem (2.3.8) admits a unique solution $e \in C^{1,\alpha}(\overline{\Omega})$: existence and uniqueness are a consequence of Theorems 2.3.4 and 2.3.7, while regularity is due to Corollaries 2.2.3 and 2.2.8. Moreover, (f_1) guarantees there exists $\delta > 0$ such that

$$f(x,s) \geq 1 \quad \forall\, (x,s) \in \Omega \times (0,\delta).$$

Set $M := \max\{\|e\|_\infty, \delta\}$ and $\underline{u} := \frac{\delta}{M}\, e$. We get

$$\underline{u} \leq \frac{\delta}{M} \|e\|_\infty \leq \delta$$

so, by homogeneity of the p-Laplacian,

$$-\Delta_p \underline{u} = \left(\frac{\delta}{M}\right)^{p-1} (-\Delta_p e) \leq 1 \leq f(x, \underline{u}) \quad \text{in } \Omega. \tag{2.3.9}$$

We also observe that Remark 2.1.9 implies the existence of $l > 0$ such that $e \geq l\, d$ in Ω, being d the distance function; so

$$\underline{u} \geq \frac{\delta}{M}\, l\, d =: m\, d \quad \text{in } \Omega. \tag{2.3.10}$$

Claim 2: under (f_1)–(f_2), we claim that there exists $\overline{u} \in C^{1,\alpha}(\overline{\Omega})$ super-solution to (2.3.7). For this purpose, consider the auxiliary problem

$$\begin{cases} -\Delta_p E = d^{-\gamma} & \text{in } \Omega, \\ E > 0 & \text{in } \Omega, \\ E = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3.11}$$

which admits a unique solution $E \in C^{1,\alpha}(\overline{\Omega})$: indeed, existence and uniqueness follow from Theorems 2.3.4 and 2.3.7, while regularity is guaranteed by Theorem 2.2.10. According to Remark 2.1.9, taking a smaller $l > 0$ if necessary, we have $E \geq ld$ in Ω. Setting $\overline{u} := \mu E$, for $\mu > 0$ sufficiently large we obtain

$$\begin{aligned} f(x, \overline{u}) &\leq c_1\, \mu^{-\gamma} E^{-\gamma} + c_2\, \mu^{q-1} E^{q-1} \leq c\, (\mu^{-\gamma} d^{-\gamma} + \mu^{q-1}) \\ &\leq c\, (\mu^{-\gamma} + \mu^{q-1})\, d^{-\gamma} \leq \mu^{p-1} d^{-\gamma} = -\Delta_p \overline{u}. \end{aligned} \tag{2.3.12}$$

Having this example in mind, we prove a sub-super-solution theorem for the Dirichlet boundary value problem

$$\begin{cases} -\operatorname{div} \mathcal{A}(x, u, \nabla u) = \mathcal{B}(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.3.13}$$

where Ω is a bounded domain of class $C^{0,1}$. Here $\mathcal{A} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ and $\mathcal{B} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory functions and $\mathcal{A}$ satisfies the structural conditions

(A_1) $$|\mathcal{A}(x, s, \xi)| \leq a_1 |\xi|^{p-1} + a_2 |s|^{p-1} + \psi(x),$$

(A_2) $$\mathcal{A}(x, s, \cdot) \text{ is strictly monotone,}$$

(A_3) $$\mathcal{A}(x, s, \xi) \cdot \xi \geq a_3 |\xi|^p - \eta(x),$$

for all $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, being $\psi \in L^{p'}(\Omega)$ and $\eta \in L^1(\Omega)$. Our statement is similar to the one of [31, Theorem 3.17], and the proof seems to be slightly simpler. Anyway, the two statements are not comparable: indeed, we admit that the coefficients of the local growth condition (see (B) below) belong to Lebesgue spaces; on the other hand, we cannot handle a p-linear growth in the gradient variable. Proving a sub-super-solution theorem in presence of p-linear reaction terms requires a penalization argument, and it can be done following [31].

Theorem 2.3.10 (Sub-super-solution theorem). *Suppose* (A_1)–(A_3) *to be satisfied, and let* $\underline{u}, \overline{u}$ *be a sub- and a super-solution to* (2.3.13) *satisfying* $\underline{u} \leq \overline{u}$. *Moreover, suppose the following local growth condition:*

(B) $$|\mathcal{B}(x,s,\xi)| \leq k(x)|\xi|^r + f(x),$$

for all $(x,s,\xi) \in \Omega \times [\underline{u},\overline{u}] \times \mathbb{R}^N$, *being* $r \in [0, p-1)$, $f \in L^{q'}(\Omega)$ *with* $q \in (1,p^*)$, $k \in L^t(\Omega)$ *with* $t \in (1,+\infty]$, *and*

$$\frac{1}{t} + \frac{r}{p} + \frac{1}{q} < 1. \tag{2.3.14}$$

Then (2.3.13) *admits a weak solution* $u \in W^{1,p}_0(\Omega)$ *such that* $u \in [\underline{u},\overline{u}]$ *(i.e.,* $\underline{u} \leq u \leq \overline{u}$ *in* Ω*).*

Proof. Consider the *truncation operator* $T : W^{1,p}_0(\Omega) \to W^{1,p}_0(\Omega)$ defined as

$$T(u)(x) := \begin{cases} \underline{u}(x) & \text{if } u(x) < \underline{u}(x), \\ u(x) & \text{if } \underline{u}(x) \leq u(x) \leq \overline{u}(x), \\ \overline{u}(x) & \text{if } u(x) > \overline{u}(x). \end{cases}$$

T is well defined since $\underline{u} \leq 0 \leq \overline{u}$ on $\partial\Omega$. Due to [31, Lemma 2.89], T is bounded and continuous. We define also the nonlinear operators $A_T : W^{1,p}_0(\Omega) \to W^{-1,p'}(\Omega)$, $B_T : W^{1,p}_0(\Omega) \to L^{q'}(\Omega)$, and $i^* : L^{q'}(\Omega) \to W^{-1,p'}(\Omega)$ as

$$\begin{aligned} \langle A_T(u), \varphi \rangle &= \int_\Omega \mathcal{A}(x, T(u), \nabla u) \cdot \nabla \varphi \, \mathrm{d}x \quad \forall \varphi \in W^{1,p}_0(\Omega), \\ \langle B_T(u), \varphi \rangle &= \int_\Omega \mathcal{B}(x, T(u), \nabla T(u)) \, \varphi \, \mathrm{d}x \quad \forall \varphi \in L^q(\Omega), \\ \langle i^*(u), \varphi \rangle &= \int_\Omega u \, \varphi \, \mathrm{d}x \quad \forall \varphi \in W^{1,p}_0(\Omega). \end{aligned}$$

These operators are well defined thanks to (A_1), (B), and $q \in (1,p^*)$. Consider the functional equation (in the variable u)

$$\Psi(u) := A_T(u) - (i^* \circ B_T)(u) = 0. \tag{2.3.15}$$

We would like to apply Theorem 2.3.3 to get a solution of (2.3.15). First we observe that A_T is bounded, continuous, and of type (S_+), according to [31, Theorem 2.109]. On the other hand, i^* is linear and compact (and, in particular, bounded and continuous), since it is the adjoint of the embedding operator $i : W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$, which is compact by Rellich-Kondrachov's theorem.

We verify that B_T is bounded and continuous. Due to (B) and (2.3.14), besides Young's inequality, the following estimate holds true for any $u \in W_0^{1,p}(\Omega)$ and $\varepsilon > 0$:

$$
\begin{aligned}
|\mathcal{B}(x, T(u), \nabla T(u))|^{q'} &\leq (k(x)|\nabla T(u)|^r + f(x))^{q'} \\
&\leq c(k(x)^{q'}|\nabla u|^{rq'} + f(x)^{q'}) \\
&\leq \varepsilon|\nabla u|^p + c_\varepsilon(k(x)^t + 1) + f(x)^{q'}.
\end{aligned}
\tag{2.3.16}
$$

Integrating (2.3.16) proves that B_T is bounded. To prove continuity, let $\{u_n\} \subseteq W_0^{1,p}(\Omega)$ and $u \in W_0^{1,p}(\Omega)$ such that $u_n \to u$ in $W_0^{1,p}(\Omega)$. Reasoning up to subsequences, [23, Theorem 4.9] ensures that $u_n \to u$ and $\nabla u_n \to \nabla u$ a.e. in Ω, as well as $|\nabla u_n| \leq U$ a.e. in Ω for some $U \in L^p(\Omega)$. Hence, using (2.3.16), we have

$$
|\mathcal{B}(x, T(u_n), \nabla T(u_n))|^{q'} \leq \varepsilon U^p + c_\varepsilon(k(x)^t + 1) + f(x)^{q'}. \tag{2.3.17}
$$

Thus, using (2.3.16)–(2.3.17) and the continuity of T, we can apply the dominated convergence theorem to $|\mathcal{B}(x, T(u_n), \nabla T(u_n)) - \mathcal{B}(x, T(u), \nabla T(u))|^{q'}$, proving the continuity of B_T. Summarizing, $i^* \circ B_T$ is a completely continuous operator. According to Lemma 2.3.2, applied with $A := A_T$ and $B := i^* \circ B_T$, we deduce that Ψ is pseudo-monotone.
Now we prove that Ψ is coercive. From (B) and (2.3.14), besides the Hölder, Sobolev, and Young inequalities, we derive that

$$
\begin{aligned}
&\int_\Omega |\mathcal{B}(x, T(u), \nabla T(u))||u| \,\mathrm{d}x \\
&\leq \int_\Omega k(x)|\nabla u|^r|u| \,\mathrm{d}x + \int_\Omega f(x)|u| \,\mathrm{d}x \\
&\leq c\|k\|_t\|\nabla u\|_p^r\|u\|_q + \|f\|_{q'}\|u\|_q \\
&\leq c(\|k\|_t\|\nabla u\|_p^{r+1} + \|f\|_{q'}\|\nabla u\|_p) \\
&\leq \varepsilon\|\nabla u\|_p^p + c_\varepsilon(\|k\|_t^{\frac{p}{p-r-1}} + \|f\|_{q'}^{p'}).
\end{aligned}
\tag{2.3.18}
$$

Using (A_3) and (2.3.18) we get

$$
\langle \Psi(u), u \rangle \geq (a_3 - \varepsilon)\|\nabla u\|_p^p - \|\eta\|_1 - c_\varepsilon(\|k\|_t^{\frac{p}{p-r-1}} + \|f\|_{q'}^{p'}).
$$

To conclude, it suffices to choose $\varepsilon < a_3$.
Now we are in the position to apply Theorem 2.3.3 and get a solution $u \in W_0^{1,p}(\Omega)$ to (2.3.15). It remains to prove that $u \in [\underline{u}, \overline{u}]$. We will prove that

$u \leq \overline{u}$ in Ω, since the other inequality is analogous. To this end, recalling that $\overline{u}$ is a super-solution to (2.3.13) and u is a solution to (2.3.15) we have

$$\int_\Omega \mathcal{A}(x,\overline{u},\nabla\overline{u}) \cdot \nabla\varphi \,\mathrm{d}x \geq \int_\Omega \mathcal{B}(x,\overline{u},\nabla\overline{u})\, \varphi \,\mathrm{d}x \tag{2.3.19}$$

and

$$\int_\Omega \mathcal{A}(x,T(u),\nabla u) \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega \mathcal{B}(x,T(u),\nabla(T(u)))\, \varphi \,\mathrm{d}x \tag{2.3.20}$$

for any $\varphi \in W^{1,p}_0(\Omega)$, $\varphi \geq 0$ in Ω. Testing (2.3.19)–(2.3.20) with $\varphi = (u-\overline{u})^+$ and subtracting the former from the latter yield

$$\begin{aligned}&\int_\Omega (\mathcal{A}(x,\overline{u},\nabla u) - \mathcal{A}(x,\overline{u},\nabla\overline{u})) \cdot \nabla(u-\overline{u})^+ \,\mathrm{d}x \\ &\leq \int_\Omega (\mathcal{B}(x,\overline{u},\nabla\overline{u}) - \mathcal{B}(x,\overline{u},\nabla\overline{u}))\,(u-\overline{u})^+ \,\mathrm{d}x = 0.\end{aligned}$$

Hence, recalling (A_2), one has $\|\nabla(u-\overline{u})^+\|_p^p \leq 0$, whence $u \leq \overline{u}$ in Ω. □

As a consequence of Theorem 2.3.10 and the preceding arguments, our example problem (2.3.7) admits a weak solution provided $p^* > \frac{1}{1-\gamma}$. Indeed, the inequality $\underline{u} \leq \overline{u}$ follows from Theorem 2.3.6, together with (2.3.9) and (2.3.12) which yield

$$-\Delta_p \overline{u} = \mu^{p-1} d^{-\gamma} \geq \mu^{p-1} (\operatorname{diam}\Omega)^{-\gamma} \geq 1 \geq -\Delta_p \underline{u}$$

for $\mu > 0$ large enough; moreover, (f_2) and (2.3.10) imply

$$f(x,s) \leq c_1\, \underline{u}^{-\gamma} + c_2\, \overline{u}^{q-1} \leq c_1\, m^{-\gamma} d^{-\gamma} + c_2\, \mu^{q-1} \|E\|_\infty^{q-1} \leq c\, d^{-\gamma} \quad \text{in } \Omega\times[\underline{u},\overline{u}].$$

Accordingly, (B) is satisfied with $k \equiv 0$ and $f = c\,d^{-\gamma} \in L^{q'}(\Omega)$, with q' that can be chosen within $((p^*)', 1/\gamma)$ (see (2.2.28)). Actually, the restriction $p^* > \frac{1}{1-\gamma}$ can be eliminated, but it is necessary to use variational methods: see paragraphs 3.1.1 and 4.1.1.

Despite the generality of Theorem 2.3.10, often it is not simple to construct both a sub- and a super-solution; we address to paragraph 3.1.1 for an existence result whose proof uses only a sub-solution and is based on variational methods.

Sub- and super-solutions can be employed also for systems of differential equations, but in this case a relation between the sub- and the super-solution occurs, in the spirit of the local hypothesis (B) above, that allows to treat a wide class of reaction terms without requiring too much restrictive growth conditions. For the sake of simplicity, we restrict our attention to Dirichlet boundary value problems with two variables (see paragraph 3.2.1 for an application), but the definition can be extended also for Neumann systems, following Definition 2.1.10 and Remark 2.1.11; a sub-super-solution theorem about Neumann systems is presented in paragraph 4.2.1.

Definition 2.3.11. We say that $(\underline{u}, \underline{v}), (\overline{u}, \overline{v}) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is a sub-super-solution pair for

$$\begin{cases} -\operatorname{div} \mathcal{A}_1(x, u, v, \nabla u, \nabla v) = \mathcal{B}_1(x, u, v, \nabla u, \nabla v) & \text{in } \Omega, \\ -\operatorname{div} \mathcal{A}_2(x, u, v, \nabla u, \nabla v) = \mathcal{B}_2(x, u, v, \nabla u, \nabla v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

if

$$\begin{aligned} &\underline{u} \leq \overline{u} \quad \text{in } \Omega \quad \text{and} \quad \underline{u} \leq 0 \leq \overline{u} \quad \text{on } \partial\Omega, \\ &\underline{v} \leq \overline{v} \quad \text{in } \Omega \quad \text{and} \quad \underline{v} \leq 0 \leq \overline{v} \quad \text{on } \partial\Omega, \end{aligned}$$

and the following inequalities hold true for any $(\varphi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$, $\varphi, \psi \geq 0$ in Ω, and $(w, z) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$:

$$\begin{aligned} \int_\Omega \mathcal{A}_1(x, \underline{u}, z, \nabla \underline{u}, \nabla z) \cdot \nabla \varphi \, \mathrm{d}x &\leq \int_\Omega \mathcal{B}_1(x, \underline{u}, z, \nabla \underline{u}, \nabla z) \, \varphi \, \mathrm{d}x, \\ \int_\Omega \mathcal{A}_1(x, \overline{u}, z, \nabla \overline{u}, \nabla z) \cdot \nabla \varphi \, \mathrm{d}x &\geq \int_\Omega \mathcal{B}_1(x, \overline{u}, z, \nabla \overline{u}, \nabla z) \, \varphi \, \mathrm{d}x, \\ \int_\Omega \mathcal{A}_2(x, w, \underline{v}, \nabla w, \nabla \underline{v}) \cdot \nabla \psi \, \mathrm{d}x &\leq \int_\Omega \mathcal{B}_2(x, w, \underline{v}, \nabla w, \nabla \underline{v}) \, \psi \, \mathrm{d}x, \\ \int_\Omega \mathcal{A}_2(x, w, \overline{v}, \nabla w, \nabla \overline{v}) \cdot \nabla \psi \, \mathrm{d}x &\geq \int_\Omega \mathcal{B}_2(x, w, \overline{v}, \nabla w, \nabla \overline{v}) \, \psi \, \mathrm{d}x. \end{aligned}$$

The 'rectangle' $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ is called *trapping region*.

Appearing for the first time probably in [30], the theory of trapping regions has been developed in the last decades, mainly by Carl and Motreanu; it led to several existence results: see, e.g., [32] and [33]. Nevertheless, in the context of singular quasilinear convective elliptic systems, we recall the aforementioned work by Motreanu, Moussaoui, and Zhang [127] (vide Section 1.2), regarding singular convective systems allowing singularities in the convection terms. References of [127] provide several other examples of application of the trapping region theory.

2.4 Variational methods

The structure of the principal part of equation (P) (see paragraph 2.1.3), namely, $\mathcal{A}$ can be very general but, as we have seen in Sections 2.2–2.3, there are some customary hypotheses to be imposed in order to develop the whole theory. To retain the generality of the operator, but to ensure also the availability of all the tools developed above, in the last decades a particular Uhlenbeck operator has been considered and extensively studied: see, for instance, [135, 93, 90].

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain of class $C^{0,1}$ and $p, q \in (1, +\infty)$ with $p > q$. Consider $a \in C^1(\mathbb{R}^N; \mathbb{R}^N)$ with Uhlenbeck structure (2.2.44), and suppose that (a_1')–(a_2') of paragraph 2.2.3 hold true.

Remark 2.4.1. The operator $\mathcal{A}(x, u, \nabla u) := a(\nabla u)$ satisfies all the general conditions described in Sections 2.2–2.3, precisely:

- (A_1)–(A_2) of paragraph 2.2.1 (Moser's technique);
- (A_1')–(A_3'), (2.2.27) of paragraph 2.2.2 (Hölder regularity);
- (a_1')–(a_2') of paragraph 2.2.3 (Lipschitz regularity);
- (A_1)–(A_2), (a) of paragraph 2.3.2 (Maximum and comparison principles);
- (A_1)–(A_3) of paragraph 2.3.3 (Sub-super-solutions and trapping region).

Indeed, take $\omega(s) := sa_0(s)$ and verify (2.2.27) through (a_1'): according to

$$\frac{s\omega'(s)}{\omega(s)} = \frac{sa_0'(s) + a_0(s)}{a_0(s)} = \frac{sa_0'(s)}{a_0(s)} + 1 \in [i_a + 1, s_a + 1] \subseteq (0, +\infty), \tag{2.4.1}$$

valid for all $s \in (0, +\infty)$, one can choose $C_1 := i_a + 1$ and $C_2 := s_a + 1$. Dividing (2.2.27) by s and integrating in $[1, s]$ for any $s \geq 1$, one gets

$$(i_a + 1) \log s \leq \log \omega(s) - \log \omega(1) \leq (s_a + 1) \log s, \tag{2.4.2}$$

whence

$$\omega(1) s^{i_a+1} \leq \omega(s) \leq \omega(1) s^{s_a+1} \quad \forall s \in [1, +\infty). \tag{2.4.3}$$

Analogously, integrating (2.2.27) in $[s, 1]$ for any $s \in (0, 1]$, one obtains

$$\omega(1) s^{s_a+1} \leq \omega(s) \leq \omega(1) s^{i_a+1} \quad \forall s \in (0, 1]. \tag{2.4.4}$$

Incidentally, (a) of paragraph 2.3.2 is guaranteed by (2.4.4) and

$$\omega'(s) = a_0(s)\left(\frac{sa_0'(s)}{a_0(s)} + 1\right) > (i_a + 1)a_0(s) > 0, \tag{2.4.5}$$

according to (2.4.1). We notice that (A_3') of paragraph 2.2.2 is trivially satisfied, while (A_1')–(A_2') are equivalent to

$$\gamma_1 \frac{\omega(|\xi|)}{|\xi|} \le \lambda_{\min}(\xi) \le \lambda_{\max}(\xi) \le \gamma_2 \frac{\omega(|\xi|)}{|\xi|}, \tag{2.4.6}$$

where $\lambda_{\min}(\xi)$ and $\lambda_{\max}(\xi)$ are, respectively, the minimum and the maximum eigenvalues of $\nabla a(\xi)$. Let us compute $\nabla a(\xi)$: we obtain

$$\nabla a(\xi) = |\xi| a_0'(|\xi|) \frac{\xi}{|\xi|} \otimes \frac{\xi}{|\xi|} + a_0(|\xi|) I_N,$$

where I_N is the $N \times N$ identity matrix, while the symbol $\otimes$ denotes the tensor product in $\mathbb{R}^N$. Thus, eigenvalues of $\nabla a(\xi)$ are $|\xi| a_0'(|\xi|) + a_0(|\xi|)$ (with eigenspace generated by ξ) and $a_0(|\xi|)$ (whose eigenspace is the orthogonal complement of ξ). Observe that $\frac{\omega(|\xi|)}{|\xi|} = a_0(|\xi|)$. Reasoning as in (2.4.5), as well as exploiting (2.4.1), it is readily seen that (2.4.6) holds true with $\gamma_1 := \min\{1, i_a + 1\}$ and $\gamma_2 := \max\{1, s_a + 1\}$. Concerning ($\mathrm{A}_2$) of paragraph 2.3.3, by the mean value theorem and (2.4.6) we have

$$\begin{aligned}
(a(\xi_1) - a(\xi_2)) \cdot (\xi_1 - \xi_2) &= (\xi_1 - \xi_2)^T \nabla a(\xi^*)(\xi_1 - \xi_2) \\
&\ge \lambda_{\min}(\xi^*)|\xi_1 - \xi_2|^2 \\
&\ge \gamma_1 \frac{\omega(|\xi^*|)}{|\xi^*|}|\xi_1 - \xi_2|^2 > 0,
\end{aligned}$$

for some $\xi^* \in [\xi_1, \xi_2]$. All the other conditions are a direct consequence of (a_2'). It is worth noticing that the set $\mathscr{A}$ of conditions on a assumed in the aforementioned works (we cited [135, 93, 90] just to give an example) is actually equivalent to hypotheses (a_1')–(a_2'). In particular, we highlight that the assumptions

$$\lim_{s \to 0^+} s a_0(s) = 0, \quad \lim_{s \to 0^+} \frac{s a_0'(s)}{a_0(s)} > -1, \tag{2.4.7}$$

belonging to $\mathscr{A}$, are redundant for the set $\mathscr{A}$. Indeed, by (2.4.6) and (2.4.3)–(2.4.4) we get

$$0 \le s a_0(s) \le \gamma_2 \omega(s) \le \gamma_2 \omega(1)(s^{s_a+1} + s^{i_a+1}) \to 0 \quad \text{as } s \to 0^+$$

and

$$\frac{s a_0'(s)}{a_0(s)} + 1 = \frac{s a_0'(s) + a_0(s)}{a_0(s)} \ge \frac{\gamma_1}{\gamma_2} > 0 \quad \forall s \in (0, +\infty).$$

Variational methods consist in studying a functional J of class C^1 whose critical points are solutions to the differential problem under consideration; this functional is called *energy functional.* Given a bounded domain $\Omega \subseteq \mathbb{R}^N$, consider the Dirichlet problem

$$\begin{cases} -\operatorname{div} a(\nabla u) = b(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4.8}$$

being $a \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ as above and $b : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function obeying the *critical growth condition*

$$|b(x,s)| \leq c|s|^{p^*-1} + f(x), \quad \text{for all } (x,s) \in \Omega \times \mathbb{R}, \tag{2.4.9}$$

where $f \in L^{(p^*)'}(\Omega)$ and $c > 0$. The energy functional associated with (2.4.8) is

$$J(u) := \int_\Omega A(\nabla u)\,\mathrm{d}x - \int_\Omega B(x,u)\,\mathrm{d}x \quad \text{for all } u \in W^{1,p}_0(\Omega), \tag{2.4.10}$$

where

$$A(\xi) := \int_0^{|\xi|} t a_0(t)\,\mathrm{d}t, \quad B(x,s) := \int_0^s b(x,t)\,\mathrm{d}t. \tag{2.4.11}$$

Set

$$\Phi(u) := \int_\Omega A(\nabla u)\,\mathrm{d}x, \quad \Psi(u) := \int_\Omega B(x,u)\,\mathrm{d}x. \tag{2.4.12}$$

Under (2.4.9), the functional J is well defined and of class C^1. Indeed, observe that

$$\nabla A(\xi) = |\xi| a_0(|\xi|)\frac{\xi}{|\xi|} = a(\xi) \tag{2.4.13}$$

and, by (a_2'),

$$|a(\xi)| = a_0(|\xi|)|\xi| \leq c_2(|\xi|^{p-1} + 1). \tag{2.4.14}$$

Thus the mean value theorem yields, for a suitable $\tau \in (0,t)$,

$$\begin{aligned} \lim_{t\to 0^+} \frac{1}{t}\int_\Omega (A(\nabla u + t\nabla v) - A(\nabla u))\,\mathrm{d}x &= \lim_{t\to 0^+} \int_\Omega a(\nabla u + \tau\nabla v)\cdot\nabla v\,\mathrm{d}x \\ &= \int_\Omega a(\nabla u)\cdot\nabla v\,\mathrm{d}x, \end{aligned}$$

where the dominated convergence theorem is applied, thanks to (2.4.14); hence $\Phi(u)$ is Gateaux-differentiable. Using [23, Exercise 4.16] and (2.4.14)

again, it is readily seen that Φ' is continuous, so $\Phi \in C^1(W_0^{1,p}(\Omega))$. Regarding Ψ, it is of class C^1 on $L^{p^*}(\Omega)$, according to [56, Theorem 2.8] and (2.4.9). Sobolev's embedding theorem (see Theorem 2.1.5) then ensures that $\Psi \in C^1(W_0^{1,p}(\Omega))$.
Notice that J encompasses the boundary conditions of problem (2.4.8); anyway, the definition of J can be adapted to other settings, such as Neumann or Robin problems, as well as problems in the whole $\mathbb{R}^N$.

In the sequel we will treat variational methods for regular problems, because regularity of the energy functional is necessary in the whole theory; indeed, the equivalence between the solutions to (2.4.8) and the critical points of the functional J defined in (2.4.10) requires J to be differentiable. In the context of singular problems, it is customary to truncate the energy functional J at the level of a sub-solution $\underline{u}$, to deal with a C^1 functional $\tilde{J}$. Once a critical point u for $\tilde{J}$ is found, it can be proved that $u \geq \underline{u}$ via weak comparison principle (Theorem 2.3.6), exploiting the fact that u is a solution to the Euler-Lagrange equation of $\tilde{J}$. For a first example concerning this procedure, see paragraph 3.1.1.

2.4.1 The Weierstrass-Tonelli theorem

Let us start with the following abstract result, which can be found in [153, Theorem 1.1].

Theorem 2.4.2. *Let X be a Hausdorff topological space and let $J : X \to \mathbb{R} \cup \{+\infty\}$ satisfy the following condition: for any $c \in \mathbb{R}$ the sub-level set $J^c := \{u \in X : J(u) \leq c\}$ is compact. Then J is bounded from below and attains its infimum, that is, there exists $u^* \in X$ such that $J(u^*) = \min_X J > -\infty$. The conclusion remains valid if compactness of J^c is replaced by sequential compactness.*

Theorem 2.4.2 is very general, since we can look for a suitable topology on X ensuring that the sub-level sets J^c are compact. On the other hand, compactness of each J^c implies that J is lower semi-continuous. Hence we have two competing conditions: compactness of sub-level sets and lower semi-continuity of J. Indeed, a coarse topology possesses several compact sets, while a fine topology easily ensures that J is lower semi-continuous. A good balance is often represented by the weak topology on X, leading to the following theorem, known as Weierstrass-Tonelli's theorem (see [153, Theorem 1.2]).

Theorem 2.4.3 (Weierstrass-Tonelli)**.** *Let $(X, \|\cdot\|)$ be a reflexive Banach space and $C \subseteq X$ a weakly closed set (endowed with the induced topology).*

Suppose $J : C \to \mathbb{R} \cup \{+\infty\}$ to be coercive on C and weakly lower semi-continuous on C. Then J is bounded from below on C and attains its infimum in C. The conclusion remains valid if C is a weakly sequentially closed set and J is weakly sequentially lower semi-continuous on C.

Remark 2.4.4. If b satisfies the *sub-critical growth condition*

$$|b(x,s)| \le c|s|^{q-1} + f(x) \quad \text{for all } (x,s) \in \Omega \times \mathbb{R}, \tag{2.4.15}$$

being $q \in (1, p^*)$, $f \in L^{(p^*)'}(\Omega)$, and $c > 0$, then the functional (2.4.10) is weakly sequentially lower semi-continuous on $W_0^{1,p}(\Omega)$. Indeed, recalling (2.4.13) and (2.4.6), the Hessian of A is semi-definite positive; thus A is a convex function, and so is Φ. Since $\Phi \in C^1(W_0^{1,p}(\Omega))$, its sub-level sets are closed, while convexity of Φ ensures that they are also convex; so they are weakly closed, guaranteeing the weak (sequential) lower semi-continuity of Φ. Concerning Ψ, Rellich-Kondrachov's theorem and Lebesgue's dominated convergence theorem, together with (2.4.15), ensure that it is weakly sequentially continuous (one can use either [23, Exercise 4.16] or [23, Theorem 4.9]).

If b satisfies the (more restrictive) *p-sub-linear growth condition*

$$|b(x,s)| \le c|s|^{q-1} + f(x), \quad \text{for all } (x,s) \in \Omega \times \mathbb{R}, \tag{2.4.16}$$

being $q \in (1, p)$, $f \in L^{(p^*)'}(\Omega)$, and $c > 0$, then the functional (2.4.10) is coercive on $W_0^{1,p}(\Omega)$. Indeed, by (a_2') we have

$$a(\xi) \cdot \xi = a_0(|\xi|)|\xi|^2 \ge c'|\xi|^p, \tag{2.4.17}$$

while (2.4.16) entails

$$\begin{aligned} |B(x,s)| &\le \int_{-|s|}^{|s|} |b(x,t)| \, \mathrm{d}t \le c \int_{-|s|}^{|s|} |t|^{q-1} \, \mathrm{d}t + 2\, f(x)|s| \\ &\le c'' \left(|s|^q + f(x)|s|\right). \end{aligned} \tag{2.4.18}$$

Using Torricelli's theorem and (2.4.17)–(2.4.18), as well as the Hölder and Sobolev inequalities, we get

$$\begin{aligned} J(u) &\ge \int_\Omega \left(\int_0^1 a(t\nabla u) \cdot \nabla u \, \mathrm{d}t \right) \mathrm{d}x - \int_\Omega |B(x,u)| \, \mathrm{d}x \\ &\ge c'\|\nabla u\|_p^p - c''\|u\|_q^q - c''\|f\|_{(p^*)'}\|u\|_{p^*} \\ &\ge c'\|\nabla u\|_p^p - c''\|\nabla u\|_p^q - c''\|f\|_{(p^*)'}\|\nabla u\|_p \xrightarrow{\|\nabla u\|_p \to \infty} +\infty. \end{aligned} \tag{2.4.19}$$

Coercivity of J is guaranteed also in the p-linear case $q = p$ (cf. (2.4.16)), provided c in (2.4.16) is sufficiently small with respect to the first eigenvalue of the p-Laplacian in $W_0^{1,p}(\Omega)$, which is variationally characterized by the Rayleigh quotient

$$\lambda_1(p,\Omega) := \inf_{\substack{u\in W_0^{1,p}(\Omega)\\ u\neq 0}} \frac{\|\nabla u\|_p^p}{\|u\|_p^p}.$$

Since $J \in C^1(W_0^{1,p}(\Omega))$, the minimizer given by Theorem 2.4.3 is a critical point of J, so it satisfies the Euler-Lagrange equation for J, which corresponds to (2.4.8). Hence Theorem 2.4.3 produces a solution to (2.4.8).

2.4.2 The Mountain-Pass theorem

In paragraph 2.4.1 a solution to (2.4.8) is found by Theorem 2.4.3 under the p-sub-linearity growth assumption (2.4.16). When this condition is not met, for instance in p-super-linear problems, one can look for critical points of J either among constrained minima, applying Theorem 2.4.3 on a convex closed set $C \subseteq X$ to be chosen, or among saddle points, exploiting (for instance) the theorem we are going to present. The method of constrained minima is often applied by taking the *Nehari manifold* as the set C or, more generally, considering a *natural constraint*; we do not enter into details, addressing to [3, 49] for an introduction on the topic. The second method is based on the Mountain-Pass theorem, also called Ambrosetti-Rabinowitz theorem, that chiefly exploits a deformation theorem to obtain a particular saddle point, called of mountain-pass type; generalizations along this direction (not treated here, for the sake of brevity) are represented by linking theorems; see [3, 49, 123].

