Chapter

Simple Oscillating Systems

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1. Introduction

A wave is a disturbance (deviation from equilibrium) that propagates through space. This disturbance can be caused by an impulse excitation (shock wave) or by a time-varying excitation (e.g., the sound generated by vocal cords). The most frequent cause of wave excitation is a source with periodical time dependence—*oscillations*. The simplest one is *harmonic oscillation*. For example, we can express any periodic function of time as a sum of the harmonic functions (*Fourier series*). Thus, the excitation by the harmonic oscillations is a matter of specific interest. In this chapter, attention is, therefore, paid to the description of the physical nature of oscillations and their properties, see also Halliday [1].

Oscillations represent a very wide group of processes, which are generally characterised by their regular state repeating caused by the internal dynamics of a system. Such systems, whose internal couplings allow oscillations, are called *oscillating systems*. From the energy point of view, the oscillations are conditioned by the existence of two conservative forms of energy, which can *reversibly* exchange due to the internal dynamics of the system. There is, for example, potential energy—kinetic energy (oscillations of mass on a spring) or electric field energy of capacitor—magnetic field energy of inductor (an oscillating LC circuit). A special case represents the 'oscillations' in a rotating system, such as the movement of a conical pendulum where energy exchanges between two perpendicular kinetic components of $\frac{1}{2}mv_x^2$ and $\frac{1}{2}mv_y^2$, or the precession of a rotating body where energy exchanges between two perpendicular rotational components of kinetic energy $\frac{1}{2}J_x\omega_x^2$ and $\frac{1}{2}J_y\omega_y^2$. Some of these cases will be described below as examples.

If the oscillating system is isolated from external influences, it oscillates spontaneously after the initial energy supply (*excitation*). Thus, we are talking about *self-sustained oscillations*. The oscillation amplitude remains almost constant if the energy losses of the oscillations in the system are negligibly small. The oscillations of the ideal lossless system are called *undamped self-oscillations* and represent only theoretical idealisation. There exist loss mechanisms in each real system. They cause the *irreversible* transformation of the conservative form of the system energy into another *non-conservative* one, for example, friction, heat losses due to internal friction, energy losses of the electrical system by radiation, etc. As the total energy of the conservative components decreases, the amplitude of oscillations gradually decreases over time too. We call them *damped self-oscillations*. Due to the damping, these self-oscillations disappear after some time. Thus, they are a transient phenomenon in the system, for example, the vibrations of the string of the musical instrument fade; a swinging of pendulum stops after a certain time; oscillations of an LC circuit gradually disappear, etc.

The system may oscillate permanently without damping if there is a mechanism capable to cover the energy losses from an energy storage device. Such systems are different types of *oscillators*. Examples are pendulum clocks with weights or watches with a spring. In modern watches, a precisely sharpened crystal represents the

oscillating system. In this case, a small lithium cell covers the energy losses caused by damping.

Specific phenomena arise when the system is exposed to periodic force. When such a force acts, the system, after attenuating the transient event, enters a steady state, characterised by oscillations with a constant amplitude and a period equal to the excitation period. These oscillations are called *forced oscillations* of the system. The magnitude of the response to periodic excitation depends on the period or frequency of excitation. Significant is the *resonance* phenomenon that occurs when the excitation frequency is equal to the frequency of the system's undamped oscillations.

The following sections focus on the different types of oscillations in simple systems, that is, in systems in which two conservative forms of energy occur in oscillations. This textbook presents a summary of the knowledge with an emphasis on application. A more detailed analysis of the mentioned phenomena can be found in physics textbooks.

1.1 Undamped self-oscillations

As the basic model of the oscillating system, we use a particle bound to the equilibrium position by the reversing conservative force of the springs (**Figure 1**).

At the top of the figure, the particle is in equilibrium, and the resulting force acting on it is zero. If the particle shifts from the equilibrium position by the displacement of x, there arises a force of $F(x)$ which depends on the x displacement, is reversible, and has the opposite direction as the displacement.

If the particle is displaced from the equilibrium position and released, it starts to move back to the equilibrium position. Its velocity is a derivative of the displacement

$$v = \frac{dx}{dt} = \dot{x} \text{ (a dot over } x \text{ describes the time derivative)}. \tag{1}$$

Displacing particle from equilibrium by x, we perform a work of W, which represents the potential energy of the particle

$$E_p(x) = W = \int_0^x F(\xi)d\xi. \tag{2}$$

A moving particle has a kinetic energy

$$E_k(v) = \frac{1}{2}mv^2. \tag{3}$$

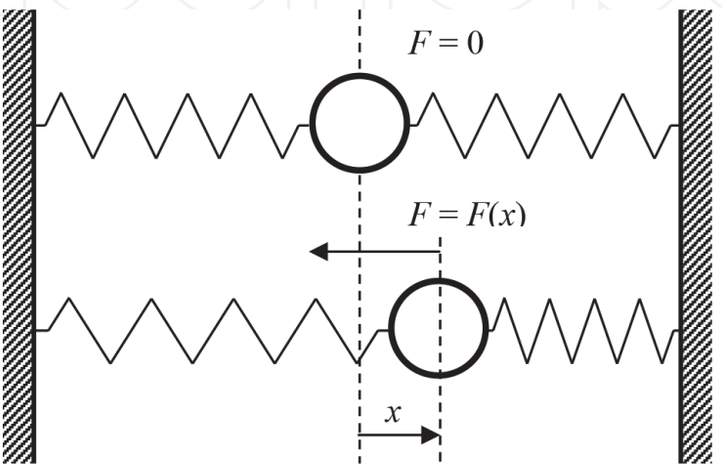


Figure 1.
Particle bound to the equilibrium by the restoring force.

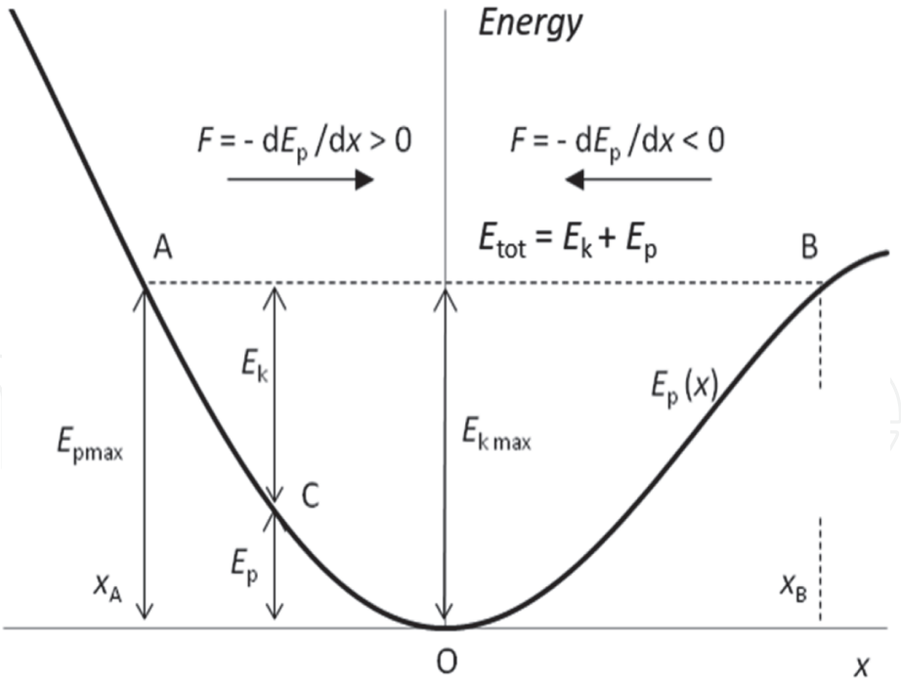


Figure 2.
Total potential and kinetic energy of a particle moving along the x*-axis under conservative forces (law of conservation of mechanical energy).*

Reversible energy exchange occurs between the energy components of E_p and E_k. It is a system with two degrees of freedom. If we consider no loss mechanisms, the sum of the E_p + E_k energy components remains constant. The situation is in **Figure 2**. It indicates the potential energy as a function of the displacement x (solid line). We can see that in the equilibrium position (x = 0) the potential energy is minimal, and thus, at a constant sum, $E_{\text{tot}} = E_p + E_k$, the kinetic energy is maximal. It means that the particle moves due to inertia until its kinetic energy drops to zero. The particle thus moves periodically between the extreme positions of A and B, which are given by the total energy E_{tot}.

We can express a function of the potential energy $E_p(x)$ near the minimum, that is, equilibrium position, by the Taylor power series

$$E_p(x) = \frac{1}{2!}kx^2 + \frac{1}{3!}lx^3 + \frac{1}{4!}nx^4 + \cdots, \tag{4}$$

where $k = \left.\frac{d^2E_p}{dx^2}\right|_{x=0}$ represents the *stiffness* of the system, $l = \left.\frac{d^3E_p}{dx^3}\right|_{x=0}$ expresses *asymmetry* of the potential energy function regarding the equilibrium position. Other coefficients such as $n = \left.\frac{d^4E_p}{dx^4}\right|_{x=0}$ and higher (odd and even derivatives) have similar characteristics but they change function course in the larger distance of x from the equilibrium position. The first power term of x is zero because it is the local minimum of the $E_p(x)$.

A negative potential energy gradient defines the force acting on a particle as follows

$$F(x) = -\frac{dE_p(x)}{dx} = -kx - \frac{1}{2!}lx^2 - \frac{1}{3!}nx^3 - \cdots. \tag{5}$$

We distinguish the *linear* and *non-linear* oscillating systems depending on the number of force or energy terms that are considered in the motion. If the displacement x approaches zero, the higher powers of x^n decrease faster than the first one.

Then, the higher terms of the function $F(x)$ are negligibly small, and the system appears to be linear. However, the terms with higher powers apply, and the system behaves as non-linear if displacement x *significantly increases*.

1.1.1 Undamped self-oscillations of linear system

A system is *linear* if the restoring force is a linear function of the displacement of x from the equilibrium position. According to Eqs. (4) or (5), it follows

$$F(x) \approx -kx, \text{ or } E_p(x) \approx \frac{1}{2}kx^2. \tag{6}$$

Potential energy is a quadratic function of the displacement x and is called a *quadratic potential well*. Its graph is a quadratic parabola. **Figure 3** shows a replacement of the real function $E_p(x)$ by a quadratic function. This replacement fits well only in the near vicinity of the minimum, that is, only for small variations of x around the equilibrium. Oscillations within the range of the fitted region are sometimes called *small oscillations*.

Equation of motion of the particle $ma = F$, where $a = \ddot{x}$ is an acceleration, has form for the linear system as follows

$$\ddot{x} + \omega_0^2 x = 0, \text{where } \omega_0 = \sqrt{\frac{k}{m}}. \tag{7}$$

The solution of this equation is the function

$$x(t) = x_m \sin(\omega_0 t + \alpha), \tag{8}$$

where x_m and α are integration constants and their values are determined from initial conditions $x(0) = x_0$ and $v(0) = \dot{x}(0) = v_0$ at $t = 0$.

It is a harmonic motion where the x_m is an amplitude of oscillations and α is a phase constant (initial phase).

Oscillations with harmonic time dependence are called *harmonic oscillations*. It follows from the previous description that harmonic oscillations occur when a particle (body) moves in a quadratic potential well.

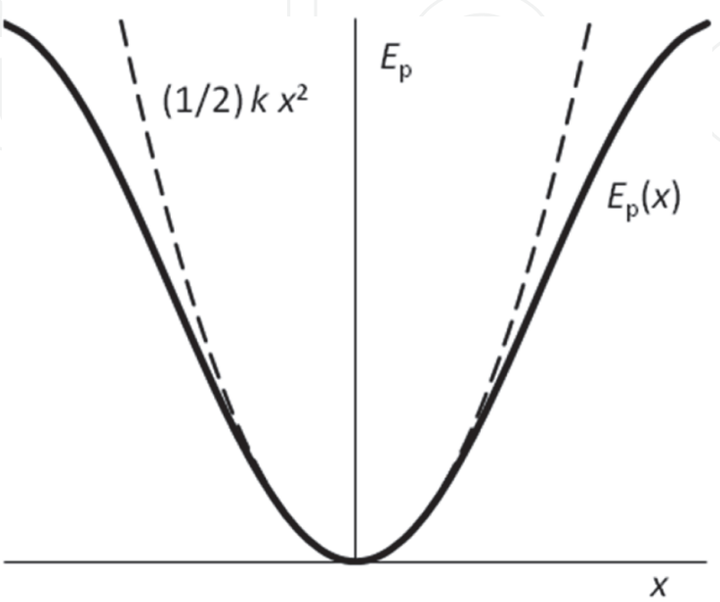


Figure 3.
Quadratic fitting (dashed) of the potential energy function.

The quantity

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0} \tag{9}$$

represents the angular frequency, where f_0 is frequency and T_0 is the period of the *undamped self-oscillations* of the system. According to Eq. (7), these quantities depend on the stiffness k of the system, and the inertia given by the mass m of the particle. As the stiffness increases, the frequency f_0 increases as well, and the period T_0 decreases. With the mass increase, the frequency f_0 decreases, and the period T_0 increases. For example, as a body hung on the spring oscillates with the period of order seconds, an atom in the crystal lattice with the period of the order of 10^{-14} s.

Example 1. Oscillations in an electrical LC circuit.

Let us assume one loop electrical circuit consisting of an inductor L and a capacitor C. Electrical current $i(t)$ flows through this circuit, and we can express the energy of an electrical field of the capacitor and a magnetic field of the inductor as follows:

$$E = \frac{1}{2}\ \frac{1}{C}\ Q^2 + \frac{1}{2} L\ i^2,$$

where the electrical charge Q of the capacitor relates to the current i of the inductor $i = \dot{Q}$. We can see the analogy between electrical and mechanical systems, in case $x \to Q$, $k \to 1/C$ and $m \to L$. If we do not consider the power losses, the energy E is constant, and by differentiating it, we get the equation

$$\frac{1}{C}\ Q\ \frac{\mathrm{d}Q}{\mathrm{d}t} + L\ i\ \frac{\mathrm{d}i}{\mathrm{d}t} = 0.$$

Dividing the equation by $i = \mathrm{d}Q/\mathrm{d}t$, we get

$$\frac{d^2Q}{dt^2} + \frac{1}{LC} Q = 0,$$

which has the same form as (7). Then, the solution is

$$Q(t) = Q_m \sin(\omega_0 t + \alpha), \text{where } \omega_0 = \sqrt{\frac{1}{LC}}.$$

The capacitor voltage is

$$u_C(t) = \frac{Q(t)}{C} = \frac{Q_m}{C} \sin(\omega_0 t + \alpha) = U_m \sin(\omega_0 t + \alpha)$$

and the inductor current equals to

$$i(t) = \frac{dQ(t)}{dt} = \omega_0 Q_m \cos(\omega_0 t + \alpha) = I_m \sin\left(\omega_0 t + \alpha + \frac{\pi}{2}\right).$$

Thus, there are the harmonic undamped oscillations of the circuit quantities with the angular frequency of ω_0.