However, the methods mentioned above are strongly susceptible of the geometry of the functional; for this reason, despite the generality of Theorem 2.4.3, here we have to require some regularity and compactness properties on J. Hence, from here now, we suppose $J : X \to \mathbb{R}$ to be a functional of class C^1 satisfying the Palais-Smale condition (Definitions 2.4.5–2.4.6 below; cf. [153, p.70]).

Definition 2.4.5. Let $J \in C^1(X)$. A sequence $\{u_n\} \subseteq X$ is said to be a *Palais-Smale sequence* (briefly, PS-sequence) if $\{J(u_n)\}$ is bounded in X and $J'(u_n) \to 0$ (strongly) in X^* as $n \to \infty$.

Definition 2.4.6. A functional $J \in C^1(X)$ satisfies the Palais-Smale condition (briefly, (PS)) if any PS-sequence admits a strongly convergent subsequence.

The following lemma and its corollary are a good way to verify condition (PS) in the context of variational problems stemming from differential equations, since the derivative of the energy functional is decomposed into the sum of a principal part, having good monotonicity properties, and a reaction term, which is usually compact.

Lemma 2.4.7. *Let X be a reflexive Banach space and $J \in C^1(X)$ such that $J' = \Phi + \Psi$, where Φ is a (S_+) operator, while Ψ is a compact operator. Then any bounded PS-sequence admits a strongly convergent subsequence.*

Proof. Let $\{u_n\}$ be a bounded sequence such that $\{J(u_n)\}$ is bounded in X and $J'(u_n) \to 0$ in X^*. By reflexivity of X one has $u_n \rightharpoonup u$ for some $u \in X$, and by compactness of Ψ one gets $\Psi(u_n) \to \psi$ for some $\psi \in X^*$. Thus, recalling also that $J'(u_n) \to 0$ in X^*,

$$\langle \Phi(u_n), u_n - u \rangle = \langle J'(u_n), u_n - u \rangle - \langle \Psi(u_n), u_n - u \rangle \to 0.$$

The (S_+) property of Φ then ensures $u_n \to u$ in X (up to subsequences). □

Corollary 2.4.8. *Suppose that the assumptions of Lemma 2.4.7 are satisfied, and also that J is coercive. Then J satisfies condition* (PS)*.*

Proof. It suffices to observe that, for any PS-sequence $\{u_n\}$, the boundedness of $\{J(u_n)\}$ implies that $\{u_n\}$ is bounded. Then Lemma 2.4.7 applies. □

A particular class of non-coercive integral functionals, arising from superlinear problems, satisfies the (PS)-condition: it is constructed moving from the Ambrosetti-Rabinowitz condition; see Lemma 2.4.10 below.

Definition 2.4.9. Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function. f satisfies the *Ambrosetti-Rabinowitz condition* (briefly, (AR)) if there exist $\mu > p$ and $R > 0$ such that

$$\operatorname{ess\,inf}_{x\in\Omega} F(x,s) > 0 \quad \text{and} \quad \mu F(x,s) \le f(x,s)\, s \quad \text{for a.a. } x \in \Omega,\ |s| > R,$$

where

$$F(x,s) := \int_0^s f(x,t)\,\mathrm{d}t. \tag{2.4.20}$$

See [129] for an interesting discussion about the requirement on the essential infimum of F. The next result can be easily derived by [61, Theorems 15 and 16] (see also Theorem 4.1.9).

Lemma 2.4.10. *Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the sub-critical growth condition*

$$|f(x,s)| \le c|s|^{q-1} + \psi(x) \quad \textit{for all } (x,s) \in \Omega \times \mathbb{R},$$

being $q \in (1, p^*)$ *and* $\psi \in L^{(p^*)'}(\Omega)$, *and let* F *be as in* (2.4.20). *If* f *satisfies the* (AR)*-condition, then the functional* $J : W^{1,p}(\Omega) \to \mathbb{R}$ *defined as*

$$J(u) := \frac{1}{p}\|u\|_{1,p}^p - \int_\Omega F(x,u)\,\mathrm{d}x$$

satisfies the (PS)*-condition and is unbounded from below.*

Before going on, we highlight that some other extensions of the (PS)-condition are often considered; see, e.g., [123, Section 5.1]. For instance, a localized version of (PS), called $(\mathrm{PS})_c$ for $c \in \mathbb{R}$, requires that a PS-sequence satisfies $J(u_n) \to c$ instead of $\{J(u_n)\}$ to be bounded; this generalization allows to work with some particular sub-level sets of J. Another variant consists in the Cerami condition, briefly (C), that requires $(1+\|u_n\|)\,J'(u_n) \to 0$ in X^*, instead of $J'(u_n) \to 0$; also condition (C) admits the localized version $(\mathrm{C})_c$. Nevertheless, a variant of the (AR)-condition is used, especially dealing with singular problems: the unilateral version, called $(\mathrm{AR})_+$; for its definition and an application to singular problems, see paragraph 4.1.1.

The core of this paragraph is represented by the following theorem.

Theorem 2.4.11 (Ambrosetti-Rabinowitz)**.** *Let* X *be a Banach space,* $J \in C^1(X)$, $r > 0$, *and* $u_0, u_1 \in X$ *such that* $\|u_0 - u_1\| > r$. *Suppose that* J *satisfies the following condition, known as* mountain pass geometry*:*

$$\max\{J(u_0), J(u_1)\} \le \inf\{J(u) : \|u - u_0\| = r\} =: b.$$

Let

$$\Gamma := \{\gamma \in C^0([0,1]; X) : \gamma(0) = u_0, \gamma_1 = u_1\}$$

be the set of all the paths joining u_0 *and* u_1, *and set*

$$c := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)). \tag{2.4.21}$$

If J *satisfies the* (PS)*-condition, then* $c \ge b$ *and* c *is a critical value for* J, *i.e., there exists* $u \in X$ *such that* $J(u) = c$ *and* $J'(u) = 0$. *Moreover, if* $c = b$, *then* u *can be taken on* $\partial B_r(u_0)$.

Remark 2.4.12. The conclusions of Theorem 2.4.11 remain valid also if the (PS)-condition is replaced by the more general $(\mathrm{C})_c$-condition, being c defined in (2.4.21). Generalizations of Theorem 2.4.11 and related theorems can be found in [123, Section 5.3].

2.4.3 The variational principle

In this last paragraph concerning variational methods, we present a key tool to treat problems depending on a parameter, such as

$$\begin{cases} -\Delta_p u = \lambda f(x,u) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.4.22}$$

retaining the hypotheses of (2.3.7) and assuming $\lambda > 0$. Problem (2.4.22) admits an energy functional of the form $J = \Phi - \lambda\Psi$ as in (2.4.10), being Φ and Ψ as in (2.4.12) (with the opportune changes).

The theorem we are going to present has a history of more than twenty years: starting from a basic principle established by Ricceri [145] in 2000, in the following years Bonanno et al. produced several variants (cf. [7, 13, 16, 19, 14]) which are very useful in applications to partial differential equations (see, e.g., [7, 15] for the so-called *three-solution theorems* and [17, 20, 18] for some results about existence of infinitely many solutions in different settings). The version presented here can be found in [19] (taking $j \equiv 0$ on X); for a different proof (cf. ibid., Remark 2.2) see [115].

Theorem 2.4.13. *Let X be a reflexive Banach space, $\Phi, \Psi : X \to \mathbb{R}$, and $\lambda > 0$. Set $J_\lambda := \Phi - \lambda\Psi$. Suppose that Φ is coercive and sequentially weakly lower semi-continuous, while Ψ is sequentially weakly upper semi-continuous. For any $r > \inf_X \Phi$ define*

$$\varphi(r) := \inf_{u \in \Phi^{-1}((-\infty,r))} \frac{\sup_{v \in \Phi^{-1}((-\infty,r))} \Psi(v) - \Psi(u)}{r - \Phi(u)} \tag{2.4.23}$$

and

$$\gamma := \liminf_{r \to +\infty} \varphi(r), \qquad \delta := \liminf_{r \to (\inf_X \Phi)^+} \varphi(r).$$

Then:

(a) *for every $r > \inf_X \Phi$ and every $\lambda \in (0, \frac{1}{\varphi(r)})$, the restriction of the functional J_λ to $\Phi^{-1}((-\infty,r))$ admits a global minimum, which is a local minimum of J_λ in X.*

(b) *if $\gamma < +\infty$ then, for each $\lambda \in (0, \frac{1}{\gamma})$, the following alternative holds: either*

  (b$_1$) *J_λ possesses a global minimum, or*

  (b$_2$) *there exists a sequence $\{u_n\}$ of local minima of J_λ such that $\Phi(u_n) \to +\infty$.*

(c) *if* $\delta < +\infty$ *then, for each* $\lambda \in (0, \frac{1}{\delta})$, *the following alternative holds: either*

   (c$_1$) *there exists a global minimum of* Φ *which is local minimum of* J_λ, *or*

   (c$_2$) *there exists a sequence* $\{u_n\}$ *of pairwise distinct local minima of* J_λ *such that* $\Phi(u_n) \to \inf_X \Phi$ *and weakly converging to a global minimum of* Φ.

Remark 2.4.14. It is worth noticing that, if $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$, then (choosing $u \equiv 0$ in (2.4.23))

$$\varphi(r) \leq \frac{\sup_{v \in \Phi^{-1}((-\infty, r))} \Psi(v)}{r} =: \psi(r)$$

for any $r > 0$. Using ψ in place of φ in Theorem 2.4.13 has the disadvantage that the ranges of λ considered in the theorem are smaller, but it has the big advantage of an easier estimate of the parameters γ and δ: indeed, it suffices to estimate Ψ from above to get information on γ, δ. This trick will be used in paragraph 4.1.1.

2.5 Topological and set-valued methods

As we have seen, variational methods are a powerful tool to get existence and multiplicity results for partial differential equations in divergence form. Despite this wide range of applicability and the large amount of information about solutions they give (energy estimates, classification via critical groups, etc.), they have a big limitation: the differential equation cannot contain convection terms (i.e., terms that depend on the gradient of solution); in other words, convection terms 'destroy' the variational structure of the equation. For this reason, a different approach is necessary, as exploiting fixed-point and set-valued methods, although the best way to attack convection problems is to combine the latter methods with the variational ones. To give an idea of how to perform this combination, we are going to illustrate a procedure called *freezing technique*.

Let us consider the problem (cf. (2.4.8))

$$\begin{cases} -\operatorname{div} a(\nabla u) = b(x,u,\nabla u) & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5.1}$$

being $a \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ as in Section 2.4 and $b : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function obeying the p-sub-linear growth condition

$$|b(x,s,\xi)| \leq c_1|\xi|^{q-1} + c_2|s|^{q-1} + c_3 \tag{2.5.2}$$

for all $(x,s,\xi) \in \Omega\times\mathbb{R}\times\mathbb{R}^N$, where $q \in (1,p)$ and $c_1, c_2, c_3 > 0$. We freeze the convection term ∇u and consider, in dependence of $w \in C^1(\overline{\Omega})$, the auxiliary problem

$$\begin{cases} -\operatorname{div} a(\nabla u) = b_w(x,u) & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.5.3}$$

where $b_w(x,s) := b(x,s,\nabla w(x))$ for all $(x,s) \in \Omega \times \mathbb{R}$. Now problem (2.5.3) is in the form (2.4.8), and (2.5.2) implies (2.4.16) for b_w instead of b. Hence Theorem 2.4.3 (recall also Remark 2.4.4) produces a solution $u_w \in W_0^{1,p}(\Omega)$ to (2.5.3). Testing (2.5.3) with u_w and exploiting (2.4.17), besides using the Young, Hölder, and Sobolev inequalities, we get the energy estimate

$$\begin{aligned} c' \|\nabla u_w\|_p^p &\leq \int_\Omega a(\nabla u_w)\cdot\nabla u_w \,\mathrm{d}x \\ &\leq c_1 \int_\Omega |\nabla w|^{q-1}|u_w| \,\mathrm{d}x + c_2 \int_\Omega |u_w|^q \,\mathrm{d}x + c_3 \int_\Omega |u_w| \,\mathrm{d}x \\ &\leq c_1 \|\nabla w\|_q^q + (c_1+c_2)\|u_w\|_q^q + c_3\|u_w\|_1 \\ &\leq c_1' \|\nabla w\|_p^q + (c_1'+c_2')\|\nabla u_w\|_p^q + c_3'\|\nabla u_w\|_p\,, \end{aligned} \tag{2.5.4}$$

guaranteeing an upper bound for $\|\nabla u_w\|_p$. Hence Theorems 2.2.2 and 2.2.7 (jointly with Remark 2.2.9, which will be tacitly used each time we recall 2.2.7) ensure $u_w \in C^{1,\alpha}(\overline{\Omega})$. Now let us consider the map $T : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ associating to each w the function u_w just found. If $u \in C^1(\overline{\Omega})$ is a solution of the fixed point equation $u = T(u)$, then it is a solution to (2.5.1).

The biggest issue in this argument is that the map T has been defined for each w regardless of local properties, like continuity, or global ones, like compactness, essential ingredients in fixed point theory (as we will see in paragraph 2.5.1). Unfortunately there is no hope, in general, to guarantee continuity or compactness for such a T; anyway, since problem (2.5.3) may admit multiple solutions, we can try to re-define T in a suitable way in order to apply a fixed point theorem; this will be done in paragraph 2.5.2. We anticipate that further modifications on T are needed when one deals with singular problems; see Remark 2.5.12.

2.5.1 Fixed-point theorems

Given a set X, a subset $\emptyset \neq Y \subseteq X$, and a function $T : Y \to X$, we define the *fixed point set* of T as

$$\operatorname{Fix} T := \{u \in Y : T(u) = u\}.$$

Theorem 2.5.1 (Schauder)**.** *Let X be a Banach space and $C \subseteq X$ be convex. Suppose that $T : C \to C$ is continuous and $\overline{T(C)}$ is compact. Then* $\operatorname{Fix} T \neq \emptyset$.

Theorem 2.5.1 is a refinement of the classical Schauder theorem [81, Theorem 11.1 and Corollary 11.2]; for a proof, see [88, p.119]. The concept of 'compact map' used in [88] should not be confused with the one of 'compact operator' from Definition 2.1.1.

Another fixed point theorem, know as Schaefer's theorem or Leray-Schauder alternative, is of great interest, especially in the setting of partial differential equations; see [66, p.504].

Theorem 2.5.2 (Schaefer)**.** *Let X be a Banach space and $T : X \to X$ be a completely continuous operator. If the set*

$$\Lambda(T) := \{u \in X : u = \lambda T(u) \text{ for some } \lambda \in [0,1]\} \tag{2.5.5}$$

is bounded, then $\operatorname{Fix} T \neq \emptyset$.

Remark 2.5.3. Following [66, p.504], we observe three things.

- The importance of Schaefer's theorem in partial differential equations is the following: it reflects the heuristic principle that a priori estimates, proved under the assumption that a solution exists, actually imply that this solution do exist. Indeed, a priori estimates on T furnish a bound on $\|u\|_X$, ensuring that $\Lambda(T)$ is bounded.

- The advantage of Schaefer's theorem over Schauder's one, especially in applications, is that it is not necessary to identify an explicit convex set C such that $T(C) \subseteq C$ and $T(C)$ is relatively compact.

- Another advantage of Schaefer's theorem is that it requires to study T only 'along' the set $\{u = \lambda T(u)\}$; this means, informally, that it suffices to look at the graph of T only at the points of type $\left(u, \frac{u}{\lambda}\right)$, neglecting all the possible issues far away from them.

To show the applicability of Theorems 2.5.1–2.5.2, as well as to practically see the observations of Remark 2.5.3, let us re-consider problem (2.5.1) under condition (2.5.2), and suppose that the operator $T : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ constructed above is completely continuous (as already said, this topic will be treated in paragraph 2.5.2 below). We want to apply Schaefer's theorem, so it remains to prove that the set $\Lambda(T)$ defined in (2.5.5) is bounded in $C^1(\overline{\Omega})$. Let us take $u \in \Lambda(T)$. Then, recalling that $T(u)$ is a solution to (2.5.3) with $w = u$, besides $u = \lambda T(u)$ and $\lambda \in [0,1]$, the energy estimate (2.5.4) reads as

$$\begin{aligned} c' \|\nabla(T(u))\|_p^p &\le c_1' \|\nabla u\|_p^q + (c_1' + c_2') \|\nabla(T(u))\|_p^q + c_3' \|\nabla(T(u))\|_p \\ &\le c_1' \lambda^q \|\nabla(T(u))\|_p^q + (c_1' + c_2') \|\nabla(T(u))\|_p^q + c_3' \|\nabla(T(u))\|_p \\ &\le c_1' \|\nabla(T(u))\|_p^q + (c_1' + c_2') \|\nabla(T(u))\|_p^q + c_3' \|\nabla(T(u))\|_p , \end{aligned}$$

proving that there exists $K > 0$, independent of u, such that

$$\|\nabla u\|_p = \lambda \|\nabla T(u)\|_p \le \|\nabla(T(u))\|_p \le K \quad \text{for all } u \in \Lambda(T).$$

Reasoning as above, we have

$$\begin{aligned} |b_u(x, T(u))| &\le c_1 |\nabla u|^{q-1} + c_2 |T(u)|^{q-1} + c_3 \\ &= c_1 \lambda^{q-1} |\nabla(T(u))|^{q-1} + c_2 |T(u)|^{q-1} + c_3 \\ &\le c_1 |\nabla(T(u))|^{q-1} + c_2 |T(u)|^{q-1} + c_3 \\ &\le c_1 |\nabla(T(u))|^{p-1} + c_2 |T(u)|^{p-1} + c_1 + c_2 + c_3. \end{aligned} \tag{2.5.6}$$

Thus, Theorem 2.2.2 furnishes $L = L(K) > 0$ such that

$$\|u\|_\infty = \lambda \|T(u)\|_\infty \le \|T(u)\|_\infty \le L \quad \text{for all } u \in \Lambda(T). \tag{2.5.7}$$

Finally, Theorem 2.2.7, jointly with (2.5.6)–(2.5.7), ensures the existence of $M = M(L) > 0$ such that

$$\|u\|_{C^{1,\alpha}} = \lambda\|T(u)\|_{C^{1,\alpha}} \leq \|T(u)\|_{C^{1,\alpha}} \leq M \quad \text{for all } \ u \in \Lambda(T),$$

proving the boundedness of $\Lambda(T)$ in $C^1(\overline{\Omega})$. Schaefer's theorem then produces $u \in C^1(\overline{\Omega})$ such that $T(u) = u$, whence u is a solution to (2.5.1); actually $u \in C^{1,\alpha}(\overline{\Omega})$, since the image of T is contained in $C^{1,\alpha}(\overline{\Omega})$.

2.5.2 The solution map

In this paragraph we fill the gap in the theory exposed at the beginning of the section, regarding the construction of an operator $T : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ which associates to each w a solution u_w to (2.5.3), and that satisfies suitable continuity and compactness conditions; in particular, we will be able to prove that the T we construct is a completely continuous operator (see Definition 2.1.1).

First of all, fixed $w \in C^1(\overline{\Omega})$, we have to take into account all the solutions to (2.5.3), in order to choose the most 'appropriate'. To this end, let us consider the multi-function $\mathscr{S} : C^1(\overline{\Omega}) \to 2^{C^1(\overline{\Omega})}$ defined as

$$\mathscr{S}(w) := \{u \in C^1(\overline{\Omega}) : \ u \ \text{ is a solution to } \ (2.5.3)\}.$$

As already proved via Theorem 2.4.3, the multi-function $\mathscr{S}$ has non-empty values (that is, $\operatorname{dom} \mathscr{S} = C^1(\overline{\Omega})$). The operator T will be constructed as a continuous selection of $\mathscr{S}$, obtained through order-theoretic arguments which mainly rely on the following basic lemma.

Lemma 2.5.4. *Let $w \in C^1(\overline{\Omega})$. Suppose that $u_1, u_2 \in C^1(\overline{\Omega})$ are two sub-solutions (resp., super-solutions) to* (2.5.3)*. Then $u := \max\{u_1, u_2\}$ (resp., $u := \min\{u_1, u_2\}$) is a sub-solution (resp., super-solution) to* (2.5.3).

Proof. We will prove only that if u_1, u_2 are sub-solutions to (2.5.3), then $u := \max\{u_1, u_2\}$ is a sub-solution to (2.5.3); the other part of the statement can be verified similarly. Choose $\eta \in C^\infty(\mathbb{R})$ as in (2.2.4) and set

$$\eta_\varepsilon(x) := \eta\left(\frac{u_1 - u_2}{\varepsilon}\right), \quad \overline{\eta}_\varepsilon(x) := 1 - \eta_\varepsilon(x).$$

Take any $\varphi \in C^1_0(\overline{\Omega})$, $\varphi \geq 0$. Since u_i, $i = 1, 2$, is a sub-solution to (2.5.3) we get

$$\int_\Omega a(\nabla u_1) \cdot \nabla(\eta_\varepsilon \varphi) \,\mathrm{d}x \leq \int_\Omega b_w(x, u_1)\, \eta_\varepsilon \varphi \,\mathrm{d}x,$$
$$\int_\Omega a(\nabla u_2) \cdot \nabla(\overline{\eta}_\varepsilon \varphi) \,\mathrm{d}x \leq \int_\Omega b_w(x, u_2)\, \overline{\eta}_\varepsilon \varphi \,\mathrm{d}x,$$

that can be rewritten as

$$\begin{aligned}
&\int_\Omega \eta_\varepsilon\, a(\nabla u_1)\cdot\nabla\varphi\,\mathrm{d}x + \frac{1}{\varepsilon}\int_\Omega \eta'\left(\frac{u_1-u_2}{\varepsilon}\right)\varphi\, a(\nabla u_1)\cdot\nabla(u_1-u_2)\,\mathrm{d}x \\
&\qquad\qquad \le \int_\Omega b_w(x,u_1)\,\eta_\varepsilon\varphi\,\mathrm{d}x, \\
&\int_\Omega \overline{\eta}_\varepsilon\, a(\nabla u_2)\cdot\nabla\varphi\,\mathrm{d}x - \frac{1}{\varepsilon}\int_\Omega \eta'\left(\frac{u_1-u_2}{\varepsilon}\right)\varphi\, a(\nabla u_2)\cdot\nabla(u_1-u_2)\,\mathrm{d}x \\
&\qquad\qquad \le \int_\Omega b_w(x,u_2)\,\overline{\eta}_\varepsilon\varphi\,\mathrm{d}x.
\end{aligned} \tag{2.5.8}$$

Observe that, since $\varphi\ge 0$, $\eta'\ge 0$, and $a:\mathbb{R}^N\to\mathbb{R}^N$ is a monotone operator, then

$$\int_\Omega \eta'\left(\frac{u_1-u_2}{\varepsilon}\right)\varphi\,[a(\nabla u_1)-a(\nabla u_2)\cdot(\nabla u_1-\nabla u_2)]\,\mathrm{d}x \ge 0. \tag{2.5.9}$$

Adding (2.5.8) term by term, besides using (2.5.9), yields

$$\begin{aligned}
&\int_\Omega \eta_\varepsilon\, a(\nabla u_1)\cdot\nabla\varphi\,\mathrm{d}x + \int_\Omega \overline{\eta}_\varepsilon\, a(\nabla u_2)\cdot\nabla\varphi\,\mathrm{d}x \\
&\le \int_\Omega b_w(x,u_1)\,\eta_\varepsilon\varphi\,\mathrm{d}x + \int_\Omega b_w(x,u_2)\,\overline{\eta}_\varepsilon\varphi\,\mathrm{d}x.
\end{aligned} \tag{2.5.10}$$

Notice that $\eta_\varepsilon\to\chi_{\{u_1>u_2\}}$ and $\overline{\eta}_\varepsilon\to\chi_{\{u_1\le u_2\}}$ as $\varepsilon\to 0^+$. Passing to the limit in (2.5.10) via Lebesgue's dominated convergence theorem produces

$$\begin{aligned}
&\int_{\{u_1>u_2\}} a(\nabla u_1)\cdot\nabla\varphi\,\mathrm{d}x + \int_{\{u_1\le u_2\}} a(\nabla u_2)\cdot\nabla\varphi\,\mathrm{d}x \\
&\le \int_{\{u_1>u_2\}} b_w(x,u_1)\varphi\,\mathrm{d}x + \int_{\{u_1\le u_2\}} b_w(x,u_2)\varphi\,\mathrm{d}x.
\end{aligned} \tag{2.5.11}$$

Recalling the definition of u, (2.5.11) reads as

$$\int_\Omega a(\nabla u)\cdot\nabla\varphi\,\mathrm{d}x \le \int_\Omega b_w(x,u)\,\varphi\,\mathrm{d}x.$$

A density argument then proves that u is a sub-solution to (2.5.3). $\square$

Let us introduce some basic concepts about ordering; concerning fixed point theory in ordered sets and its applications to differential equations, we refer to the monograph [29].

Definition 2.5.5. A partially ordered set $(A, \leq)$ is said to be *downward* (resp., *upward*) *directed* if, for any $a, b \in A$, there exists $c \in A$ such that $c \leq a$ (resp., $c \geq a$) and $c \leq b$ (resp., $c \geq b$). The set A is said to be *directed* if it both is downward and upward directed.

Incidentally, we notice that $C^1(\overline{\Omega})$ is an ordered Banach space with the ordering

$$u_1 \leq u_2 \quad \Leftrightarrow \quad u_1(x) \leq u_2(x) \quad \text{for all } \ x \in \overline{\Omega}. \tag{2.5.12}$$

Lemma 2.5.6. *Let $(A, \leq)$ be a partially ordered set which is downward (resp., upward) directed. If $a \in A$ is a minimal (resp., maximal) element, then $a = \min A$ (resp., $a = \max A$).*

Proof. We suppose A to be downward directed; the dual statement can be proved with the same argument. Take any $b \in A$. Since A is downward directed, there exists $c \in A$ such that $c \leq a$ and $c \leq b$. By minimality, $c = a$. Hence $a \leq b$ for an arbitrary $b \in A$, as desired. $\square$

Theorem 2.5.7. *For any $w \in C^1(\overline{\Omega})$, the set $\mathscr{S}(w)$ admits minimum and maximum.*

Proof. First we prove that $\mathscr{S}(w)$ is directed. Let us take u_1, u_2 solutions to (2.5.3), and set $\overline{u} := \min\{u_1, u_2\}$. According to Lemma 2.5.4, $\overline{u}$ is a super-solution to (2.5.3). Consider the auxiliary problem

$$\begin{cases} -\operatorname{div} a(\nabla u) = b_w(x, \min\{u, \overline{u}\}) & \text{in } \ \Omega, \\ \qquad\qquad u = 0 & \text{on } \ \partial\Omega. \end{cases} \tag{2.5.13}$$

As for problem (2.5.3), Theorems 2.4.3, 2.2.2, and 2.2.7, besides an energy estimate similar to (2.5.4), produce $u \in C^{1,\alpha}(\overline{\Omega})$ solution to (2.5.13). Reasoning as in the final part of the proof of Theorem 2.3.10 shows that $u \leq \overline{u}$ in Ω; hence u is a solution to (2.5.3) and it satisfies both $u \leq u_1$ and $u \leq u_2$. This proves that $\mathscr{S}(w)$ is downward directed. A dual argument, performed by using $\underline{u} := \max\{u_1, u_2\}$, proves that $\mathscr{S}$ is upward directed.

In order to apply Lemma 2.5.6, we use Zorn's lemma to guarantee that $\mathscr{S}(w)$ admits minimal and maximal elements. To this end, let us consider a chain $\mathcal{C} \subseteq \mathscr{S}(w)$ and a sequence $\{u_n\} \subseteq \mathcal{C}$ which is decreasing in $C^1(\overline{\Omega})$ (see (2.5.12)). Energy estimate (2.5.4), jointly with Theorems 2.2.2 and 2.2.7, ensure that $\{u_n\}$ is bounded in $C^{1,\alpha}(\overline{\Omega})$; thus, Theorem 2.1.3 guarantees that $u_n \to u$ in $C^1(\overline{\Omega})$ for a suitable $u \in C^1(\overline{\Omega})$; in particular, $u \leq u_n$ for all $n \in \mathbb{N}$. Passing to the limit in the weak formulation of (2.5.3) (through uniform convergence) reveals that $u \in \mathscr{S}(w)$. Then Zorn's lemma produces

$u_* \in \mathscr{S}(w)$ minimal element for $\mathscr{S}(w)$. In the same way, it can be proved that there exists a maximal element $u^* \in \mathscr{S}(w)$. Now Lemma 2.5.6 yields $u_* = \min \mathscr{S}(w)$ and $u^* = \max \mathscr{S}(w)$, completing the proof. $\square$

According to Theorem 2.5.7 define $T : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$, selection of $\mathscr{S}$, as

$$T(w) := \min \mathscr{S}(w). \tag{2.5.14}$$

Now we investigate some further properties of the multi-function $\mathscr{S}$, and we see how these properties transfer, in a suitable form, to its selection T defined in (2.5.14).

Theorem 2.5.8. *The multi-function $\mathscr{S}$ is compact.*

Proof. It suffices to prove sequential compactness. Take any bounded sequence $\{w_n\} \subseteq C^1(\overline{\Omega})$ and pick an arbitrary sequence $\{u_n\}$ such that $u_n \in \mathscr{S}(w_n)$. Estimate (2.5.4) ensures that $\{u_n\}$ is bounded in $W^{1,p}_0(\Omega)$, which implies $u_n \to u$ in $C^1(\overline{\Omega})$ for a suitable $u \in C^1(\overline{\Omega})$, through (2.5.2), regularity theory, and Ascoli-Arzelà's theorem (reasoning as in paragraph 2.5.1 and Theorem 2.5.7). Arbitrariness of $\{u_n\}$ concludes the proof. $\square$

Corollary 2.5.9. *The operator T defined in* (2.5.14) *is a compact operator.*

Proof. It suffices to apply Theorem 2.5.8 and observe that the selections of compact multi-functions are compact operators. $\square$

Theorem 2.5.10. *The multi-function $\mathscr{S}$ is lower semi-continuous.*

Proof. Take any $\{w_n\} \subseteq C^1(\overline{\Omega})$ and $w \in C^1(\overline{\Omega})$ such that $w_n \to w$ in $C^1(\overline{\Omega})$; pick also $u \in \mathscr{S}(w)$. Consider the following family of auxiliary problems, depending on $n, m \in \mathbb{N}$ and defined by induction on m:

$$\begin{cases} -\operatorname{div} a(\nabla u_n^m) = b(x, u_n^{m-1}, \nabla w_n) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u_n^0 = u & \text{for all } n \in \mathbb{N}. \end{cases} \tag{2.5.15}$$

Problems (2.5.15) are well defined: the right-hand side is in $L^\infty(\Omega)$, due to (2.5.2), (2.5.4), and Corollary 2.2.3 (applied recursively), so Theorem 2.3.4 can be applied, furnishing u_n^m unique solution to (2.5.15) for any fixed $m, n \in \mathbb{N}$. Reasoning as for (2.5.4), using the Hölder, Poincaré, and Young inequalities, besides recalling that $\|\nabla w_n\|_p \le C$ for a suitable $C > 0$ by hypothesis, we deduce the following energy estimate:

$$\begin{aligned}
c' \, \|\nabla u_n^m\|_p^p &\leq \int_\Omega (c_1 \, |\nabla w_n|^{q-1} + c_2 \, |u_n^{m-1}|^{q-1} + c_3) \, |u_n^m| \, \mathrm{d}x \\
&\leq c \, (\|\nabla w_n\|_q^{q-1} + \|u_n^{m-1}\|_q^{q-1} + 1) \, \|u_n^m\|_q \\
&\leq c \, (\|\nabla w_n\|_p^{q-1} + \|\nabla u_n^{m-1}\|_p^{q-1} + 1) \, \|\nabla u_n^m\|_p \\
&\leq c \, (C^{q-1} + \varepsilon \|\nabla u_n^{m-1}\|_p^{p-1} + c_\varepsilon) \, \|\nabla u_n^m\|_p.
\end{aligned}$$

Dividing by $c' \, \|\nabla u_n^m\|_p$ we get

$$a_m \leq \alpha a_{m-1} + \beta, \tag{2.5.16}$$

with $a_m := \|\nabla u_n^m\|_p^{p-1}$ for all $m \in \mathbb{N}$, $\alpha := \frac{c}{c'} \, \varepsilon$, and $\beta := \frac{c}{c'} \, (C^{q-1} + c_\varepsilon)$. Choosing $\varepsilon \in \left(0, \frac{c'}{c}\right)$ and proceeding inductively we get

$$\begin{aligned}
a_m &\leq \alpha a_{m-1} + \beta \leq \alpha^2 a_{m-2} + \beta \, (1 + \alpha) \leq \ldots \leq \alpha^m a_0 + \beta \sum_{i=0}^{m-1} \alpha^i \\
&= \alpha^m a_0 + \beta \, \frac{1 - \alpha^m}{1 - \alpha} \leq a_0 + \frac{\beta}{1 - \alpha} \, .
\end{aligned} \tag{2.5.17}$$

Using (2.5.17) into (2.5.16), as well as recalling that $u_n^0 = u$ for any $n \in \mathbb{N}$, yields

$$\|\nabla u_n^m\|_p^{p-1} \leq \|\nabla u\|_p^{p-1} + \frac{\beta}{1 - \alpha} \quad \text{for all } \; m, n \in \mathbb{N}. \tag{2.5.18}$$

According to (2.5.18), both sequences $\{u_n^m\}_n$ and $\{u_n^m\}_m$ are bounded in $W_0^{1,p}(\Omega)$, whence (by (2.5.2), nonlinear regularity, and Ascoli-Arzelà's theorem)

$$\lim_{n \to \infty} u_n^m = u^m \quad \text{for all } \; m \in \mathbb{N}, \quad \lim_{m \to \infty} u_n^m = u_n \quad \text{for all } \; n \in \mathbb{N}, \tag{2.5.19}$$

for suitable $u^m, u_n \in C^1(\overline{\Omega})$; the limits in (2.5.19) have to be understood in $C^1(\overline{\Omega})$ sense. Passing to the limit (through uniform convergence) in (2.5.15) and using the two relations in (2.5.19), besides recalling the uniqueness of u_n^m solution to (2.5.15), we get

$$u^m = u \quad \text{for all } \; m \in \mathbb{N}, \quad u_n \in \mathscr{S}(w_n) \quad \text{for all } \; n \in \mathbb{N}. \tag{2.5.20}$$

The double limit lemma [72, Proposition A.2.35], together with (2.5.19)–(2.5.20), finally gives

$$\lim_{n \to \infty} u_n = \lim_{n \to \infty} \lim_{m \to \infty} u_n^m = \lim_{m \to \infty} \lim_{n \to \infty} u_n^m = \lim_{m \to \infty} u^m = u.$$

□

Corollary 2.5.11. *The operator T defined in* (2.5.14) *is a continuous operator.*

Proof. Let $\{w_n\} \subseteq C^1(\overline{\Omega})$ and $w \in C^1(\overline{\Omega})$ such that $w_n \to w$ in $C^1(\overline{\Omega})$. Set $u_n := T(w_n)$ for all $n \in \mathbb{N}$ and $u := T(w)$. Corollary 2.5.9 thus furnishes $\hat{u} \in C^1(\overline{\Omega})$ such that $u_n \to \hat{u}$ in $C^1(\overline{\Omega})$. Passing to the limit in (2.5.3) produces $\hat{u} \in \mathscr{S}(w)$. Theorem 2.5.10 provides a sequence $\{\hat{u}_n\}$ such that $\hat{u}_n \in \mathscr{S}(w_n)$ for all $n \in \mathbb{N}$ and $\hat{u}_n \to u$ in $C^1(\overline{\Omega})$. By minimality, $u_n \leq \hat{u}_n$ for any $n \in \mathbb{N}$ and $u \leq \hat{u}$. Hence

$$u \leq \hat{u} = \lim_{n\to\infty} u_n \leq \lim_{n\to\infty} \hat{u}_n = u,$$

whence $\hat{u} = u$. $\square$

Corollaries 2.5.9 and 2.5.11 ensure that T is a completely continuous operator, as required by Schaefer's theorem (see Theorem 2.5.2); hence the argument shown at the end of paragraph 2.5.1 can be used to get a solution of (2.5.1).

Remark 2.5.12. Dealing with singular problems requires a modification in the definition of $\mathscr{S}$, arising from the fact that a sub-solution is often needed to ensure that $\mathscr{S}(w) \neq \emptyset$ for all $w \in C^1(\overline{\Omega})$. In particular, if there exists $\underline{u}$, independent of w, sub-solution to (2.5.3), then $\mathscr{S}$ can be defined as

$$\mathscr{S}(w) := \{u \in C^1(\overline{\Omega}) : \ u \ \text{ is a solution to (2.5.3)}, \ u \geq \underline{u}\}.$$

An example of application is given in paragraphs 4.1.2 and 4.2.1.
We also point out that, even in the case of singular problems, an adaptation of the arguments made in this section has the following advantage on the sub-super-solution methods (see paragraph 2.3.3): it is not necessary to produce both the sub- and the super-solution. Obviously, this advantage has to be paid in terms of growth conditions on the reaction term: indeed, here we had to impose the p-sub-linear condition (2.5.2), which is in particular a *condition at infinity*, while sub-super-solution methods would have required only the local condition (B) stated in Theorem 2.3.10.

3 Sub-super-solutions or shifting?

This chapter is devoted to the study of some basic singular problems in bounded domains subjected to Dirichlet boundary conditions. The analysis of such problems should highlight the differences between different techniques of approach. In particular we are interested in investigating two main methods to obtain existence results for singular problems, both allowing to avoid singularities: the sub-super-solution technique, relying on the methods discussed in paragraph 2.3.3, and the shifting method, whose main idea is translating the singular terms by a small quantity $\varepsilon \in (0,1)$, solving the corresponding shifted problems, and then letting $\varepsilon \to 0^+$ in the sequence of the found solutions. We will try to understand the nature of hypotheses that 'couple well' with each method, in order to 'suggest' a focused approach to deal with a given singular problem. We will analyze both equations ('scalar case'; see Section 3.1) and systems ('vectorial case'; vide Section 3.2), commenting the obtained results in Section 3.3.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with $\partial\Omega$ of class C^2. We consider the following scalar problem:

$$\begin{cases} -\operatorname{div} a(\nabla u) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.0.1}$$

where the operator $u \mapsto -\operatorname{div} a(\nabla u)$ is the one introduced in Section 2.4, and $f : \Omega \times (0,+\infty) \to (0,+\infty)$ is a Carathéodory function obeying

$$f(x,s) \le c_1 s^{-\gamma} + c_2 s^{r-1}, \tag{H$_1$}$$

where $0 < \gamma < 1 \le r < p$ and $c_1, c_2 > 0$. We also make the following assumption:

$$\liminf_{s\to 0^+} f(x,s) > 0 \quad \text{uniformly w.r.t. } x \in \Omega. \tag{H$_2$}$$

Incidentally, observe that (H$_2$) is a quite natural assumption for problems that may admit a singularity near the origin; indeed, autonomous (i.e., f is independent of x) singular problems satisfy the stronger condition

$$\lim_{s\to 0^+} f(s) = +\infty.$$

We consider also the following vectorial problem:

$$\begin{cases} -\Delta_p u = f(x,u,v) & \text{in } \Omega, \\ -\Delta_q v = g(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.0.2}$$

where $p, q \in (1, N)$ and $f, g : \Omega \times (0, +\infty)^2 \to (0, +\infty)$ are Carathéodory functions satisfying

$$\begin{aligned} m_1 s^{\alpha_1} t^{\beta_1} &\le f(x, s, t) \le M_1(s^{\alpha_1} t^{\beta_1} + s^{\gamma_1} + t^{\delta_1}), \\ m_2 s^{\alpha_2} t^{\beta_2} &\le g(x, s, t) \le M_2(s^{\alpha_2} t^{\beta_2} + s^{\gamma_2} + t^{\delta_2}), \end{aligned} \tag{H$_3$}$$

being $m_i, M_i > 0$, while $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{R}$ are such that

$$\begin{aligned} &-1 < \alpha_1 + \beta_1 \le |\alpha_1| + |\beta_1| < p - 1, \quad \gamma_1, \delta_1 \in [0, p - 1), \\ &-1 < \alpha_2 + \beta_2 \le |\alpha_2| + |\beta_2| < q - 1, \quad \gamma_2, \delta_2 \in [0, q - 1). \end{aligned} \tag{C$_1$}$$

In paragraph 3.2.2 we replace (C_1) with

$$\begin{aligned} &-1 < \alpha_1 < 0 < \beta_1, \quad \max\left\{\frac{\beta_1}{q^*}, \frac{\gamma_1}{p^*}, \frac{\delta_1}{q^*}\right\} + \frac{p}{p^*} < 1, \\ &-1 < \beta_2 < 0 < \alpha_2, \quad \max\left\{\frac{\alpha_2}{p^*}, \frac{\gamma_2}{p^*}, \frac{\delta_2}{q^*}\right\} + \frac{q}{q^*} < 1, \end{aligned} \tag{C$_2'$}$$

and

$$\max\{\beta_1, \delta_1\} \max\{\alpha_2, \gamma_2\} < (p - 1 - \gamma_1)(q - 1 - \delta_2). \tag{C$_2''$}$$

The prototype of (possible) singular terms for systems is

$$f(x, s, t) = s^{\alpha_1} t^{\beta_1}, \quad g(x, s, t) = s^{\alpha_2} v^{\beta_2}, \tag{3.0.3}$$

where $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$: indeed, (3.0.3) encompasses both cooperative and competitive structures. By extension, hereafter system (3.0.2) is called 'cooperative' if (H_3) holds with $\beta_1, \alpha_2 > 0$ and 'competitive' if it holds with $\beta_1, \alpha_2 < 0$; the system is also called 'singular' whenever $\alpha_1, \beta_2 \in (-1, 0)$. Notice that, even in (3.0.3), f, g can vanish near the origin (also in the singular case, along suitable curves in the (s, t)-plane), so hypotheses like (H_2) are not natural in this context.
We observe that in the vectorial case we are compelled to consider only homogeneous operators, as the p-Laplacian: indeed, as we will see, we exploit homogeneity in the sub-solution technique while, concerning the shifting method, we use a particular consequence of the weak Harnack inequality, extendable only for weakly coercive operators (see [54, Section 5]).
We also point out that, in the vectorial case, we are able to apply the shifting method only for systems having cooperative structure (see the sign assumptions in (C_2')), due to the lack of Sobolev uniform estimates in other situations. It is, however, possible to apply the shifting method also in other situations, as systems having competitive structures (but no sub-linear

terms): see [126], in which the shifting method is combined with the truncation technique. In other words, the sub-super-solution method covers some cases that are not encompassed by the shifting method, while the latter is applicable in some situations in which the former is not available. A detailed discussion about this topic is reported in Section 3.3.

3.1 Scalar case

3.1.1 Sub-solution technique

First of all, observe that (H_1) gives $\varepsilon, \delta \in (0,1)$ such that

$$f(x,s) > \varepsilon \quad \forall\,(x,s) \in \Omega \times (0,\delta). \tag{3.1.1}$$

Now a sub-solution is constructed by generalizing the arguments used in paragraph 2.3.3 for the homogeneous problem (2.3.7). Fix any $\sigma \in (0,1)$ and consider the following torsion problem:

$$\begin{cases} -\mathrm{div}\, a(\nabla u) = \sigma & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.2}$$

Minty-Browder's theorem, nonlinear regularity theory, and the strong maximum principle (see Corollary 2.3.4 and Theorems 2.2.2, 2.2.7, 2.3.7) ensure that there exists a unique solution $u_\sigma \in C^{1,\alpha}(\overline{\Omega})$ to (3.1.2), with $u_\sigma > 0$ in Ω. Since the a priori estimates furnished by Theorems 2.2.2 and 2.2.7 are uniform in σ, then the Ascoli-Arzelà theorem (vide Theorem 2.1.3) gives $u_\sigma \to u$ in $C^1(\overline{\Omega})$ as $\sigma \to 0^+$, for a suitable $u \in C^1(\overline{\Omega})$. Passing to the limit in (3.1.2) via uniform convergence produces $u \equiv 0$ in Ω. Hence we can choose σ such that

$$\sigma \in (0,\varepsilon), \quad \|u_\sigma\|_\infty < \delta, \tag{3.1.3}$$

being ε, δ as in (3.1.1). Set $\underline{u} := u_\sigma$ with σ as in (3.1.3). By (3.1.1)–(3.1.3), $\underline{u}$ is a sub-solution: indeed,

$$-\mathrm{div}\, a(\nabla \underline{u}) = \sigma \le \varepsilon \le f(x,\underline{u}). \tag{3.1.4}$$

The construction of a super-solution to (3.0.1) seems to be hard, although the p-sub-linear condition (H_1) holds true: indeed, the inhomogeneity of the operator prevents to exploit (2.3.11)–(2.3.12). As a consequence, we are not in the position to apply the sub-super-solution theorem (see Theorem 2.3.10); thus, we truncate the problem at level of $\underline{u}$, dealing with a regular problem, and use the variational methods described in Section 2.4 to solve it.

Consider the auxiliary problem

$$\begin{cases} -\mathrm{div}\, a(\nabla u) = \hat{f}(x,u) & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1.5}$$

where $\hat{f} : \Omega \times \mathbb{R} \to (0,+\infty)$ is a Carathéodory function defined as

$$\hat{f}(x,s) = f(x, \max\{s, \underline{u}(x)\}).$$

The energy functional $\hat{J} : W_0^{1,p}(\Omega) \to \mathbb{R}$ of (3.1.5) can be written as in (2.4.10), i.e.,

$$\hat{J}(u) = \int_\Omega A(\nabla u)\,\mathrm{d}x - \int_\Omega \hat{F}(x,u)\,\mathrm{d}x, \tag{3.1.6}$$

where A is defined in (2.4.11) and $\hat{F}$ is a primitive of $\hat{f}$ (cf. (2.4.11)). Since $\underline{u} \in C^{1,\alpha}(\overline{\Omega})$, Lemma 2.1.8 and Remark 2.1.9 produce $ld \le \underline{u} \le Ld$ in $\overline{\Omega}$ for suitable $l, L > 0$. Hence, by (H_1) we get

$$\begin{aligned} 0 \le \hat{f}(x,s) &\le c_1(\max\{s,\underline{u}\})^{-\gamma} + c_2(\max\{s,\underline{u}\})^{r-1} \\ &\le c_1\underline{u}^{-\gamma} + c_2\left(|s|^{r-1} + \underline{u}^{r-1}\right) \\ &\le c_2|s|^{r-1} + c_2L^{r-1}d^{r-1} + c_1l^{-\gamma}d^{-\gamma} \\ &\le c(|s|^{r-1} + d^{-\gamma}). \end{aligned} \tag{3.1.7}$$

Observe that Weierstrass-Tonelli's theorem (Theorem 2.4.3) can be applied also when the reaction term fulfills an estimate like (3.1.7); see Lemma 4.1.3 below. Accordingly, Theorem 2.4.3 furnishes $u \in W_0^{1,p}(\Omega)$ global minimizer of $\hat{J}$, so u is a solution to (3.1.5). Reasoning as in the final part of the proof of Theorem 2.3.10 gives $u \ge \underline{u}$, so is a solution to (3.0.1). Nonlinear regularity theory ensures that $u \in C^{1,\alpha}(\overline{\Omega})$: indeed, by (2.4.17), ($\mathrm{H}_1$), and Hölder's inequality, we estimate

$$\begin{aligned} c'\|\nabla u\|_p^p &\le \int_\Omega a(\nabla u)\cdot\nabla u\,\mathrm{d}x = \int_\Omega f(x,u)u\,\mathrm{d}x \le c_1\int_\Omega u^{1-\gamma}\,\mathrm{d}x + c_2\int_\Omega u^r\,\mathrm{d}x \\ &\le c\left(\|u\|_p^r + 1\right) \le c\left(\|\nabla u\|_p^r + 1\right), \end{aligned}$$

ensuring a bound on $\|\nabla u\|_p$. Thus, recalling also (H_1), Theorem 2.2.2 yields $u \in L^\infty(\Omega)$. To guarantee that $u \in C^{1,\alpha}(\overline{\Omega})$, let us consider the auxiliary problem

$$\begin{cases} -\Delta v = f(x,u(x)) & \text{in } \Omega, \\ \quad v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1.8}$$

and notice that $0 \le f(x,u(x)) \le cd^{-\gamma}$ for a suitable $c > 0$, thanks to (3.1.7) and $u \in L^\infty(\Omega)$. Exploiting Corollary 2.3.4 and Theorem 2.2.10, we deduce that problem (3.1.8) admits a unique solution $v \in C^{1,\alpha}(\overline{\Omega})$. Observe that u solves

$$\begin{cases} -\mathrm{div}(a(\nabla u) - \nabla v(x)) = 0 & \text{in } \Omega, \\ \qquad\qquad\qquad\quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.9}$$

From $\nabla v \in C^{0,\alpha}(\overline{\Omega})$, we have that the operator $u \mapsto \mathrm{div}(a(\nabla u) - \nabla v(x))$ satisfies the hypotheses of Remark 2.2.9; hence, $u \in C^{1,\alpha}(\overline{\Omega})$.

3.1.2 Shifting method

The shifting method for (3.0.1) is based on the study of the auxiliary problems

$$\begin{cases} -\operatorname{div} a(\nabla u) = f_\sigma(x,u) & \text{in } \Omega, \\ \qquad\qquad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1.10}$$

where $\sigma \in (0,1)$ and $f_\sigma : \Omega \times \mathbb{R} \to (0,+\infty)$ is a Carathéodory function defined as

$$f_\sigma(x,s) := f(x, s^+ + \sigma).$$

As in paragraph 3.1.1, the energy functional $J_\sigma : W_0^{1,p}(\Omega) \to \mathbb{R}$ associated with (3.1.10) is

$$J_\sigma(u) = \int_\Omega A(\nabla u)\,\mathrm{d}x - \int_\Omega F_\sigma(x,u)\,\mathrm{d}x, \tag{3.1.11}$$

where $F_\sigma(x,s)$ is a primitive of f_σ. Thanks to (H_1), besides recalling $\sigma \in (0,1)$, we estimate

$$0 \le f_\sigma(x,s) \le c_1\,(s^+ + \sigma)^{-\gamma} + c_2\,(s^+ + \sigma)^{r-1} \le 2^{r-2}c_2|s|^{r-1} + c_1\sigma^{-\gamma} + 2^{r-2}c_2,$$

so (2.4.16) holds true with f_σ in place of f. The same arguments exploited in paragraph 3.1.1 provide $u_\sigma \in C^{1,\alpha}(\overline{\Omega})$, $u_\sigma > 0$ in Ω, global minimizer of J_σ and, consequently, solution to (3.1.10).

We would like to let $\sigma \to 0^+$ in $\{u_\sigma\}_\sigma$ (at least for a subsequence), and verify that the limit solves (3.0.1); to do this, we need uniform estimates. Testing (3.1.10) with u_σ, as well as using (2.4.17) and the Sobolev inequality, yields

$$\begin{aligned} c'\|\nabla u_\sigma\|_p^p &\le \int_\Omega a(\nabla u_\sigma)\cdot\nabla u_\sigma\,\mathrm{d}x = \int_\Omega f_\sigma(x,u_\sigma)\,u_\sigma\,\mathrm{d}x \\ &\le c_1\int_\Omega (u_\sigma+\sigma)^{-\gamma}u_\sigma\,\mathrm{d}x + c_2\int_\Omega (u_\sigma+\sigma)^{r-1}u_\sigma\,\mathrm{d}x \\ &\le c_1\int_\Omega u_\sigma^{1-\gamma}\,\mathrm{d}x + 2^{r-2}c_2\int_\Omega (u_\sigma^r + u_\sigma)\,\mathrm{d}x \\ &\le c\,(\|\nabla u_\sigma\|_p^r + 1), \end{aligned} \tag{3.1.12}$$

for a suitable $c = c(r,c_1,c_2,\Omega) > 0$. Since $p > r$, we get an uniform bound on $\{u_\sigma\}$ in $W_0^{1,p}(\Omega)$ leading, via Theorem 2.2.2 and (H_1), to

$$\|u_\sigma\|_\infty \le M \quad \forall \sigma \in (0,1), \tag{3.1.13}$$

for a suitable $M > 0$ independent of σ. Hence, reasoning up to subsequences, there exists $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $u \geq 0$ a.e. in Ω, such that

$$u_\sigma \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega), \quad u_\sigma \to u \quad \text{in } L^p(\Omega), \quad u_\sigma \to u \quad \text{a.e. in } \Omega. \tag{3.1.14}$$

The lack of comparison with the distance function, usually achieved via sub-solution arguments (that we do not use here, since we want to present a 'pure shifting' approach), prevents us to get $u \in C^{1,\alpha}(\overline{\Omega})$. Accordingly, now we prove that $u > 0$ in Ω and $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$, and then pass to the limit in (3.1.10) in the sense of distributions.
Consider the functional J_0 defined as in (3.1.11), with the position

$$F_0(x,s) := \int_0^{s^+} f(x,t)\,\mathrm{d}t.$$

By Lebesgue's dominated convergence theorem and (H_1) we have

$$\begin{aligned} F_\sigma(x,u_\sigma) &= F_\sigma(x,u_\sigma) - F_\sigma(x,u) + F_\sigma(x,u) \\ &= \int_{u(x)}^{u_\sigma(x)} f(t+\sigma)\,\mathrm{d}t + \int_0^{u_\sigma(x)} f(t+\sigma)\,\mathrm{d}t \to F_0(x,u) \end{aligned}$$

a.e. in Ω, as $\sigma \to 0^+$. Observe that the estimate

$$\begin{aligned} F_\sigma(x,u_\sigma) &= \int_0^{u_\sigma(x)} f(x,t+\sigma)\,\mathrm{d}t \\ &\leq \frac{c_1}{1-\gamma} u_\sigma(x)^{1-\gamma} + \frac{2^{r-1}}{q} c_2 (u_\sigma(x)^r + 1) \end{aligned} \tag{3.1.15}$$

is uniform in $\sigma \in (0,1)$, according to (3.1.13). Hence we get

$$\lim_{\sigma \to 0^+} \int_\Omega F_\sigma(x,u_\sigma)\,\mathrm{d}x = \int_\Omega F_0(x,u)\,\mathrm{d}x.$$

Recalling also that the principal part is weakly lower semi-continuous, mainly derived by convexity of A (vide Remark 2.4.4), we get

$$J_0(u) \leq \liminf_{\sigma \to 0^+} J_\sigma(u_\sigma). \tag{3.1.16}$$

Fix any $\varphi \in C_c^\infty(\Omega)$, $\varphi \geq 0$ in Ω. Using Torricelli's theorem (as in (2.4.19)), (2.4.4), and (3.1.1), there exists $s \in (0, \frac{\delta}{2}\|\varphi\|_{C^1}^{-1})$ (depending only on δ, ε, and φ) such that, for any $\sigma \in (0, \frac{\delta}{2})$,

$$\begin{aligned} J_\sigma(s\varphi) &\leq s^q \|\nabla\varphi\|_q^q - \int_\Omega \left(\int_0^{s\varphi(x)} f(x,t+\sigma)\,\mathrm{d}t \right) \mathrm{d}x \\ &\leq s^q \|\nabla\varphi\|_q^q - \varepsilon s \|\varphi\|_1 < -\tau, \end{aligned} \tag{3.1.17}$$

for a suitable $\tau = \tau(\varphi, \delta, \varepsilon) > 0$. Exploiting (3.1.16)–(3.1.17), besides recalling that u_σ is a minimizer of J_σ, we deduce

$$J_0(u) \le \liminf_{\sigma\to 0^+} J_\sigma(u_\sigma) \le \liminf_{\sigma\to 0^+} J_\sigma(s\varphi) < -\tau,$$

which proves that $u \not\equiv 0$. Take any $v \in W_0^{1,p}(\Omega)$. Reasoning as above we obtain

$$\begin{aligned} J_0(u) &\le \liminf_{\sigma\to 0^+} J_\sigma(u_\sigma) \le \liminf_{\sigma\to 0^+} J_\sigma(v) \\ &= J_0(v) + \liminf_{\sigma\to 0^+} (J_\sigma(v) - J_0(v)) \\ &= J_0(v) + \liminf_{\sigma\to 0^+} \int_\Omega (F_0(v) - F_\sigma(v))\,\mathrm{d}x = J_0(v), \end{aligned} \tag{3.1.18}$$

proving that u is a minimizer of J_0. Now pick any $\varphi \in W_0^{1,p}(\Omega)$ such that $\varphi \ge 0$ in Ω. Then (3.1.18) applied with $v = u + t\varphi$, $t > 0$, produces

$$\begin{aligned} 0 &\le \frac{1}{t}(J_0(u + t\varphi) - J(u)) \\ &= \frac{1}{t}\left[\int_\Omega (A(\nabla u + t\nabla\varphi) - A(\nabla u))\,\mathrm{d}x - \int_\Omega (F_0(x, u + t\varphi) - F_0(x, u))\,\mathrm{d}x\right] \\ &\le \frac{1}{t}\int_\Omega (A(\nabla u + t\nabla\varphi) - A(\nabla u))\,\mathrm{d}x \le \int_\Omega a(\nabla u)\cdot\nabla\varphi\,\mathrm{d}x, \end{aligned}$$

since $F_0(x, \cdot)$ is non-decreasing and A is convex. Hence we infer

$$-\mathrm{div}\, a(\nabla u) \ge 0.$$

Recalling that u is non-trivial, the strong maximum principle (Theorem 2.3.7) ensures

$$u > 0 \quad \text{in } \Omega. \tag{3.1.19}$$

According to [82, Theorem 7.6], jointly with the uniform estimates (3.1.15) and (3.1.13), $\{u_\sigma\}$ is bounded in $C^{0,\alpha}_{\mathrm{loc}}(\Omega)$, so we can suppose $u_\sigma \to u$ in $C^0_{\mathrm{loc}}(\Omega)$, by virtue of Ascoli-Arzelà's theorem (see Theorem 2.1.3) and (3.1.14). This fact, together with (3.1.19), (3.1.13), and $u \in C^0_{\mathrm{loc}}(\Omega)$, allows to suppose

$$m \le u_\sigma(x) \le M \quad \forall \sigma \in (0,1), \quad \forall x \in \Omega',$$

for all $\Omega' \Subset \Omega$ and opportune $m = m(\Omega') > 0$. Thus $f_\sigma(\cdot, u_\sigma)$ is uniformly bounded in $L^\infty_{\mathrm{loc}}(\Omega)$, whence

$$u_\sigma \to u \quad \text{in } C^{1,\alpha}_{\mathrm{loc}}(\Omega), \tag{3.1.20}$$

because of Theorem 2.2.7. Passing to the limit in the distributional formulation of (3.1.10) via (3.1.20) proves that $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ is a distributional solution to (3.0.1).

3.2 Vectorial case

3.2.1 Sub-super-solution technique

Set $(r_1, r_2) := (p, q)$. For $i = 1, 2$, we consider the auxiliary systems

$$\begin{cases} -\Delta_{r_i} w_i = d^{\alpha_i+\beta_i} & \text{in } \Omega, \\ \quad w_i = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta_{r_i} z_i = d^{\min\{\alpha_i+\beta_i,\gamma_i,\delta_i\}} & \text{in } \Omega, \\ \quad z_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2.1}$$

where $\alpha_i, \beta_i, \gamma_i, \delta_i$ satisfy (C_1). Minty-Browder's theorem (Corollary 2.3.4), Theorem 2.2.10, and the strong maximum principle (Theorem 2.3.7), together with (H_3), furnish $w_i, z_i \in C^{1,\alpha}(\overline{\Omega})$, $w_i, z_i > 0$ in Ω, solutions to (3.2.1). According to Remark 2.1.9 and Lemma 2.1.8 we can find $L > 1$ such that

$$L^{-1}d \le \min\{w_i, z_i\} \le \max\{w_i, z_i\} \le Ld \quad \text{in } \overline{\Omega}. \tag{3.2.2}$$

Now set $(\underline{u}, \underline{v}) := (C^{-1}w_1, C^{-1}w_2)$, $(\overline{u}, \overline{v}) := (Cz_1, Cz_2)$, being $C > L$ a constant to be determined; notice that $C > L$ implies $\underline{u} \le \overline{u}$ and $\underline{v} \le \overline{v}$. Consider

$$K = \{(u,v) \in C^1(\overline{\Omega})^2 : \ \underline{u} \le u \le \overline{u}, \ \underline{v} \le v \le \overline{v}\},$$

which is a convex, closed subset of $C^1(\overline{\Omega})^2$. Pick any $(U, V) \in K$. Reasoning as above, there exists $(u_{U,V}, v_{U,V}) \in C^{1,\alpha}(\overline{\Omega})^2$ solving the following problem:

$$\begin{cases} -\Delta_p u = f(x, U, V) & \text{in } \Omega, \\ -\Delta_q v = g(x, U, V) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2.3}$$

Thus we can consider the map $T : K \to K$ defined as

$$T(U, V) = (u_{U,V}, v_{U,V}).$$

We claim that this operator is well defined (i.e., K is invariant under T) and completely continuous. To fix the ideas, let $\alpha_1 < 0 < \beta_1$; observe that, by (3.2.2) and (H_3), besides recalling $(U, V) \in K$, for a sufficiently large C we have

$$\begin{aligned} -\Delta_p \underline{u} &= C^{-(p-1)} d^{\alpha_1+\beta_1} \le C^{-(p-1)} (L^{-1}z_1)^{\alpha_1} (Lw_2)^{\beta_1} \\ &= C^{-\alpha_1+\beta_1-(p-1)} L^{-\alpha_1+\beta_1} \overline{u}^{\alpha_1} \underline{v}^{\beta_1} \le m_1 \overline{u}^{\alpha_1} \underline{v}^{\beta_1} \\ &\le f(x, U, V) = -\Delta_p u_{U,V} \\ &\le M_1 (\underline{u}^{\alpha_1} \overline{v}^{\beta_1} + \overline{u}^{\gamma_1} + \overline{v}^{\delta_1}) \\ &= M_1 (C^{-\alpha_1+\beta_1} w_1^{\alpha_1} z_2^{\beta_1} + C^{\gamma_1} z_1^{\gamma_1} + C^{\delta_1} z_2^{\delta_1}) \\ &\le M_1 ((CL)^{-\alpha_1+\beta_1} d^{\alpha_1+\beta_1} + (CL)^{\gamma_1} d^{\gamma_1} + (CL)^{\delta_1} d^{\delta_1}) \\ &\le C^{p-1} d^{\min\{\alpha_1+\beta_1,\gamma_1,\delta_1\}} = -\Delta_p \overline{u}. \end{aligned}$$

Similar computations can be done in the remaining cases. Hence, the weak comparison principle guarantees that $\underline{u} \leq u_{U,V} \leq \overline{u}$ and $\underline{v} \leq v_{U,V} \leq \overline{v}$, as desired. Passing to the limit in (3.2.3) through uniform convergence proves the continuity of T, while compactness follows from nonlinear regularity theory and Ascoli-Arzelà's theorem. An application of Schauder's fixed point theorem (vide Theorem 2.5.1) ensures the existence of $(u,v) \in C^{1,\alpha}(\overline{\Omega})^2$ solution to (3.0.2).

To conclude, we notice that Theorem 2.3.10 cannot be applied in this context, since the reaction terms of problem (3.0.2) possess a low summability, even when restricted within the trapping region $[\underline{u},\overline{u}] \times [\underline{v},\overline{v}]$: indeed, in this region they behave like $d^{\alpha_i+\beta_i}$, and possibly $\alpha_i + \beta_i < 0$ (cf. 2.2.28).

3.2.2 Shifting method

Hereafter we suppose (C_2')–(C_2'') in place of (C_1). For any $\sigma \in (0,1)$, we consider the system

$$\begin{cases} -\Delta_p u = f_\sigma(x,u,v) & \text{in } \Omega, \\ -\Delta_q v = g_\sigma(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2.4}$$

where $f_\sigma, g_\sigma : \Omega \times \mathbb{R}^2 \to (0,+\infty)$ are defined by

$$f_\sigma(x,s,t) = f(x,s^+ + \sigma, t^+), \quad g_\sigma(x,s,t) = g(x,s^+,t^+ + \sigma).$$

Now fix any $(U,V) \in C^1(\overline{\Omega})^2$ and notice that the 'frozen' problem

$$\begin{cases} -\Delta_p u = f_\sigma(x,U,V) & \text{in } \Omega, \\ -\Delta_q v = g_\sigma(x,U,V) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2.5}$$

admits a unique positive solution $(u_{\sigma,U,V}, v_{\sigma,U,V}) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)$, according to Minty-Bowder's theorem and the strong maximum principle.
In order to apply Schauder's fixed point theorem, thus finding a solution to (3.2.4), we prove the following energy estimate for (3.2.5) by using (H_3) and (C_2'), besides the Hölder and Sobolev inequalities:

$$\begin{aligned} \|\nabla u_{\sigma,U,V}\|_p^p &\leq M_1 \sigma^{\alpha_1} \int_\Omega (|V|^{\beta_1} + (|U|+1)^{\gamma_1} + |V|^{\delta_1})\, u_{\sigma,U,V} \,\mathrm{d}x \\ &\leq c_\sigma \left(\|V\|_{q^*}^{\beta_1} + \|U\|_{p^*}^{\gamma_1} + \|V\|_{q^*}^{\delta_1} + 1\right) \|u_{\sigma,U,V}\|_{p^*} \\ &\leq c_\sigma \left(\|\nabla U\|_p + \|\nabla V\|_q + 1\right)^{\max\{\beta_1,\gamma_1,\delta_1\}} \|\nabla u_{\sigma,U,V}\|_p, \end{aligned}$$

being $c_\sigma = c_\sigma(\sigma, \alpha_1, \beta_1, \gamma_1, \delta_1, p, \Omega) > 0$ a suitable constant; a similar inequality holds also for $v_{\sigma,U,V}$. Since $\max\{\beta_i, \gamma_i, \delta_i\} < r_i - 1$, $i = 1, 2$, we obtain

$$\|\nabla U\|_p + \|\nabla V\|_q \leq C_1 \Rightarrow \|\nabla u_{\sigma,U,V}\|_p + \|\nabla v_{\sigma,U,V}\|_q \leq C_1, \tag{3.2.6}$$

for a sufficiently big $C_1 > 0$, depending also on σ.
Regularity theory furnishes a uniform $C^{1,\alpha}(\overline{\Omega})^2$-estimate for $(u_{\sigma,U,V}, v_{\sigma,U,V})$ in terms of the Sobolev norm of (U, V), that is,

$$\|\nabla U\|_p + \|\nabla V\|_q \leq C_1 \Rightarrow \|u_{\sigma,U,V}\|_{C^{1,\alpha}} + \|v_{\sigma,U,V}\|_{C^{1,\alpha}} \leq C_2, \tag{3.2.7}$$

where $C_2 > 0$ is a constant, depending also on C_1.
Hence the map $T_\sigma : C^1(\overline{\Omega})^2 \to C^1(\overline{\Omega})^2$ defined as

$$T_\sigma(U, V) = (u_{\sigma,U,V}, v_{\sigma,U,V})$$

possesses the invariant closed, convex set

$$K_\sigma = \{(U, V) \in C^1(\overline{\Omega})^2 : \|\nabla U\|_p + \|\nabla V\|_q \leq C_1, \ \|U\|_{C^1} + \|V\|_{C^1} \leq C_2\},$$

according to (3.2.6)–(3.2.7); in addition, reasoning as in paragraph 3.2.1, T_σ is continuous and compact. We get $(u_\sigma, v_\sigma) \in C^{1,\alpha}(\overline{\Omega})^2$ positive solution to (3.2.4).