Example 2. Pendulum.

Consider a small body suspended on a long fibre (**Figure 4**). After the initial excitation, the body oscillates around the equilibrium position, and thus performs a

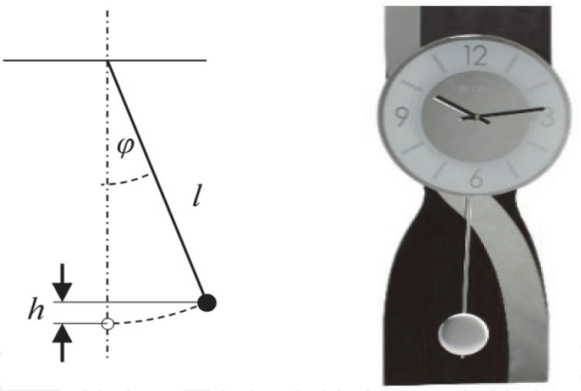


Figure 4.
Pendulum.

circular motion with a radius equal to the fibre length of the l. If we displace the fibre by an angle φ from the equilibrium position, then the potential energy of the body changes as

$$E_{\mathrm{p}}(\varphi) = mgh = mgl\ \ (1 - \cos\varphi).$$

The $E_p(\varphi)$ function is not quadratic, and therefore, we can use a decomposition using the power series

$$E_{\mathrm{p}}(\varphi) = mgl\ \ \left(\frac{\varphi^2}{2!} - \frac{\varphi^4}{4!} + \cdots\right).$$

We can neglect the series terms of the higher order for $\varphi << 1$, and then the potential energy is

$$E_{\mathrm{p}}(\varphi) = \frac{1}{2} mgl\ \ \varphi^2.$$

Additionally, the kinetic energy is

$$E_{\mathrm{k}}(v) = \frac{1}{2} m\ \ v^2 = \frac{1}{2} m\ \ l^2\ \ \dot{\varphi}^2.$$

We can see the analogy to (7) again, if $x \rightarrow \varphi$, $k \rightarrow mgl$ and $m \rightarrow ml^2$.
Pendulum displacement is described by the function

$$\phi(t) = \phi_m \sin(\omega_0 t + \alpha), \text{where } \omega_0 = \sqrt{\frac{g}{l}}.$$

Hence, the body oscillates with the period

$$T_0 = 2\pi\sqrt{\frac{l}{g}}.$$

By measuring the oscillation period, it is possible to determine the length of the pendulum if we do not have a measuring tool. Alternatively, with a pendulum of a certain length, we can realise a periodic movement with the required period, such as in the case of the pendulum clock.

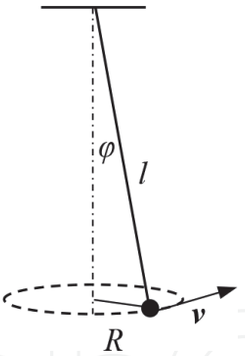


Figure 5.
Cone pendulum.

Example 3. Cone pendulum.

Consider the same case as in the previous example, but let the body move along a circle in the horizontal plane (x, y). The pendulum copies a conical surface as it moves. Thus, the deviation angle from the vertical axis is φ, as shown in **Figure 5**. The radius of motion of the body is $R = l \sin \varphi$. The kinetic energy of the body is

$$E_k = \frac{1}{2}mv^2 = \frac{1}{2}m\left(v_x^2 + v_y^2\right),$$

where v is the velocity of the circular motion.

The force acts on the body and equals to $F_d = -m\omega^2 r$, where the angular velocity is $\omega = v/r$. The centrifugal force composes of gravitational force $F_g = mg$, while the resultant force has the direction of the pendulum fibre that means $tg\varphi = F_d/F_g$. For small displacement, when $\varphi << 1$ rad, it follows that $tg\varphi \approx \sin\varphi = r/l$. From the $F_d/F_g = r/l$, we obtain $\omega = \sqrt{g/l}$, which is similar to the previous example.

Potential energy connected with the centrifugal force is given as

$$E_p = \frac{1}{2}m\omega^2 r^2 = \frac{1}{2}m\omega^2\left(x^2 + y^2\right).$$

And finally, the total energy can be expressed

$$E_k + E_p = \left(\frac{1}{2}mv_x^2 + \frac{1}{2}m\omega^2 x^2\right) + \left(\frac{1}{2}mv_y^2 + \frac{1}{2}m\omega^2 y^2\right).$$

In this case, the motion can be considered as a superposition of two mutually perpendicular oscillations in the x- and y-direction, which are phase-shifted by $\pi/2$ rad. The total energy $(E_k + E_p)$ is constant and is the sum of the total energy of oscillations in the x- and y-direction.

Example 4. Precession of magnetic dipole in the magnetic field.

Another specific case of periodic movement is the precession (rotating axis of a rotating body). We can observe it looking at a children's toy, such as a spinning top.

By spinning and laying it on the pad, the toy axis rotates, see the illustration. The precession occurs due to the gravitational force.

Similarly, the magnetic dipole, here the proton, is affected by an external magnetic field. One of the proton parameters is the angular momentum $\boldsymbol{L}$, which describes its mechanical rotation. The rotation of the charged particle is associated with the accompanying magnetic field. Thus, the proton behaves like an elemental magnet (the magnetic dipole) with a magnetic moment $\boldsymbol{m}$. The ratio of magnetic moment to mechanical angular momentum is called the gyromagnetic ratio $\gamma = \boldsymbol{m}/\boldsymbol{L}$ (see **Table 1**). If the dipole is in an external magnetic field, then the moment of the force acting on it is.

$\boldsymbol{M} = \boldsymbol{m} \times \boldsymbol{B}$, where $\boldsymbol{B}$ is the magnetic induction.

The moment of force determines the dynamics of the dipole movement. The basic equation of rotational motion (impulse theorem II) has the form

$$\frac{d\boldsymbol{L}}{dt} = \boldsymbol{M}.$$

Combining both equations, we get $\boldsymbol{dL} = \boldsymbol{m} \times \boldsymbol{B}\ \boldsymbol{dt} = \gamma\ \boldsymbol{L} \times \mathbf{B}\ \mathrm{d}t$.

The d$\boldsymbol{L}$ vector is perpendicular to the vector $\boldsymbol{L}$, and therefore its magnitude does not change but the direction only.

As shown in **Figure 6**, the end of the $\boldsymbol{L}$ vector moves along a circle with a radius equal to $L \sin\alpha$. The angle dφ over the time dt determines the magnitude of change of $\mathrm{d}L = L \sin\alpha\ \mathrm{d}\varphi$. The magnitude of the d$L$ change according to the equation of motion is $\mathrm{d}L = mB \sin\alpha\ \mathrm{d}t$. By comparing these two expressions, we get the angular velocity of the endpoint of the $\boldsymbol{L}$ vector

$$\omega_L = \frac{d\phi}{dt} = \gamma B.$$

Core	s (spin)	γ [$\times 10^8\ \mathrm{s^{-1} \cdot T^{-1}}$]
^{1}H (proton)	1/2	2.68
^{13}C	1/2	0.67
^{19}F	1/2	2.52
^{31}P	1/2	1.08
Free electron	1/2	−1758

Table 1.
Properties of selected nuclei of atoms and electron.

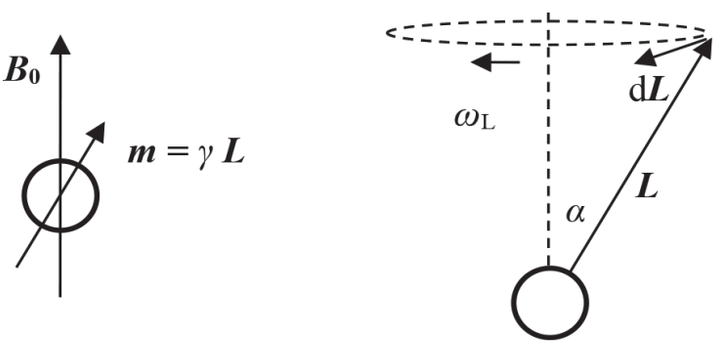


Figure 6.
Precession movement of a magnetic dipole in a magnetic field.

Thus, the dipole axis performs a circular (funnel) motion in the magnetic field, called the Larmor's precession. The frequency of $f_L = \omega_L/2\pi$ of this motion depends on the type of particle represented by its gyromagnetic ratio γ and the induction B of the magnetic field but does not depend on the angle α. As we show later, this phenomenon is used in magnetic resonance imaging and magnetic resonance spectroscopy. The nature of the phenomenon is like that of a conical pendulum.

Example 5. Ion oscillations in the crystal.

Crystals represent a simple or more complex regular arrangement of atoms of solids. For example, aluminium consists of an arranged lattice of positive ions. There are Al^+ ions and electron gas. The ions are subjected to electric forces by the surrounding particles. The equilibrium ion position is given by the zero resultant force or by the minimum value of the potential energy. If the ion deviates from the equilibrium position, it begins to oscillate around it.

As a simple model, consider three monovalent ions, of which two are fixed, and the third can move between the other two. In equilibrium, the distance of the central ion from the extreme ones is a (see **Figure 7**).

Let us move the central ion displacing it from the equilibrium. Then, the force acting on the ion is

$$F = F_1 - F_2 = k\ \frac{e^2}{a+x} - \ k\ \frac{e^2}{a-x} = -k\ \frac{2ae^2}{a^2-x^2}\ x,$$

where $k \approx 9.0 \times 10^9$ m·F^{-1} is Coulomb's law constant and $e \approx 1.6 \times 10^{-19}$ C is the elementary charge.

If the displacement is $x << a$, then we can express the resultant force by the linear approximation as

$$F \approx -\frac{2ke^2}{a}\ x = -\ K\ x.$$

As shown, if a particle with a mass m exerted by a reversing force proportional to the displacement x, the particle oscillates around an equilibrium position with a frequency

$$f = \frac{1}{2\pi}\ \sqrt{\frac{K}{m}} = \frac{1}{2\pi}\ \sqrt{\frac{2ke^2}{ma}}.$$

The oscillating of a charged particle is the source of the electromagnetic wave at this frequency and the wavelength of this wave is

$$\lambda = \frac{c}{f} = 2\pi\ c\ \sqrt{\frac{ma}{2k\ e^2}}.$$

For example, if we use the typical values for aluminium: $m \approx 4.5 \times 10^{-26}$ kg, $a \approx 2.8 \times 10^{-10}$ m, while $c \approx 3.0 \times 10^8$ m·s^{-1}, we get $\lambda \approx 31$ μm.

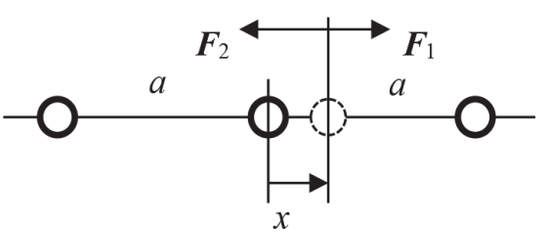


Figure 7.
Three monovalent ions.

The result corresponds to the wavelength of infrared (thermal) radiation. Oscillations of crystal lattice ions are the cause of the thermal radiation of the bodies.

There are many similar examples of oscillating systems, all of which have a similar physical nature. It is always a periodic exchange of energy between the various conservative forms of energy caused by the internal dynamics of the system.

1.1.2 Undamped self-oscillations of non-linear system

We find the system as non-linear if we cannot neglect its non-linearity. This means that we consider other higher terms in the expression of force by the power series [Eq. (5)]. Since the terms of the series generally gradually decrease with an increasing exponent of power, we can now consider the first higher non-zero member only. If the $E_p(x)$ function is odd (it means asymmetric potential well), we consider the term with the l coefficient, that is., quadratic term in the force expression. If the potential well is symmetrical, it is $l = 0$, the first non-linear term of the series is a cubic one. Accordingly, we are solving single cases by using this simplification.

In the following section, we analyse the case of oscillations in an *asymmetric potential well*, for which we express the force acting on a particle in the form

$$F(x) \approx -k\ x - \frac{1}{2!}\ l\ x^2. \tag{10}$$

Then the equation of motion is

$$ma = -kx - \frac{1}{2}\ l\ x^2. \tag{11}$$

We can rewrite the equation to the form

$$\ddot{x} + \omega_0^2 (1 + \lambda x) x = 0, \tag{12}$$

where $\omega_0^2 = \frac{k}{m}$.

The coefficient $\lambda = \frac{l}{2k}$ is the degree of asymmetry of the potential well.

Figure 8 shows an example of the asymmetric potential well and it illustrates the fitting of the well by a quadratic function (dashed line). This function fits the well only in the near vicinity of the equilibrium position. Cubic function correction is

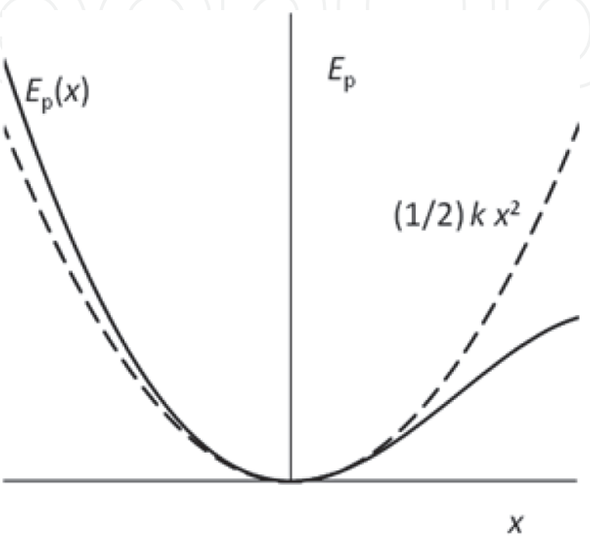


Figure 8.
Asymmetric potential well with asymmetry of the type $\lambda < 0$ *or* $\lambda < 0$.

positive on the left side and negative on the right side, which means that the asymmetry coefficient is $l < 0$.

Equation (12) represents a non-linear differential equation. When solving it, we use the physical nature of the phenomenon, which means the particle motion is periodic with an unknown angular frequency ω. We know, the periodic function can be expressed in the form of a Fourier series. If we choose for the start time $t = 0$ the moment when the particle displacement crosses the extreme value, then we can describe the course of the time dependence as an even function (symmetrical around the beginning $t = 0$).

For an even function, the Fourier series contains only even (cosine) terms.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t. \tag{13}$$

The a_0 value represents the mean value of the particle displacement, the a_n are amplitudes of the individual harmonics with frequencies of $n\omega$.

The solution procedure is such that we substitute the function (13) into the differential equation and arrange the terms according to the angular frequency. If the expression on the left side is to be equal to the right side of the equation (i.e., zero), all terms must be zero at corresponding frequencies—harmonics with angular frequencies ω, 2ω, etc. So, we get a set of equations for unknown parameters ω, a_0, and a_n for $n = 1, 2, \ldots$

Assuming the weak non-linearity of the system, which is given by $\lambda x_m \ll 1$, where x_m is the maximal displacement from the equilibrium position, we get

$$\omega = \omega_0 \sqrt{1 - \frac{5}{6}(\lambda a_1)^2}, a_0 \approx -\frac{1}{2}(\lambda a_1)a_1, a_2 \approx \frac{1}{6}(\lambda a_1)a_1 \text{ etc.}, \tag{14}$$

where a_1 is the amplitude of the first harmonic with the frequency of ω. Thus, non-linearity influences the frequency of the self-oscillations. It causes the shift of the mean value of the position a_0, and it causes the higher harmonics involved in oscillations.

Example of the derivation:

After substituting into the differential equation, we get the equation

$$-\sum_{n=1}^{\infty} a_n n^2\omega^2 \cos n\omega t + \omega_0^2 a_0 + \omega_0^2 \sum_{n=1}^{\infty} a_n \cos n\omega t$$
$$+\omega_0^2\lambda\left(a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t\right)\left(a_0 + \sum_{k=1}^{\infty} a_k \cos k\omega t\right) = 0$$

and then

$$\omega_0^2 a_0 + \omega_0^2\lambda a_0^2 + \sum_{n=1}^{\infty}(\omega_0^2 - n^2\omega^2)a_n \cos n\omega t + 2\omega_0^2\lambda a_0 \sum_{n=1}^{\infty} a_n \cos n\omega t +$$
$$+\omega_0^2\lambda \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} a_k a_n \cos k\omega t \cos n\omega t = 0$$

and then

$$\omega_0^2 a_0(1 + \lambda a_0) + \sum_{n=1}^{\infty}(\omega_0^2 - n^2\omega^2 + 2\omega_0^2\lambda a_0)a_n \cos n\omega t +$$
$$+\omega_0^2\lambda \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{2}a_k a_n[\cos(k+n)\omega t + \cos(k-n)\omega t] = 0$$

Equality must be met separately for each harmonic component and for the constant component.

For constant terms of the equation, we have

$$\omega_0^2 a_0(1+\lambda\ a_0)+\frac{1}{2}\omega_0^2\lambda\ \sum_{n=1}^{\infty}a_n^2\ \ =0.$$

For terms with a fundamental angular frequency ω, we get the equation

$$\left(\omega_0^2-\omega^2+2\omega_0^2\lambda a_0\right)\ a_1+\omega_0^2\lambda\ \sum_{n=1}^{\infty}a_n a_{n+1}=0.$$

Then, the terms with the frequency of 2ω (second harmonic)

$$\left(\omega_0^2-4\omega^2+2\omega_0^2\lambda a_0\right)a_2+\frac{1}{2}\omega_0^2\lambda a_1^2+\omega_0^2\lambda\sum_{n=1}^{\infty}a_n a_{n+2}=0,\text{ etc.}$$

From the second equation, we get after neglecting higher terms

$$\left(\omega_0^2-\omega^2+2\omega_0^2\lambda a_0+\omega_0^2\lambda\ a_2\right)\ a_1=0,$$

from where

$$\omega^2=\omega_0^2\ \ [1+\lambda(2a_0+a_2)]\approx\omega_0^2.$$

It yields from the first equation

$$a_0(1+\lambda a_0)+\frac{1}{2}\lambda a_1^2=0,\text{ and approximately } a_0\approx\ -\frac{1}{2}\lambda a_1^2.$$

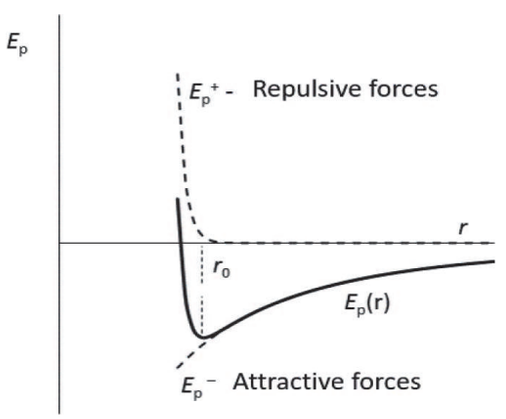
From the third equation, we obtain

$$\left(\omega_0^2-4\omega^2\right)a_2+\frac{1}{2}\omega_0^2\lambda a_1^2=0\text{ and }a_2\approx\frac{1}{6}\lambda a_1^2.$$

By substituting a_0 and a_2 to relation for ω^2, we get a more precise result in the form

$$\omega^2=\omega_0^2[1+\lambda(2a_0+a_2)]=\omega_0^2\left[1-\frac{5}{6}(\lambda a_1)^2\right].$$

Example 6. Thermal expansion of substances.



Atoms or molecules of solids or liquids are arranged in ordered structures. Attractive electric forces ensure the consistency of the substance. Approaching or moving the molecules or atoms together causes repulsive forces, which, along with attractive forces, provide equilibrium distances. The potential energy of the particle relative to the adjacent particle is shown in the figure. We can see that the potential well is asymmetrical. The minimum potential energy corresponds to the equilibrium distance of the particles of the substance. If we supply the particles with energy (e.g., in the form of heat), the amplitude of the oscillations of the particles increases.

Moreover, due to the non-linearity of the binding potential, the mean interatomic distance also increases. It means the macroscopic elongation of the material. According to the Eq. (14), the displacement of the mean distance of a_0 is proportional to the square a_1^2 of the amplitude of the fundamental harmonic. This amplitude square is proportional to the energy of the oscillations and the temperature. Hence, the thermal expansion of the substances is

$$\frac{\Delta l}{l_0} = \alpha(T - T_0), \text{resp.} l = l_0[1 + \alpha(T - T_0)],$$

where α is the coefficient of the length thermal expansion.

1.2 Oscillations in the linear system with viscous damping

In real systems, oscillation damping occurs because of irreversible energy loss of the system during the oscillation process. The loss mechanism describes the force that depends on the movement state of the system. In mechanical systems, it is mainly friction or resistance of the environment. In electrical circuits, there are Joule losses when current is passing through a resistor or emitting EM waves to the surrounding space. **Figure 9** shows an example of a damped oscillation model.

Let us consider the loss mechanism that often occurs in oscillating systems, which is a viscous resistance. A resistive force proportional to the velocity of movement characterises it, or in other words, the viscous resistance depends on power

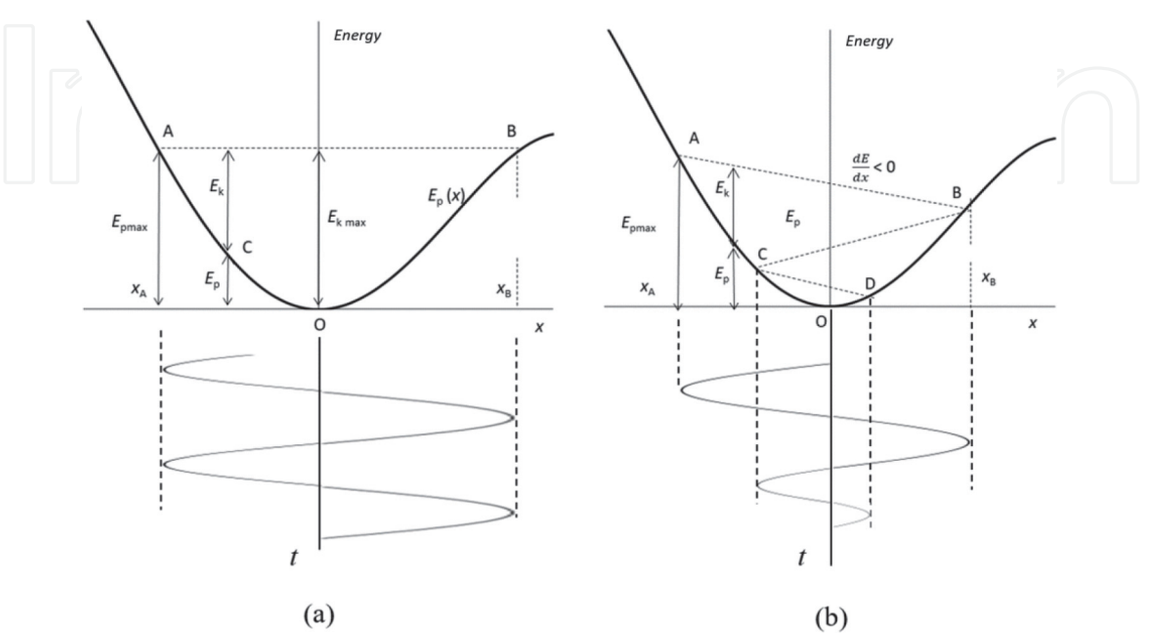


Figure 9.
Comparison of damped (a) and undamped oscillations (b).

dissipation proportional to the square of the velocity (in electric circuits, it is the square of the current). Thus,

$$F_o = -rv \tag{15}$$

where r is the coefficient of resistance. In the case of mechanical resistance, the viscous resistance depends on the dimension and shape of the body. It depends on the surrounding medium viscosity in which the body moves.

Power of the resistive force (power dissipation)

$$P_s = F_o v = -rv^2 \tag{16}$$

is a quadratic function of the velocity. In the case of the electrical circuit, the power dissipation is expressed as $P = Ri^2$. If the electrical current is analogous to the speed of motion, see Example 1, then this equation is analogous to Eq. (16) for viscous losses.

Motion equation $ma = F$ for the linear system with viscous damping has a form

$$ma = -kx - rv, \tag{17}$$

which can be rearranged to

$$\ddot{x} + 2b\dot{x} + \omega_0^2 x = 0, \tag{18}$$

where $v = \dot{x}$ and $b = r/(2\,m)$ is damping coefficient.

This equation is a linear differential equation with constant coefficients, and we find the solution in a form of the exponential function $e^{\lambda t}$. We obtain the values of the λ from the characteristic equation

$$\lambda^2 + 2b\lambda + \omega_0^2 = 0, \tag{19}$$

which solution is

$$\lambda_{1,2} = -b \pm \sqrt{b^2 - \omega_0^2}. \tag{20}$$

The type of motion of this oscillatory system depends on the ratio of the b, and ω_0 values, which defines the quality factor

$$Q = \frac{\omega_0}{2b}. \tag{21}$$

1.2.1 Underdamped oscillation system

Underdamping occurs in systems with a quality factor of $Q > 1/2$. Characteristic equation solution corresponds to a complex number $\lambda = -b \pm j\omega$, where $\omega = \sqrt{\omega_0^2 - b^2}$. Then, the solution can be expressed as

$$x(t) = A\mathrm{e}^{-bt}\cos\,(\omega t + \alpha), \tag{22}$$

where A and α are integration constants and they depend on the initial conditions of the movement, which are the initial particle displacement of x_0 and initial velocity of v_0 in time $t = 0$

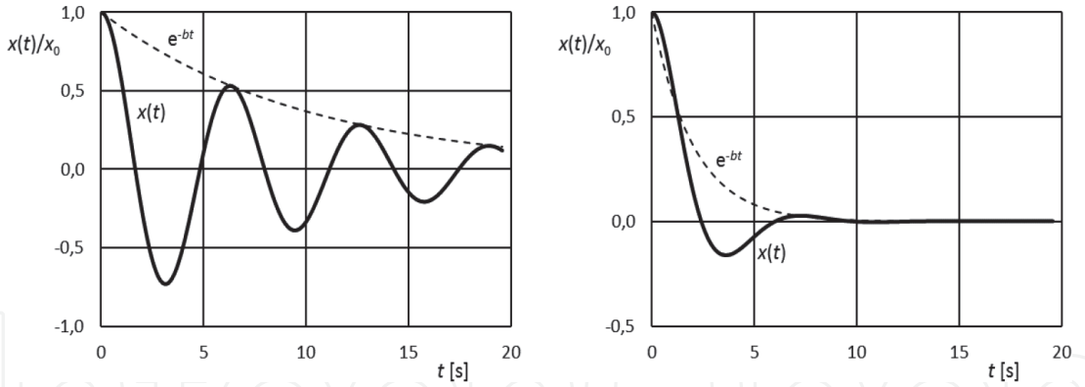


Figure 10.
Time course of subcritical damped oscillations for two values of the attenuation on the left is ω_o = 1.0 rad s^{-1}, b = 0.1 s^{-1}, v_o = 0 m s^{-1}, on the right ω_o = 1.0 rad s^{-1}, b = 0.5 s^{-1}, v_o = 0 m s^{-1}.

$$x_0 = A\cos\alpha \text{ and } v_0 = -bA\cos\alpha - \omega A\sin\alpha, \tag{23}$$

from where $\tan\alpha = -\frac{1}{\omega}\left(\frac{v_0}{x_0} + b\right)$ and $A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2\left(1 + \frac{bx_0}{v_0}\right)^2}$.

See **Figure 10** as example, where are underdamped oscillations for different values of the attenuation coefficients b = 0.1 s^{-1} and b = 0.5 s^{-1}, or for quality factors Q = 5 and Q = 1, respectively, at ω_0 = 1 rad•s^{-1}, and initial conditions $x_0 > 0$ and v_0 = 0 m•s^{-1}.

In the case of the underdamped system, the particle displacement overshoots the zero value (see the negative values in the graphs).

The value of

$$\tau = \frac{1}{b} = \frac{2Q}{\omega_0} \tag{24}$$

is *damping time constant* and it indicates the time when the e^{-bt} function decreases to the value of $1/e \approx 0.37$(=37%). This constant provides information about the time when the oscillations disappear. Usually, we consider the disappearance time of 3τ, when the maximal particle displacement reaches $e^{-3} \approx 5\%$ of its initial value, or the time of 5τ, at which the displacement drops to $e^{-5} < 1\%$.