Now we produce some estimates for $\{u_\sigma, v_\sigma\}_\sigma$, uniform in $\sigma \in (0, 1)$. Through Hölder's and Sobolev's inequalities we estimate

$$\begin{aligned}
\|\nabla u_\sigma\|_p^p &\leq M_1 \int_\Omega (u_\sigma^{\alpha_1} v_\sigma^{\beta_1} + (u_\sigma + 1)^{\gamma_1} + v_\sigma^{\delta_1})\, u_\sigma \,\mathrm{d}x \\
&\leq c \left[\|u_\sigma\|_{p^*}^{1+\alpha_1} \|v_\sigma\|_{q^*}^{\beta_1} + \|u_\sigma\|_{p^*}^{\gamma_1+1} + \|u_\sigma\|_{p^*} + \|v_\sigma\|_{q^*}^{\delta_1} \|u_\sigma\|_{p^*} \right] \\
&\leq c \max\{1, \|u_\sigma\|_{p^*}\}^{\gamma_1+1} \max\{1, \|v_\sigma\|_{q^*}\}^{\max\{\beta_1,\delta_1\}} \\
&\leq c \max\{1, \|\nabla u_\sigma\|_p\}^{\gamma_1+1} \max\{1, \|\nabla v_\sigma\|_q\}^{\max\{\beta_1,\delta_1\}}
\end{aligned} \tag{3.2.8}$$

for some $c > 0$; analogously,

$$\|\nabla v_\sigma\|_q^q \leq c \max\{1, \|\nabla v_\sigma\|_q\}^{\delta_2+1} \max\{1, \|\nabla u_\sigma\|_p\}^{\max\{\alpha_2,\gamma_2\}}. \tag{3.2.9}$$

If $\min\{\|\nabla u_\sigma\|_p, \|\nabla v_\sigma\|_q\} \leq 1$, then the uniform Sobolev estimate is given directly by (3.2.8)–(3.2.9). Hence, suppose $\min\{\|\nabla u_\sigma\|_p, \|\nabla v_\sigma\|_q\} \geq 1$ and re-write the estimates as

$$\begin{cases} \|\nabla u_\sigma\|_p^{p-1-\gamma_1} \leq c \|\nabla v_\sigma\|_q^{\max\{\beta_1,\delta_1\}}, \\ \|\nabla v_\sigma\|_q^{q-1-\delta_2} \leq c \|\nabla u_\sigma\|_p^{\max\{\alpha_2,\gamma_2\}}, \end{cases}$$

leading to

$$\begin{cases} \|\nabla u_\sigma\|_p^{(p-1-\gamma_1)(q-1-\delta_2)} \le c\|\nabla u_\sigma\|_p^{\max\{\beta_1,\delta_1\}\max\{\alpha_2,\gamma_2\}}, \\ \|\nabla v_\sigma\|_q^{(p-1-\gamma_1)(q-1-\delta_2)} \le c\|\nabla v_\sigma\|_q^{\max\{\beta_1,\delta_1\}\max\{\alpha_2,\gamma_2\}}. \end{cases}$$

The uniform estimate of (u_σ, v_σ) in $W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ is a consequence of (C_2'') and the Poincaré inequality. By reflexivity we can suppose that (3.1.14) holds true for a suitable $u \in W_0^{1,p}(\Omega)$; the same thing can be done for $\{v_\sigma\}$, converging to a certain $v \in W_0^{1,q}(\Omega)$ in all the sense prescribed in (3.1.14). To get uniform bounds, consider the set $\Omega_1 := \{x \in \Omega : u(x) \ge 1\}$. Hypothesis ($H_3$) provides

$$f(x, u_\sigma, v_\sigma) \le M_1(v_\sigma^{\beta_1} + u_\sigma^{\gamma_1} + v_\sigma^{\delta_1}) \quad \forall x \in \Omega_1. \tag{3.2.10}$$

According to (C_2'),

$$v_\sigma^{\beta_1}, v_\sigma^{\delta_1} \in L^r(\Omega) \quad \text{with } r > \left(\frac{p^*}{p}\right)' = \frac{N}{p}. \tag{3.2.11}$$

Adapting the proof of Lemma 2.2.1, under (3.2.10), and applying Theorem 2.2.2, with (3.2.11), produce

$$\|u_\sigma\|_\infty \le M \quad \forall \sigma \in (0,1), \tag{3.2.12}$$

for an opportune $M > 0$ independent of σ. The same argument can be repeated for the second equation to get $\|v_\sigma\|_\infty \le M$ for all $\sigma \in (0,1)$.

Now we prove that for any $B_\rho \Subset \Omega$ there exists $\omega_\rho > 0$ such that $u_\sigma, v_\sigma > \omega_\rho$ in B_ρ for any $\sigma \in (0,1)$; the proof is based on a consequence of the weak Harnack inequality [143, Theorem 7.1.2]. Fix any $B_\rho \Subset \Omega$. According to (H_3) and (3.2.12), for any σ we have

$$\begin{aligned} f_\sigma(\cdot, u_\sigma, v_\sigma) &\ge m_1(M+1)^{\alpha_1} v_\sigma^{\beta_1}, \\ g_\sigma(\cdot, u_\sigma, v_\sigma) &\ge m_2 u_\sigma^{\alpha_2}(M+1)^{\beta_2} \end{aligned}$$

a.e. in Ω. From [63, Theorem 3.1] it thus follows

$$\begin{aligned} \left(\inf_{B_\rho} u_\sigma\right)^{p-1} &\ge \frac{c}{|B_\rho|}\int_{B_\rho} v_\sigma^{\beta_1}\mathrm{d}x \ge c\left(\inf_{B_\rho} v_\sigma\right)^{\beta_1}, \\ \left(\inf_{B_\rho} v_\sigma\right)^{q-1} &\ge \frac{c}{|B_\rho|}\int_{B_\rho} u_\sigma^{\alpha_2}\mathrm{d}x \ge c\left(\inf_{B_\rho} u_\sigma\right)^{\alpha_2}, \end{aligned}$$

for an opportune $c = c(\rho, M) > 0$. We get

$$\inf_{B_\rho} u_\sigma \le c \left(\inf_{B_\rho} u_\sigma \right)^{\frac{(p-1)(q-1)}{\alpha_2 \beta_1}}, \quad \inf_{B_\rho} v_\sigma \le c \left(\inf_{B_\rho} v_\sigma \right)^{\frac{(p-1)(q-1)}{\alpha_2 \beta_1}}.$$

Condition (C_2'') then ensures $\alpha_2\beta_1 < (p-1)(q-1)$, implying $u_\sigma, v_\sigma \ge \omega_\rho$ in B_ρ for an opportune $\omega_\rho > 0$. Hence, recalling (H_3), $f_\sigma(\cdot, u_\sigma, v_\sigma)$ and $g_\sigma(\cdot, u_\sigma, v_\sigma)$ are uniformly bounded $L^\infty_{\mathrm{loc}}(\Omega)$; so Theorem 2.2.7 ensures that

$$(u_\sigma, v_\sigma) \to (u, v) \quad \text{in } \; C^{1,\alpha}_{\mathrm{loc}}(\Omega).$$

The conclusion follows reasoning as in paragraph 3.1.2.

3.3 Final comments

Regarding the scalar case, analyzed in Section 3.1, hypotheses (H_1)–(H_2) allow to get existence of a solution to (3.0.1) with both methods: (H_1) has been used to deduce all the estimates for the auxiliary problems (3.1.5) and (3.1.10); (H_2) has been exploited to construct a sub-solution, in paragraph 3.1.1, or to prove that $u = \lim_\sigma u_\sigma$ is non-trivial, in paragraph 3.1.2. The biggest difference between the two methods concerns regularity of solutions up to the boundary: indeed, as already observed in paragraph 3.1.2, a comparison between the distance function is necessary to derive global regularity, as well as to ensure that the found solution is weak and not merely distributional. Anyway, hypotheses (H_1)–(H_2) seem general enough to expect no differences about the application of the two methods at any 'higher level', e.g., in presence of super-linear or convection terms. Indeed, considering super-linear reaction terms introduces difficulties only in solving (3.1.5) and (3.1.10), respectively, but the main strategies of Section 3.1 seem not to be affected; nevertheless, the functionals (3.1.6) and (3.1.11) look very similar, so that these difficulties can be overcome often in the same way (exploiting, for instance, the variational tools presented in Section 2.4). Also adding convection terms does not interact with the core of the two methods: indeed, once the gradient terms are frozen, truncation and shifting act as above, and then in any case 'un-freezing' the convection terms is a matter of topological methods (as described in Section 2.5).

Doubtless, a comparison of the two methods is more interesting in the context of systems, which have been investigated in Section 3.2. Indeed, the methods work with hypotheses of different nature: more precisely, the sub-super-solution technique exploits some geometric features of problem (3.0.2), such as the p-sub-linear behavior at infinity expressed in (H_3), while the shifting method requires some regularity assumptions on (3.0.2), usually guaranteed by summability conditions like (C_2').

As anticipated at the beginning of the chapter, now we would like to perform a comparison between the sub-super-solution technique and the shifting method for a particular class of systems in the form of (3.0.2): we require that they possess a cooperative structure (in the generalized sense, that is, assuming no monotonicity condition on f, g but only sign conditions on α_i, β_i in (H_3)), and that they do not exhibit any additional sub-linear terms: specifically, retaining the notation used above, we assume (H_3) with

$$\alpha_1 < 0 < \beta_1 < p - 1, \quad \gamma_1 = \delta_1 = 0,$$
$$\beta_2 < 0 < \alpha_2 < q - 1, \quad \gamma_2 = \delta_2 = 0.$$

Notice that these assumptions ensure (C_2''), which is the only mixed condition

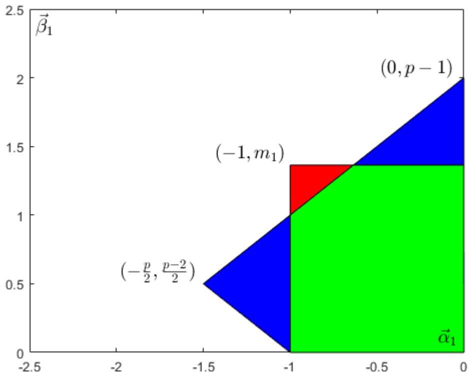


Figure 1: *Plot of* (M_1')–(M_2') *in the* (α_1, β_1)*-plane, assuming* $0 < p-2 < pq^*/N < p-1$ *(chosen parameters:* $N = 8$*,* $p = 3$*,* $q = \frac{5}{2}$*). In the blue region* $\Xi_1 \setminus \Sigma_1$ *condition* (M_1') *is satisfied but not* (M_2')*, while in the red region* $\Sigma_1 \setminus \Xi_1$ *condition* (M_2') *is verified but not* (M_1')*; both conditions hold true within the green region* $\Xi_1 \cap \Sigma_1$*.*

appearing in Section 3.2. For this reason, as a preliminary step, we can compare (C_1) and (C_2') for the first equation, with growth exponents $\alpha := \alpha_1 < 0 < \beta_1 =: \beta$; then the symmetry of (C_1) and (C_2') allows us to invert the role of α and β, as well as p and q, in order to study the second equation of (3.0.2). In this framework we re-write hypotheses (C_1) and (C_2'), in their parts concerning α_1 and β_1, as follows:

$$(\mathrm{M}_1') \qquad -1 < \alpha_1 + \beta_1 < \beta_1 - \alpha_1 < p - 1.$$

$$(\mathrm{M}_2') \qquad -1 < \alpha_1 < 0 < \beta_1 < \min\left\{\tfrac{pq^*}{N}, p-1\right\} =: m_1.$$

We define

$$\Xi_1 := \{(\alpha_1, \beta_1) \in (-p/2, 0) \times (0, p-1) : (\mathrm{M}_1') \text{ holds true}\},$$

and

$$\Sigma_1 := \{(\alpha_1, \beta_1) \in (-1, 0) \times (0, p-1) : (\mathrm{M}_2') \text{ holds true}\}.$$

We highlight that Ξ_1 and Σ_1 depend on N, p, q. Figure 1 shows that Ξ_1 is a rectangle whose diagonals are parallel to the axes, while Σ_1 is a rectangle with sides parallel to axes; moreover we can characterize, in dependence of N, p, q, whether the blue (resp., red) region is not empty (hereafter we make use of the logical connectives of conjunction '$\wedge$' and disjunction '$\vee$'):

$$\begin{aligned}
\Xi_1 \setminus \Sigma_1 \neq \emptyset \quad &\Leftrightarrow \quad p-1 > m_1 \vee \frac{p}{2} > 1 \quad &&\Leftrightarrow \quad q^* < \frac{N}{p'} \vee p > 2, \\
\Sigma_1 \setminus \Xi_1 \neq \emptyset \quad &\Leftrightarrow \quad m_1 + 1 > p-1 &&\Leftrightarrow \quad q^* > N\left(1 - \frac{2}{p}\right).
\end{aligned}$$

In particular, inequality $m_1 + 1 > p - 1$ stems from the fact that $\Sigma_1 \setminus \Xi_1$ is not empty provided the point $(\alpha_1, \beta_1) = (-1, m_1)$ stays above the line $\beta_1 - \alpha_1 = p - 1$. Solving with respect to q we get

$$\begin{aligned}
\Xi_1 \setminus \Sigma_1 \neq \emptyset \quad &\Leftrightarrow \quad q < \psi_1(p) \vee p > 2, \\
\Sigma_1 \setminus \Xi_1 \neq \emptyset \quad &\Leftrightarrow \quad q > \psi_2(p),
\end{aligned}$$

where ψ_1, ψ_2 are strictly increasing functions defined as

$$\psi_1(r) = \frac{N(r-1)}{2r-1} \quad \forall\, r \in (1, N) \tag{3.3.1}$$

and

$$\psi_2(r) = \frac{N(r-2)}{2r-2} \quad \forall\, r \in (1, N). \tag{3.3.2}$$

Arguing similarly for the second equation, we can re-write the parts of (C_1) and (C_2') concerning α_2 and β_2 as

(M_1'') $$-1 < \alpha_2 + \beta_2 < \alpha_2 - \beta_2 < q - 1,$$

(M_2'') $$-1 < \beta_2 < 0 < \alpha_2 < \min\left\{\tfrac{qp^*}{N}, q-1\right\} =: m_2.$$

Hence, (C_1) is equivalent to $(\mathrm{M}_1') \wedge (\mathrm{M}_1'')$, while (C_2') is equivalent to $(\mathrm{M}_2') \wedge (\mathrm{M}_2'')$. We also define the regions

$$\begin{aligned}
\Xi_2 &:= \{(\alpha_2, \beta_2) \in (0, q-1) \times (-q/2, 0) : (\mathrm{M}_1'') \text{ holds true}\}, \\
\Sigma_2 &:= \{(\alpha_2, \beta_2) \in (0, q-1) \times (-1, 0) : (\mathrm{M}_2'') \text{ holds true}\}.
\end{aligned}$$

The symmetry between (M_i') and $(\mathrm{M}_\mathrm{i}'')$, $i = 1, 2$, yields

$$\begin{aligned}
\Xi_2 \setminus \Sigma_2 \neq \emptyset \quad &\Leftrightarrow \quad p < \psi_1(q) \vee q > 2, \\
\Sigma_2 \setminus \Xi_2 \neq \emptyset \quad &\Leftrightarrow \quad p > \psi_2(q).
\end{aligned}$$

As already pointed out, the regions Ξ_i and Σ_i, for $i = 1, 2$, depend in particular on p and q. Now we set $\Xi := \Xi_1 \times \Xi_2$ and $\Sigma := \Sigma_1 \times \Sigma_2$. We are interested in characterizing, in dependence of N, p, q, the strict inclusions $\Xi \subsetneq \Sigma$ and $\Sigma \subsetneq \Xi$; informally speaking, these inclusions correspond to the heuristic proposition 'the sub-super-solution technique is better than the shifting method' or vice-versa, since the range of applicability of one method properly includes the range of the other one.

Using De Morgan's laws, we have

$$\begin{aligned}
\Xi \setminus \Sigma \neq \emptyset \quad &\Leftrightarrow \quad \Xi_1 \setminus \Sigma_1 \neq \emptyset \,\wedge\, \Xi_2 \setminus \Sigma_2 \neq \emptyset \\
&\Leftrightarrow \quad (q < \psi_1(p) \,\vee\, p > 2) \,\wedge\, (p < \psi_1(q) \,\vee\, q > 2) \\
&\Leftrightarrow \quad (q < \psi_1(p) \,\wedge\, p < \psi_1(q)) \,\vee\, 2 < q < \psi_1(p) \\
&\qquad\quad \vee\, 2 < p < \psi_1(q) \,\vee\, \min\{p,q\} > 2 \\
&\Leftrightarrow \quad (q < \psi_1(p) \,\wedge\, p < \psi_1(q)) \,\vee\, p < 2 < q < \psi_1(p) \\
&\qquad\quad \vee\, q < 2 < p < \psi_1(q) \,\vee\, \min\{p,q\} > 2.
\end{aligned}$$

If $2 \leq N \leq 5$ then $\psi_1(r) < r$ for all $r \in (1, N)$; thus,

$$\begin{aligned}
q < \psi_1(p) \,\wedge\, p < \psi_1(q) \quad &\Rightarrow \quad q < \psi_1(p) < p < \psi_1(q) < q \quad &\Rightarrow \quad \text{absurd}, \\
p < 2 < q < \psi_1(p) \quad &\Rightarrow \quad p < 2 < q < \psi_1(p) < p \quad &\Rightarrow \quad \text{absurd}, \\
q < 2 < p < \psi_1(q) \quad &\Rightarrow \quad q < 2 < p < \psi_1(q) < q \quad &\Rightarrow \quad \text{absurd}.
\end{aligned}$$

Accordingly,

$$\Xi \setminus \Sigma \neq \emptyset \;\Leftrightarrow\; \min\{p,q\} > 2 \qquad \text{provided } \; 2 \leq N \leq 5.$$

In the case $N \geq 6$ we have $\psi_1(r) \geq \psi(2) = \frac{N}{3} \geq 2$ for any $r \geq 2$; hence,

$$\begin{aligned}
p < 2 < q < \psi_1(p) \quad &\Rightarrow \quad p < \psi_1(q) \quad &\Rightarrow \quad q < \psi_1(p) \,\wedge\, p < \psi_1(q), \\
q < 2 < p < \psi_1(q) \quad &\Rightarrow \quad q < \psi_1(p) \quad &\Rightarrow \quad q < \psi_1(p) \,\wedge\, p < \psi_1(q).
\end{aligned}$$

So

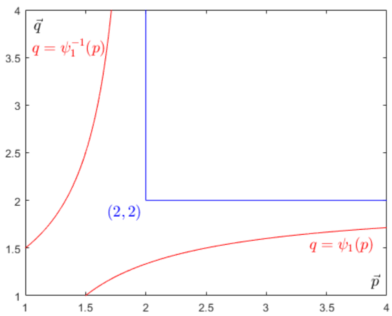
$$\Xi \setminus \Sigma \neq \emptyset \;\Leftrightarrow\; (q < \psi_1(p) \wedge p < \psi_1(q)) \vee \min\{p,q\} > 2 \qquad \text{provided } \; N \geq 6.$$

Notice that, if $(p^*, q^*) := (\frac{7}{4}, \frac{7}{4}) \in (1,2)^2$, then $\psi_1(p^*) = \frac{3}{10}N > p^*$; this shows that condition '$q < \psi_1(p) \wedge p < \psi_1(q)$' does not imply '$\min\{p,q\} > 2$'. Moreover, since $\psi_1(p) < N$ for all $p \in (1,N)$, the aforementioned conditions are mutually independent. Figure 2 collects the results obtained above.
A similar argument leads to

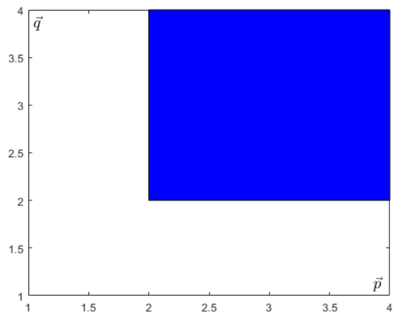
$$\begin{aligned}
\Sigma \setminus \Xi \neq \emptyset \quad &\Leftrightarrow \quad \Sigma_1 \setminus \Xi_1 \neq \emptyset \,\wedge\, \Sigma_2 \setminus \Xi_2 \neq \emptyset \\
&\Leftrightarrow \quad q > \psi_2(p) \,\wedge\, p > \psi_2(q).
\end{aligned}$$

As shown in Figure 3, dimension 12 represents a threshold for ψ_2: indeed, $\psi_2(p) < p$ for all $p \in (1,N)$ provided $N < 12$.
Writing the dependencies $\Xi = \Xi(p,q)$ and $\Sigma = \Sigma(p,q)$ explicitly, we get

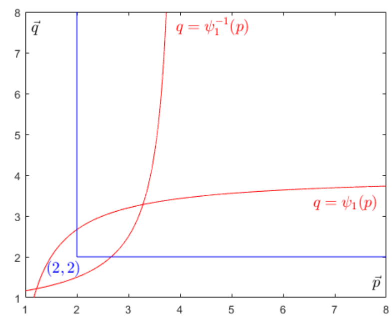
$$\begin{aligned}
\Xi(p,q) \subseteq \Sigma(p,q) \quad &\Leftrightarrow \quad (q \geq \psi_1(p) \,\vee\, p \geq \psi_1(q)) \,\wedge\, \min\{p,q\} \leq 2, \\
\Sigma(p,q) \subseteq \Xi(p,q) \quad &\Leftrightarrow \quad q \leq \psi_2(p) \,\vee\, p \leq \psi_2(q).
\end{aligned} \tag{3.3.3}$$



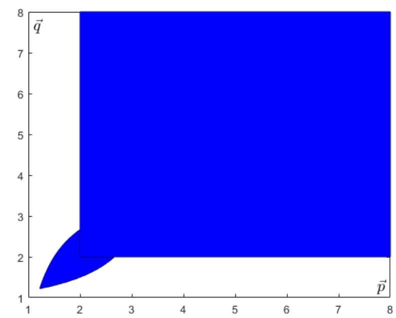
(a) *Plot of* ψ_1 *when* $N \leq 5$.



(b) *Plot of* $\Xi \setminus \Sigma \neq \emptyset$ *when* $N \leq 5$.

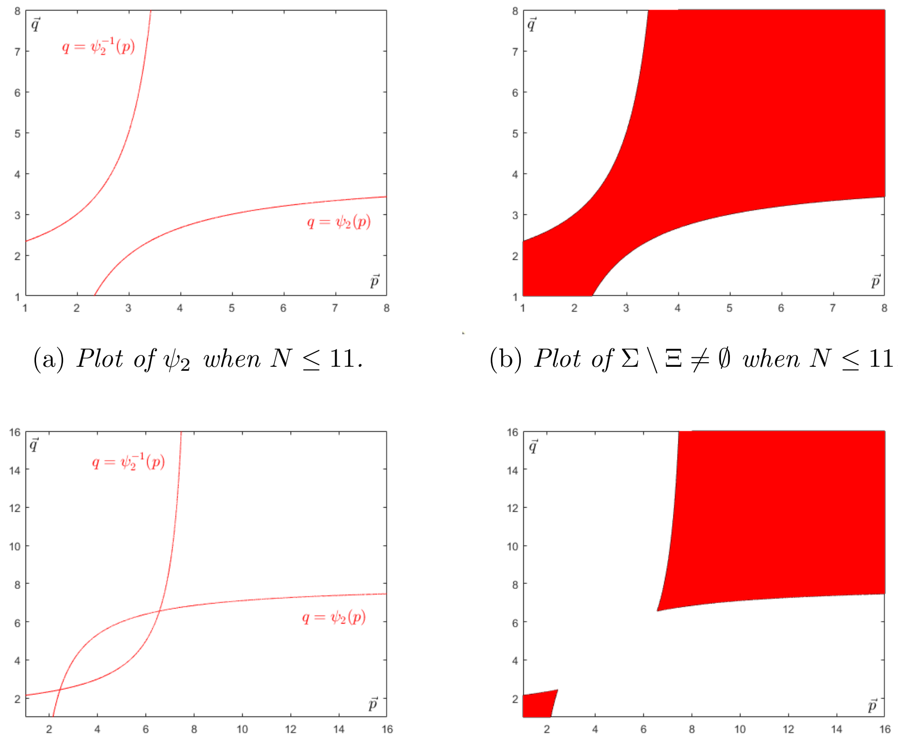


(c) *Plot of* ψ_1 *when* $N \geq 6$.



(d) *Plot of* $\Xi \setminus \Sigma \neq \emptyset$ *when* $N \geq 6$.

Figure 2: *Plots related to condition* $\Xi \setminus \Sigma \neq \emptyset$ *(chosen parameters:* $N = 4$ *for* (a)–(b) *and* $N = 8$ *for* (c)–(d)*).*



(a) *Plot of* ψ_2 *when* $N \leq 11$.

(b) *Plot of* $\Sigma \setminus \Xi \neq \emptyset$ *when* $N \leq 11$.

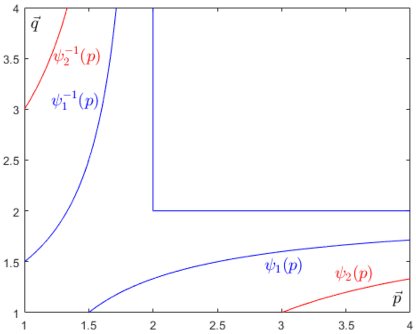
(c) *Plot of* ψ_2 *when* $N \geq 12$.

(d) *Plot of* $\Sigma \setminus \Xi \neq \emptyset$ *when* $N \geq 12$.

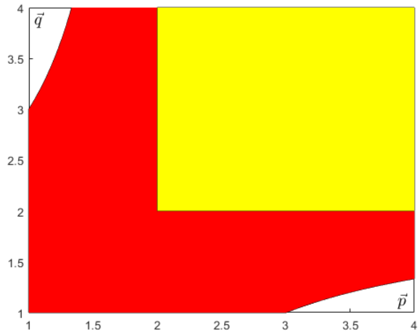
Figure 3: *Plots related to condition* $\Sigma \setminus \Xi \neq \emptyset$ *(chosen parameters:* $N = 8$ *for* (a)–(b) *and* $N = 16$ *for* (c)–(d)*).*

A numerical analysis of the conditions in (3.3.3) shows three different scenarios: the outcome is displayed in Figure 4. We observe that the (p, q)-region in which $\Sigma(p, q) \subsetneq \Xi(p, q)$ (colored in blue) is empty in low dimension, i.e., $N \leq 6$; anyway, this region enlarges when the dimension N increases. On the other hand, the red region $\Xi(p, q) \subsetneq \Sigma(p, q)$ tends to disappear as $N \to \infty$.

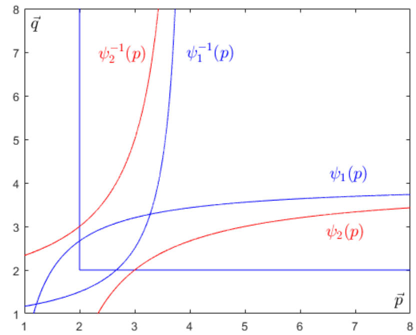
Remark 3.3.1. For the sake of completeness, we say that our analysis does not pretend to give optimal results, nor to definitively answer to the question whether the sub-super-solution technique is better than the shifting method or not (for a given triple (N, p, q)): indeed, most likely, the proofs furnished in Sections 3.1–3.2 could be adapted to encompass situations more general than the ones described by (C_1) and (C_2')–(C_2''). The comparison offered in this section would like to give only the flavor of the differences between these two very powerful methods, and tries to show - in a simple case - a possible procedure to compare existence results whose applicability heavily relies on the growth of reaction terms.



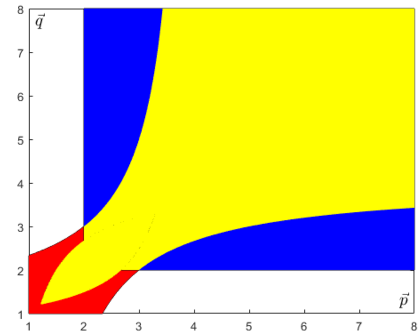
(a) *Plot of ψ_1 and ψ_2 when $2 \le N \le 6$.*



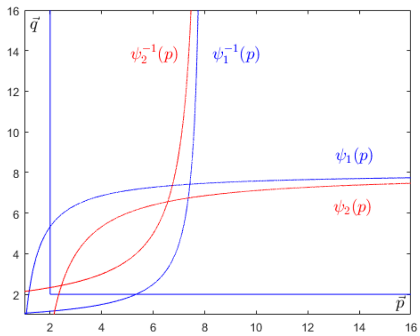
(b) *Comparison plot for $2 \le N \le 6$.*



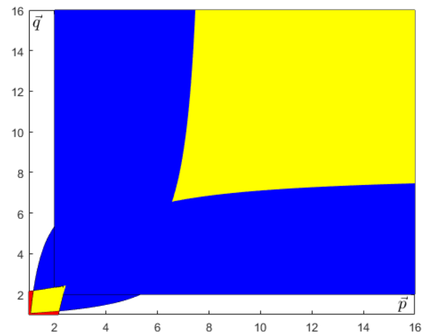
(c) *Plot of ψ_1 and ψ_2 when $7 \le N \le 11$.*



(d) *Comparison plot for $7 \le N \le 11$.*



(e) *Plot of ψ_1 and ψ_2 when $N \ge 12$.*



(f) *Comparison plot for $N \ge 12$.*

Figure 4: *Concerning the comparison plots* (b), (d), *and* (f), *we represented the results in the following way: in the blue region we have $\Sigma \subsetneq \Xi$, while in the red region it holds $\Xi \subsetneq \Sigma$; the two methods are not comparable in the yellow region (i.e., $\Sigma \not\subseteq \Xi$ and $\Xi \not\subseteq \Sigma$), while in the white region we have $\Xi = \Sigma$. (chosen parameters: $N = 4$ for* (a)–(b)*, $N = 8$ for* (c)–(d)*, and $N = 16$ for* (e)–(f)*).*

4 Existence, uniqueness, and multiplicity results

This chapter is dedicated to the analysis of problems more sophisticated than the ones presented in Chapter 3: indeed, the former problems can exhibit super-linear or convection terms, can be subjected to Neumann or Robin boundary conditions, or their setting can be the whole space $\mathbb{R}^N$. The results contained in the present chapter have been published in international journals: see [26, 90, 89, 91]. Nevertheless, in paragraph 4.2.2 we present a more general version of [91]: indeed, retaining the notation of [91], we obtain existence of a weak solution to (4.2.13) without supposing neither $p, q > 2 - \frac{1}{N}$ nor hypothesis H_3; we also removed the assumption $a_1 \in L^{s_p}_{\mathrm{loc}}(\mathbb{R}^N)$ (resp., $a_2 \in L^{s_q}_{\mathrm{loc}}(\mathbb{R}^N)$) with $s_p > p'N$ (resp., $s_q > q'N$).

We introduce the following notation: if Y is a real function space on a set $\Omega \subseteq \mathbb{R}^N$ and $u, v \in Y$, then $u \leq v$ means $u(x) \leq v(x)$ for almost every $x \in \Omega$; moreover, $Y_+ := \{u \in Y : u \geq 0\}$.