The ratio

$$\frac{\tau}{T} = \frac{\omega}{2\pi b} = \frac{1}{\pi}\ Q\ \sqrt{1 - \frac{1}{4Q^2}} \tag{25}$$

represents the oscillation count during the time of τ. We can see that there are no oscillations in the system if $Q \leq 1/2$.

1.2.2 Critical damping

Critical damping occurs if $b = \omega_0$, and Eq. (19) has only one double solution. In this case, the solution of the equation is

$$x(t) = (A_1 + A_2 t)\ e^{-bt}, \tag{26}$$

where initial conditions are $x = x_0$ and $v = v_0$ at t = 0 s to determine A_1 and A_2.

Figure 11 shows typical time courses for different initial conditions.

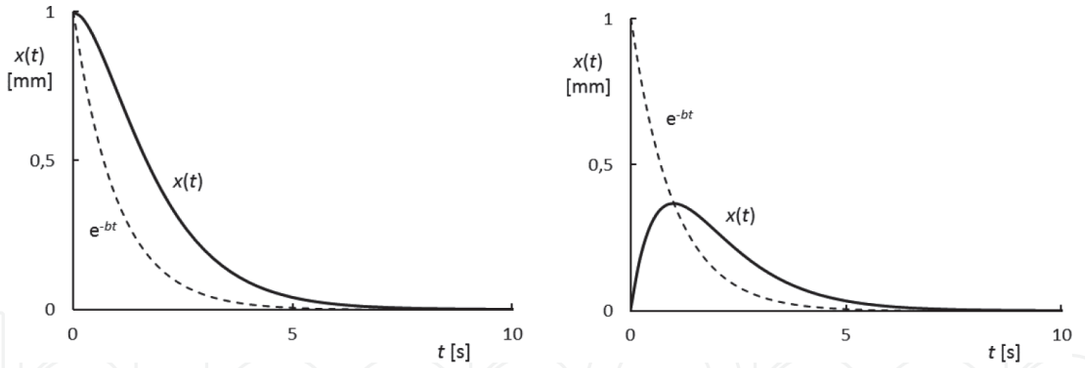


Figure 11.
Time course of critical damped oscillations for $\omega_o = b = 1\ s^{-1}$ (on the left $x_o = 1$ mm, $v_o = 0$, on the right $x_o = 0$, $v_o = 1\ mm\ s^{-1}$).

The importance of critical damping is that the system returns from the non-equilibrium state to the equilibrium fast and without overshooting. Various systems utilise critical damping, for example, shock absorbers for vehicles such as cars, motorcycles, etc. Critical damping is also used in the impulse electrical circuits to minimise distortion of the rising and falling edges of the impulse signal.

1.2.3 Overdamped oscillation system

Overdamping is given by $b > \omega_0$. If we denote $a = \sqrt{b^2 - \omega_0^2}$, then the solution of the Eq. (18) has a form as

$$x(t) = \mathrm{e}^{-bt}(A_1\mathrm{e}^{at} + A_2\mathrm{e}^{-at}), \tag{27}$$

where A_1 and A_2 result from the initial conditions. The particle displacement over the time consists of two exponential functions while one function has a short relaxation time $\tau_1 = 1/(b + a)$ and the second function has a time of $\tau_2 = 1/(b - a)$. **Figure 12** shows examples of critical damped systems for different initial conditions.

Dashed lines in the graphs indicate both exponential components with different time constants. We can see that this is an aperiodic event with no overshoot through the equilibrium position.

Viscous damping occurs especially in the case of small oscillations of a mass in the liquid, when there is laminar flow, or in the case of capillary damping devices. Linear damping is also typical for oscillations of atoms due to heat exchange, or for damping of oscillations in electrical circuits. In the case of the mass movement in a

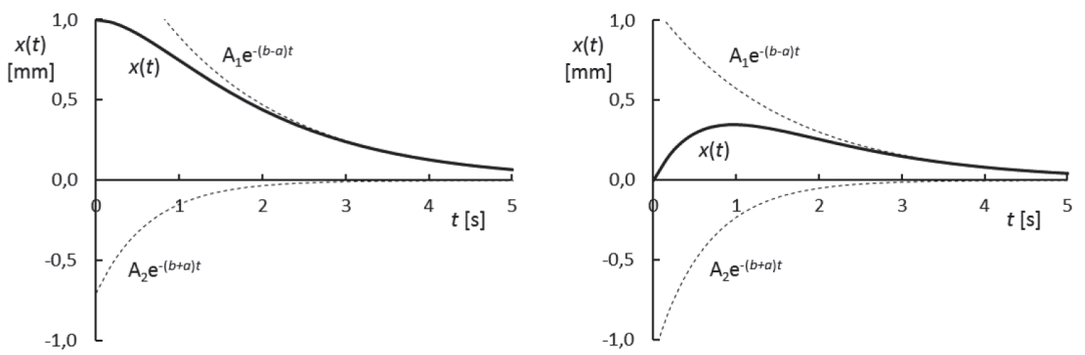


Figure 12.
Overdamped oscillation system for $\omega_o = 1.00\ rad\ s^{-1}$, $b = 1.10\ s^{-1}$ (on the left $x_o = 1$ mm, $v_o = 0\ mm\ s^{-1}$, on the right $x_o = 0$ mm, $v_o = 1\ mm\ s^{-1}$).

gaseous medium, for example, the pendulum in the air, the aerodynamic drag force $F \sim v^2$ usually applies, which is characterised by a quadratic dependence on speed. This means that it is no longer a linear system, and the solution leads to a non-linear differential equation even at small oscillations.

Example 7. Pendulum in a liquid.

Consider a pendulum (Example 2), whereby the suspended ball moves in water in a dish. For low velocities, the viscous resistance force for the ball-shaped body is given by the Stokes relation

$$F = -6\pi\eta r v,$$

where η is the dynamic viscosity of the liquid, r is the ball radius and v is the velocity of the motion.

The attenuation coefficient follows from (18):

$$b = \frac{3\pi\eta r}{m} = \frac{9\eta}{4\rho r^2},$$

where ρ is the ball density.

For example, the water has η = 1.0 × 10^{-3} Pa·s (at 20°C), the density of steel is 7.8 × 10^3 kg·m^{-3} and the ball radius r = 5.0 mm. Then, we get $b \approx 1.2 \times 10^{-2}$ s^{-1}.

If the ball hangs on the thread of the length l = 1.0 m, then $\omega_0 \approx 3.1$ s^{-1}.

The Q-factor is $Q \approx 130$. It is, therefore, subcritical damping and according to (25)

$$\frac{\tau}{T} = \frac{\omega}{2\pi b} = \frac{1}{\pi}\ Q\ \sqrt{1 - \frac{1}{4Q^2}} \approx 41.$$

It means that the oscillations are damped to the ratio of $1/e \approx 37\%$ after 41 periods.

Example 8. Oscillation damping in electrical *RLC* circuit.

Consider a single loop of series-connected elements of an inductor L, a capacitor C, and a resistor R. Assume that initially, the capacitor was charged to a U_0 voltage, and the current in the circuit was zero (*RL* connection to the charged C capacitor).

The energy of the conservative energy components is then

$$E = \frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}Li^2.$$

The time change of this energy is equal to the power of Joule's losses

$$\frac{dE}{dt} = \frac{Q}{C}\dot{Q} + Li\dot{i} = -Ri^2,$$

where $\dot{Q} = i$. If we divide the equation by the current i, and knowing the $\dot{i} = \ddot{Q}$, we get

$$\frac{Q}{C} + L\ \ddot{Q} = -\ R\ \dot{Q}, \text{resp.} \ddot{Q} + 2\ \frac{R}{2L}\ \dot{Q} + \frac{1}{LC}\ Q = 0,$$

where $\omega_0 = \frac{1}{\sqrt{LC}}$ and $b = \frac{R}{2L}$.

For example, L = 50 mH, C = 20 μF, and R = 10 Ω, we obtain $\omega_0 \approx 1.0 \times 10^3$ s^{-1} and $b \approx 1.0 \times 10^2$ s^{-1}.

Thus, there are subcritically damped oscillations.

The charge time response is then

$$Q(t) = Q_0 \, \mathrm{e}^{-bt} \quad \cos \quad \omega t,$$

where $\omega = \sqrt{\omega_0^2 - b^2} \approx 0.99 \times 10^3 \, \mathrm{s}^{-1}$.

$$\frac{\tau}{T} = \frac{\omega}{2\pi b} = \frac{1}{\pi} \; Q \; \sqrt{1 - \frac{1}{4Q^2}} \approx 1.6.$$

We can see that the angular frequency ω differs only slightly from the angular frequency ω_0 of the non-attenuated oscillations. However, the motion is significantly attenuated. The relative decrease to $1/e \approx 37\%$ of the initial value occurs after 1.6 periods of oscillations.

1.3 Oscillation of damped system with harmonic excitation

If an external periodic excitation force acts on the oscillation system, the system responds, after the transient process has disappeared, with a periodic answer. If the excitation is harmonic and the system is linear, then the steady answer is also harmonic with the same frequency. Any periodic stimulus of the linear system represents a superposition of harmonic components in terms of the Fourier series. Therefore, we will pay special attention to the response of the linear oscillation system to the external harmonic excitation.

1.3.1 Spectral characteristics of linear system with harmonic excitation

The harmonic force acting on linear oscillation system with viscous damping is given by the equation of motion

$$F_\mathrm{v} = F_\mathrm{m} \sin \Omega t, \tag{28}$$

where F_m is force amplitude and Ω is its angular frequency.
Then the equation of motion has a form

$$m a = - \; kx - rv + F_\mathrm{v}, \tag{29}$$

which we can rewrite to

$$\ddot{x} + 2b\dot{x} + \omega_0^2 x = f_\mathrm{m} \sin \Omega \, t, \tag{30}$$

where $f_m = F_m/m$ is external force amplitude related to the mass of the system.
The solution of the homogeneous equation corresponds to some of the results of the section 1.2 depending on the type of the system damping. A particular solution respects the right side. The homogeneous solution is a transient that fades out over time. The particular solution represents a process that lasts as long as the exciting force acts. There are steady harmonic oscillations in the system. In the case of harmonic excitation, the particular solution has a form

$$x_\mathrm{p}(t) = x_\mathrm{m} \sin (\Omega \;\; t + \beta), \tag{31}$$

where

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$$x_m = \frac{f_m}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + (2b\Omega)^2}}, \text{and } \beta = -\text{arctg}\left(\frac{2b\Omega}{\omega_0^2 - \Omega^2}\right). \tag{32}$$

The x_m is the amplitude of oscillations and β is the phase shift of the response compared to the phase of the excitation force (28). These results can be convinced by directly substituting the solution (31) into the Eq. (30). The linear oscillation system must respond to a harmonic response with the same angular frequency. As can be seen from the previous relationships, the amplitude and phase shift of the response depends on the Ω angular frequency of the excitation.

A special case is the excitation response with an angular frequency which is equal to the angular frequency $\Omega_r = \omega_0$ of the undamped system. This case is called *resonance*. For the resonance state, we get values from relations (32)

$$x_{\text{mr}} = \frac{f_{\text{m}}}{2b\omega_0} = x_0 Q, \text{ and } \beta_r = -\frac{\pi}{2}\text{rad}, \tag{33}$$

where $x_0 = F_m/k$ is the displacement from the equilibrium while the constant force F_m acts on the system (zero angular frequency $\Omega = 0$). In the case of a system with a high Q-factor of $Q \gg 1$, the amplitude of the response in the resonance state is significantly greater than the displacement of x_0 caused by the constant force. The response is phase-delayed by $\pi/2$ rad compared to excitation. **Figure 13** shows the frequency response characteristics for different Q-factor values.

We can see from these characteristics that if the resonant amplification of the system oscillations is undesirable, it is necessary to choose critical or overcritical damping. In this case, however, considerable energy losses occur in the system because of the resistance force. On the other hand, there are systems with low internal losses and characterised by a very high Q-factor (in hundreds to thousands). In the case of high values of the quality factor ($Q \gg 1$), the frequency bandwidth of the resonant maximum can be determined at a level of 3db decrease relative to the maximum value (i.e., decrease to approximately 0.707 x_{mr})

$$\Delta\Omega = \frac{\omega_0}{Q}. \tag{34}$$

The resonant maximum increases proportionally with the Q-factor and narrows inversely with it. Therefore, the systems with a very high Q-factor have high selectivity, and we can use them, for example, for the spectral analysis of an unknown signal or for controlling pendulum clocks, resonant crystal clocks, atomic

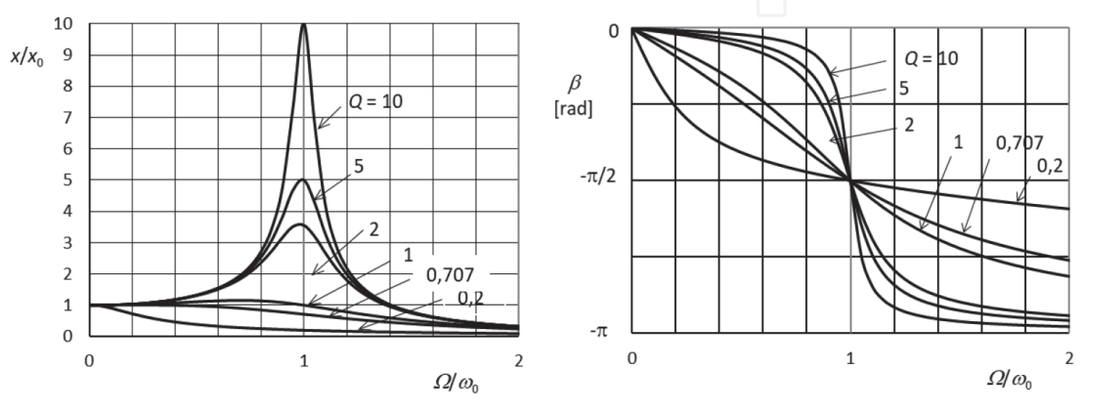

Figure 13.
Amplitude and phase shift response of the oscillating system versus a relative angular frequency for different values of the quality factor.

clocks, etc. An undesirable consequence of resonance in mechanical devices can be the occurrence of vibrations, for example, when the engine rpm corresponds to the resonant frequency of the mechanical system. These phenomena are not limited to mechanical systems only. Similarly, resonance phenomena occur in electrical circuits or electromagnetic systems. An example, to be mentioned later, is magnetic resonance used in medical diagnostics. In a very simplified view, the human auditory organ is a complicated resonant system too that allows different sound frequencies (pitch of tones) to be distinguished.