4.1 Equations

4.1.1 Parametric singular p-Laplacian problems

In this paragraph we study the existence of at least two weak solutions for the following Dirichlet problem:

$$\begin{cases} -\Delta_p u = \lambda f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$P_{\lambda,f}$}$$

where $N \geq 2$, $1 < p < N$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ denotes the p-Laplacian operator, $\lambda > 0$ is a parameter, and $f : \Omega \times (0,+\infty) \to [0,+\infty)$ is a Carathéodory function satisfying the following assumptions:

$\underline{\mathrm{H(f)_1}}$: $\lim\limits_{s\to 0^+} f(x,s) = +\infty$, uniformly w.r.t. $x \in \Omega$.

$\underline{\mathrm{H(f)_2}}$: There exist $s_0, b_1 > 0$, $b_2 \geq 0$, $\gamma \in (0,1)$, $q \in (1, p^*)$, and $a \in L^{q'}(\Omega)_+$ such that:

$$\text{(i)} \quad f(x,s)s^\gamma \leq b_1 \quad \text{for a.a. } x \in \Omega \quad \text{and for all } s \in (0, s_0);$$
$$\text{(ii)} \quad f(x,s) \leq a(x) + b_2 s^{q-1} \quad \text{for a.a. } x \in \Omega \text{ and for all } s \in [s_0, +\infty).$$

Moreover, in order to obtain a second solution in the super-linear case, that is, $p < q < p^*$, we also require that the function f fulfills a unilateral version of the Ambrosetti-Rabinowitz condition (cf. [134, pag. 154] and also [140, 141]):

(AR^+) : There exist $\mu > p$, $R > s_0$, and $s_1 \in [s_0, R)$ such that

$$0 < \inf_{x\in\Omega} \int_0^R f(x,t)dt, \quad 0 < \mu \int_{s_1}^s f(x,t)dt \leq f(x,s)s,$$

for a.a. $x \in \Omega$ and for all $s \geq R$, being s_0 as in $\mathrm{H(f)}_2$.

In this paragraph we denote with $\|\cdot\|$ the standard equivalent norm in $W_0^{1,p}(\Omega)$, that is $\|u\| := \|\nabla u\|_p$ for all $u \in W_0^{1,p}(\Omega)$.
We denote with λ_1 and φ_p the first eigenvalue and the corresponding (L^p-normalized) positive eigenfunction associated to the negative $p-$Laplacian in $W_0^{1,p}(\Omega)$; in other words, φ_p solves problem $(P_{\lambda_1,|u|^{p-2}u})$. Theorem 2.1.8 and Remark 2.1.9 ensure that

$$\varphi_p \in C^{1,\alpha}(\overline{\Omega}), \quad 0 < \alpha \leq 1, \quad \text{and} \quad \tilde{l}d(x) \leq \varphi_p(x) \leq \tilde{L}d(x) \tag{4.1.1}$$

for all $x \in \overline{\Omega}$, being $\tilde{l}$ and $\tilde{L}$ two positive constants, and d as in paragraph 2.1.3.

In order to fix an auxiliary variational setting, the first step is showing that problem $(P_{\lambda,f})$ admits a sub-solution: the proof is essentially the same of the one given at the beginning of paragraph 2.3.3.

Lemma 4.1.1. *Let* $\mathrm{H(f)}_1$ *hold true and* s_0 *be as in* $\mathrm{H(f)}_2$(i)*. Then, for every* $\lambda > 0$*, there exist* $0 < \delta < s_0$*,* $l = l(\lambda) > 0$*, and* $\underline{u} \in C^{1,\alpha}(\overline{\Omega})$*, being* α *as in* (4.1.1)*, such that* $\underline{u}$ *is a sub-solution to problem* $(P_{\lambda,f})$*, that is,*

$$\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla v dx \leq \lambda \int_\Omega f(x,\underline{u})v dx \quad \forall v \in W_0^{1,p}(\Omega)_+ \tag{4.1.2}$$

and

$$ld(x) \leq \underline{u}(x) \leq \delta \quad \forall x \in \overline{\Omega}. \tag{4.1.3}$$

Take any $\underline{u}$ as in Lemma 4.1.1 and the following Carathéodory function $f^* : \Omega \times \mathbb{R} \to \mathbb{R}$, defined as

$$f^*(x,s) = \begin{cases} f(x,s), & \text{if } s \geq \underline{u}, \\ f(x,\underline{u}), & \text{if } |s| < \underline{u}, \\ f(x,-s), & \text{if } s \leq -\underline{u}. \end{cases} \tag{4.1.4}$$

Exploiting (4.1.3), besides $\mathrm{H(f)}_2$, we get

$$f^*(x,s) \leq \begin{cases} b_1\underline{u}^{-\gamma} \leq \tilde{b}_1 d^{-\gamma}, & \text{if } |s| < s_0, \\ a(x) + b_2|s|^{q-1}, & \text{if } |s| \geq s_0, \end{cases} \tag{4.1.5}$$

where $\tilde{b}_1 = b_1 l^{-\gamma}$. Hence, using (2.1.7), the Hölder inequality, and (2.1.4), one gets

$$\begin{aligned}
&\left|\int_\Omega f^*(x,u)v dx\right| \le \int_\Omega f^*(x,u)|v|dx \\
&\le \tilde{b}_1 \int_\Omega |v| d(x)^{-\gamma} dx + b_2 \int_\Omega |u|^{q-1}|v|dx + \int_\Omega a(x)|v|dx \\
&\le c_1 \left(\|v\| + \|u\|_q^{q-1}\|v\|_q + \|a\|_{q'}\|v\|_q\right) \\
&\le c_2 \left(1 + \|u\|^{q-1} + \|a\|_{q'}\right)\|v\| < +\infty,
\end{aligned} \tag{4.1.6}$$

for every $u, v \in W_0^{1,p}(\Omega)$. Therefore, problem (P_{λ,f^*}) admits the equivalent weak formulation

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \lambda \int_\Omega f^*(x,u)v dx \quad \forall v \in W_0^{1,p}(\Omega). \tag{4.1.7}$$

Reasoning as in the final part of the proof of Theorem 2.3.10, it is possible to prove what follows.

Lemma 4.1.2. *For any $\lambda > 0$, any solution to (P_{λ,f^*}) is a solution of $(P_{\lambda,f})$.*

Set

$$F : \Omega \times \mathbb{R} \to \mathbb{R}, \quad F(x,s) = \int_0^s f^*(x,t)dt, \tag{4.1.8}$$

and

$$\Psi : W_0^{1,p}(\Omega) \to \mathbb{R}, \quad \Psi(u) = \int_\Omega F(x,u)dx. \tag{4.1.9}$$

Clearly, F is a Carathéodory function. Bearing in mind (4.1.5), one has

$$f^*(x,s) \le a_1(x) + b_2|s|^{q-1}, \tag{4.1.10}$$

for all $x \in \Omega$, being $a_1 = c_1 d^{-\gamma} + a \in L^1(\Omega)$ and c_1 a suitable constant. The function a_1 does not satisfy the standard summability condition $a_1 \in L^{q'}(\Omega)$, which ensures the Nemyskii operator $N_F u := F(x,u)$ associated to F to be of class C^1 in $L^q(\Omega)$; thus we collect in the following lemma some properties of the integral functional Ψ, and for the sake of completeness we also give a sketch of their proofs.

Lemma 4.1.3. *Under hypotheses* $\mathrm{H(f)}_1$ *and* $\mathrm{H(f)}_2$*, the functional Ψ, introduced in* (4.1.9)*, is well defined, of class C^1, and weakly sequentially continuous, with*

$$\langle \Psi'(u), v\rangle = \int_\Omega f^*(x,u)v dx$$

for all $u, v \in W_0^{1,p}(\Omega)$. Moreover, the operator $\Psi' : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is strongly continuous.

Proof. Inequality (4.1.5), besides (2.3.10) and Young's inequality, leads to

$$\begin{aligned} |F(x,s)| &\le \int_0^{|s|} f^*(x,t)dt \\ &\le b_1 \int_0^{s_0} \max\{t,\underline{u}(x)\}^{-\gamma} dt + \frac{b_2}{q}|s|^q + a(x)|s| \\ &= b_1 \left(\int_0^{\underline{u}(x)} \underline{u}(x)^{-\gamma} dt + \int_{\underline{u}(x)}^{s_0} t^{-\gamma} dt \right) + \frac{b_2}{q}|s|^q + a(x)|s| \\ &\le \left(1 + \frac{1}{1-\gamma}\right) b_1 s_0^{1-\gamma} + \frac{1}{q'} a(x)^{q'} + \frac{b_2+1}{q}|s|^q, \end{aligned} \tag{4.1.11}$$

for all $x \in \Omega$ and $s \in \mathbb{R}$. The last member of (4.1.11) for $s = u^+(x)$ is a $L^1(\Omega)$ function; thus, Ψ is well defined.

In order to prove the regularity of Ψ, let us compute its Gâteaux derivative. Take any $v \in W_0^{1,p}(\Omega)$.

$$\lim_{t\to 0^+} \frac{\Psi(u+tv)-\Psi(u)}{t} = \lim_{t\to 0^+} \int_\Omega \frac{F(x,(u+tv)) - F(x,u)}{t} dx. \tag{4.1.12}$$

Fix any $t > 0$. According to Torricelli's theorem, one has

$$\begin{aligned} F(x,(u+tv)) - F(x,u) &= tv \left(\int_0^1 \left(\frac{d}{ds} F(x,u+stv) \right) ds \right) \\ &= tv \left(\int_0^1 f^*(x,u+stv) ds \right). \end{aligned} \tag{4.1.13}$$

On the other hand, (4.1.10) furnishes

$$f^*(x,y+z) \le c_1 d(x)^{-\gamma} + a(x) + b_2 2^{q-1}(|y|^{q-1} + |z|^{q-1}), \tag{4.1.14}$$

for every $x \in \Omega$ and $y,z \in \mathbb{R}$. Plugging (4.1.13) into (4.1.12), besides exploiting Fubini's theorem, inequality (2.1.7), estimate (4.1.14), Lebesgue's dominated convergence theorem (argue as in (4.1.6)), we have

$$\begin{aligned} \langle \Psi'(u), v \rangle &= \lim_{t\to 0^+} \frac{\Psi(u+tv)-\Psi(u)}{t} \\ &= \lim_{t\to 0^+} \int_\Omega \left(\int_0^1 v f^*(x,u+stv) ds \right) dx \\ &= \lim_{t\to 0^+} \int_0^1 \left(\int_\Omega v f^*(x,u+stv) dx \right) ds \\ &= \int_0^1 \left(\int_\Omega v \lim_{t\to 0^+} \left(f^*(x,u+stv) \right) dx \right) ds, \\ &= \int_\Omega f^*(x,u) v dx, \end{aligned} \tag{4.1.15}$$

for all $u, v \in W_0^{1,p}(\Omega)$.

Observe that, for any $u \in W_0^{1,p}(\Omega)$ and $\{u_n\} \subseteq W_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, the compactness of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $q \in [1, p^*)$ implies that there exists $w \in L^q(\Omega)$ such that

$$u_n \to u \quad \text{in} \quad L^q(\Omega), \quad u_n \to u \quad \text{a.e. in } \Omega, \quad \text{and } |u_n| \le w \tag{4.1.16}$$

for every $n \in \mathbb{N}$ (see [23, Theorem 4.9]). Now we prove that $\Psi' : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is a continuous operator. To this aim, let $u_n \to u$ in $W_0^{1,p}(\Omega)$. According to (4.1.10) we have

$$f^*(x, u_n) \le c_1 d(x)^{-\gamma} + a(x) + b_2|w|^{q-1} \tag{4.1.17}$$

for all $x \in \Omega$ and $n \in \mathbb{N}$. Therefore, as above, Lebesgue's dominated convergence theorem and continuity of $f^*(x, \cdot)$ for a.a. $x \in \Omega$ ensure that

$$\lim_{n\to\infty} \langle \Psi'(u_n), v\rangle = \int_\Omega \lim_{n\to\infty} f^*(x, u_n) v dx = \int_\Omega f^*(x, u) v dx = \langle \Psi'(u), v\rangle.$$

We conclude that Ψ is of class C^1.
In order to prove that Ψ' is strongly continuous, let $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Observe that

$$\lim_{n\to\infty} |\langle \Psi'(u_n) - \Psi'(u), v\rangle| \le \lim_{n\to\infty} \int_\Omega |f^*(x, u_n) - f^*(x, u)||v| dx \tag{4.1.18}$$

for any $v \in W_0^{1,p}(\Omega)$. Owing to (4.1.16)-(4.1.17), Lebesgue's dominated convergence theorem applies; thus, (4.1.18) gives $\Psi'(u_n) \to \Psi'(u)$, as desired.
The weak sequential continuity of Ψ is a direct consequence of [164, Corollary 41.9]. $\square$

Remark 4.1.4. We point out that estimate (4.1.11) implies that for any $R > 0$ and $u \in W_0^{1,p}(\Omega)$ there exists a positive constant $\Pi(R)$ such that

$$\int_{\Omega(|u|\le R)} |F(x, u(x))| dx \le \Pi(R), \tag{4.1.19}$$

being $\Omega(|u| \le R) := \{x \in \Omega : -R \le u(x) \le R\}$. Analogously, estimates (4.1.5) and (4.1.3) ensure that for any $R > 0$ and $u \in W_0^{1,p}(\Omega)$ there exists a positive constant $\Pi'(R)$ such that

$$\int_{\Omega(|u|\le R)} f^*(x, u(x))|u(x)| dx \le \Pi'(R). \tag{4.1.20}$$

The following three positive constants A, B, and r_λ^* play a crucial role in our existence result:

$$A = \left(1 + \frac{1}{1-\gamma}\right) b_1 s_0^{1-\gamma} |\Omega| + \frac{1}{q'} \|a\|_{q'}^{q'}, \tag{4.1.21}$$

$$B = \left(\frac{b_2+1}{q}\right) S_q^q p^{\frac{q}{p}}, \tag{4.1.22}$$

with S_q arising from (2.1.4), and

$$r_\lambda^* \begin{cases} > r : \lambda B r^{\frac{q}{p}} - r + \lambda A = 0, & \text{if } 1 < q < p, \\ > \frac{\lambda A}{1-\lambda B}, & \text{if } q = p, \\ = \left(\frac{A}{B}\frac{p}{q-p}\right)^{\frac{p}{q}}, & \text{if } p < q < p^*. \end{cases} \tag{4.1.23}$$

Our first main result is the following

Theorem 4.1.5. *Suppose that* $\mathrm{H(f)}_1$ *and* $\mathrm{H(f)}_2$ *hold. Set*

$$\lambda^* = \begin{cases} +\infty, & \text{if } 1 < q < p, \\ \frac{1}{B}, & \text{if } q = p, \\ \frac{1}{q}\left(\frac{q-p}{A}\right)^{1-\frac{p}{q}} \left(\frac{p}{B}\right)^{\frac{p}{q}}, & \text{if } p < q < p^*, \end{cases} \tag{4.1.24}$$

where A, B, and r_λ^* are defined in (4.1.21)-(4.1.22)-(4.1.23). *Then for any $\lambda \in (0, \lambda^*)$ there exists a weak solution $u^* = u^*(\lambda)$ to $(P_{\lambda,f})$ with $\|u^*\| \leq (pr_\lambda^*)^{\frac{1}{p}}$.*

Proof. Taking into account Lemma 4.1.2, our goal is to apply Theorem 2.4.13 (through Remark 2.4.14) to the energy functional $J_\lambda = \Phi - \lambda\Psi$ associated with problem (P_{λ,f^*}), being

$$\Phi : W_0^{1,p}(\Omega) \to \mathbb{R}, \quad \Phi(u) = \frac{1}{p}\|u\|^p,$$

and

$$\Psi : W_0^{1,p}(\Omega) \to \mathbb{R}, \quad \Psi(u) = \int_\Omega F(x,u)dx,$$

where f^* and F are defined in (4.1.4) and (4.1.8), respectively. Bearing in mind Lemma 4.1.3, the functionals Φ and Ψ satisfy all the assumptions of Theorem 2.4.13, as well as that the critical points of J_λ are weak solutions of problem (P_{λ,f^*}).

Fix $r > 0$. Owing to (4.1.11), (4.1.21), and (4.1.22), one has

$$\frac{1}{r}\sup_{\|u\|\le(pr)^{\frac{1}{p}}}\Psi(u)\le\frac{A}{r}+Br^{\frac{q}{p}-1}=:\frac{1}{h(r)}. \tag{4.1.25}$$

Notice that

$$\lim_{r\to0^+}h(r)=0 \tag{4.1.26}$$

for any $q\in(1,p^*)$,

$$\lim_{r\to+\infty}h(r)=\begin{cases}+\infty, & \text{if } 1<q<p,\\ \frac{1}{B}, & \text{if } q=p,\\ 0, & \text{if } p<q<p^*,\end{cases} \tag{4.1.27}$$

and h is strictly increasing whenever $q\in(1,p]$.
If $p<q<p^*$, a direct computation of the (unique) critical point of h shows that

$$\sup_{r>0}h(r)=h\left(\left(\frac{A}{B}\frac{p}{q-p}\right)^{\frac{p}{q}}\right)=\frac{1}{q}\left(\frac{q-p}{A}\right)^{1-\frac{p}{q}}\left(\frac{p}{B}\right)^{\frac{p}{q}}.$$

Thus we have

$$\sup_{r>0}h(r)=\lambda^*\le\frac{r^*_\lambda}{\sup_{\Phi^{-1}([0,r^*_\lambda])}\Psi},$$

being r^*_λ as in (4.1.23). Finally, fix $\lambda\in(0,\lambda^*)$; our conclusion follows from Theorem 2.4.13 and Remark 2.4.14, once we take $r=r^*_\lambda$. □

Remark 4.1.6. It is worth noticing that the constants A and B are related to both the estimate (4.1.11) and the geometry of problem $(P_{\lambda,f})$ (see also (2.1.4)). On the other hand, r^*_λ gives the smallest radius of the ball in $W^{1,p}_0(\Omega)$ where a local minimum for the energy functional J_λ, associated with problem (P_{λ,f^*}), is located. Finally, we point out that in the case $q\le p$ the functional J_λ turns out to be coercive.

Remark 4.1.7. The condition

(j) $\limsup_{s\to0^+}f(x,s)s^\gamma\le b_1$, uniformly w.r.t. $x\in\Omega$,

clearly means that there exists $\rho>0$ such that $f(x,s)s^\gamma\le b_1$ for every $0<s<\rho$. If, in addition, we have that

(jj) the function

$$M_\rho:\Omega\to[0,\infty),\quad M_\rho(x)=\sup_{s\in[\rho,s_0]}f(x,s),$$

belongs to $L^{q'}(\Omega)$,

then it follows that H(f)$_2$(ii) continue to hold whenever the function a is replaced with the function $M_\rho + a \in L^{q'}(\Omega)$.

Of course, (jj) is guaranteed in the autonomous case, i.e., when f does not depend on x.

Example 4.1.8. Let $h \in L^\infty(\Omega)$, $\operatorname{ess\,inf} h > 0$, $\gamma \in (0,1)$, $g_1 : \Omega \times \mathbb{R} \to [0, +\infty)$ a Carathéodory function such that

$$g_1(x,s) \leq c_1 + c_2 s^{q-1},$$

for all $x \in \Omega$ and $s \geq 0$, with $c_1, c_2 \geq 0$ and $q > 1$. Let $g_2 : \mathbb{R} \to [0, +\infty)$ be a continuous function such that

$$\limsup_{s \to 0^+} \frac{g_2(s)}{s^\gamma} = +\infty, \quad \limsup_{s \to 0^+} g_2(s) \leq b_1, \quad \limsup_{s \to +\infty} \frac{g_2(s)}{s^{q-1+\gamma}} < +\infty,$$

for some $b_1 \geq 0$ and $q \in (1, p^*)$.

Nonlinearities that satisfy hypotheses H(f)$_1$–H(f)$_2$ are, e.g.,

$$f(x,s) = \frac{h(x)}{s^\gamma} + g_1(x,s),$$

and

$$f(x,s) = \frac{g_2(s)}{s^\gamma}.$$

Now we expose our main result concerning the existence of two weak solutions for problem $(P_{\lambda,f})$.

Lemma 4.1.9. *Suppose that* H(f)$_1$, H(f)$_2$, *and* (AR^+) *hold. Then the functional* J_λ *satisfies* (PS) *and is unbounded from below.*

Proof. This proof is patterned after [61, Theorems 15 and 16]. Let us consider a sequence $\{u_n\} \subseteq W_0^{1,p}(\Omega)$ such that $\{J_\lambda(u_n)\}$ is bounded and $\{J'_\lambda(u_n)\}$ converges to zero. In other words, for any $n \in \mathbb{N}$,

$$\frac{1}{p}\|u_n\|^p - \lambda \int_\Omega F(x, u_n^+) dx \leq c_1 \tag{4.1.28}$$

and

$$\left| \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla v dx - \lambda \int_\Omega f^*(x, u_n) v dx \right| \leq \|v\| \quad \forall v \in W_0^{1,p}(\Omega) \tag{4.1.29}$$

hold true for an opportune $c_1 > 0$, independent of n. We will prove that $\{u_n\}$ is bounded by showing the property for $\{u_n^-\}$ and $\{u_n^+\}$.

Exploiting (4.1.29) with $v = -u_n^-$ leads to

$$\|u_n^-\|^p \leq \|u_n^-\|^p + \lambda \int_\Omega f(x, u_n) u_n^- dx \leq \|u_n^-\|,$$

and hence $\|u_n^-\| \leq 1$ for all $n \in \mathbb{N}$.
Let $\Omega_n := \{x \in \Omega : u_n(x) \geq R\}$ and $\Omega_n' := \Omega \setminus \Omega_n$ for any $n \in \mathbb{N}$. According to (4.1.19), one has

$$\int_{\Omega_n'} F(x, u_n^+)dx \leq \Pi(R), \tag{4.1.30}$$

while (AR^+) gives

$$\begin{aligned}\int_{\Omega_n} F(x, u_n^+)dx &= \int_{\Omega_n} \left(F(x, s_1) + \int_{s_1}^{u_n^+} f(x,t)dt \right) dx \\ &\leq \Pi(s_1) + \frac{1}{\mu} \int_{\Omega_n} f(x, u_n^+)u_n^+ dx \\ &\leq \Pi(R) + \frac{1}{\mu} \int_{\Omega} f^*(x, u_n)u_n^+ dx - \frac{1}{\mu} \int_{\Omega_n'} f^*(x, u_n)u_n^+ dx.\end{aligned} \tag{4.1.31}$$

On the other hand, from (4.1.20) we get

$$\left| \int_{\Omega_n'} f^*(x, u_n)u_n^+ dx \right| \leq \Pi'(R). \tag{4.1.32}$$

Then (4.1.31) becomes

$$\int_{\Omega_n} F(x, u_n^+)dx \leq \Pi(R) + \frac{1}{\mu} \int_{\Omega} f^*(x, u_n)u_n^+ dx + \frac{1}{\mu}\Pi'(R). \tag{4.1.33}$$

From (4.1.28) we infer

$$\begin{aligned}&\frac{1}{p}\|u_n^+\|^p \leq \frac{1}{p}\|u_n\|^p \leq c_1 + \lambda \left(\int_{\Omega_n} F(x, u_n^+)dx + \int_{\Omega_n'} F(x, u_n^+)dx \right) \\ &\leq c_1 + \lambda \left(2\Pi(R) + \frac{1}{\mu} \int_{\Omega} f^*(x, u_n)u_n^+ dx + \frac{1}{\mu}\Pi'(R) \right) \\ &=: \frac{\lambda}{\mu} \int_{\Omega} f^*(x, u_n)u_n^+ dx + c_2,\end{aligned} \tag{4.1.34}$$

while (4.1.29), tested with $v = u_n^+$, yields

$$-\frac{1}{\mu}\|u_n^+\|^p - \frac{1}{\mu}\|u_n^+\| \leq -\frac{\lambda}{\mu} \int_{\Omega} f^*(x, u_n)u_n^+ dx. \tag{4.1.35}$$

Adding term by term (4.1.34) and (4.1.35), we conclude

$$\left(\frac{1}{p} - \frac{1}{\mu} \right) \|u_n^+\|^p - \frac{1}{\mu}\|u_n^+\| \leq c_2,$$

hence $\{u_n^+\}$ is bounded in $W_0^{1,p}(\Omega)$.
Up to subsequences, we have $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$. Since Ψ' is strongly continuous (see Lemma 4.1.9), (4.1.29) leads to

$$\lim_{n\to\infty} \langle \Phi'(u_n), u_n - u\rangle = \lim_{n\to\infty} \langle J_\lambda(u_n), u_n - u\rangle + \lambda \lim_{n\to\infty} \langle \Psi'(u_n), u_n - u\rangle = 0.$$

Property (S_+) of the p-Laplacian operator forces $u_n \to u$ in $W_0^{1,p}(\Omega)$. In conclusion, J_λ satisfies (PS).
Now we prove that J_λ is unbounded from below.
For all $M \geq 1$, set $\Omega_M = \{x \in \Omega : M\varphi_p(x) \geq R\}$ and $\Omega'_M := \Omega \setminus \Omega_M$. Fix $\overline{M} > 0$ such that $|\Omega_{\overline{M}}| > 0$ (this choice is possible because $\Omega_M \uparrow \Omega$ for $M \to +\infty$). Obviously, $\Omega_{\overline{M}} \subseteq \Omega_M$ for all $M \geq \overline{M}$, and hence $|\Omega_M| > 0$. From (AR^+) we have, for any $\sigma \geq R$,

$$\frac{\mu}{\sigma} \leq \frac{f(x,\sigma)}{\int_{s_1}^{\sigma} f(x,t)dt} = \frac{\frac{\partial}{\partial\sigma}\left(\int_{s_1}^{\sigma} f(x,t)dt\right)}{\int_{s_1}^{\sigma} f(x,t)dt}. \tag{4.1.36}$$

Let $s \geq R$. Integrating (4.1.36) in σ on $[R,s]$ we get

$$\log\left[\left(\frac{s}{R}\right)^{\mu}\right] \leq \log\left(\frac{\int_{s_1}^{s} f(x,t)dt}{\int_{s_1}^{R} f(x,t)dt}\right).$$

Hence,

$$\int_{s_1}^{s} f(x,t)dt \geq \left(R^{-\mu}\int_{s_1}^{R} f(x,t)dt\right) s^{\mu} =: G(x)s^{\mu} \quad \forall s \geq R, \tag{4.1.37}$$

with $G(x) = R^{-\mu}\int_{s_1}^{R} f(x,t)dt > 0$ for all $x \in \Omega$, by virtue of (AR^+). Moreover, $H(f)_2$(ii) ensures that $G \in L^1(\Omega)$.
We claim that $\lim_{M\to+\infty} J_\lambda(M\varphi_p) = -\infty$. To this end, we observe that we have

$$\begin{aligned}\int_{\Omega_M} F(x, M\varphi_p)dx &= \int_{\Omega_M}\left(F(x,s_1) + \int_{s_1}^{M\varphi_p} f(x,t)dt\right)dx \\ &\geq -\Pi(s_1) + M^{\mu}\int_{\Omega_M} G(x)\varphi_p^{\mu}dx \\ &\geq M^{\mu}\left(\frac{R}{\overline{M}}\right)^{\mu}\int_{\Omega_{\overline{M}}} G(x)dx - \Pi(R)\end{aligned} \tag{4.1.38}$$

for $M \geq \overline{M}$. Notice that $\int_{\Omega_{\overline{M}}} G(x)dx > 0$, because $G > 0$ in Ω and $|\Omega_{\overline{M}}| > 0$. On the other hand, (4.1.19) gives

$$\left|\int_{\Omega'_M} F(x, M\varphi_p)dx\right| \leq \Pi(R). \tag{4.1.39}$$

Plugging (4.1.38)-(4.1.39) together, we find

$$
\begin{aligned}
& J_\lambda(M\varphi_p) \\
& = \frac{1}{p}\|M\varphi_p\|^p - \lambda\left(\int_{\Omega_M} F(x, M\varphi_p)dx + \int_{\Omega'_M} F(x, M\varphi_p)dx\right) \\
& \le \frac{\lambda_1}{p}M^p - \lambda\left(\frac{R}{\overline{M}}\right)^\mu M^\mu \int_{\Omega_{\overline{M}}} G(x)dx + 2\lambda\Pi(R) \to -\infty
\end{aligned}
\tag{4.1.40}
$$

for $M \to +\infty$. The claim is proved. □

Theorem 4.1.10. *Suppose that* H(f)$_1$ *and* H(f)$_2$ *hold. Let f be satisfying (AR^+). Then for any $\lambda \in (0, \lambda^*)$ problem $(P_{\lambda,f})$ admits at least two weak solutions $u^*(\lambda)$ and $\tilde{u}(\lambda)$, where λ^* is defined in* (4.1.24)*.*

Proof. Fix $\lambda \in (0, \lambda^*)$ and consider r^*_λ defined in (4.1.23). Set $r' := (pr^*_\lambda)^{\frac{1}{p}}$, and

$$B(0, r') := \left\{u \in W_0^{1,p}(\Omega) : \|u\| < r'\right\}.$$

Existence of a solution $u^* \in \overline{B}(0, r')$ is guaranteed by Theorem 4.1.5, and u^* is a minimizer of the restriction to $\overline{B}(0, r')$ of the functional J_λ. Without loss of generality, we can suppose u^* to be a proper minimizer, i.e.,

$$J_\lambda(u^*) < J_\lambda(u) \quad \forall u \in \overline{B}(0, r') \setminus \{u^*\}. \tag{4.1.41}$$

According to Lemma 4.1.9, the functional J_λ satisfies (PS) and is unbounded from below. Then, for any $M > 0$ sufficiently large, (4.1.40)-(4.1.41) imply

$$J_\lambda(M\varphi_p) < J_\lambda(u^*) \le \inf_{\partial B(0,r')} J_\lambda. \tag{4.1.42}$$

Taking a larger M if necessary, we can also suppose

$$\|M\varphi_p - u^*\| \ge r'. \tag{4.1.43}$$

Owing to (4.1.42)-(4.1.43), Theorem 2.4.11 ensures the existence of $\tilde{u} \in W_0^{1,p}(\Omega)$ such that $J_\lambda(\tilde{u}) > J_\lambda(u^*)$ (and consequently $\tilde{u} \ne u^*$) and $J'_\lambda(\tilde{u}) = 0$ (so $\tilde{u}$ is a solution to $(P_{\lambda,f})$). □

Remark 4.1.11. It is worth pointing out that all the above results remain true even if $q = 1$ (cf. H(f)$_2$); in this case, function a is assumed to be essentially bounded.

4.1.2 Singular convective Robin problems

Let $\Omega \subseteq \mathbb{R}^N$ ($N \geq 3$) be a bounded domain with a C^2-boundary $\partial\Omega$ and let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$, $g : \Omega \times (0, +\infty) \to [0, +\infty)$ be two Carathéodory functions. In this paragraph, we study existence and uniqueness of solutions to the following Robin problem:

$$\begin{cases} -\operatorname{div} a(\nabla u) = f(x, u, \nabla u) + g(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \partial_\nu u = -\beta |u|^{p-2} u & \text{on } \partial\Omega, \end{cases} \tag{4.1.44}$$

being $a : \mathbb{R}^N \to \mathbb{R}^N$ as in Section 2.4, $\beta > 0$, and $1 < p < +\infty$. The symbol ∂_ν denotes the co-normal derivative associated with a.

We assume the following hypotheses on the reaction terms.

H(f) $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$ is a Carathéodory function. Moreover, to every $M > 0$ there correspond $c_M, d_M > 0$ such that

$$f(x, s, \xi) \leq c_M + d_M |s|^{p-1} \quad \forall\, (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N \text{ with } |\xi| \leq M.$$

H(g) $g : \Omega \times (0, +\infty) \to [0, +\infty)$ is a Carathéodory function having the properties:

(g_1) $g(x, \cdot)$ turns out non-increasing on $(0, 1]$ whatever $x \in \Omega$, and $g(\cdot, 1) \not\equiv 0$.

(g_2) There exist $c, d > 0$ such that

$$g(x, s) \leq c + d s^{p-1} \quad \forall\, (x, s) \in \Omega \times (1, +\infty).$$

(g_3) With appropriate $\theta \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and $\varepsilon_0 > 0$, the map $x \mapsto g(x, \varepsilon\theta(x))$ belongs to $L^{p'}(\Omega)$ for any $\varepsilon \in (0, \varepsilon_0)$.

The paper [113] contains meaningful examples of functions g that satisfy H(g).

In this paragraph we will make use of the norms

$$\|u\|_{1,p} := \left(\|u\|_p^p + \|\nabla u\|_p^p\right)^{\frac{1}{p}} \ ; \quad \|u\|_{\beta,1,p} := \left(\beta \|u\|_{p,\partial\Omega}^p + \|\nabla u\|_p^p\right)^{\frac{1}{p}} .$$

Remark 4.1.12. If $\beta > 0$, then $\|\cdot\|_{\beta,1,p}$ is a norm on $W^{1,p}(\Omega)$ equivalent to $\|\cdot\|_{1,p}$. In particular, there exists $c_1 = c_1(p, \beta, \Omega) \in (0, 1)$ such that

$$c_1 \|u\|_{1,p} \leq \|u\|_{\beta,1,p} \leq \frac{1}{c_1} \|u\|_{1,p} \quad \forall\, u \in W^{1,p}(\Omega). \tag{4.1.45}$$

For the proof we refer to [136].

Fix $w \in C^1(\overline{\Omega})$. We first focus on the singular problem (without convection terms)

$$\begin{cases} -\operatorname{div} a(\nabla u) = f(x,u,\nabla w) + g(x,u) & \text{in } \Omega, \\ \qquad u > 0 & \text{in } \Omega, \\ \qquad \partial_\nu u = -\beta |u|^{p-2}u & \text{on } \partial\Omega. \end{cases} \tag{P_w}$$

The set of sub-solutions to (P_w) will be denoted by $\underline{U}_w$, while $\overline{U}_w$ is the super-solution set. The solution set of (P_w) will be denoted by U_w; obviously, $U_w = \overline{U}_w \cap \underline{U}_w$.

Lemma 4.1.13. *Let* H(f) *and* H(g) *be satisfied. Then there exists a sub-solution* $\underline{u} \in \operatorname{int}(C^1(\overline{\Omega})_+)$ *to* (P_w) *independent of* w *and such that* $\|\underline{u}\|_\infty \leq 1$.

Proof. Given any $\delta > 0$, consider the problem

$$\begin{cases} -\operatorname{div} a(\nabla u) = \tilde{g}(x,u) & \text{in } \Omega, \\ \qquad \partial_\nu u = -\beta |u|^{p-2}u & \text{on } \partial\Omega, \end{cases} \tag{4.1.46}$$

where $\tilde{g}(x,s) := \min\{g(x,s), \delta\}$, $(x,s) \in \Omega \times (0,+\infty)$. Weierstrass-Tonelli's theorem, the weak maximum principle, regularity up to the boundary, and the strong maximum principle (see Theorems 2.4.3, 2.3.5, 2.3.7, and Remark 2.2.9) furnish $\underline{u} \in \operatorname{int}(C^1(\overline{\Omega})_+)$ solution to (4.1.46). If $u_\delta \in C^{1,\alpha}(\overline{\Omega})_+$ is the solution to (3.1.2) with $\sigma := \delta$, we have that $\|u_\delta\|_\infty \leq 1$ whenever $\delta > 0$ is sufficiently small (cf. (3.1.3)). Hence, using the comparison principle (vide Theorem 2.3.6), one has

$$\|\underline{u}\|_\infty \leq 1 \tag{4.1.47}$$

once δ is small enough. Let θ and ε_0 be as in (g_3). Since $\underline{u}, \theta \in \operatorname{int}(C^1(\overline{\Omega})_+)$, there exists $\varepsilon \in (0, \varepsilon_0)$ such that $\underline{u} - \varepsilon\theta \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Via ($\mathrm{g}_1$), (4.1.47), and ($\mathrm{g}_3$), we thus infer

$$0 \leq g(\cdot, \underline{u}) \leq g(\cdot, \varepsilon\theta) \in L^{p'}(\Omega). \tag{4.1.48}$$

The conclusion is achieved by verifying that $\underline{u} \in \underline{U}_w$ for any $w \in C^1(\overline{\Omega})$. Pick such a w, test (4.1.46) with $v \in W^{1,p}(\Omega)_+$, and recall the definition of $\tilde{g}$, to arrive at

$$\begin{aligned} &\int_\Omega a(\nabla \underline{u}) \cdot \nabla v \, \mathrm{d}x + \beta \int_{\partial\Omega} \underline{u}^{p-1} v \, \mathrm{d}\sigma = \int_\Omega \tilde{g}(\cdot, \underline{u}) v \, \mathrm{d}x \\ &\leq \int_\Omega g(\cdot, \underline{u}) v \, \mathrm{d}x \leq \int_\Omega [f(\cdot, u, \nabla w) + g(\cdot, \underline{u})] v \, \mathrm{d}x, \end{aligned}$$

as desired. □

Remark 4.1.14. This proof shows that the sub-solution $\underline{u}$ constructed in Lemma 4.1.13 enjoys the further property:

$$\begin{aligned}&\int_\Omega a(\nabla\underline{u})\cdot\nabla v\mathrm{d}x+\beta\int_{\partial\Omega}|\underline{u}|^{p-2}\underline{u}v\mathrm{d}\sigma\\&\le\int_\Omega g(\cdot,\underline{u})v\mathrm{d}x\ \ \forall\, v\in W^{1,p}(\Omega)_+.\end{aligned}\tag{4.1.49}$$

Given $w\in C^1(\overline{\Omega})$, consider the truncated problem

$$\begin{cases}-\operatorname{div}a(\nabla u)=\hat{f}(x,u)+\hat{g}(x,u) & \text{in }\ \Omega,\\ \qquad u>0 & \text{in }\ \Omega,\\ \qquad \partial_\nu u=-\beta|u|^{p-2}u & \text{on }\ \partial\Omega,\end{cases}\tag{4.1.50}$$

where

$$\hat{f}(x,s):=\begin{cases}f(x,\underline{u}(x),\nabla w(x)) & \text{if }\ s\le\underline{u}(x),\\ f(x,s,\nabla w(x)) & \text{otherwise,}\end{cases}\tag{4.1.51}$$

$$\hat{g}(x,s):=\begin{cases}g(x,\underline{u}(x)) & \text{if }\ s\le\underline{u}(x),\\ g(x,s) & \text{otherwise.}\end{cases}\tag{4.1.52}$$

The energy functional corresponding to (4.1.50) is

$$\mathscr{E}_w(u):=\frac{1}{p}\int_\Omega G(\nabla u)\mathrm{d}x+\frac{\beta}{p}\int_{\partial\Omega}|u|^p\mathrm{d}\sigma-\int_\Omega\hat{F}(\cdot,u)\mathrm{d}x-\int_\Omega\hat{G}(\cdot,u)\mathrm{d}x$$

for all $u\in W^{1,p}(\Omega)$, with

$$\hat{F}(x,s):=\int_0^s\hat{f}(x,t)\mathrm{d}t,\quad \hat{G}(x,s):=\int_0^s\hat{g}(x,t)\mathrm{d}t.$$

Hypotheses H(f)–H(g) ensure that $\mathscr{E}_w$ is of class C^1 and weakly sequentially lower semi-continuous; see Remark 2.4.4. Under the additional condition

$$d_M+d<c_1^pc_2\quad\forall\, M>0,\tag{4.1.53}$$

it turns out also coercive, as the next lemma shows.

Lemma 4.1.15. *Let $\mathscr{B}$ be a nonempty bounded set in $C^1(\overline{\Omega})$. If* H(f), H(g), *and* (4.1.53) *hold true then there exist $\alpha_1\in(0,1)$, $\alpha_2>0$ such that*

$$\mathscr{E}_w(u)\ge\frac{\alpha_1}{p}\|u\|_{1,p}^p-\alpha_2(1+\|u\|_{1,p})\quad\forall\,(u,w)\in W^{1,p}(\Omega)\times\mathscr{B}.$$

Proof. Put $\hat{M} := \sup_{w\in\mathscr{B}} \|w\|_{C^1(\overline{\Omega})}$. By (4.1.51)–(4.1.52) and (2.4.17) we get

$$\begin{aligned}\mathscr{E}_w(u) \geq \frac{c_2}{p}\|\nabla u\|_p^p + \frac{\beta}{p}\|u\|_{p,\partial\Omega}^p - \int_\Omega [f(\cdot,\underline{u},\nabla w) + g(\cdot,\underline{u})]\underline{u}\mathrm{d}x \\ - \int_{\Omega(u>\underline{u})}\left(\int_{\underline{u}}^u f(\cdot,t,\nabla w)\mathrm{d}t\right)\mathrm{d}x - \int_{\Omega(u>\underline{u})}\left(\int_{\underline{u}}^u g(\cdot,t)\mathrm{d}t\right)\mathrm{d}x.\end{aligned}$$

Hypothesis H(f) along with Hölder's inequality imply

$$\begin{aligned}\int_{\Omega(u>\underline{u})}\left(\int_{\underline{u}}^u f(\cdot,t,\nabla w)\mathrm{d}t\right)\mathrm{d}x &\leq \int_{\Omega(u>\underline{u})}\left(\int_0^u f(\cdot,t,\nabla w)\mathrm{d}t\right)\mathrm{d}x \\ &\leq c_{\hat{M}}|\Omega|^{\frac{1}{p'}}\|u\|_p + \frac{d_{\hat{M}}}{p}\|u\|_p^p \\ &\leq c_{\hat{M}}|\Omega|^{\frac{1}{p'}}\|u\|_{1,p} + \frac{d_{\hat{M}}}{p}\|u\|_{1,p}^p.\end{aligned}$$

Exploiting (4.1.47), (g_2), and Hölder's inequality again, we have

$$\begin{aligned}&\int_{\Omega(u>\underline{u})}\left(\int_{\underline{u}}^u g(\cdot,t)\mathrm{d}t\right)\mathrm{d}x \\ &\leq \int_{\Omega(u>\underline{u})}\left(\int_{\underline{u}}^1 g(\cdot,t)\mathrm{d}t\right)\mathrm{d}x + \int_{\Omega(u>1)}\left(\int_1^u g(\cdot,t)\mathrm{d}t\right)\mathrm{d}x \\ &\leq \int_{\Omega(u>\underline{u})} g(\cdot,\underline{u})\mathrm{d}x + \int_{\Omega(u>1)}\left(\int_1^u (c+dt^{p-1})\mathrm{d}t\right)\mathrm{d}x \\ &\leq \int_\Omega g(\cdot,\underline{u})\mathrm{d}x + c|\Omega|^{\frac{1}{p'}}\|u\|_p + \frac{d}{p}\|u\|_p^p \\ &\leq \int_\Omega g(\cdot,\underline{u})\mathrm{d}x + c|\Omega|^{\frac{1}{p'}}\|u\|_{1,p} + \frac{d}{p}\|u\|_{1,p}^p.\end{aligned}$$

Hence, through (4.1.45) we easily arrive at

$$\begin{aligned}\mathscr{E}_w(u) &\geq \frac{c_2}{p}\|u\|_{\beta,1,p}^p - \frac{d_{\hat{M}}+d}{p}\|u\|_{1,p}^p - (c_{\hat{M}}+c)|\Omega|^{\frac{1}{p'}}\|u\|_p - K \\ &\geq \frac{c_1^p c_2 - d_{\hat{M}} - d}{p}\|u\|_{1,p}^p - (c_{\hat{M}}+c)|\Omega|^{\frac{1}{p'}}\|u\|_{1,p} - K \\ &\geq \frac{c_1^p c_2 - d_{\hat{M}} - d}{p}\|u\|_{1,p}^p - \max\{(c_{\hat{M}}+c)|\Omega|^{\frac{1}{p'}}, K\}(1+\|u\|_{1,p}),\end{aligned}$$

where

$$\begin{aligned}K &:= \int_\Omega [f(\cdot,\underline{u},\nabla w)] + g(\cdot,\underline{u})]\underline{u}\mathrm{d}x + \int_\Omega g(\cdot,\underline{u})\mathrm{d}x \\ &\leq \int_\Omega (c_{\hat{M}} + d_{\hat{M}})\mathrm{d}x + 2\int_\Omega g(\cdot,\varepsilon\theta)\mathrm{d}x \leq (c_{\hat{M}} + d_{\hat{M}})|\Omega| + 2\|g(\cdot,\varepsilon\theta)\|_{p'}|\Omega|^{\frac{1}{p}}\end{aligned}$$

due to H(f) and (4.1.47)–(4.1.48). Now, the conclusion follows from (4.1.53). □

Moser's iteration (Theorem 2.2.2) and regularity up to the boundary (Remark 2.2.9) ensure that any solution to (4.1.50) is actually Hölder-continuous up to the boundary. Arguing as in the final part of the proof of Theorem 2.3.10, the following lemma can be demonstrated.

Lemma 4.1.16. *Let* H(f)*,* H(g)*, and* (4.1.53) *be satisfied. Then*

$$\emptyset \neq \mathrm{Crit}(\mathscr{E}_w) \subseteq U_w \cap \{u \in C^1(\overline{\Omega}) : u \geq \underline{u}\}.$$

For every $w \in C^1(\overline{\Omega})$ we define

$$\mathscr{S}(w) := \{u \in C^1(\overline{\Omega}) : u \in U_w,\, u \geq \underline{u},\, \mathscr{E}_w(u) < 1\}.$$

Reasoning as in paragraph 2.5.2, it is possible to prove that, for any $w \in C^1(\overline{\Omega})$, the set $\mathscr{S}(w)$ admits minimum (cf. Theorem 2.5.7). Hence, it is possible to define $\Gamma : C^1(\overline{\Omega}) \to C^1(\overline{\Omega})$ given by

$$\Gamma(w) := \min \mathscr{S}(w) \quad \forall\, w \in C^1(\overline{\Omega}).$$

Moreover, under the condition

$$d_M + d < \frac{c_1^p c_2}{p} \quad \forall\, M > 0, \tag{4.1.54}$$

which is more restrictive than 4.1.53, it can be shown that $\mathscr{S}$ is a compact, lower semi-continuous multi-valued operator (Theorems 2.5.8 and 2.5.10), so Γ is a completely continuous operator (Corollaries 2.5.9 and 2.5.11).

To establish our main result, the stronger version below of H(f) will be employed.

<u>H′(f)</u> $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to [0, +\infty)$ is a Carathéodory function such that

$$f(x, s, \xi) \leq c_3 + c_4|s|^{p-1} + c_5|\xi|^{p-1} \quad \forall\, (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

with appropriate $c_3, c_4, c_5 > 0$.

Condition (4.1.53) is substituted by

$$c_4 + (2p-1)c_5 + d < c_1^p c_2\,. \tag{4.1.55}$$

Remark 4.1.17. Assumption H′(f) clearly implies H(f), with $c_M := c_3 + c_5 M^{p-1}$ and $d_M := c_4$. Likewise, (4.1.55) forces (4.1.53) while (4.1.54) reads as

$$c_4 + d < \frac{c_1^p c_2}{p}\,. \tag{4.1.56}$$

Remark 4.1.18. Conditions (4.1.53)–(4.1.56) can be omitted if $q < p-1$.

Theorem 4.1.19. *Let* $\mathrm{H}'(\mathrm{f})$, $\mathrm{H}(\mathrm{g})$, *and* (4.1.55)–(4.1.56) *be satisfied. Then problem* (4.1.44) *possesses a solution* $u \in \mathrm{int}(C^1(\overline{\Omega})_+)$. *The set of solutions to* (4.1.44) *is compact in* $C^1(\overline{\Omega})$.

Proof. Define

$$\Lambda(\Gamma) := \{u \in C^1(\overline{\Omega}) : u = \tau\,\Gamma(u) \text{ for some } \tau \in (0,1)\}.$$

Claim: $\Lambda(\Gamma)$ is bounded in $W^{1,p}(\Omega)$.
To see this, pick any $u \in \Lambda(\Gamma)$. Since $\frac{u}{\tau} = \Gamma(u) \in \mathscr{S}(u)$, one has $\mathscr{E}_u\left(\frac{u}{\tau}\right) < 1$. Assumption $\mathrm{H}'(\mathrm{f})$, combined with Young's and Hölder's inequalities, produces

$$\begin{aligned}
&\int_{\Omega(\frac{u}{\tau} > \underline{u})} \left(\int_{\underline{u}}^{\frac{u}{\tau}} f(\cdot, t, \nabla u) \mathrm{d}t \right) \mathrm{d}x \\
&\quad \le \int_\Omega \left(\int_0^{\frac{u}{\tau}} (c_3 + c_4 t^{p-1} + c_5 |\nabla u|^{p-1}) \mathrm{d}t \right) \mathrm{d}x \\
&\quad \le c_3 \left\| \frac{u}{\tau} \right\|_1 + \frac{c_4}{p} \left\| \frac{u}{\tau} \right\|_p^p + c_5 \int_\Omega |\nabla u|^{p-1} \left| \frac{u}{\tau} \right| \mathrm{d}x \\
&\quad \le c_3 |\Omega|^{\frac{1}{p'}} \left\| \frac{u}{\tau} \right\|_p + \frac{c_4}{p} \left\| \frac{u}{\tau} \right\|_p^p + c_5 \left(\frac{\left\| \frac{u}{\tau} \right\|_p^p}{p} + \frac{\|\nabla u\|_p^p}{p'} \right) \\
&\quad \le c_3 |\Omega|^{\frac{1}{p'}} \left\| \frac{u}{\tau} \right\|_{1,p} + \frac{c_4 + c_5}{p} \left\| \frac{u}{\tau} \right\|_{1,p}^p + \frac{c_5}{p'} \|u\|_{1,p}^p.
\end{aligned}$$

Analogously, on account of (4.1.47),

$$\begin{aligned}
\int_\Omega f(\cdot, \underline{u}, \nabla u) \underline{u} \mathrm{d}x &\le \int_\Omega \left(c_3 \underline{u} + c_4 \underline{u}^p + c_5 |\nabla u|^{p-1} \right) \underline{u} \mathrm{d}x \\
&\le \left(c_3 + c_4 + \frac{c_5}{p} \right) |\Omega| + \frac{c_5}{p'} \|\nabla u\|_p^p \\
&\le \left(c_3 + c_4 + \frac{c_5}{p} \right) |\Omega| + \frac{c_5}{p'} \|u\|_{1,p}^p.
\end{aligned}$$

Reasoning as in Lemma 4.1.15 and recalling that $\tau \in (0,1)$, we thus achieve

$$\begin{aligned}
1 &> \mathscr{E}_u \left(\frac{u}{\tau} \right) \\
&\ge \frac{c_1^p c_2 - c_4 - (2p-1)c_5 - d}{p} \left\| \frac{u}{\tau} \right\|_{1,p}^p - (c_3 + c) |\Omega|^{\frac{1}{p'}} \left\| \frac{u}{\tau} \right\|_{1,p} - K',
\end{aligned}$$

where

$$K' := \left(c_3 + c_4 + \frac{c_5}{p}\right)|\Omega| + 2\|g(\cdot, \varepsilon\theta)\|_{p'}|\Omega|^{\frac{1}{p}}.$$

Thanks to (4.1.55), the above inequalities force

$$\|u\|_{1,p} \leq \left\|\frac{u}{\tau}\right\|_{1,p} \leq K^*,$$

with $K^* > 0$ independent of u and τ. Thus, the claim is proved.

By regularity (see Remark 2.2.9), the set $\Lambda(\Gamma)$ turns out bounded in $C^1(\overline{\Omega})$. Hence, due to the complete continuity of Γ, Theorem 2.5.2 applies, which entails $\mathrm{Fix}(\Gamma) \neq \emptyset$. Let $u \in \mathrm{Fix}(\Gamma)$. From $u = \Gamma(u) \in \mathscr{S}(u)$ we deduce both $u \geq \underline{u}$ and $u \in U_u$. Accordingly,

$$\hat{f}(\cdot, u) = f(\cdot, u, \nabla u), \quad \hat{g}(\cdot, u) = g(\cdot, u),$$

namely the function u solves problem (4.1.44). Further, $u \in \mathrm{int}(C^1(\overline{\Omega})_+)$ because of the strong maximum principle.

Finally, arguing as in Lemma 3.2.2 ensures that each solution to (4.1.44) lies in $C^{1,\alpha}(\overline{\Omega})$. Since $C^{1,\alpha}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ compactly and the solution set of (4.1.44) is closed in $C^1(\overline{\Omega})$, the conclusion follows. $\square$

Remark 4.1.20. The same techniques can be applied for finding solutions to the Neumann problem

$$\begin{cases} -\operatorname{div} a(\nabla u) + |u|^{p-2}u = f(x, u, \nabla u) + g(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases}$$

Indeed, it is enough to replace the norm $\|\cdot\|_{\beta,1,p}$ with the standard one $\|\cdot\|_{1,p}$.

Now we investigate uniqueness of solutions to (4.1.44) in the particular case $p = 2$. Retaining H(f)-H(g), the following further conditions will be posited:

(a_4) There exists $c_6 \in (0, 1]$ such that

$$(a(\xi) - a(\eta)) \cdot (\xi - \eta) \geq c_6|\xi - \eta|^2 \quad \forall\, \xi, \eta \in \mathbb{R}^N.$$

<u>H″(f)</u> With appropriate $c_7, c_8 > 0$ one has

$$[f(x, s, \xi) - f(x, t, \xi)](s - t) \leq c_7|s - t|^2 \tag{4.1.57}$$

$$|f(x, t, \xi) - f(x, t, \eta)| \leq c_8|\xi - \eta| \tag{4.1.58}$$

in $\Omega \times \mathbb{R} \times \mathbb{R}^N$.

H′(g) There is $c_9 > 0$ such that

$$[g(x,s) - g(x,t)](s-t) \le c_9|s-t|^2 \;\; \forall\, x \in \Omega, \;\; s,t \in [1,+\infty). \tag{4.1.59}$$

Moreover,

$$g(x,s) \le g(x,1) \;\; \text{in} \;\; \Omega \times (1,+\infty). \tag{4.1.60}$$

Example 4.1.21. The parametric $(2,q)$-Laplacian $\Delta + \mu\Delta_q$, where $1 < q < 2$, $\mu \ge 0$, satisfies H(a) and (a_4); cf. [139, Lemma A.0.5].

Theorem 4.1.22. *Under the above assumptions, problem* (4.1.44) *admits a unique solution provided*

$$c_7 + c_1c_8 + c_9 < c_1^2c_6. \tag{4.1.61}$$

Proof. Suppose u, v solve (4.1.44), test with $u - v$, and subtract to arrive at

$$\begin{aligned} &\int_\Omega (a(\nabla u) - a(\nabla v)) \cdot \nabla(u-v)\mathrm{d}x + \beta \int_{\partial\Omega} |u-v|^2 \mathrm{d}\sigma \\ &= \int_\Omega [f(\cdot,u,\nabla u) - f(\cdot,v,\nabla v)](u-v)\mathrm{d}x \\ &+ \int_\Omega [g(\cdot,u) - g(\cdot,v)](u-v)\mathrm{d}x. \end{aligned} \tag{4.1.62}$$

The left-hand side of (4.1.62) can easily be estimated from below via (a_4) as follows:

$$\int_\Omega (a(\nabla u) - a(\nabla v)) \cdot \nabla(u-v)\mathrm{d}x + \beta \int_{\partial\Omega} |u-v|^2 \mathrm{d}\sigma \ge c_6\|u-v\|_{\beta,1,2}^2. \tag{4.1.63}$$

Using (4.1.57)–(4.1.58) and Hölder's inequality we get

$$\begin{aligned} &\int_\Omega [f(\cdot,u,\nabla u) - f(\cdot,v,\nabla v)](u-v)\mathrm{d}x \\ &\quad = \int_\Omega [f(\cdot,u,\nabla u) - f(\cdot,v,\nabla u)](u-v)\mathrm{d}x \\ &\qquad\quad + \int_\Omega [f(\cdot,v,\nabla u) - f(\cdot,v,\nabla v)](u-v)\mathrm{d}x \\ &\quad \le c_7 \int_\Omega |u-v|^2 \mathrm{d}x + c_8 \int_\Omega |\nabla u - \nabla v||u-v|\mathrm{d}x \\ &\quad \le c_7\|u-v\|_2^2 + c_8\|\nabla(u-v)\|_2\|u-v\|_2 \\ &\quad \le \frac{c_7}{c_1^2}\|u-v\|_{\beta,1,2}^2 + \frac{c_8}{c_1}\|u-v\|_{\beta,1,2}^2. \end{aligned} \tag{4.1.64}$$

Observe now that

$$
\begin{aligned}
&\int_\Omega [g(\cdot,u)-g(\cdot,v)](u-v)\mathrm{d}x \\
&= \int_{\Omega(\max\{u,v\}\le 1)} [g(\cdot,u)-g(\cdot,v)](u-v)\mathrm{d}x \\
&+ \int_{\Omega(\min\{u,v\}>1)} [g(\cdot,u)-g(\cdot,v)](u-v)\mathrm{d}x \\
&+ \int_{\Omega(u\le 1<v)} [g(\cdot,u)-g(\cdot,v)](u-v)\mathrm{d}x \\
&+ \int_{\Omega(v\le 1<u)} [g(\cdot,u)-g(\cdot,v)](u-v)\mathrm{d}x.
\end{aligned}
\tag{4.1.65}
$$

By hypothesis (g_1) in H(g) one has

$$
\int_{\Omega(\max\{u,v\}\le 1)} [g(\cdot,u)-g(\cdot,v)](u-v)\mathrm{d}x \le 0. \tag{4.1.66}
$$

Inequality (4.1.59) entails

$$
\begin{aligned}
&\int_{\Omega(\min\{u,v\}>1)} [g(\cdot,u)-g(\cdot,v)](u-v)\mathrm{d}x \\
&\le c_9\|u-v\|_2^2 \le \frac{c_9}{c_1^2}\|u-v\|_{\beta,1,2}^2.
\end{aligned}
\tag{4.1.67}
$$

Thanks to (g_1) again and (4.1.60) we obtain

$$
\begin{aligned}
&\int_{\Omega(u\le 1<v)} [g(\cdot,u)-g(\cdot,v)](u-v)\mathrm{d}x \\
&\le \int_{\Omega(u\le 1<v)} [g(\cdot,1)-g(\cdot,v)](u-v)\mathrm{d}x \le 0.
\end{aligned}
\tag{4.1.68}
$$

Likewise,

$$
\int_{\Omega(v\le 1<u)} [g(\cdot,u)-g(\cdot,v)](u-v)\mathrm{d}x \le 0. \tag{4.1.69}
$$

Plugging (4.1.66)–(4.1.69) into (4.1.65) and (4.1.63)–(4.1.65) into (4.1.62) yields

$$
c_6\|u-v\|_{\beta,1,2}^2 \le \left(\frac{c_7}{c_1^2}+\frac{c_8}{c_1}+\frac{c_9}{c_1^2}\right)\|u-v\|_{\beta,1,2}^2.
$$

On account of (4.1.61), this directly leads to $u=v$, as desired. □

Remark 4.1.23. The conditions that guarantee existence or uniqueness, namely (4.1.55), (4.1.56), and (4.1.61), represent a balance between data (growth or variation of reaction terms) and structure (principal operator and domain) of the problem.

4.2 Systems

4.2.1 Singular convective Neumann systems

In this paragraph we investigate the following homogeneous Neumann problem:

$$\begin{cases} -\Delta_p u = f(x,u,v,\nabla u,\nabla v) & \text{in } \Omega, \\ -\Delta_q v = g(x,u,v,\nabla u,\nabla v) & \text{in } \Omega, \\ \quad u,v > 0 & \text{in } \Omega, \\ \quad \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.2.1}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$, is a bounded domain with C^2-boundary $\partial\Omega$ having outer normal ν, $1 < p,q < +\infty$, and $f,g : \Omega \times (0,+\infty)^2 \times \mathbb{R}^{2N} \to \mathbb{R}$ are Carathéodory functions.

If Z is a real function space on Ω and $v,w \in Z$, then

$$v^+ := \max\{0,v\}, \quad [v,w] := \{z \in Z : v \leq z \leq w\}, \quad Z_+ := \{z \in Z : 0 \leq z\}.$$

Let $Z^2 := Z \times Z$ and let $(v_1,v_2),(w_1,w_2) \in Z^2$. By definition, one has

$$(v_1,v_2) \leq (w_1,w_2) \iff v_1 \leq w_1 \text{ and } v_2 \leq w_2.$$

If $\|\cdot\|_Z$ is a norm on Z, then we put $B_Z(\rho) := \{z \in Z : \|z\|_Z \leq \rho\}$, $\rho > 0$, as well as

$$\|(z_1,z_2)\|_{Z^2} := \|z_1\|_Z + \|z_2\|_Z \quad \forall\,(z_1,z_2) \in Z^2.$$

We set $W^{1,r}_b(\Omega) := W^{1,r}(\Omega) \cap L^\infty(\Omega)$.

Pick any $w := (w_1,w_2) \in C^1(\overline{\Omega})^2$ and consider problem (4.2.1) with 'frozen' gradients, i.e.,

$$\begin{cases} -\Delta_p u = h_1(x,u,v) & \text{in } \Omega, \\ -\Delta_q v = h_2(x,u,v) & \text{in } \Omega, \\ \quad u,v > 0 & \text{in } \Omega, \\ \quad \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega, \end{cases} \tag{P$_\mathrm{w}$}$$

where

$$\begin{aligned} h_1(x,s,t) &:= f(x,s,t,\nabla w_1(x),\nabla w_2(x)), \\ h_2(x,s,t) &:= g(x,s,t,\nabla w_1(x),\nabla w_2(x)). \end{aligned} \tag{4.2.2}$$

The assumption below will be posited.

$\underline{\text{(H)}}$ There exist $\varepsilon > 0$, $(\underline{u},\underline{v}),(\overline{u},\overline{v}) \in W^{1,p}_b(\Omega) \times W^{1,q}_b(\Omega)$ such that

$$(\varepsilon,\varepsilon) \leq (\underline{u},\underline{v}) \leq (\overline{u},\overline{v}).$$

Moreover, if $K := C^1(\overline{\Omega})^2 \cap ([\underline{u},\overline{u}] \times [\underline{v},\overline{v}])$, then:

(i) For appropriate $\rho_1, \rho_2, C > 0$ one has

$$|f(\cdot, u, v, \nabla w)| \le \rho_1, \quad |g(\cdot, u, v, \nabla w)| \le \rho_2$$

whenever $(u, v, w) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}] \times D$, where

$$D := \{w \in K : \|\nabla w\|_{L^\infty(\Omega)^2} \le C\}. \tag{4.2.3}$$

(ii) For every fixed $w \in D$ the pair $(\underline{u}, \underline{v}), (\overline{u}, \overline{v})$ is a sub-super-solution to problem ($\mathrm{P_w}$), namely

$$\begin{cases} \int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla \varphi \, \mathrm{d}x & \le \int_\Omega h_1(\cdot, \underline{u}, v) \varphi \, \mathrm{d}x, \\ \int_\Omega |\nabla \underline{v}|^{q-2} \nabla \underline{v} \nabla \psi \, \mathrm{d}x & \le \int_\Omega h_2(\cdot, u, \underline{v}) \psi \, \mathrm{d}x, \end{cases}$$
$$\begin{cases} \int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla \varphi \, \mathrm{d}x & \ge \int_\Omega h_1(\cdot, \overline{u}, v) \varphi \, \mathrm{d}x, \\ \int_\Omega |\nabla \overline{v}|^{q-2} \nabla \overline{v} \nabla \psi \, \mathrm{d}x & \ge \int_\Omega h_2(\cdot, u, \overline{v}) \psi \, \mathrm{d}x \end{cases} \tag{4.2.4}$$

whenever $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$, $(\varphi, \psi) \in W^{1,p}_b(\Omega)_+ \times W^{1,q}_b(\Omega)_+$.

Now, given $(u, v) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$, we define

$$T_p(u)(x) := \begin{cases} \underline{u}(x) & \text{if } u(x) < \underline{u}(x), \\ u(x) & \text{if } \underline{u}(x) \le u(x) \le \overline{u}(x), \\ \overline{u}(x) & \text{if } u(x) > \overline{u}(x), \end{cases} \qquad x \in \Omega,$$

$$T_q(v)(x) := \begin{cases} \underline{v}(x) & \text{if } v(x) < \underline{v}(x), \\ v(x) & \text{if } \underline{v}(x) \le v(x) \le \overline{v}(x), \\ \overline{v}(x) & \text{if } v(x) > \overline{v}(x). \end{cases} \qquad x \in \Omega.$$

Lemma 2.89 of [31] ensures that the operators $T_p : W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ and $T_q : W^{1,q}(\Omega) \to W^{1,q}(\Omega)$ are continuous.