1.3.2 Non-linear oscillating system with harmonic excitation

The situation is more complex in the case of a non-linear oscillating system exposed to external harmonic excitation. As an example, consider the non-linear system with the asymmetric potential well described in Section 1.1.2, with harmonic excitation and viscous damping. The equation of motion expressed in a standard form, see Eqs. (12) and (30), has a form

$$\ddot{x} + 2b\dot{x} + \omega_0^2 (1 + \lambda x)x = f_{\mathrm{m}} \sin \Omega \ t. \tag{35}$$

We are interested again in the steady-state response of the system described by the particular solution of the differential equation. The response of a non-linear system to harmonic excitation is no longer harmonic but remains periodic with the same angular frequency. The periodic response function is expressed as a superposition of harmonic components using the Fourier series

$$x(t) = x_0 + \sum_{n=1}^{\infty} x_{\mathrm{m}n} \sin (n\Omega t + \beta_n). \tag{36}$$

After substituting this assumed solution into the differential equation, we obtain the values of the individual quantities. In the case of weak non-linearity ($\lambda x_{\mathrm{m}1} << 1$), the results have the form of

$$x_{\mathrm{m}1} = \frac{f_{\mathrm{m}}}{\sqrt{\left(\omega_0^2 - \Omega^2 + 2\lambda\omega_0^2 B_0\right)^2 + (2b\,\Omega)^2}} \approx \frac{f_{\mathrm{m}}}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 + (2b\,\Omega)^2}}, \tag{37}$$

$$x_0 = -\frac{1}{2}\lambda x_{\mathrm{m}1}^2 \tag{38}$$

$$x_{\mathrm{m}2} \approx -\frac{\lambda\omega_0^2}{2} \frac{x_{\mathrm{m}1}^2}{\sqrt{\left(\omega_0^2 - 4\Omega^2\right)^2 + (4b\,\Omega)^2}} \tag{39}$$

$$x_{\mathrm{m}3} \approx \frac{\lambda\omega_0^2}{2} \frac{B_{\mathrm{m}1}}{\sqrt{\left(\omega_0^2 - 9\Omega^2\right)^2 + (6b\Omega)^2}} x_{\mathrm{m}2}, \text{etc.} \tag{40}$$

The fundamental harmonic having amplitude $x_{\mathrm{m}1}$ dominates. Its properties are similar to those of the linear system. Also, the mid-position x_0 is shifted due to the system's non-linearity. Furthermore, the resonance occurs at subharmonic frequencies, an integer fraction of the fundamental harmonic frequency ($\Omega_n = \omega_0/n$). They are expressed by response amplitudes $x_{\mathrm{m}n}$. The subharmonics components have an origin caused by excitation having a specific subharmonic frequency Ω_n. But the response has the fundamental resonance frequency ω_0 since there is the response of

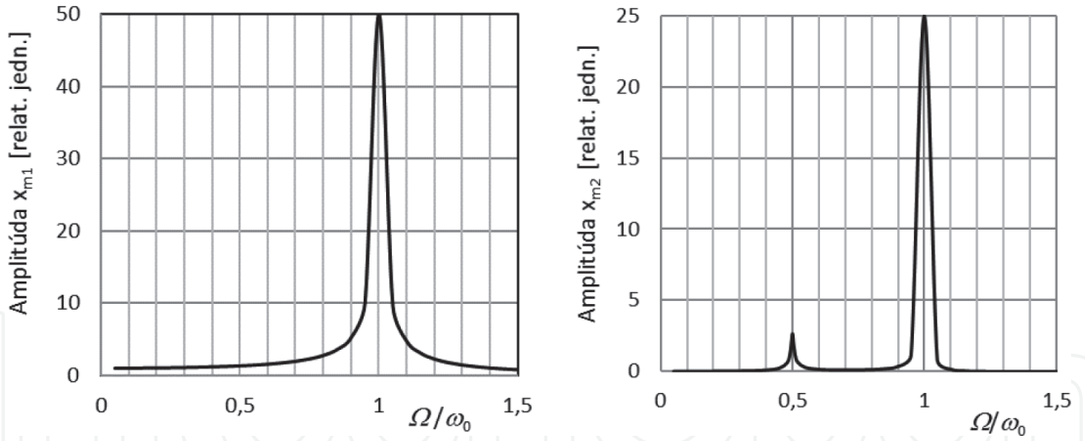

Figure 14.
Frequency amplitude characteristics of the first and second harmonic components for values of $\lambda = 0.06\ m^{-1}$, $Q = 50$.

specific harmonic defined as $n\Omega_n = \omega_0$. **Figure 14** shows the frequency amplitude characteristics of the first and second harmonics.

The subharmonic resonance is important to explain the perception of musical chords by the non-linear system of the auditory organ. For example, if we hear two tones with frequencies in the ratio 2:1 (octave), the tone with the angular frequency ω_{01} produces a signal with the second harmonic of $2\omega_{01}$. If this frequency is not equal to the frequency of ω_{02}, the auditory organ sensitively detects the difference between ω_{02} and $2\omega_{01}$ and evokes a feeling of non-tuned music interval.

1.3.3 Harmonic interaction in a non-linear oscillating system

In practice, we encounter cases, in which the oscillating system is simultaneously excited by several harmonic signals with different frequencies.

As a simple example, we will excite the system with two harmonic signals and determine its response to this excitation.

The equation of the response has the form

$$m\ddot{x} + r\dot{x} + k\ x + (l/2)\,x^2 = F_{\mathrm{m}1}\sin\Omega_1 t + F_{\mathrm{m}2}\sin\Omega_2 t, \tag{41}$$

where the Ω_1 and Ω_2 are angular frequencies of the harmonic components of the excitation.

We can rewrite the equation to

$$\ddot{x} + 2\,b\,\dot{x} + \omega_0\,(\omega_0 + \lambda x)x = f_{\mathrm{m}1}\sin\Omega_1 t + f_{\mathrm{m}2}\sin\Omega_2 t, \tag{42}$$

where $f_{\mathrm{m}1} = F_{\mathrm{m}1}/m$ and $f_{\mathrm{m}2} = F_{\mathrm{m}2}/m$.

Since the excitation signal is periodic, the response must also be periodic.

Considering the weak non-linearity when $\lambda \ll \omega_0 x_{\mathrm{m}}$, harmonic components with excitation angular frequencies dominate in response. Therefore, the steady response has the dominant components

$$x_1 = x_{\mathrm{m}1}\sin\ (\Omega_1 t + \beta_1) + x_{\mathrm{m}2}\sin\ (\Omega_2 t + \beta_2). \tag{43}$$

If we substitute this function into a quadratic term in the Eq. (42), there are elements with combinational frequencies $\Omega_1 \pm \Omega_2$ on the left side of the equation

$$x_1^2 = \frac{1}{2}\,x_{\mathrm{m}1}^2\,[1 - \cos\ (2\,\Omega_1 t + 2\beta_1)] + \frac{1}{2}\,x_{\mathrm{m}2}^2\,[1 - \cos\ (2\,\Omega_2 t + 2\beta_2)] +$$
$$+ x_{\mathrm{m}1}x_{\mathrm{m}2}\,[\cos\ ((\Omega_1 - \Omega_2)\,t + \beta_1 - \beta_2) - \cos\ ((\Omega_1 + \Omega_2)\,t + \beta_1 + \beta_2)]\ .$$

Due to non-linearity, components with frequencies $2\,\Omega_1$, $2\,\Omega_2$, $\Omega_1 + \Omega_2$, and $\Omega_1 - \Omega_2$ appear in the equation. Including these components together with the original components in the overall system response, the non-linearity (quadratic term) results in the second generation of components with twice the frequencies and with all combinations, for example, $3\Omega_1$, $3\Omega_2$, $2\,\Omega_1 \pm \Omega_2$, $\Omega_1 \pm 2\,\Omega_2$. The solution is very complex, and therefore, we will focus on the approximate determination of combination components of the first generation.

In the time response of the system, we consider only the most significant components

$$x(t) = x_0 + x_{\mathrm{m1,0}} \sin\left(\Omega_1 t + \beta_{1,0}\right) + x_{\mathrm{m0,1}} \sin\left(\Omega_2 t + \beta_{0,1}\right) +$$
$$+ x_{\mathrm{m2,0}} \sin\left(2\Omega_1 t + \beta_{2,0}\right) + x_{\mathrm{m0,2}} \sin\left(2\Omega_2 t + \beta_{0,2}\right) +$$
$$+ x_{\mathrm{m1,1}} \sin\left((\Omega_1 + \Omega_2)t + \beta_{1,1}\right) + x_{\mathrm{m1,-1}} \sin\left((\Omega_1 - \Omega_2)t + \beta_{1,-1}\right),$$

where numbered indices correspond to frequency combinations, for example, $Q_{\mathrm{m}k,l}$ relates to the frequency of $\Omega_{k,l} = k\Omega_1 + l\Omega_2$.

If we substitute these components into the equation of motion and separate the corresponding harmonic elements on the left and right sides, we get a response for the amplitudes of the harmonic components

$$x_{\mathrm{m1,0}} = f^*_{\mathrm{m}\,1}\ \frac{1}{\sqrt{\left(\omega_0^2 - \Omega_1^2 + 2\lambda\omega_0 Q_0\right)^2 + \left(2\,b\,\Omega_1\right)^2}}$$

$$x_{\mathrm{m}\ 0,1} = f^*_{\mathrm{m}\,2}\frac{1}{\sqrt{\left(\omega_0^2 - \Omega_2^2 + 2\lambda\omega_0 Q_0\right)^2 + \left(2\,b\,\Omega_2\right)^2}}\ ,$$

which corresponds to the frequency response of the linear system (resonant characteristics with a maximum at the resonant frequency of ω_0).

Considering the dominant components with angular frequencies Ω_1 and Ω_2, the constant component is

$$x_0 \approx -\frac{1}{2}\lambda\frac{x^2_{\mathrm{m}\,1,0} + x^2_{\mathrm{m0,1}}}{\omega_0 + \lambda Q_0} \approx -\left(\frac{\lambda}{2\,\omega_0}\right)\ \left(x^2_{\mathrm{m}\,1,0} + x^2_{\mathrm{m}\,0,1}\right).$$

Components with double frequencies of $2\,\Omega_1$ and $2\,\Omega_2$ shall be determined as in the case of simple harmonic excitation and with the same results.

For the lowest combination frequencies, we get a relationship

$$x_{\mathrm{m1},\pm1} \approx \lambda\omega_0 \frac{x_{\mathrm{m}\,1,0}\ \ x_{\mathrm{m}\,0,1}}{\sqrt{\left[\omega_0^2 - \left(\Omega_1 \pm \Omega_2\right)^2 + 2\lambda\omega_0 Q_0\right]^2 + \left[2\,b\,\left(\Omega_1 \pm \Omega_2\right)\right]^2}}.$$

From this relationship, we can see that the cause of combination frequencies is non-linearity, which occurs in the result as $\lambda\omega_0$. Similarly, we can determine the amplitudes of the response components with higher combinational frequencies. As with resonance at *subharmonic* frequencies, resonance occurs when combinational frequencies are

$$\left|\Omega_1 \pm \Omega_2\right| = \sqrt{\omega_0^2 + 2\lambda\omega_0\ Q_0} \approx \omega_0.$$

In this case, the response amplitude with an angular frequency of ω_0 is given by equation

$$x_{\mathrm{m}1,\pm1} \approx \left(\frac{\omega_0}{2b}\right)\left(\frac{\lambda}{\omega_0}\right) x_{\mathrm{m}1,0} x_{\mathrm{m}0,1}.$$

System resonances also occur at higher combinational frequencies. Non-linearity and the resulting response components with combinational frequencies increase at higher excitation. There are systems where the combinational frequencies are undesirable. For example, the acoustic loudspeakers are load overrated, which means that the effects of system non-linearity under operating loads are negligible.

As indicated in Section 1.3.2, resonances with combinational frequencies or resonances at subharmonic frequencies are important, for example, in explaining the perception of musical chords by the non-linear system of the human auditory organ. For example, if we hear two tones with frequencies in the ratio of 2 (octave 2:1), the tone with the angular frequency of ω_{01} produces a signal with the second harmonic of $2\omega_{01}$. If this frequency is not equal to the frequency of ω_{02}, the auditory organ sensitively detects the difference of $\omega_{02} - 2\omega_{01}$ and creates a sense of non-tuned music interval. It is like other music intervals such as small third 6:5, big third 5:4, fourth 4:3, fifth 3:2, small sixth 8:5, big sixth 5:3, small seventh 16:9, and big seventh 15:8. These music intervals have the ratio of frequencies equal to the integer ratio. Due to the non-linearity of the auditory organ, the music listener can distinguish a pure (harmonic) or impure (disharmonic) chord and thus perceive the beauty of musical compositions.

1.3.4 Power losses and the nature of spectroscopy

The oscillations relate to the exchange of energy between conservative elements of the system. The total energy of the system is equal to the sum of kinetic and potential energy or their equivalents. In the case of forced harmonic oscillations of a linear system with the fundamental frequency of ω_0 and excitation frequency of Ω, the total energy of the system at a steady state is

$$E = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 = \frac{1}{2} m v_{\mathrm{m}}^2 \cos^2(\Omega t + \beta) + \frac{1}{2} k x_{\mathrm{m}}^2 \sin^2(\Omega t + \beta) =$$
$$= \frac{1}{4} m\left(\Omega^2 + \omega_0^2\right) x_{\mathrm{m}}^2 + \frac{1}{4} m\left(\Omega^2 - \omega_0^2\right) \cos(2\Omega t + 2\beta).$$

The mean value of this energy is

$$\langle E \rangle = \frac{1}{4} m\left(\Omega^2 + \omega_0^2\right) x_{\mathrm{m}}^2 = \frac{1}{4}\left(1 + \frac{\Omega^2}{\omega_0^2}\right) k x_{\mathrm{m}}^2. \tag{44}$$

The second (alternating) component is directly proportional to the difference $\Omega^2 - {\omega_0}^2$ and corresponds to the periodic energy exchange between the source and the system with an angular frequency of $2\,\Omega$.