Truncating reactions allows to neglect the singular behavior in zero as well as possible super-critical growths at infinity. Hence, we add a potential term in both sides, which makes the differential operator strictly monotone, and truncate the reaction terms, thus coming to the problem

$$\begin{cases} -\Delta_p u + |u|^{p-2} u = k_1(x, u, v) & \text{in } \Omega, \\ -\Delta_q v + |v|^{q-2} v = k_2(x, u, v) & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\tilde{\mathrm{P}}_{\mathrm{w}}$}$$

where

$$k_1(\cdot,u,v) := h_1(\cdot,T_p(u),T_q(v)) + |T_p(u)|^{p-2}T_p(u),$$
$$k_2(\cdot,u,v) := h_2(\cdot,T_p(u),T_q(v)) + |T_q(v)|^{q-2}T_q(v). \tag{4.2.5}$$

Solutions of $(\tilde{\mathrm{P}}_{\mathrm{w}})$ will be sought by freezing reactions again. Accordingly, bear in mind (4.2.3), and, for every fixed $(u,v,w) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega) \times D$, consider the variational problem

$$\begin{cases} -\Delta_p \hat{u} + |\hat{u}|^{p-2}\hat{u} = k_1(x,u(x),v(x)) & \text{in } \Omega, \\ -\Delta_q \hat{v} + |\hat{v}|^{q-2}\hat{v} = k_2(x,u(x),v(x)) & \text{in } \Omega, \\ \partial_\nu \hat{u} = \partial_\nu \hat{v} = 0 & \text{on } \partial\Omega. \end{cases} \tag{$\tilde{\mathrm{P}}_{(\mathrm{u,v,w})}$}$$

Remark 4.2.1. Hypothesis (H)(i) evidently forces

$$k_1(\cdot,u,v), k_2(\cdot,u,v) \in L^\infty(\Omega).$$

So, through Moser's iteration technique (Corollary 2.2.3), we see that any solution $(\hat{u},\hat{v})$ of $(\tilde{\mathrm{P}}_{(\mathrm{u,v,w})})$ turns out essentially bounded. Lieberman's regularity theory up to the boundary (Corollary 2.2.8), yields $\alpha \in (0,1)$ and $R > 0$ (depending only on p,q,Ω,ρ_1,ρ_2) such that

$$(\hat{u},\hat{v}) \in B_{C^{1,\alpha}(\overline{\Omega})^2}(R) \subseteq B_{C^1(\overline{\Omega})^2}(R).$$

By Minty-Browder's theorem (vide Corollary 2.3.4) we achieve the following result.

Lemma 4.2.2. *Let* (H)(i) *be satisfied and let* $(u,v,w) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega) \times D$*. Then problem* $(\tilde{\mathrm{P}}_{(\mathrm{u,v,w})})$ *possesses a unique solution* $(\hat{u},\hat{v}) \in B_{C^{1,\alpha}(\overline{\Omega})^2}(R)$*.*

Next, pick $w \in D$. For every $(u,v) \in B_{C^1(\overline{\Omega})^2}(R)$ we set

$$\Phi(u,v) := (\hat{u},\hat{v}), \tag{4.2.6}$$

where $(\hat{u},\hat{v})$ is as in Lemma 4.2.2. Since

$$B_{C^{1,\alpha}(\overline{\Omega})^2}(R) \overset{c}{\hookrightarrow} B_{C^1(\overline{\Omega})^2}(R), \tag{4.2.7}$$

the operator $\Phi : B_{C^1(\overline{\Omega})^2}(R) \to B_{C^1(\overline{\Omega})^2}(R)$ defined by (4.2.6) is compact. It will play a basic role to prove the following

Lemma 4.2.3. *If* (H) *holds and* $w \in D$*, then* $(\mathrm{P}_{\mathrm{w}})$ *admits solutions in* K*.*

Proof. We claim that Φ is continuous. Indeed, let $\{(u_n,v_n)\}_n \subseteq B_{C^1(\overline{\Omega})^2}(R)$ satisfy $(u_n,v_n)\to(u,v)$ in $C^1(\overline{\Omega})^2$ and let $(\hat{u}_n,\hat{v}_n):=\Phi(u_n,v_n)$, $n\in\mathbb{N}$. The compactness of Φ forces $(\hat{u}_n,\hat{v}_n)\to(\hat{u},\hat{v})$ in $C^1(\overline{\Omega})^2$, where a subsequence is considered when necessary. On the other hand, each $(\hat{u}_n,\hat{v}_n)$ solves $(\tilde{\mathrm{P}}_{(\mathrm{u_n,v_n,w})})$, whence $(\hat{u},\hat{v})$ turns out a solution to $(\tilde{\mathrm{P}}_{(\mathrm{u,v,w})})$, as we easily see once $n\to\infty$ in $(\tilde{\mathrm{P}}_{(\mathrm{u_n,v_n,w})})$. By uniqueness one has $(\hat{u},\hat{v})=\Phi(u,v)$, thus showing the continuity of Φ.

Now, Schauder's fixed point theorem (see Theorem 2.5.1) gives $(u,v)\in B_{C^1(\overline{\Omega})^2}(R)$ such that $(u,v)=\Phi(u,v)$, namely (u,v) solves $(\tilde{\mathrm{P}}_{\mathrm{w}})$. Reasoning as in the final part of the proof of Theorem 2.3.10 leads the conclusion. □

Define, for every $w\in D$,

$$\mathcal{S}(w):=\{(u,v)\in K:(u,v)\text{ is a solution to }(\mathrm{P_w})\}.$$

Reasoning as in paragraph 2.5.2, it can be proved that, for any $w\in C^1(\overline{\Omega})$, the set $\mathcal{S}(w)$ admits minimum (cf. Theorem 2.5.7). Hence, it is possible to define $\Gamma:D\to K$ given by

$$\Gamma(w):=\min\mathcal{S}(w)\quad\forall\, w\in D. \tag{4.2.8}$$

Moreover, it can be shown that $\mathcal{S}$ is a compact, lower semi-continuous multivalued operator (Theorems 2.5.8 and 2.5.10), so Γ is a completely continuous operator (Corollaries 2.5.9 and 2.5.11).

By Theorem 2.2.16 and Hölder's inequality, any solution $(u,v)\in K$ to $(\mathrm{P_w})$ satisfies the gradient estimates

$$\begin{aligned}\|\nabla u\|_{L^\infty(\Omega)}&\le\eta_1\|f(\cdot,u,v,\nabla w)\|_{L^\infty(\Omega)}^{\frac{1}{p-1}},\\ \|\nabla v\|_{L^\infty(\Omega)}&\le\eta_2\|g(\cdot,u,v,\nabla w)\|_{L^\infty(\Omega)}^{\frac{1}{q-1}},\end{aligned} \tag{4.2.9}$$

where $\eta_1,\eta_2>0$ denote suitable constants. Evidently, there is no loss of generality in assuming $\eta_1,\eta_2\ge1$.

Our main result requires a further condition on the reaction terms, which however complies with various meaningful cases; see Theorems 4.2.6–4.2.7 below. Hereafter, we suppose that

$$\rho_1\le\left(\frac{C}{\eta_1}\right)^{p-1},\quad\rho_2\le\left(\frac{C}{\eta_2}\right)^{q-1}, \tag{4.2.10}$$

where ρ_1,ρ_2,C come from (H), while η_1,η_2 are as in (4.2.9).

Theorem 4.2.4. *If* (H) *and* (4.2.10) *hold, then problem* (4.2.1) *possesses a solution belonging to* $C^{1,\alpha}(\overline{\Omega})^2\cap K$.

Proof. Let Γ be given by (4.2.8). Condition (4.2.10) and (4.2.9) guarantee that $\Gamma(D) \subseteq D$. Thus, recalling that Γ is a completely continuous operator, Schauder's fixed point theorem can be applied, which entails $(u, v) = \Gamma(u, v)$ for some $(u, v) \in D$. Through (4.2.2) and Remark 4.2.1 we easily verify that (u, v) satisfies the conclusion. $\square$

As a byproduct of Theorem 4.2.4, we prove a result about existence of infinitely many solutions to (4.2.1), in two different contexts: sub-linear case and super-linear case.
In the sub-linear setting we make the hypotheses below.

(F_1') There exist $\alpha_1 < 0 < \beta_1$, $\gamma_1, \delta_1 \in [0, p-1)$, and $a_1, b_1, c_1 \in L^\infty(\Omega)$ such that

$$|f(x, s, t, \xi_1, \xi_2)| \leq a_1(x) s^{\alpha_1} t^{\beta_1} + b_1(x)(|\xi_1|^{\gamma_1} + |\xi_2|^{\delta_1}) + c_1(x)$$

for all $(x, s, t, \xi_1, \xi_2) \in \Omega \times (0, +\infty)^2 \times \mathbb{R}^{2N}$.

(G_1') There exist $\beta_2 < 0 < \alpha_2$, $\gamma_2, \delta_2 \in [0, q-1)$, and $a_2, b_2, c_2 \in L^\infty(\Omega)$ such that

$$|g(x, s, t, \xi_1, \xi_2)| \leq a_2(x) s^{\alpha_2} t^{\beta_2} + b_2(x)(|\xi_1|^{\gamma_2} + |\xi_2|^{\delta_2}) + c_2(x)$$

for all $(x, s, t, \xi_1, \xi_2) \in \Omega \times (0, +\infty)^2 \times \mathbb{R}^{2N}$.

(S_1) There exist $\{h_n\}_n, \{\hat{h}_n\}_n, \{k_n\}_n, \{\hat{k}_n\}_n, \{C_n\}_n \subseteq (0, +\infty)$, with $C_n \to +\infty$, satisfying $h_n < k_n < h_{n+1}$, $\hat{h}_n < \hat{k}_n < \hat{h}_{n+1}$, and

$$\begin{aligned} f(x, k_n, t, \xi_1, \xi_2) \leq 0 \leq f(x, h_n, t, \xi_1, \xi_2), \\ g(x, s, \hat{k}_n, \xi_1, \xi_2) \leq 0 \leq g(x, s, \hat{h}_n, \xi_1, \xi_2) \end{aligned} \tag{S$'$}$$

for all $(x, s, t, \xi_1, \xi_2) \in \Omega \times [h_n, k_n] \times [\hat{h}_n, \hat{k}_n] \times B_{\mathbb{R}^N}(C_n)^2$, $n \in \mathbb{N}$. Further,

$$\begin{aligned} \|a_1\|_{L^\infty(\Omega)} \limsup_{n\to\infty} \frac{h_n^{\alpha_1} \hat{k}_n^{\beta_1}}{C_n^{p-1}} < \eta_1^{1-p}, \\ \|a_2\|_{L^\infty(\Omega)} \limsup_{n\to\infty} \frac{k_n^{\alpha_2} \hat{h}_n^{\beta_2}}{C_n^{q-1}} < \eta_2^{1-q}, \end{aligned} \tag{S$''$}$$

where $\eta_1, \eta_2 \geq 1$ stem from estimates (4.2.9).

Remark 4.2.5. One can take $\gamma_1, \delta_1 \in [0, p-1]$ provided

$$\|a_1\|_{L^\infty(\Omega)} \limsup_{n\to\infty} \frac{h_n^{\alpha_1} \hat{k}_n^{\beta_1}}{C_n^{p-1}} + 2\|b_1\|_{L^\infty(\Omega)} < \eta_1^{1-p},$$

which implies the first inequality in (S$''$). A similar comment applies to γ_2, δ_2.

Theorem 4.2.6. *Let* (F_1'), (G_1'), *and* (S_1) *be satisfied. Then problem* (P) *admits a sequence of solutions* $\{(u_n, v_n)\}_n \subseteq C^1(\overline{\Omega})^2$ *such that* $(u_n, v_n) < (u_{n+1}, v_{n+1})$ *for all* $n \in \mathbb{N}$. *Moreover,* $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = +\infty$ *uniformly in* $\overline{\Omega}$ *once* $h_n, \hat{h}_n \to +\infty$.

Proof. Define

$$K_n := C^1(\overline{\Omega})^2 \cap ([h_n, k_n] \times [\hat{h}_n, \hat{k}_n]),$$

as well as

$$D_n := \{w \in K_n : \|\nabla w\|_{L^\infty(\Omega)^2} \le C_n\}.$$

If $(u, v, w) \in [h_n, k_n] \times [\hat{h}_n, \hat{k}_n] \times D_n$, then through (F_1') and (S'') we obtain

$$\begin{aligned} &|f(\cdot, u, v, \nabla w)| \\ &\quad \le \|a_1\|_{L^\infty(\Omega)} h_n^{\alpha_1} \hat{k}_n^{\beta_1} + \|b_1\|_{L^\infty(\Omega)} (C_n^{\gamma_1} + C_n^{\delta_1}) + \|c_1\|_{L^\infty(\Omega)} \\ &\quad \le \left(\frac{C_n}{\eta_1}\right)^{p-1} \end{aligned} \tag{4.2.11}$$

for any $n \in \mathbb{N}$ large enough. Likewise, (G_1') and (S'') yield

$$\begin{aligned} &|g(\cdot, u, v, \nabla w)| \\ &\quad \le \|a_2\|_{L^\infty(\Omega)} k_n^{\alpha_2} \hat{h}_n^{\beta_2} + \|b_2\|_{L^\infty(\Omega)} (C_n^{\gamma_2} + C_n^{\delta_2}) + \|c_2\|_{L^\infty(\Omega)} \\ &\quad \le \left(\frac{C_n}{\eta_2}\right)^{q-1}. \end{aligned} \tag{4.2.12}$$

Hence, from (4.2.9), with $K := K_n$, it follows $\Gamma(D_n) \subseteq D_n$, where Γ is given by (4.2.8). Observe that, thanks to (S'),

$$\begin{aligned} f(\cdot, k_n, v, \nabla w) \le 0 \le f(\cdot, h_n, v, \nabla w), \\ g(\cdot, u, \hat{k}_n, \nabla w) \le 0 \le g(\cdot, u, \hat{h}_n, \nabla w), \end{aligned}$$

which easily force (4.2.4). So, hypothesis (H) of Theorem 4.2.4 is fulfilled. Thus, for every $n \in \mathbb{N}$, problem (P) possesses a solution $(u_n, v_n) \in K_n$. Since $k_n < h_{n+1}$ and $\hat{k}_n < \hat{h}_{n+1}$, we evidently have $(u_n, v_n) < (u_{n+1}, v_{n+1})$. Finally, if $h_n, \hat{h}_n \to +\infty$ then $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = +\infty$ uniformly in $\overline{\Omega}$. $\square$

Now we investigate the super-linear setting, assuming the following conditions.

(F_2') There exist $\alpha_1 < 0 < \beta_1$, $\gamma_1, \delta_1 \in (p-1, +\infty)$, and $a_1, b_1 \in L^\infty(\Omega)$ such that

$$|f(x, s, t, \xi_1, \xi_2)| \le a_1(x) s^{\alpha_1} t^{\beta_1} + b_1(x)(|\xi_1|^{\gamma_1} + |\xi_2|^{\delta_1})$$

for all $(x, s, t, \xi_1, \xi_2) \in \Omega \times (0, +\infty)^2 \times \mathbb{R}^{2N}$.

(G_2') There exist $\beta_2 < 0 < \alpha_2$, $\gamma_2, \delta_2 \in (q-1, +\infty)$, and $a_2, b_2 \in L^\infty(\Omega)$ such that
$$|g(x,s,t,\xi_1,\xi_2)| \le a_2(x)s^{\alpha_2}t^{\beta_2} + b_2(x)(|\xi_1|^{\gamma_2} + |\xi_2|^{\delta_2})$$
for all $(x,s,t,\xi_1,\xi_2) \in \Omega \times (0,+\infty)^2 \times \mathbb{R}^{2N}$.

(S_2) There exist $\{h_n\}_n, \{\hat{h}_n\}_n, \{k_n\}_n, \{\hat{k}_n\}_n, \{C_n\}_n \subseteq (0,+\infty)$, with $C_n \to 0$, satisfying $k_{n+1} < h_n < k_n$, $\hat{k}_{n+1} < \hat{h}_n < \hat{k}_n$ and such that (S')–(S'') are true for all $(x,s,t,\xi_1,\xi_2) \in \Omega \times [h_n,k_n] \times [\hat{h}_n,\hat{k}_n] \times B_{\mathbb{R}^N}(C_n)^2$, $n \in \mathbb{N}$.

Remark 4.2.5 can be adapted to (F_2')–(G_2').

Theorem 4.2.7. *Under assumptions* (F_2'), (G_2'), *and* (S_2), *problem* (P) *has a sequence of solutions* $\{(u_n,v_n)\}_n \subseteq C^1(\overline{\Omega})^2$ *such that* $(u_{n+1},v_{n+1}) < (u_n,v_n)$ *for every* $n \in \mathbb{N}$. *Moreover,* $\lim_{n\to\infty} u_n = \lim_{n\to\infty} v_n = 0$ *uniformly in* $\overline{\Omega}$ *once* $k_n, \hat{k}_n \to 0$.

Proof. The argument is analogous to Theorem 4.2.6, because (4.2.11)–(4.2.12), written for $c_1 \equiv c_2 \equiv 0$, hold whenever $n \in \mathbb{N}$ is sufficiently large. □

Remark 4.2.8. Conditions (F_i') and (G_i'), $i = 1,2$, above have been formulated on the whole $\Omega \times (0,+\infty)^2 \times \mathbb{R}^{2N}$ just to avoid cumbersome statements. Indeed, consider, e.g., Theorem 4.2.7. Since C_n is arbitrary small for n large while, through (S''), the term $h_n^{\alpha_1}\hat{k}_n^{\beta_1}$ can be controlled by C_n^{p-1}, it suffices to require (F_2') in $\Omega \times (0,\delta]^2 \times B_{\mathbb{R}^N}(\delta)^2$ with $\delta > 0$ appropriate, and the same arguments work. So, we can actually treat reactions f, g having arbitrary behavior far from the origin. A 'dual' comment holds for Theorem 4.2.6.

Example 4.2.9. Define, provided $(x,s,t,\xi_1,\xi_2) \in \Omega \times (0,+\infty)^2 \times \mathbb{R}^{2N}$,
$$f(x,s,t,\xi_1,\xi_2) = \sin s + \frac{1}{2}\cos t, \quad g(x,s,t,\xi_1,\xi_2) = \frac{1}{2}\sin s + \cos t.$$
In this case, $a_i \equiv b_i \equiv 0$, $i = 1,2$, so (S'') holds true. Choosing $h_n = \frac{\pi}{2} + 2\pi n$, $k_n = \frac{3}{2}\pi + 2\pi n$, $\hat{h}_n = 2\pi n$, $\hat{k}_n = \pi + 2\pi n$, and $C_n = n$, also (S') is fulfilled. Hence, f and g comply with (S_1).

Finally, an example of nonlinearities, with both singular and convective terms, that satisfy (S_2) is the following.

Example 4.2.10. Set, for every $(x,s,t,\xi_1,\xi_2) \in \Omega \times (0,+\infty)^2 \times \mathbb{R}^{2N}$,
$$f(x,s,t,\xi_1,\xi_2) = \sin\frac{1}{s}\left(s^{\alpha_1}t^{\beta_1} - |\xi_1|^{\gamma_1} - |\xi_2|^{\delta_1}\right),$$
$$g(x,s,t,\xi_1,\xi_2) = \cos\frac{1}{t}\left(s^{\alpha_2}t^{\beta_2} - |\xi_1|^{\gamma_2} - |\xi_2|^{\delta_2}\right),$$

where

$$\min\{\gamma_1,\delta_1\} > \alpha_1+\beta_1 > p-1, \quad \min\{\gamma_2,\delta_2\} > \alpha_2+\beta_2 > q-1.$$

To check (S_2), simply pick $h_n = \left(\frac{\pi}{2}+2\pi n\right)^{-1}$, $k_n = \left(-\frac{\pi}{2}+2\pi n\right)^{-1}$, $\hat{h}_n = (2\pi+2\pi n)^{-1}$, $\hat{k}_n = (\pi+2\pi n)^{-1}$, and $C_n = \frac{1}{n}$.

Finally, we would like to mention a corollary of Theorem 4.2.6.

Corollary 4.2.11. *Let $h:\mathbb{R}\to\mathbb{R}$ be a continuous, periodic function, and $\alpha:\Omega\to\mathbb{R}$ such that*

$$\inf_{\mathbb{R}} h \le \alpha(x) \le \sup_{\mathbb{R}} h \quad \text{for a.a. } x\in\Omega.$$

Then the problem

$$\begin{cases} -\Delta_p u = h(u)-\alpha(x) & \text{in } \Omega,\\ u>0 & \text{in } \Omega,\\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases}$$

admits infinitely many solutions.

4.2.2 Singular convective systems in $\mathbb{R}^N$

In this paragraph we deal with the problem

$$\begin{cases} -\Delta_p u = f(x,u,v,\nabla u,\nabla v) & \text{in } \mathbb{R}^N,\\ -\Delta_q v = g(x,u,v,\nabla u,\nabla v) & \text{in } \mathbb{R}^N,\\ u,v>0 & \text{in } \mathbb{R}^N, \end{cases} \tag{4.2.13}$$

where $N\ge 3$ and $1<p,q<N$, while $f,g:\mathbb{R}^N\times(0,+\infty)^2\times\mathbb{R}^{2N}\to(0,+\infty)$ are Carathéodory functions satisfying the following assumptions:

$\underline{H_1(f)}$ There exist $\alpha_1\in(-1,0]$, $\beta_1,\delta_1\in[0,q-1)$, $\gamma_1\in[0,p-1)$, $m_1,\hat{m}_1>0$, and $a_1\in L^1(\mathbb{R}^N)\cap L^{\zeta_1}(\mathbb{R}^N)$, with $\zeta_1>N$, such that

$$m_1a_1(x)s_1^{\alpha_1}s_2^{\beta_1} \le f(x,s_1,s_2,\mathbf{t}_1,\mathbf{t}_2) \le \hat{m}_1a_1(x)\left(s_1^{\alpha_1}s_2^{\beta_1}+|\mathbf{t}_1|^{\gamma_1}+|\mathbf{t}_2|^{\delta_1}\right)$$

in $\mathbb{R}^N\times(0,+\infty)^2\times\mathbb{R}^{2N}$. Moreover, $\operatorname*{ess\,inf}_{B_\rho} a_1>0$ for all $\rho>0$.

H$_1$(g) There exist $\beta_2 \in (-1,0]$, $\alpha_2, \gamma_2 \in [0,p-1)$, $\delta_2 \in [0,q-1)$, $m_2, \hat{m}_2 > 0$, and $a_2 \in L^1(\mathbb{R}^N) \cap L^{\zeta_2}(\mathbb{R}^N)$, with $\zeta_2 > N$, such that

$$m_2 a_2(x) s_1^{\alpha_2} s_2^{\beta_2} \leq g(x, s_1, s_2, \mathbf{t}_1, \mathbf{t}_2) \leq \hat{m}_2 a_2(x) \left(s_1^{\alpha_2} s_2^{\beta_2} + |\mathbf{t}_1|^{\gamma_2} + |\mathbf{t}_2|^{\delta_2} \right)$$

in $\mathbb{R}^N \times (0,+\infty)^2 \times \mathbb{R}^{2N}$. Moreover, $\operatorname*{ess\,inf}_{B_\rho} a_2 > 0$ for all $\rho > 0$.

H$_1$(a) The numbers ζ_1, ζ_2 occurring in hypotheses H$_1$(f)–H$_1$(g) fulfill

$$\frac{1}{\zeta_1} < 1 - \frac{p}{p^*} - \theta_1, \quad \frac{1}{\zeta_2} < 1 - \frac{q}{q^*} - \theta_2,$$

with

$$\theta_1 := \max\left\{ \frac{\beta_1}{q^*}, \frac{\gamma_1}{p}, \frac{\delta_1}{q} \right\} < 1 - \frac{p}{p^*}, \quad \theta_2 := \max\left\{ \frac{\alpha_2}{p^*}, \frac{\gamma_2}{p}, \frac{\delta_2}{q} \right\} < 1 - \frac{q}{q^*}.$$

H$_2$ If $\eta_1 := \max\{\beta_1, \delta_1\}$ and $\eta_2 := \max\{\alpha_2, \gamma_2\}$ then

$$\eta_1 \eta_2 < (p-1-\gamma_1)(q-1-\delta_2).$$

Example 4.2.12. H$_1$(a) is fulfilled once $a_1, a_2 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and

$$\max\left\{ \frac{\beta_1}{q^*}, \frac{\gamma_1}{p}, \frac{\delta_1}{q} \right\} < 1 - \frac{p}{p^*}, \quad \max\left\{ \frac{\alpha_2}{p^*}, \frac{\gamma_2}{p}, \frac{\delta_2}{q} \right\} < 1 - \frac{q}{q^*}.$$

Indeed, it suffices to choose $\zeta_1 := \zeta_2 := +\infty$.

Remark 4.2.13. By interpolation (see, e.g., [114, Proposition 2.1]), condition H$_1$(a) entails $a_i \in L^{\sigma_{i,j}}(\mathbb{R}^N)$, $i = 1,2$, where:

(i) $\sigma_{1,j} := \frac{1}{1-t_j}$, $j = 1,2,3,4$, with

$$t_1 = \frac{\alpha_1 + 1}{p^*} + \frac{\beta_1}{q^*}, \quad t_2 = \frac{1}{p^*} + \frac{\beta_1}{q^*}, \quad t_3 = \frac{1}{p^*} + \frac{\gamma_1}{p}, \quad t_4 = \frac{1}{p^*} + \frac{\delta_1}{q};$$

(ii) $\sigma_{2,j} := \frac{1}{1-t_j}$, $j = 1,2,3,4$, with

$$t_1 = \frac{\beta_2 + 1}{q^*} + \frac{\alpha_2}{p^*}, \quad t_2 = \frac{1}{q^*} + \frac{\alpha_2}{p^*}, \quad t_3 = \frac{1}{q^*} + \frac{\gamma_2}{p}, \quad t_4 = \frac{1}{q^*} + \frac{\delta_2}{q}.$$

Let $Z := Z(\Omega)$ be a real-valued function space on a nonempty measurable set $\Omega \subseteq \mathbb{R}^N$. Unlike the rest of the chapter, here we put

$$Z_+ := \{z \in Z : z > 0\}.$$

To avoid cumbersome expressions, define

$$X := \mathcal{D}_0^{1,p}(\mathbb{R}^N) \times \mathcal{D}_0^{1,q}(\mathbb{R}^N), \quad \|(u,v)\| := \|u\|_{1,p} + \|v\|_{1,q} \quad \forall\, (u,v) \in X,$$
$$\mathcal{C}_+^1 := X_+ \cap C^1_{\mathrm{loc}}(\mathbb{R}^N)^2, \quad \mathcal{C}_+^{1,\alpha} := X_+ \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)^2.$$

$\mathcal{C}_+^1$ and $\mathcal{C}_+^{1,\alpha}$ will be endowed with the topology induced by that of X.
Our aim is to prove the following

Theorem 4.2.14. *Under hypotheses* H_1–H_2*, problem* (4.2.13) *admits a weak solution* $(u,v) \in X$*.*

Fix $w := (w_1, w_2) \in \mathcal{C}_+^1$, $\varepsilon > 0$ and define

$$f_{w,\varepsilon} := f(\cdot, w_1 + \varepsilon, w_2, \nabla w), \quad g_{w,\varepsilon} := g(\cdot, w_1, w_2 + \varepsilon, \nabla w),$$

where $\nabla w := (\nabla w_1, \nabla w_2)$. We first focus on the auxiliary problem

$$\begin{cases} -\Delta_p u = f_{w,\varepsilon}(x) \text{ in } \mathbb{R}^N, \\ -\Delta_q v = g_{w,\varepsilon}(x) \text{ in } \mathbb{R}^N. \end{cases} \tag{P_w^ε}$$

Exploiting Minty-Browder's theorem and regularity theory (see Corollaries 2.3.4 and 2.2.8) one can prove what follows.

Lemma 4.2.15. *If* H_1 *holds then* (P_w^ε) *admits a unique solution* $(u,v) \in \mathcal{C}_+^{1,\alpha}$*, for a suitable* $\alpha \in (0,1)$*.*

Hereafter, (u,v) will denote the solution to (P_w^ε) given by Lemma 4.2.15.

Lemma 4.2.16. *Let* H_1 *be satisfied. Then there exists* $L_\varepsilon > 0$ *such that*

$$\|\nabla u\|_p^{p-1} \le L_\varepsilon(1 + \|\nabla w_1\|_p^{\gamma_1} + \|\nabla w_2\|_q^{\eta_1}),$$
$$\|\nabla v\|_q^{q-1} \le L_\varepsilon(1 + \|\nabla w_1\|_p^{\eta_2} + \|\nabla w_2\|_q^{\delta_2}),$$

where $\eta_1 := \max\{\beta_1, \delta_1\}$ *and* $\eta_2 := \max\{\alpha_2, \gamma_2\}$*.*

Proof. Test the first equation in $(\mathrm{P}^{\varepsilon}_{w})$ with u and exploit $\mathrm{H}_1(\mathrm{f})$, $\mathrm{H}_1(\mathrm{a})$, besides Theorem (2.1.5), to achieve

$$\begin{aligned}
\|\nabla u\|_p^p &= \int_{\mathbb{R}^N} f(\cdot, w_1+\varepsilon, w_2, \nabla w_1, \nabla w_2) u \mathrm{d}x \\
&\le \hat{m}_1 \int_{\mathbb{R}^N} a_1[(w_1+\varepsilon)^{\alpha_1} w_2^{\beta_1} + |\nabla w_1|^{\gamma_1} + |\nabla w_2|^{\delta_1}] u \mathrm{d}x \\
&\le \hat{m}_1 \int_{\mathbb{R}^N} a_1 \max\{1, \varepsilon^{\alpha_1}\}(w_2^{\beta_1} + |\nabla w_1|^{\gamma_1} + |\nabla w_2|^{\delta_1}) u \mathrm{d}x \qquad (4.2.14) \\
&\le c_\varepsilon \|u\|_{p^*}(\|w_2\|_{q^*}^{\beta_1} + \|\nabla w_1\|_p^{\gamma_1} + \|\nabla w_2\|_q^{\delta_1}) \\
&\le c_\varepsilon \|\nabla u\|_p(\|\nabla w_2\|_q^{\beta_1} + \|\nabla w_1\|_p^{\gamma_1} + \|\nabla w_2\|_q^{\delta_1}) \\
&\le L_\varepsilon \|\nabla u\|_p(1 + \|\nabla w_1\|_p^{\gamma_1} + \|\nabla w_2\|_q^{\eta_1}),
\end{aligned}$$

because

$$\|\nabla w_2\|_q^{\beta_1} + \|\nabla w_2\|_q^{\delta_1} \le 2(1 + \|\nabla w_2\|_q^{\eta_1}). \qquad (4.2.15)$$

This shows the first inequality. The other is verified similarly. □

Moser's technique (Corollary 2.2.3) and Corollary 2.2.14 allow to prove the following two lemmas.

Lemma 4.2.17. *Under* H_1*, there exists* $M_\varepsilon := M_\varepsilon(\|\nabla w_1\|_p, \|\nabla w_2\|_q) > 0$ *such that*

$$\max\{\|u\|_\infty, \|v\|_\infty\} \le M_\varepsilon(\|\nabla w_1\|_p, \|\nabla w_2\|_q).$$

Lemma 4.2.18. *If* H_1 *holds and* $\max\{\|w_i\|_\infty, \|\nabla w_i\|_\infty\} < +\infty$*,* $i = 1, 2$*, then*

$$\begin{aligned}
\|\nabla u\|_\infty^{p-1} &\le N_\varepsilon(\|\nabla w_1\|_p, \|\nabla w_2\|_q, \|w_2\|_\infty)(1 + \|\nabla w_1\|_\infty^{\gamma_1} + \|\nabla w_2\|_\infty^{\delta_1}), \\
\|\nabla v\|_\infty^{q-1} &\le N_\varepsilon(\|\nabla w_1\|_p, \|\nabla w_2\|_q, \|w_1\|_\infty)(1 + \|\nabla w_1\|_\infty^{\gamma_2} + \|\nabla w_2\|_\infty^{\delta_2})
\end{aligned}$$

with suitable constants $N_\varepsilon(\|\nabla w_1\|_p, \|\nabla w_2\|_q, \|w_i\|_\infty) > 0$*,* $i = 1, 2$.

Let H_1 be satisfied. Given $\varepsilon > 0$, define

$$\begin{aligned}
\mathcal{R}_\varepsilon := \{(w_1, w_2) \in \mathcal{C}^1_+ : \|\nabla w_1\|_p \le A_1,\ \|\nabla w_2\|_q \le A_2,& \\
\|w_i\|_\infty \le B_i,\ \|\nabla w_i\|_\infty \le C_i,\ i = 1, 2\},&
\end{aligned}$$

with $A_i, B_i, C_i > 0$, $i = 1, 2$, such that

$$\begin{cases}
A_1^{p-1} & \ge L_\varepsilon(1 + A_1^{\gamma_1} + A_2^{\eta_1}), \\
A_2^{q-1} & \ge L_\varepsilon(1 + A_1^{\eta_2} + A_2^{\delta_2}), \\
B_1, B_2 & \ge M_\varepsilon(A_1, A_2), \\
C_1^{p-1} & \ge N_\varepsilon(A_1, A_2, B_2)(1 + C_1^{\gamma_1} + C_2^{\delta_1}), \\
C_2^{q-1} & \ge N_\varepsilon(A_1, A_2, B_1)(1 + C_1^{\gamma_2} + C_2^{\delta_2}),
\end{cases} \qquad (4.2.16)$$

and L_ε, $M_\varepsilon(\cdot,\cdot)$, $N_\varepsilon(\cdot,\cdot,\cdot)$ stemming from Lemmas 4.2.16–4.2.18. Apropos, system (4.2.16) admits solutions. Indeed, by H_1, we can pick

$$1 < \sigma < \frac{(p-1)(q-1)}{\eta_1\eta_2}. \tag{4.2.17}$$

If $A_1 := K^{\frac{1}{\eta_2}}$ and $A_2 := K^{\frac{\sigma}{q-1}}$ then the first two inequalities of (4.2.16) become

$$K^{\frac{p-1}{\eta_2}} \geq L_\varepsilon(1 + K^{\frac{\gamma_1}{\eta_2}} + K^{\frac{\sigma\eta_1}{q-1}}), \quad K^\sigma \geq L_\varepsilon(1 + K + K^{\frac{\sigma\delta_2}{q-1}}),$$

which, due to (4.2.17), are true for any sufficiently large $K > 0$. Next, choose

$$B_1 := B_2 := M_\varepsilon(K^{\frac{1}{\eta_2}}, K^{\frac{\sigma}{q-1}}).$$

With A_i, B_i as above, set $C_1 := H^{\frac{1}{\eta_2}}$ and $C_2 := H^{\frac{\sigma}{q-1}}$. The last two inequalities in (4.2.16) rewrite as

$$\begin{aligned} H^{\frac{p-1}{\eta_2}} &\geq N_\varepsilon(A_1, A_2, B_2)(1 + H^{\frac{\gamma_1}{\eta_2}} + H^{\frac{\sigma\delta_1}{q-1}}), \\ H^\sigma &\geq N_\varepsilon(A_1, A_2, B_1)(1 + H^{\frac{\gamma_2}{\eta_2}} + H^{\frac{\sigma\delta_2}{q-1}}). \end{aligned}$$

Thanks to (4.2.17) again, they hold for every $H > 0$ big enough.