If we consider the viscous losses in the system, the energy losses in one period of T are as follows

$$W_T = \int_0^T P\,\mathrm{d}t = \int_0^T F v\,\mathrm{d}t = \int_0^T F_{\mathrm{m}} v_{\mathrm{m}} \sin \Omega t \cos(\Omega t + \beta)\,\mathrm{d}t =$$
$$= \frac{T}{2} F_{\mathrm{m}} v_{\mathrm{m}} \sin \beta = \pi F_{\mathrm{m}} x_{\mathrm{m}} \sin \beta.$$

In the state of resonance at frequency $\Omega = \omega_0$, $\beta = -\pi/2$ rad and the alternating component of energy E is zero.

Then, active power supplied to the system in the case of the steady state of forced oscillations is

$$\langle P \rangle = \frac{W_T}{T} = \frac{1}{2} F_m x_m \Omega \sin\beta = \frac{F_m^2}{2m} \frac{2b\Omega^2}{\left(\omega_0^2 - \Omega^2\right)^2 + (2b\Omega)^2} \tag{45}$$

Figure 15 shows the graph of the active power spectral function [see Eq. (45)] for two Q-values. These figures show that an oscillating system with a high Q-factor absorbs the energy of the source only in a narrow interval around the resonant frequency and changes it most often into heat.

There are various bonds of atoms and molecules in biological tissues. These bonds represent microscopic oscillating systems with characteristic resonant frequencies. If these tissues are irradiated with monochromatic electromagnetic waves with a frequency corresponding to a resonant frequency of coupling, then energy is supplied to these coupled systems. This energy can stimulate tissues at low-power applications, for example, phototherapy. At higher power, the absorbed energy increases the temperature of the tissue structures, and thus, it can lead to their destruction used, for example, in the treatment of cancer.

Example 9. Spectroscopy.

The above phenomenon explains the physical nature of *spectroscopy*. In systems with a higher Q-factor, the resonance state relates either to dynamically increased oscillations or to power absorption of the source.

The conservative forces bond the atoms of the matter and determine their equilibrium position. The oscillations around the equilibrium position are at the natural frequency and depend on the properties of the particle (mass) and the features of the bond (stiffness). Thus, differently bound particles have different oscillation frequencies. Each material has characteristic frequencies according to its composition.

When an electromagnetic wave interacts with a material, it acts on its atoms. If the EM wave frequency equals one of the resonant frequencies of the substance, then it significantly absorbs and attenuates this wave. For example, if we observe the white light of the Sun on the surface of the Earth using a spectrometer, we find in the continuous visible light spectrum black lines that correspond to the absorption of light with the appropriate frequencies of gas molecules in the atmosphere. In this way, we can measure the concentration of greenhouse gases in the atmosphere.

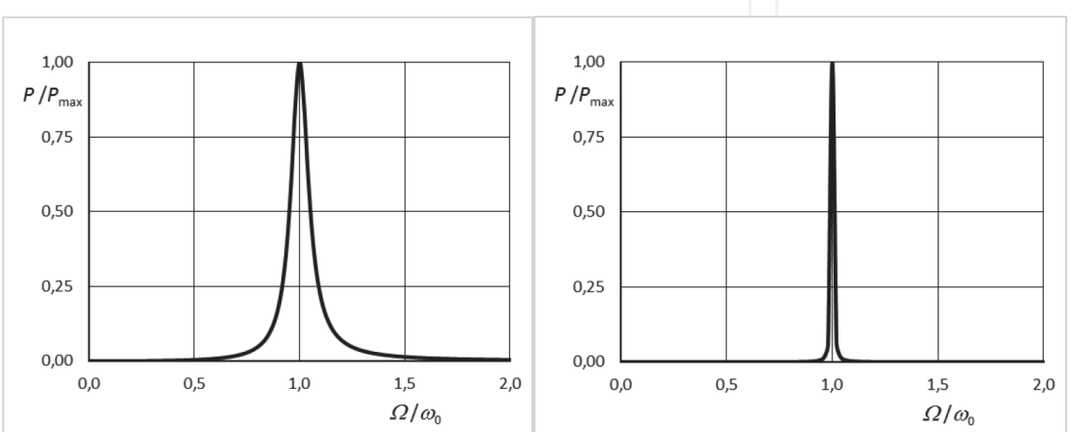

Figure 15.
Relative spectrum of the active power of oscillation system (on the right is Q = 10, *on the left* Q = 100*).*

Another example is optical spectroscopy used in biochemistry, pathology, or the investigation of blood plasma. As an example, let us pass the adjustable wavelength light through the liquid cuvette to search for wavelengths at which the liquid has a resonant absorption. Then, the found wavelengths or frequencies determine the presence of the individual substances of the material and their concentration in the solution.

Another example is magnetic resonance imaging, as discussed in the following paragraph.

1.3.5 Magnetic resonance

1.3.5.1 Nature of magnetic resonance

We talk about resonance if the frequency of external excitation on the oscillating system is the same as the frequency of its self-oscillations, and the mechanism of action can supply the oscillating system with energy. In the linear system, it is the frequency of its undamped oscillations. In the case of a magnetic dipole in a constant magnetic field, it is the *Larmor frequency* of f_L, see Example 4. Read also Vlaardingerbroek [2], Webb [3], or Hashemi [4].

If we create a rotating magnetic field in space with a rotation frequency f close to the frequency f_L, we can expect a resonance phenomenon. The external excitation magnetic field must be perpendicular to the precession axis (i.e., to the constant magnetic field $\boldsymbol{B}_0$) to interact with a magnetic dipole that performs a precession movement. We create a rotating magnetic field using two mutually perpendicular pairs of coils, which are fed by currents with the same frequency and with a mutual phase shift of $\pi/2$ rad.

Figure 16 illustrates the situation where perpendicular pairs of coils are on the left. If z is the direction of the constant magnetic field $\boldsymbol{B}_0$ and hence the axis of the dipole precession, the x and y directions are perpendicular to the z-axis. One pair of coils creates the magnetic field of B_x in the x-axis direction, the other pair the field of B_y in the y-axis direction. The coil currents and thus the magnetic field components are phase-shifted by $\pi/2$ rad, and thus

$$B_x = B_1 \sin(\omega t + \psi), \text{and } B_y = B_1 \cos(\omega t + \psi).$$

Adding both components, we get the resulting $\boldsymbol{B}_1$ vector, which has a constant value of B_1 and rotates in the x-y plane with the ω angular frequency of the coil current. The right part of the figure shows the direction of the dipole precession

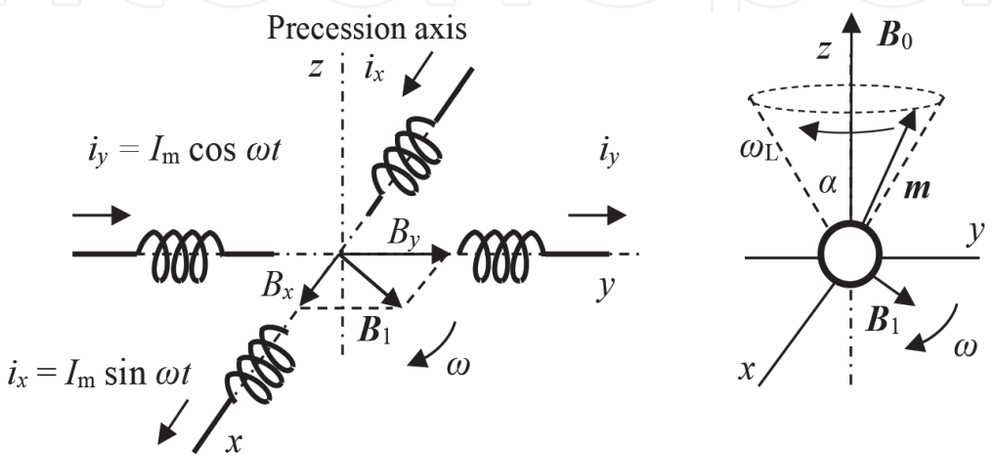

Figure 16.
Magnetic dipole in rotating magnetic field.

with the dipole moment of $\boldsymbol{m}$ and the direction of rotation of the rotating magnetic field.

Vector components of the $\boldsymbol{m}$ dipole moment are

$$m_z = m\cos\alpha, m_x = m\sin\alpha\sin(\omega_L t + \phi), \text{and } m_y = m\sin\alpha\cos(\omega_L t + \phi).$$

Magnetic dipole in the magnetic field $\boldsymbol{B}_0$ has potential energy

$$E_p = -\boldsymbol{m}\cdot\boldsymbol{B}_0 = -mB_0\cos\alpha. \tag{46}$$

If an external alternating magnetic field acts on the dipole, only the angle α can change at constant values of m and B_0. The following equation expresses the change of the potential energy dE_p of the dipole

$$dE_p = mB_0\sin\alpha\ d\alpha. \tag{47}$$

Power of external magnetic field torque is

$$\begin{aligned} P &= \boldsymbol{M}\cdot\boldsymbol{\omega}_L = (\boldsymbol{m}\times\boldsymbol{B}_1)\cdot\boldsymbol{\omega}_L = \\ &= mB_1\sin\alpha[\sin(\omega_L t+\phi)\cos(\omega t+\psi) - \cos(\omega_L t+\phi)\sin(\omega t+\psi)]\omega_L = \\ &= mB_1\omega_L\sin\alpha\ \sin[(\omega_L-\omega)t+\phi-\psi]. \end{aligned} \tag{48}$$

We can see that the power is time-varying for $\omega_L \neq \omega$ and the mean value of the power is zero. If $\omega_L = \omega$, the power has a time-invariant component, which reaches the maximum at $\varphi - \psi = \pi/2$ rad. We describe this phenomenon as *magnetic resonance*.

Then the work over the time dt is

$$\delta W = P\ dt = m\ B_1\ \omega_L\sin\alpha\ dt. \tag{49}$$

Comparing (47) and (49), we obtain for $dE_p = \delta W$

$$\frac{d\alpha}{dt} = \frac{B_1}{B_0}\omega_L = \gamma B_1. \tag{50}$$

As a result, the α angle of the 'precession funnel' varies uniformly in the magnetic resonance state with an angular velocity of $d\alpha/dt$, which depends on the amplitude of the induction of B_1 of the alternating magnetic field. These angle α changes are the periodic event, and therefore the magnetization of a substance changes periodically too. The magnetization inverts its value in the time

$$\tau_{180} = \frac{\pi}{\gamma B_1}, \tag{51}$$

or it is perpendicular to the initial direction in the time

$$\tau_{90} = \frac{\pi}{2\gamma B_1}, \text{and similar.} \tag{52}$$

It is typical for a forced oscillation of particles, and a forced precession of magnetic dipoles, that all particles oscillate synchronously with the same phase compared to the excitation signal.

1.3.5.2 FID signal origin

The paramagnetic material contains many magnetic dipoles randomly arranged due to particle thermal motion. Therefore, the resulting magnetic field of these dipoles is zero. If we insert the paramagnetic material into the $\boldsymbol{B}_0$ constant magnetic field, then the material magnetic dipoles partially arrange in the direction of the $\boldsymbol{B}_0$ vector. This behaviour better describes the magnetization vector ($\boldsymbol{M}_0 = \kappa\mu_0\boldsymbol{B}_0$), where the κ is the magnetic susceptibility of the substance. After switching on the $\boldsymbol{B}_1$ transverse alternating rotating magnetic field with an angular frequency $\omega = \omega_L$, a resonance occurs, which causes a coherent precession of the oriented magnetic dipoles. Let us apply the field $\boldsymbol{B}_1$ during the τ_{90} time. Then the $\boldsymbol{M}_0$ constant magnetization vector, parallel to the $\boldsymbol{B}_0$ vector, changes to the $\boldsymbol{M}$ vector, which has the same magnitude but rotates perpendicularly to the $\boldsymbol{B}_0$ with the angular frequency ω. The sample of a substance looks like a rotating magnet with a magnetic moment ($m^* = M_0V$), where V is the sample volume. If we place a detection coil perpendicularly to the axis of rotation, then the voltage induces in it is

$$u_{\mathrm{FID}} \sim \frac{\mathrm{d}m^*}{\mathrm{d}t} \sim \omega_{\mathrm{L}} M_0 \quad \sin \omega_{\mathrm{L}} t. \tag{53}$$

Voltage induces in the coil only in the case of synchronous dipole precession, which results in rotating magnetization, and this can only happen if magnetic resonance conditions are met. For a given $\boldsymbol{B}_0$ and ω, the resonance occurs only for certain dipoles in the substance, which satisfy the condition $\omega = \omega_L = \gamma B_0$. Thus, by measuring the induced signal, the presence of magnetic dipoles with a corresponding gyromagnetic moment γ can be detected, and their concentration determined.

If we switch off the $\boldsymbol{B}_1$ excitation field, the periodic event begins to damp due to the interaction of the dipoles with the surrounding particles of the substance. The detected signal is, therefore, attenuated (**Figure 17**). This damped signal calls the FID signal (free induction decay). In biomedicine, the magnetic resonance uses protons (with the $\gamma = 2.68 \times 10^8\ \mathrm{s}^{-1}\cdot\mathrm{T}^{-1}$), which exist mainly as nuclei of hydrogen atoms and thus in water molecules.

The organic compounds, such as biological tissues, contain hydrogen atoms too. In specific cases, the magnetic resonance uses nuclei of other biogenic elements such as isotopes of carbon ^{13}C, fluorine ^{19}F, phosphorus ^{31}P, and so on (see **Table 1**, p. 8).

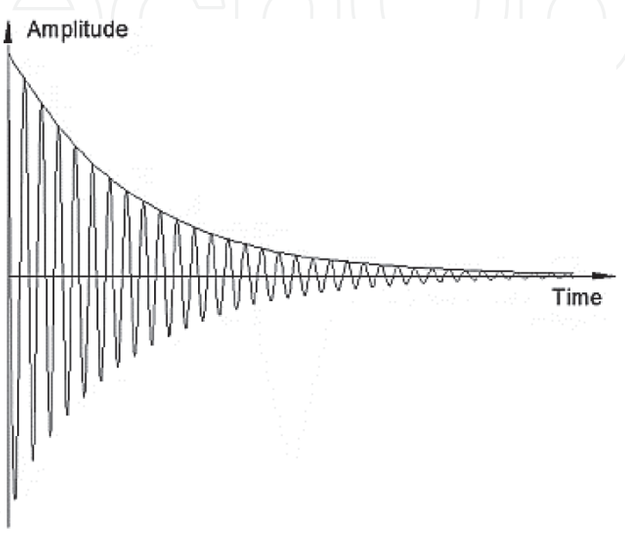

Figure 17.
FID signal after magnetic dipoles excitation.