On the *trapping region* $\mathcal{R}_\varepsilon$ we will consider the topology induced by that of X. Let us now investigate the regularized problem

$$\begin{cases} -\Delta_p u = f(x, u+\varepsilon, v, \nabla u, \nabla v) & \text{in } \mathbb{R}^N, \\ -\Delta_q v = g(x, u, v+\varepsilon, \nabla u, \nabla v) & \text{in } \mathbb{R}^N, \\ \quad u, v > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{P^ε}$$

where $\varepsilon \geq 0$. Evidently, (P^ε) reduces to (4.2.13) once $\varepsilon = 0$.

Lemma 4.2.19. *Under* H_1, *for every* $\varepsilon > 0$ *problem* (P^ε) *possesses a solution* $(u_\varepsilon, v_\varepsilon) \in \mathcal{C}^{1,\alpha}_+$.

Proof. Fix $\varepsilon > 0$ and define, provided $w \in \mathcal{R}_\varepsilon$,

$$T_\varepsilon(w) := (u,v), \text{ with } (u,v) \text{ being the unique solution to } (\mathrm{P}^\varepsilon_w);$$

cf. Lemma 4.2.15. From Lemmas 4.2.16–4.2.18, besides (4.2.16), it follows $T_\varepsilon(\mathcal{R}_\varepsilon) \subseteq \mathcal{R}_\varepsilon$.

Claim 1. $T_\varepsilon(\mathcal{R}_\varepsilon)$ is relatively compact in X.

To see this, pick $\{w_n\} \subseteq \mathcal{R}_\varepsilon$, put

$$w_n := (w_{1,n}, w_{2,n}), \quad (u_n, v_n) := T_\varepsilon(w_n), \quad n \in \mathbb{N},$$

and understand any convergence up to subsequences. Since $\{T_\varepsilon(w_n)\} \subseteq \mathcal{R}_\varepsilon$ while X is reflexive, $\{(u_n, v_n)\}$ weakly converges to a point $(u, v) \in X$. Taking any $\rho > 0$, if $Y_\rho := L^p(B_\rho) \times L^q(B_\rho)$, then one has

$$X \hookrightarrow W^{1,p}(B_\rho) \times W^{1,q}(B_\rho) \hookrightarrow Y_\rho. \tag{4.2.18}$$

Actually, the first embedding in (4.2.18) is continuous by Theorem 2.1.5 and the continuity of the restriction map $L^r(\mathbb{R}^N) \to L^r(B_\rho)$, while the other one is compact due to Rellich-Kondrachov's theorem. Thus, $X \hookrightarrow Y_\rho$ compactly, which yields $(u_n, v_n) \to (u, v)$ in Y_ρ. Let us next verify that

$$(u_n(x), v_n(x)) \to (u(x), v(x)) \text{ for almost every } x \in \mathbb{R}^N. \tag{4.2.19}$$

Indeed, $(u_n, v_n) \to (u, v)$ in Y_1 yields a sub-sequence $\{(u_n^{(1)}, v_n^{(1)})\}$ of $\{(u_n, v_n)\}$ such that

$$(u_n^{(1)}(x), v_n^{(1)}(x)) \to (u(x), v(x)) \text{ for almost all } x \in B_1\,.$$

Since $(u_n^{(1)}, v_n^{(1)}) \to (u, v)$ in Y_2, we can extract a sub-sequence $\{(u_n^{(2)}, v_n^{(2)})\}$ from $\{(u_n^{(1)}, v_n^{(1)})\}$ fulfilling

$$(u_n^{(2)}(x), v_n^{(2)}(x)) \to (u(x), v(x)) \text{ for almost every } x \in B_2\,.$$

By induction, to each $k \geq 2$ there corresponds a sub-sequence $\{(u_n^{(k)}, v_n^{(k)})\}$ of $\{(u_n^{(k-1)}, v_n^{(k-1)})\}$ such that

$$(u_n^{(k)}(x), v_n^{(k)}(x)) \to (u(x), v(x)) \text{ for almost all } x \in B_k\,.$$

Now, Cantor's diagonal procedure leads to $(u_n^{(n)}, v_n^{(n)}) \to (u, v)$ a.e. in $\mathbb{R}^N$, because $\bigcup_{k=1}^\infty B_k = \mathbb{R}^N$, and (4.2.19) follows.
Through $\mathrm{H}_1(\mathrm{f})$, besides the inclusion $\{w_n\} \subseteq \mathcal{R}_\varepsilon$, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) \mathrm{d}x \\
&= \int_{\mathbb{R}^N} f(\cdot, w_{1,n} + \varepsilon, w_{2,n}, \nabla w_n)(u_n - u) \mathrm{d}x \\
&\leq \int_{\mathbb{R}^N} f(\cdot, w_{1,n} + \varepsilon, w_{2,n}, \nabla w_n)|u_n - u| \mathrm{d}x \\
&\leq c_\varepsilon \int_{\mathbb{R}^N} a_1 |u_n - u| \mathrm{d}x \quad \forall\, n \in \mathbb{N},
\end{aligned} \tag{4.2.20}$$

with $c_\varepsilon := \hat{m}_1(\varepsilon^{\alpha_1} B_2^{\beta_1} + C_1^{\gamma_1} + C_2^{\delta_1})$. Using $T_\varepsilon(\mathcal{R}_\varepsilon) \subseteq \mathcal{R}_\varepsilon$ and (4.2.19) one has

$$a_1|u_n - u| \le 2B_1 a_1 \in L^1(\mathbb{R}^N), \quad n \in \mathbb{N}.$$

So, by (4.2.19)–(4.2.20), Lebesgue's Theorem entails

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)\mathrm{d}x \le c_\varepsilon \lim_{n\to\infty} \int_{\mathbb{R}^N} a_1|u_n - u|\mathrm{d}x = 0.$$

Now, recall (cf., e.g., [114, Proposition 2.2]) that the operator $(-\Delta_p, \mathcal{D}_0^{1,p}(\mathbb{R}^N))$ is of type (S_+) to achieve $u_n \to u$ in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$. A similar reasoning applies to $\{v_n\}$.

Claim 2. $T_\varepsilon : \mathcal{R}_\varepsilon \to \mathcal{R}_\varepsilon$ is continuous.

Let $\{w_n\} \subseteq \mathcal{R}_\varepsilon$ and $w \in \mathcal{R}_\varepsilon$ satisfy $w_n \to w$ in X. Thanks to Theorem 2.1.5, Theorem 4.9 of [23] provides

$$w_n(x) \to w(x) \ \text{ and } \ \nabla w_n(x) \to \nabla w(x) \ \text{ for almost every } \ x \in \mathbb{R}^N. \quad (4.2.21)$$

Morever, if $(u_n, v_n) := T_\varepsilon(w_n)$, $n \in \mathbb{N}$, then there exists a point $(u, v) \in X$ such that $(u_n, v_n) \to (u, v)$ in X; see the proof of Claim 1. Arguing as before, we obtain

$$u_n(x) \to u(x) \ \text{ and } \ \nabla u_n(x) \to \nabla u(x) \ \text{ for almost every } \ x \in \mathbb{R}^N. \quad (4.2.22)$$

Since $\|\nabla u_n\|_p \le A_1$ whatever n, the sequence $\{|\nabla u_n|^{p-2}\nabla u_n\} \subseteq L^{p'}(\mathbb{R}^N)$ turns out bounded. Due to (4.2.22) and [23, Exercise 4.16], this yields

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla\varphi \mathrm{d}x, \ \ \varphi \in \mathcal{D}_0^{1,p}(\mathbb{R}^N). \quad (4.2.23)$$

On the other hand,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} f(\cdot, w_{1,n} + \varepsilon, w_{2,n}, \nabla w_n)\varphi \,\mathrm{d}x = \int_{\mathbb{R}^N} f(\cdot, w_1 + \varepsilon, w_2, \nabla w)\varphi \,\mathrm{d}x \quad (4.2.24)$$

by Lebesgue's Theorem jointly with (4.2.21) and the inequality

$$f(\cdot, w_{1,n} + \varepsilon, w_{2,n}, \nabla w_n)|\varphi| \le c_\varepsilon a_1|\varphi| \in L^1(\mathbb{R}^N) \ \ \forall\, n \in \mathbb{N},$$

which easily arises from $\mathrm{H}_1(\mathrm{f})$ besides the choice of $\mathcal{R}_\varepsilon$. Finally,

$$\int_{\mathbb{R}^N} |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi \mathrm{d}x = \int_{\mathbb{R}^N} f(\cdot, w_{1,n} + \varepsilon, w_{2,n}, \nabla w_n)\varphi \mathrm{d}x, \ \ n \in \mathbb{N}, \quad (4.2.25)$$

because (u_n, v_n) solves $(\mathrm{P}^{\varepsilon}_{w_n})$. Gathering (4.2.23)–(4.2.25) together we have

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla \varphi \, \mathrm{d}x = \int_{\mathbb{R}^N} f(\cdot, w_1+\varepsilon, w_2, \nabla w)\varphi \, \mathrm{d}x \ \ \forall \varphi \in \mathcal{D}^{1,p}_0(\mathbb{R}^N).$$

The same is evidently true for v. So, (u,v) turns out a solution to $(\mathrm{P}_{\mathrm{w}})$. Uniqueness forces $(u,v) = T_\varepsilon(w)$, whence $T_\varepsilon(w_n) \to T_\varepsilon(w)$.

Now Schauder's theorem (see Theorem 2.5.1) can be applied, and T_ε admits a fixed point $(u_\varepsilon, v_\varepsilon) \in \mathcal{R}_\varepsilon$. By definition of T_ε, the pair $(u_\varepsilon, v_\varepsilon)$ solves problem (P^ε), while Lemma 4.2.15 gives $(u_\varepsilon, v_\varepsilon) \in \mathcal{C}^{1,\alpha}_+$. □

Lemma 4.2.20. *If* H_1–H_2 *hold then there exists a constant* $L > 0$*, independent of* $\varepsilon \geq 0$*, such that* $\|(u,v)\| \leq L$ *for every solution* $(u,v) \in X_+$ *to* (P^ε)*.*

Proof. Pick $\varepsilon \geq 0$ and suppose $(u,v) \in X_+$ solves (P^ε). Via H_1 and Theorem 2.1.5 one has

$$\begin{aligned}
\|\nabla u\|_p^p &= \int_{\mathbb{R}^N} f(\cdot, u+\varepsilon, v, \nabla u, \nabla v) u \mathrm{d}x \\
&\leq \hat{m}_1 \int_{\mathbb{R}^N} a_1[(u+\varepsilon)^{\alpha_1} v^{\beta_1} + |\nabla u|^{\gamma_1} + |\nabla v|^{\delta_1}] u \mathrm{d}x \\
&\leq \hat{m}_1 \int_{\mathbb{R}^N} a_1 (u^{\alpha_1+1} v^{\beta_1} + |\nabla u|^{\gamma_1} u + |\nabla v|^{\delta_1} u) \mathrm{d}x \\
&\leq c \left(\|u\|_{p^*}^{\alpha_1+1} \|v\|_{q^*}^{\beta_1} + \|\nabla u\|_p^{\gamma_1} \|u\|_{p^*} + \|\nabla v\|_q^{\delta_1} \|u\|_{p^*} \right) \\
&\leq c \left(\|\nabla u\|_p^{\alpha_1+1} \|\nabla v\|_q^{\beta_1} + \|\nabla u\|_p^{\gamma_1+1} + \|\nabla v\|_q^{\delta_1} \|\nabla u\|_p \right) \\
&\leq c \max\{1, \|\nabla u\|_p^{\gamma_1+1}\} \max\{1, \|\nabla v\|_q^{\eta_1}\}.
\end{aligned} \tag{4.2.26}$$

Likewise,

$$\|\nabla v\|_q^q \leq c \max\{1, \|\nabla v\|_q^{\delta_2+1}\} \max\{1, \|\nabla u\|_p^{\eta_2}\}. \tag{4.2.27}$$

It should be noted that the constant c does not depend on (u,v) and ε. If either $\|\nabla v\|_q \leq 1$ or $\|\nabla u\|_p \leq 1$ then (4.2.26)–(4.2.27) directly lead to the conclusion, because $\gamma_1 + 1 < p$ and $\delta_2 + 1 < q$; see H_1. Hence, we may assume $\min\{\|\nabla u\|_p, \|\nabla v\|_q\} > 1$. Dividing (4.2.26)–(4.2.27) by $\|\nabla u\|_p^{\gamma_1+1}$ and $\|\nabla v\|_q^{\delta_2+1}$, respectively, yields

$$\|\nabla u\|_p^{p-\gamma_1-1} \leq c \|\nabla v\|_q^{\eta_1}, \quad \|\nabla v\|_q^{q-\delta_2-1} \leq c \|\nabla u\|_p^{\eta_2}.$$

This clearly entails

$$\|\nabla u\|_p^{p-\gamma_1-1} \leq c \|\nabla u\|_p^{\frac{\eta_1\eta_2}{q-\delta_2-1}}, \quad \|\nabla v\|_q^{q-\delta_2-1} \leq c \|\nabla v\|_q^{\frac{\eta_1\eta_2}{p-\gamma_1-1}}.$$

The conclusion now follows from H_2. □

Another application of Moser's technique (Theorem 2.2.2) leads to the following result.

Lemma 4.2.21. *Let* H_1–H_2 *be satisfied. Then there exists* $M > 0$*, independent of* $\varepsilon \geq 0$*, such that*

$$\max\{\|u\|_\infty, \|v\|_\infty\} \leq M$$

for every solution $(u, v) \in X_+$ *to* (P^ε)*.*

Reasoning as in paragraph 3.2.2 we obtain an estimate from below for solutions to (P^ε).

Lemma 4.2.22. *Assume* H_1–H_2*. Then to every* $\rho > 0$ *there corresponds* $\sigma_\rho > 0$ *such that*

$$\min\left\{\operatorname*{ess\,inf}_{B_\rho} u, \operatorname*{ess\,inf}_{B_\rho} v\right\} \geq \sigma_\rho \tag{4.2.28}$$

for all $(u, v) \in X_+$ *distributional solution of* (P^ε)*, with* $0 \leq \varepsilon \leq 1$*.*

Lemma 4.2.23. *Under* H_1–H_2*, problem* (4.2.13) *possesses a distributional solution* $(u, v) \in X_+$*.*

Proof. Let $\varepsilon_n := \frac{1}{n}$, $n \in \mathbb{N}$. Lemma 4.2.19 furnishes a sequence $\{(u_n, v_n)\} \subseteq \mathcal{C}^1_+$ such that (u_n, v_n) solves $(\mathrm{P}^{\varepsilon_n})$ for all $n \in \mathbb{N}$. Since X is reflexive, by Lemma 4.2.20 one has $(u_n, v_n) \rightharpoonup (u, v)$ in X, where a sub-sequence is considered when necessary. As before (cf. the proof of Lemma 4.2.19), this forces

$$(u_n, v_n) \to (u, v) \quad \text{in} \;\; L^r(B_\rho) \quad \text{for all} \;\; r \in (1, p^*) \;\; \text{and} \;\; \rho > 0, \tag{4.2.29}$$

by (4.2.18), as well as (4.2.19). Moreover, $(u, v) \in X_+$ because, thanks to Lemma 4.2.22, to each $\rho > 0$ there corresponds $\sigma_\rho > 0$ satisfying

$$\min\left\{\inf_{B_\rho} u_n, \;\; \inf_{B_\rho} v_n\right\} \geq \sigma_\rho \;\; \forall\, n \in \mathbb{N}. \tag{4.2.30}$$

Claim. For every $\rho > 0$, and along a sub-sequence if necessary, one has

$$(u_n, v_n) \to (u, v) \;\; \text{in} \;\; W^{1,p}(B_\rho) \times W^{1,q}(B_\rho). \tag{4.2.31}$$

Likewise the proof of (4.2.21), this will force

$$(\nabla u_n, \nabla v_n) \to (\nabla u, \nabla v) \;\; \text{a.e. in} \;\; \mathbb{R}^N. \tag{4.2.32}$$

Let $\rho > 0$. Observe that the linear operator

$$z \in \mathcal{D}_0^{1,p}(\mathbb{R}^N) \mapsto \nabla z\lfloor_{B\rho} \in L^p(B_\rho)$$

turns out well defined and continuous in the strong topologies. Therefore,

$$\nabla u_n \rightharpoonup \nabla u \ \text{ in } \ L^p(B_\rho); \tag{4.2.33}$$

cf. [23, Theorem 3.10]. Hypothesis H_1, (4.2.30), and Lemma 4.2.21 yield, for some $r \in (1, p^*)$,

$$\begin{aligned} & f(\cdot, u_n + 1/n, v_n, \nabla u_n, \nabla v_n) \\ & \leq \hat{m}_1 a_1 \left[(u_n + 1/n)^{\alpha_1} v_n^{\beta_1} + |\nabla u_n|^{\gamma_1} + |\nabla v_n|^{\delta_1} \right] \qquad \text{in } B_{2\rho} \\ & \leq \hat{m}_1 \left(\sigma_{2\rho}^{\alpha_1} M^{\beta_1} + |\nabla u_n|^{\gamma_1} + |\nabla v_n|^{\delta_1} \right) a_1 \in L^{r'}(B_{2\rho}) \end{aligned} \tag{4.2.34}$$

whatever n. So, Theorem 2.2.19 and (4.2.33), jointly with (4.2.29) and (4.2.34), ensure that $\nabla u_n \to \nabla u$ in $L^p(B_\rho)$, extracting a subsequence if necessary. Thus,

$$|\nabla u_n|^{p-2} \nabla u_n \to |\nabla u|^{p-2} \nabla u \ \text{ in } \ L^{p'}(B_\rho). \tag{4.2.35}$$

Gathering (4.2.33) and (4.2.35) together gives

$$\lim_{n \to \infty} \int_{B_\rho} |\nabla u_n|^{p-2} \nabla u_n \nabla (u_n - u) \mathrm{d}x = 0.$$

Since $(-\Delta_p, W^{1,p}(B_\rho))$ enjoys the (S_+)-property, we easily achieve $u_n \to u$ in $W^{1,p}(B_\rho)$. A similar conclusion holds for $\{v_n\}$, which shows (4.2.31).

Now, to verify that (u, v) is a distributional solution of (4.2.13), pick any $(\varphi_1, \varphi_2) \in C_0^\infty(\mathbb{R}^N)^2$ and choose $\rho > 0$ fulfilling

$$\operatorname{supp} \varphi_1 \cup \operatorname{supp} \varphi_2 \subseteq B_\rho.$$

By (4.2.31), [23, Theorem 4.9] furnishes $(h, k) \in L^p(B_\rho) \times L^q(B_\rho)$ such that

$$|\nabla u_n| \leq h, \quad |\nabla v_n| \leq k \quad \text{a.e. in } B_\rho \text{ and for all } n \in \mathbb{N},$$

whence

$$f(\cdot, u_n + 1/n, v_n, \nabla u_n, \nabla v_n)|\varphi_1| \leq c_\rho (1 + h^{\gamma_1} + k^{\delta_1}) a_1 |\varphi_1| \in L^1(\mathbb{R}^N), \ n \in \mathbb{N},$$

through (4.2.34). So, thanks to (4.2.19) and (4.2.32), Lebesgue's Theorem entails

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} f(\cdot, u_n + 1/n, v_n, \nabla u_n, \nabla v_n) \varphi_1 \mathrm{d}x = \int_{\mathbb{R}^N} f(\cdot, u, v, \nabla u, \nabla v) \varphi_1 \mathrm{d}x.$$

On account of (4.2.35) and (4.2.32), we then get

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|\nabla u_n|^{p-2}\nabla u_n\nabla\varphi_1\mathrm{d}x=\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla\varphi_1\mathrm{d}x.$$

Recalling that each (u_n,v_n) weakly solves $(\mathrm{P}^{\varepsilon_n})$ produces

$$\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla\varphi_1\mathrm{d}x=\int_{\mathbb{R}^N}f(\cdot,u,v,\nabla u,\nabla v)\varphi_1\mathrm{d}x.$$

Likewise,

$$\int_{\mathbb{R}^N}|\nabla v|^{p-2}\nabla v\nabla\varphi_2\mathrm{d}x=\int_{\mathbb{R}^N}g(\cdot,u,v,\nabla u,\nabla v)\varphi_2\mathrm{d}x,$$

and the assertion follows. □

Lemma 4.2.24. *Let* H_1–H_2 *be satisfied and let* $(u,v)\in X_+$ *be a distributional solution to problem* (4.2.13)*. Then* (u,v) *weakly solves* (4.2.13)*.*

Proof. We evidently have, for any $\varphi\in\mathcal{D}^{1,p}_0(\mathbb{R}^N)$,

$$\varphi=\varphi^+-\varphi^-. \tag{4.2.36}$$

Due to the nature of φ^+, a localization-regularization procedure will be necessary. With this aim, fix $\theta\in C^\infty([0,+\infty))$ such that

$$\theta(t)=\begin{cases}1 & \text{if } 0\le t\le 1,\\ 0 & \text{when } t\ge 2,\end{cases}\qquad \theta \text{ is decreasing in } (1,2) \tag{4.2.37}$$

and a sequence $\{\rho_k\}\subseteq C^\infty_0(\mathbb{R}^N)$ of standard mollifiers [23, p. 108]. Define, for every $n,k\in\mathbb{N}$,

$$\theta_n(\cdot):=\theta(|\cdot|/n)\in C^\infty_0(\mathbb{R}^N),\quad \varphi_n:=\theta_n\,\varphi^+\in\mathcal{D}^{1,p}_0(\mathbb{R}^N),$$

$$\psi_{k,n}:=\rho_k*\varphi_n\in C^\infty_0(\mathbb{R}^N).$$

Using (4.2.37) we easily get $\varphi_n\uparrow\varphi^+$. Moreover, $\lim\limits_{k\to\infty}\psi_{k,n}=\varphi_n$ in $\mathcal{D}^{1,p}_0(\mathbb{R}^N)$, which entails

$$\lim_{k\to\infty}\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla\psi_{k,n}\mathrm{d}x=\int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\nabla\varphi_n\mathrm{d}x,\quad n\in\mathbb{N}. \tag{4.2.38}$$

If, to shorten notation, $\hat f:=f(\cdot,u,v,\nabla u,\nabla v)$ then the linear functional

$$\psi\in\mathcal{D}^{1,p}_0(\mathbb{R}^N)\mapsto\int_{B_{2n+2}}\hat f\psi\,\mathrm{d}x$$

turns out continuous. Indeed, Lemmas 4.2.21–4.2.22, Hölder's inequality combined with H_1(a), and Theorem 2.1.5 produce

$$\int_{B_{2n+2}} a_1 u^{\alpha_1} v^{\beta_1} |\psi| \mathrm{d}x \le \sigma_{2n+2}^{\alpha_1} M^{\beta_1} \|a_1\|_{(p^*)'} \|\psi\|_{p^*} \le c_n \|\nabla\psi\|_p.$$

Now, the assertion follows from H_1(f), because convection terms can be estimated as already made in (4.2.26).
Observe next that

$$\operatorname{supp}\psi_{k,n} \subseteq \overline{\operatorname{supp}\rho_k + \operatorname{supp}\varphi_n} \subseteq \overline{B_1 + B_{2n}} \subseteq B_{2n+2} \quad \forall\, n,k \in \mathbb{N};$$

see [23, Proposition 4.18]. Hence,

$$\begin{aligned} \lim_{k\to\infty} \int_{\mathbb{R}^N} \hat{f}\psi_{k,n} \mathrm{d}x &= \lim_{k\to\infty} \int_{B_{2n+2}} \hat{f}\psi_{k,n} \mathrm{d}x \\ &= \int_{B_{2n+2}} \hat{f}\varphi_n \mathrm{d}x = \int_{\mathbb{R}^N} \hat{f}\varphi_n \mathrm{d}x. \end{aligned} \tag{4.2.39}$$

On the other hand, the fact that $(u, v) \in X_+$ is a distributional solution to (4.2.13) evidently forces

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \psi_{k,n} \mathrm{d}x = \int_{\mathbb{R}^N} \hat{f}\psi_{k,n} \mathrm{d}x, \quad k,n \in \mathbb{N}.$$

Letting $k \to +\infty$ and exploiting (4.2.38)–(4.2.39) we thus achieve

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla \varphi_n \mathrm{d}x = \int_{\mathbb{R}^N} \hat{f}\varphi_n \mathrm{d}x \quad \forall\, n \in \mathbb{N}. \tag{4.2.40}$$

Claim. $\varphi_n \to \varphi^+$ in $\mathcal{D}_0^{1,p}(\mathbb{R}^N)$.
Indeed, for every $n \in \mathbb{N}$ one has

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla\varphi_n - \nabla\varphi^+|^p \mathrm{d}x &= \int_{\mathbb{R}^N} |\varphi^+ \nabla\theta_n + \theta_n \nabla\varphi^+ - \nabla\varphi^+|^p \mathrm{d}x \\ &\le c \left(\int_{\mathbb{R}^N} (1-\theta_n)^p |\nabla\varphi^+|^p \mathrm{d}x + \int_{B_{2n}\setminus B_n} |\nabla\theta_n|^p (\varphi^+)^p \mathrm{d}x \right) \\ &\le c \int_{\mathbb{R}^N} (1-\theta_n)^p |\nabla\varphi^+|^p \mathrm{d}x \\ &\quad + c \left(\int_{B_{2n}\setminus B_n} |\nabla\theta_n|^{\frac{pp^*}{p^*-p}} \mathrm{d}x \right)^{1-\frac{p}{p^*}} \left(\int_{B_{2n}\setminus B_n} (\varphi^+)^{p^*} \mathrm{d}x \right)^{\frac{p}{p^*}} \\ &= c \int_{\mathbb{R}^N} (1-\theta_n)^p |\nabla\varphi^+|^p \mathrm{d}x + c \|\nabla\theta_n\|_N^p \left(\int_{B_{2n}\setminus B_n} (\varphi^+)^{p^*} \mathrm{d}x \right)^{\frac{p}{p^*}}. \end{aligned} \tag{4.2.41}$$

Recall that $\varphi^+ \in \mathcal{D}_0^{1,p}(\mathbb{R}^N)$. By (4.2.37), Lebesgue's Theorem yields

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (1-\theta_n)^p |\nabla\varphi^+|^p \mathrm{d}x = 0 \tag{4.2.42}$$

while, on account of Theorem 2.1.5,

$$\lim_{n\to\infty} \int_{B_{2n}\setminus B_n} (\varphi^+)^{p^*} \mathrm{d}x = 0. \tag{4.2.43}$$

Since, due to (4.2.37) again,

$$\int_{\mathbb{R}^N} |\nabla\theta_n|^N \mathrm{d}x = \frac{1}{n^N} \int_{\mathbb{R}^N} \left|\theta'\left(\frac{|x|}{n}\right)\right|^N \mathrm{d}x = \int_{\mathbb{R}^N} |\theta'(|x|)|^N \mathrm{d}x < +\infty \quad \forall\, n \in \mathbb{N},$$

gathering (4.2.41)–(4.2.43) together shows the claim.

Consequently,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla\varphi_n \mathrm{d}x = \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla\varphi^+ \mathrm{d}x. \tag{4.2.44}$$

From $\varphi_n \uparrow \varphi^+$ and $\hat{f} \geq 0$ it then follows

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \hat{f}\varphi_n \mathrm{d}x = \int_{\mathbb{R}^N} \hat{f}\varphi^+ \mathrm{d}x \tag{4.2.45}$$

by Beppo Levi's Theorem. Through (4.2.40), (4.2.44)–(4.2.45) we thus arrive at

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla\varphi^+ \mathrm{d}x = \int_{\mathbb{R}^N} \hat{f}\varphi^+ \mathrm{d}x.$$

Likewise, one has

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla\varphi^- \mathrm{d}x = \int_{\mathbb{R}^N} \hat{f}\varphi^- \mathrm{d}x,$$

whence (cf. (4.2.36))

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla\varphi \, \mathrm{d}x = \int_{\mathbb{R}^N} f(\cdot, u, v, \nabla u, \nabla v)\varphi \, \mathrm{d}x \quad \forall\, \varphi \in \mathcal{D}_0^{1,p}(\mathbb{R}^N).$$

An analogous argument applies to the second equation in (4.2.13). $\square$

4.3 Further perspectives and open problems

At the end of this thesis, we would like to present a brief list of singular problems that could be investigated.

- Strongly singular equations in $\mathbb{R}^N$, driven by a non-homogeneous operator and exhibiting convection terms as

$$\begin{cases} -\operatorname{div} a(\nabla u) = f(x,u) + g(x,\nabla u) & \text{in } \mathbb{R}^N, \\ u, v > 0 & \text{in } \mathbb{R}^N, \end{cases}$$

with $f : \mathbb{R}^N \times (0,+\infty) \to (0,+\infty)$ and $g : \mathbb{R}^{2N} \to (0,+\infty)$ Carathéodory functions satisfying the growth conditions

$$\begin{aligned} f(x,s) &\leq h(x)s^{-\gamma}, \quad \text{with } \gamma \geq 1, \\ g(x,\xi) &\leq k(x)|\xi|^r, \quad \text{with } r \in [0,p-1), \end{aligned} \tag{4.3.1}$$

and $h, k : \mathbb{R}^N \to (0,+\infty)$ satisfying suitable summability conditions. In order to construct a sub-solution, it seems to be natural requiring also (H_2) of Chapter 3. In the spirit of [34], solutions u could be searched within the class of $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$ function satisfying the condition

$$\text{for any compact } K \subseteq \mathbb{R}^N \text{ there exists } \omega_K > 0 \text{ such that}$$

$$\operatorname*{ess\,inf}_K u \geq \omega_K.$$

- Singular parabolic equations in the form

$$\begin{cases} u_t - \Delta_p u = f(x,u) + g(x,\nabla u) & \text{in } \Omega \times (0,T), \\ u > 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

with $f : \Omega \times (0,+\infty) \to (0,+\infty)$, $g : \Omega \times \mathbb{R}^N \to (0,+\infty)$ Carathéodory functions satisfying growth conditions similar to (4.3.1), with $\gamma \in (0,1)$. Following [78], a solution could be constructed via discretization in time, provided $f(x,\cdot)$ is non-increasing, which guarantees estimates uniform in time. If u_0 lies in $W^{1,p}_0(\Omega)$, then solutions could be found in

$$\mathcal{W} := \left\{ u \in L^p(0,T;W^{1,p}_0(\Omega)) : \; u_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) \right\}.$$

Notice that $\mathcal{W} \overset{c}{\hookrightarrow} L^p(0,T;L^q(\Omega))$ for all $q \in [1,p^*)$; moreover, if $p > \frac{2N}{N+2}$, then the $\mathcal{W} \hookrightarrow C^0([0,T];L^2(\Omega))$ (see [31, Theorems 2.141 and 2.144], as well as the interesting [120]).

- Singular equations driven by a non-local operator, as the fractional p-Laplacian: for instance,

$$\begin{cases} (-\Delta_p)^s u = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

  being $s \in (0,1)$ and $f : \Omega \times (0,+\infty)$ satisfying the growth condition

$$f(x,s) \le c_1 s^{-\gamma} + c_2 s^{q-1} + c_3, \quad \gamma \in (-1,0), \;\; q \in [1,p).$$

  One can look for solutions in the classical fractional Sobolev space $W_0^{s,p}(\mathbb{R}^N)$.

- Singular problems whose reaction terms have critical growth, e.g.,

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

  with $f : \Omega \times (0,+\infty) \to (0,+\infty)$ obeying the growth condition

$$f(x,s) \le c_1 s^{-\gamma} + c_2 s^{p^*-1} + c_3, \quad \gamma \in (-1,0).$$

Until now, several open problems about solutions to singular systems in $\mathbb{R}^N$ could be raised: among the others, we mention the following ones. The example we have in mind is problem (4.2.13), possibly without convection terms.

- Uniqueness.
- Multiplicity.
- Decay.

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