1.3.5.3 Relaxation

Perpendicular magnetization is an imbalance caused by the external source of the alternating magnetic field $\boldsymbol{B}_1$. If the excitation force stops to act on the system, the aligned movement of the dipole array decays. From the viewpoint of the FID signal, the decay of the in-phase periodic precession movement at first occurs due to the inhomogeneity of the magnetic field $\boldsymbol{B}_0$, and due to the influence of surrounding dipoles, so-called spin-spin interaction. After switching-off the exciting magnetic field, the precision movement of the dipoles remains for a short time in a plane perpendicular to the $\boldsymbol{B}_0$ direction, but due to a small change in the local magnetic field, the precession of the single dipoles is out-phased, which results in an exponential decrease of the transverse magnetization, and thus an FID signal. This decrease characterises the time constant T_2. Its value is in the order of tenths of a second. The second slower mechanism of decay associates with the thermal relaxation of the imbalanced orientation of the magnetic dipoles and directs to the thermodynamic equilibrium of the dipoles, that is, to the equilibrium orientation of magnetization in the direction of the $\boldsymbol{B}_0$ vector. This process is approximately 10 times slower, and its time constant is denoted T_1.

Different substances, and thus tissues, have different values of relaxation times of T_1 and T_2. In medical applications, protons (nuclei of hydrogen) are mostly used as magnetic dipoles since the body contains many of the hydrogen atoms (especially as part of water molecules).

For illustration, see **Table 2**, which contains values of relaxation times T_1 and T_2 for water and some tissues, as well as the relative concentration of hydrogen atoms in tissues compared to the concentration in pure water.

Chemical analyses also use nuclei of other paramagnetic atoms as magnetic dipoles. Thus, we can investigate the content of specific atoms or substances in the samples by measuring the FID signal and the relaxation times.

1.3.5.4 Magnetic resonance imaging (MRI)

One of the applications of the magnetic resonance phenomenon is the *tomographic imaging* of the morphological structure of the organism. The method lies in the use of the detection of hydrogen atom nuclei, which are mainly contained in water and thus in soft tissues. Consequently, we can obtain a two-dimensional image of tissue structures by identifying different types of tissue (see **Figure 18**).

Using a relatively complicated device we call a *tomography*; it is possible to assign a specific T_1 and T_2 value or relative proton density PD to each point of the thin

Tissue	T_1 [ms]	T_2 [ms]	Relative concentration (^{1}H)
Water	4000	2000	1.00
Cerebrospinal fluid	2500	280	0.98
Edema	900	130	0.86
Grey matter	760	77	0.74
White matter	510	67	0.62
Muscle	900	50	0.50
Fat	250	60	1.00

Table 2.
Relaxation times and relative concentration of protons in water and selected tissues.

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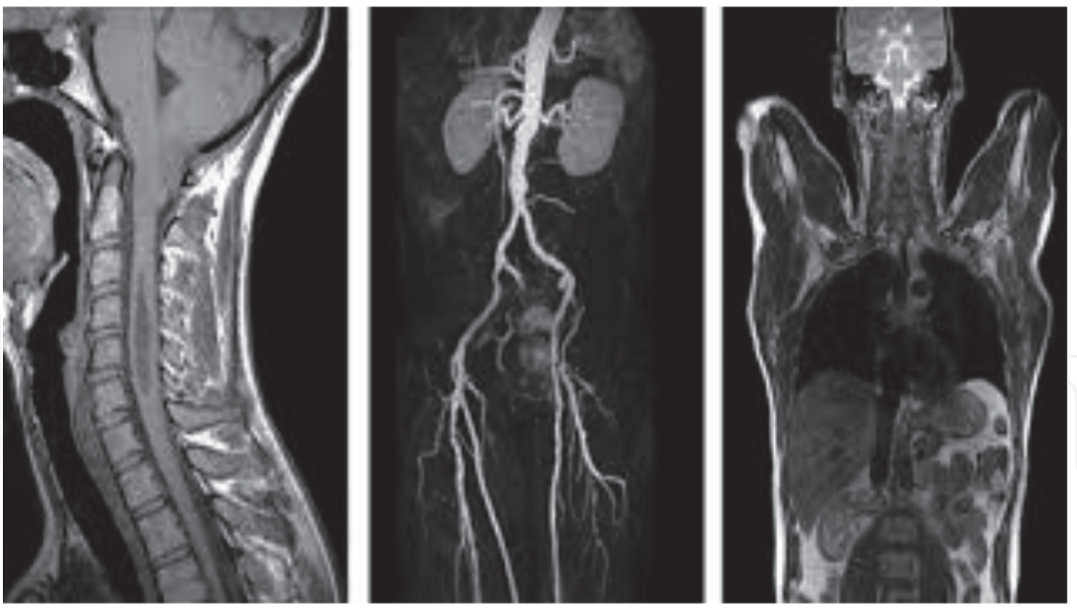

Figure 18.
MRI of the cervical spine, part of the vascular system, thorax.

transverse layer of the examined object (body) and thus to distinguish individual tissues. Different values of these quantities are assigned a certain level of grey colour when displayed on the device monitor (see **Figure 18**). In this way, we can obtain different images such as T1-image, T2-image, and PD image. Each of them has a different contrast concerning tissue differentiation and, thus, different advantages in medical diagnostics.

1.3.5.5 Magnetic resonance spectroscopy

The second application of magnetic resonance is *magnetic resonance spectroscopy* (MRS). By variation frequency ω, it is possible to select the type of atomic nucleus with the Larmor frequency of ω_L. Then, we identify the nucleus by the magnetic resonance FID signal at the frequency of $\omega = \omega_L$. The *magnetic field* $\boldsymbol{B}_0$ at the location of a given nucleus, and thus the Larmor frequency of ω_L, is slightly influenced by the magnetic field of the surrounding particles, such as electrons and other nuclei. The resonance frequency ω_L of the atom nucleus is thus slightly influenced by the chemical bonds where the magnetic dipoles (e.g., nuclei of hydrogen atoms) occur. By examining the spectrum of resonances, it is possible to identify individual hydrogen bonds in the sample under investigation, for example, O-H, C-H, C-H_2, C-H_3, N-H_2. Furthermore, we can identify the relevant organic substances (protein, enzyme, and metabolite) according to the measured resonance spectrum.

Figure 19 illustrates the organosilane spectrogram used in the manufacturing process of synthetic rubber. The horizontal axis is the offset of the resonant frequency in parts per million (ppm = 10^{-6}) relative to the reference frequency. The reference substance could be tetramethylsilane (TMS) or another proper substance. For example, hydrogen in the = CH_2 divalent group has a resonance frequency shifted by 1.3 ppm (A), in the -CH_3 monovalent group, up to 4.0 ppm (B). Each substance has a characteristic spectrogram according to which we can identify it, even at a very low concentration.

Magnetic resonance spectroscopy thus enables very sensitive biochemical diagnostics of different tissues or fluids and uses various biochemical markers to early diagnose a variety of diseases, such as epilepsy, Alzheimer's disease, Parkinson's

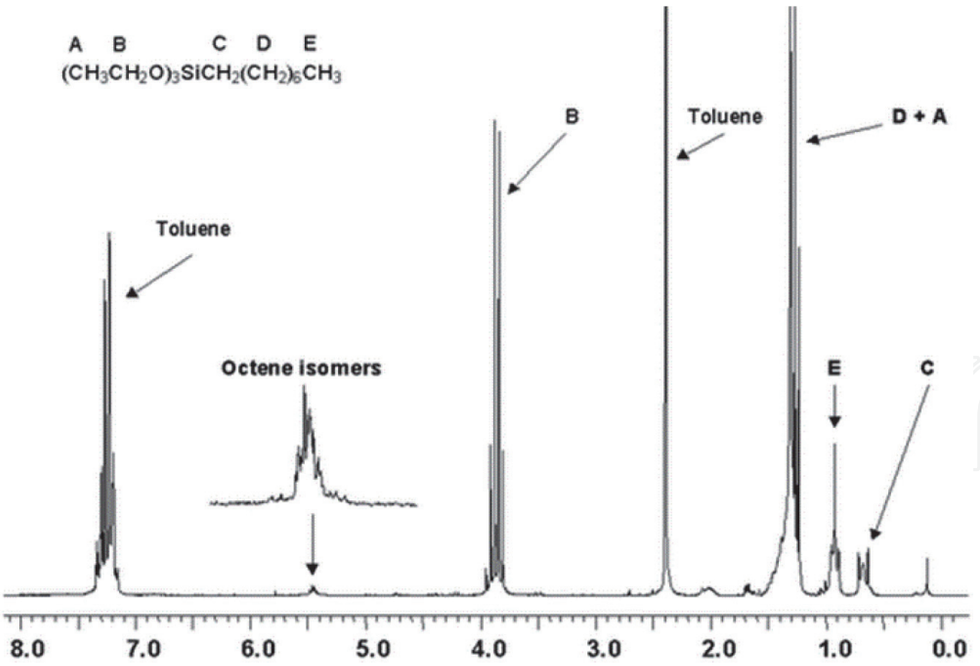

Figure 19.
Magnetic resonance spectrogram of organosilane.

disease, various cancers. Thus, magnetic resonance spectroscopy is a powerful diagnostic tool in medicine.

In specific cases, instead of hydrogen, the magnetic resonance spectroscopy uses the nuclei of other biogenic elements with an uncompensated magnetic moment such as ^{13}C, ^{19}F, ^{31}P. The MRS apparatus is quite demanding, and therefore, a special investigation of the content of other nuclei is used only rarely. Thus, the MRS uses preferably only ^{1}H (hydrogen-protons) for the determination of metabolite content, which in addition to MRI does not require additional devices and MRI and MRS images can be combined (see **Figure 20**). On the right side, it is an MRI

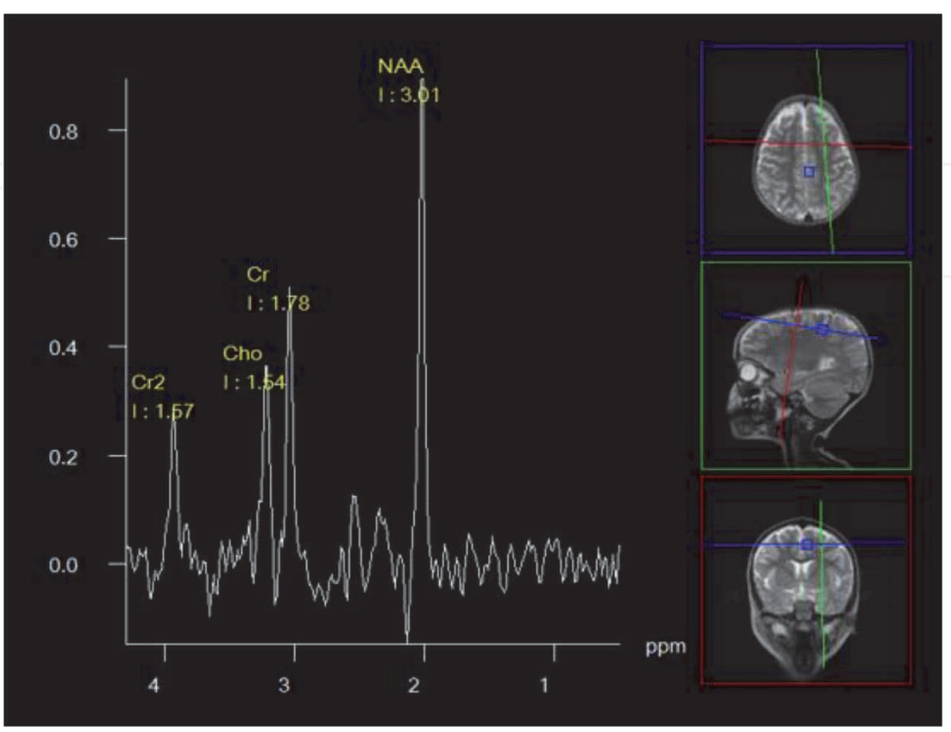

Figure 20.
Combination of MRS and MRI.

image with defined the specific location of analysis, on the left side is an MRS spectrogram of the substance at that location. From the spectral peaks typical for certain substances (here Cr-chromatin, Cho-choline, NAA-N-acetyl aspartate) and their size, it is possible to diagnose possible health disorders.

1.3.5.6 Magnetic resonance therapy

Magnetic resonance therapy (MRT) is a treatment method that uses targeted stimulation of specific structures by providing them with energy through magnetic resonance. During a resonant RF excitation pulse of the τ_{180} length, the alternating magnetic field supplies the dipole with energy [see Eq. (49)]. This energy is transferred only to the nucleus of the atoms that are in resonance with an alternating magnetic field. We can supply the energy of the electromagnetic field to specific parts of the structure that contain the resonant nucleus of the atoms. Thus, we can stimulate intracellular processes such as cell nucleus growth.

The method of magnetic resonance therapy is successfully used in the treatment of osteoarthritis and osteoporosis as we supply the energy to help cartilage and bone regeneration, as well as recovery for spinal pain following surgery (see **Figure 21**).

1.4 Oscillators

We are using various sources of periodic signals or motions, which are commonly called *oscillators*. Oscillators, mechanical or electrical, are systems with high Q-factor value and low losses having a frequency f_0 determined by the system parameters. However, each system always has, albeit small, losses that cause the oscillation to disappear at a time proportional to the quality factor [see Eq. (24)]. If the system is to oscillate continuously, we must balance its losses. This compensation consists of supplying energy equal to the losses in each period of oscillation, that is, the compensation process must be synchronous with the system's oscillations. We can achieve this by periodic power supply directly controlled by system oscillations, which means a *positive feedback method*. The classic example shows a child on a swing. If a child sits on a swing and the parent pushes it, it will swing for a while, but it will soon hang in a steady position. Children almost intuitively understand to keep the swing in motion. They must

Figure 21.
MRT—Hip joints on the left, post-hip treatment on the right.

compensate for the loss of energy by properly digging their legs in one extreme position and kicking in the other, utilising the energy of their muscle activity to increase the potential energy twice within one oscillation slightly. The child performs this activity intuitively. Thus, the child's biological energy compensates for the energy losses of the swing.

Oscillators have a precisely defined frequency by their parameters. Therefore, we can use them as a reference time signal source. Thus, they represent the essential part of the clock (mechanical with pendulum, mechanical with the rotating flywheel on spiral spring, electrical with LC circuit, electrically controlled with crystal, atomically controlled with quantum transitions in caesium atoms). The electronic clock is a part of every computer and controls the operation of such components as the processor, data storage, and data exchange with peripherals.

1.4.1 Mechanical oscillator

A commonly known mechanical oscillator is a pendulum clock. **Figure 22** shows a pendulum (dashed line) and a zoomed positive feedback step mechanism. The step wheel with inclined teeth is driven through the gearing by a force *F* generated by a weight or a spring. In the picture, the pendulum moves to the right and the right inclined tooth 'b' pushes into the stop of the escapement and supports the right-hand rotation. After reaching the extreme position, the tooth is released to the right, and the wheel rotates so that the left step-stop rests on the left oblique tooth, which pushes into the stop of the escapement and supports the pendulum moving to the left. Thus, the inclined teeth of the wheel supply energy to the pendulum via a step mechanism. The wheel drive depends on the potential energy source of the weight or spring. The system is set up to maintain a stable pendulum operation.

There are many mechanical oscillators of analogous construction, for example, a flywheel on a spring in a mechanical wristwatch, a torsional pendulum of a decorative stand clock. The pendulum clock accuracy depends on the temperature regarding the thermal expansion of the mechanical parts. A special temperature-stabilised

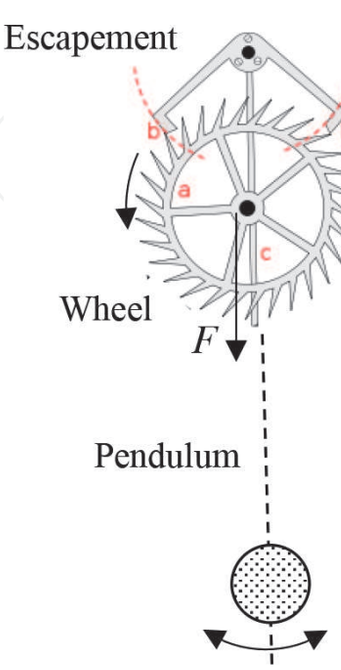

Figure 22.
Pendulum.

pendulum clock can achieve running stability of up to $\delta T/T_0 \approx 10^{-6}$ (1-second deviation in 12 days).

1.4.2 LC oscillator

The electric oscillators commonly use the LC circuit with the frequency of natural oscillations of $f_0 = \frac{1}{2\pi\sqrt{LC}}$. Due to the electrical resistance of the circuit, energy losses occur, which leads to oscillation damping. To cover energy losses and maintain the oscillations of the system, we must supply the LC circuit using a positive feedback method in connection with an amplifier. There are many LC circuit oscillators; **Figure 23** shows some examples.

These oscillators use a transistor amplifier connected with a common emitter that changes the signal phase by 180°. We must connect the output voltage of the oscillator to the input with the same phase, respectively, with offset by 2 × 180° = 360°. The input part is an LC oscillating circuit with a split capacitor: (a) Colpitts circuit, or a split inductor and (b) Hartley circuit. As shown in the figures, there is an opposite phase on split elements regarding the amplifier input and output. The (c) case shows the Meissner circuit, where phase reversal is achieved by inductive coupling with oppositely oriented windings. Figure (d) shows an example of an RC oscillator that does not use an LC circuit.

We achieve positive feedback by a three-stage RC phase shifter. The elements have a total phase shift of 3 × 60° = 180° at the desired oscillation frequency. Since the phase shifter is frequency-dependent, positive feedback occurs at only one frequency.

1.4.3 Crystal controlled oscillators

The applications demanding higher frequency stability consider the circuits mentioned above as unsatisfactory due to the used circuit elements. For example,

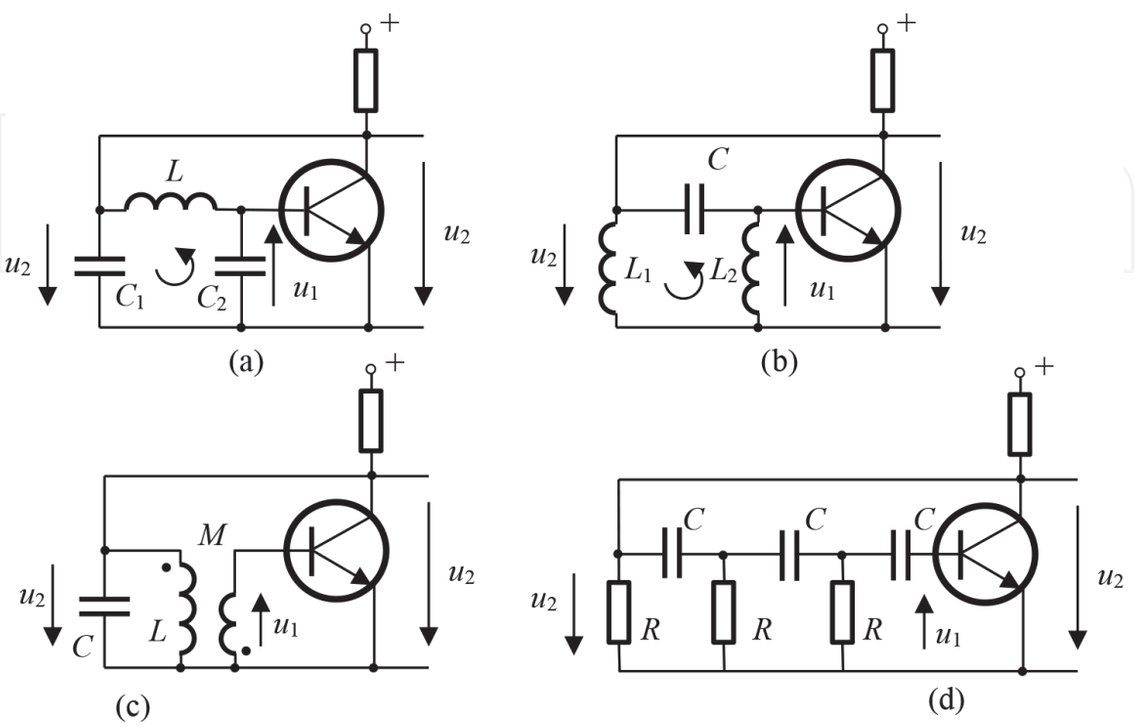

Figure 23.
Different types of electrical oscillators.

the inductors are highly temperature-dependent, or parasitic elements influence both the oscillator and amplifier circuits, and then voltage fluctuations, are applied. To suppress these parasitic effects, we are using piezoelectric crystals in the oscillating circuits instead of the inductors.

A piezoelectric crystal is an electromechanical oscillating system with a high Q-factor. This crystal is described by using the equivalent circuit diagram, as shown in **Figure 24(b)**. The inductance of the crystal depends on the mass of the crystal. The capacity corresponds to its rigidity and the resistance to the internal power losses. The capacity C_0 is the electrode capacity of $C_0 \gg C$. For the crystal as a reactance electrical circuit, the imaginary part of the complex impedance is important, and we can express it as follows:

$$X = \omega L \; \frac{\left(1 - \frac{1}{\omega^2 L C}\right)\left(1 + \frac{C}{C_0} \; - \; \omega^2 L C_0\right) - R^2 C_0}{\left(1 + \frac{C_0}{C} - \omega^2 L C_0\right)^2 + (\omega R C_0)^2}. \tag{54}$$

If we set the reactance equal to zero ($X = 0$), we can estimate the resonant frequency of the crystal. The reactance graph below (**Figure 25**) shows two resonant frequencies for the given values of the crystal (L = 100 μH, C = 100 pF, R = 1.0 Ω, C_0 = 10 nF). There is the f_s parallel and the f_p series

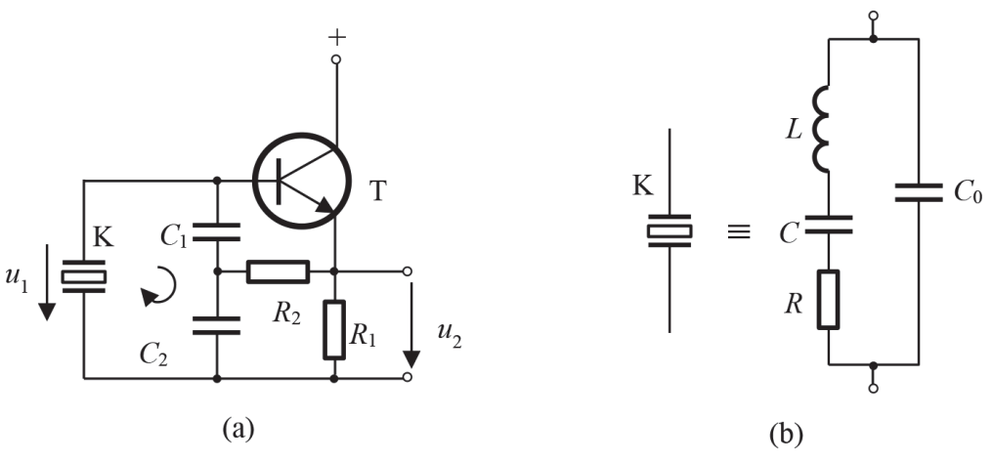

Figure 24.
Crystal controlled oscillator.

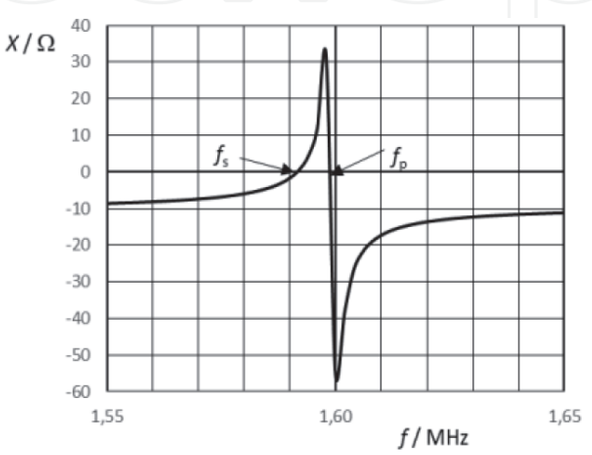

Figure 25.
Graph of the reactance X versus frequency f ot the crystal.

resonances. The equivalent circuit with a high *Q*-factor has the resonant frequencies as follows:

$$f_{\mathrm{s}} = \frac{1}{2\pi\sqrt{LC}}, \text{and } f_{\mathrm{p}} = \frac{1}{2\pi\sqrt{LCC_0/(C+C_0)}}.$$

In our case, $f_s \approx 1.5916$ MHz and $f_p \approx 1.5994$ MHz. As shown in the reactance graph, we can see a narrow interval between the f_s and f_p when $X > 0$. This means that the crystal has an inductive character. The crystal is connected in the oscillating circuit as an inductor with a parallel split capacitor with C_1 and C_2 capacitances. Thus,

Figure 24(a) shows Pierce's circuit. The split capacitor is parallel connected to the C_0, and therefore, the interval between resonant frequencies gets narrower. Thus, the oscillator can oscillate in the very narrow frequency range, which ensures high stability of the oscillator frequency. We realise positive feedback by connecting non-inverting output through the R_2 resistor to the C_2 capacitor. On the other hand, we can tune the oscillator in the range of several Hz. If we need to tune the frequency in the broader range, we must change the crystal. The crystal-controlled oscillator has high stability in order of 10^{-9}, which means the time deviation of 1 s for 30 years. Achieving this stability, we use the thermostat to stabilise the temperature of the crystal. Due to the high-stability requirement, the computer clock uses only crystal-controlled oscillators as the clock pulse generator.

1.4.4 Multivibrators

In some cases, we require a harmonic signal for biomedical applications. There are diathermy, electrotherapy, sonography, or magnetic resonance. In other applications, we need to generate periodic, but non-harmonic voltages or currents. There are pacemakers or artificial lung ventilation. In these cases, we are using rectangular or sawtooth waveforms or short repetitive pacing pulses. The primary element of non-harmonic signal generators are multivibrators serving as sources of periodic rectangular pulses. Many mechanical and thermal devices switch between two states at regular intervals. There are electrical systems, which are the most important for biomedical applications. These systems serve as periodic and non-harmonic voltage sources. As an example, **Figure 26** shows the circuit of an astable flip-flop multivibrator.

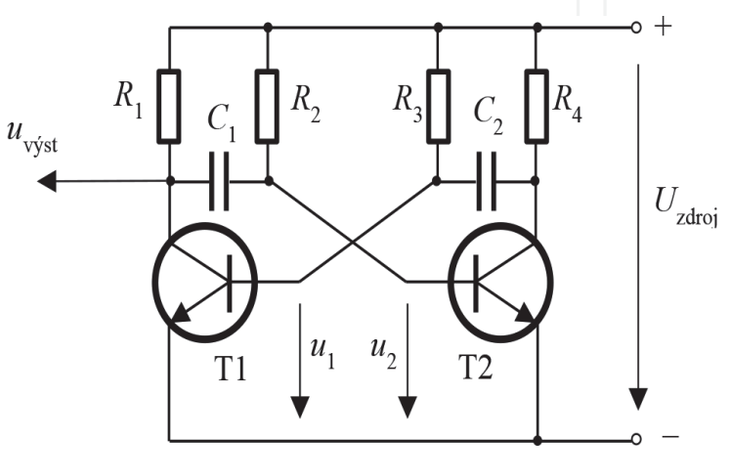

Figure 26.
Astable flip-flop circuit.

In the principle, the T1 transistor alternately switches between its ON/OFF states. If the T1 is open, the T2 is closed and vice versa. This process repeats periodically. The toggling period is given by time constants defined as R_2C_1 and R_3C_2. The output voltage is then rectangular. Connecting output to the differentiator circuit, we obtain short pulses, which can be used for the pacemakers. Using the integrator, we get a sawtooth waveform, which can be used for the generation of the linearly rising gradient field at magnetic resonance imaging.

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References

[1] Halliday D, Resnick R, Walker J. Fundamentals of Physics. Hoboken NJ: John Wiley & Sons Inc.; 2008

[2] Vlaardingerbroek MT, Boer JA. Magnetic Resonance Imaging: Theory and Practice. Berlin, Heidelberg, New York:: Springer Science & Business Media; 2013

[3] Webb AG. Introduction to Biomedical Imaging. Hoboken NJ: John Wiley & Sons; 2017

[4] Hashemi RH, Bradley G, Lisanti CHJ. MRI: The Basics. Philadelphia PA: Lippincott Williams & Wilkins; 2011