Chapter

On Parametrizations of State Feedbacks and Static Output Feedbacks and Their Applications

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Abstract

In this chapter, we provide an explicit free parametrization of all the stabilizing static state feedbacks for continuous-time Linear-Time-Invariant (LTI) systems, which are given in their state-space representation. The parametrization of the set of all the stabilizing static output feedbacks is next derived by imposing a linear constraint on the stabilizing static state feedbacks of a related system. The parametrizations are utilized for optimal control problems and for pole-placement and exact pole-assignment problems.

Keywords: control systems, continuous-time systems, state-space representation, feedback stabilization, static state feedback, static output feedback, Lyapunov equation, parametrization, optimization, optimal control, H_∞-control, H_2-control, linear-quadratic regulators, pole assignment, pole placement, robust control

1. Introduction

The solution of the problem of stabilizing static output feedback (SOF) has a great practical importance, for several reasons: they are simple, cheap, reliable, and their implementation is simple and direct. Since in practical applications, full-state measurements are not always available, the application of stabilizing state feedback (SF) is not always possible. Obviously, in practical applications, the entries of the needed SOFs are bounded with bounds known in advance, but unfortunately, the problem of SOFs with interval constrained entries is NP-hard (see [1, 2]). Exact pole assignment and simultaneous stabilization via SOF or stabilization via structured SOFs are also NP-hard problems (see [2, 3] resp.). These problems become even harder when optimal SOFs are sought, when the optimality notions can be the sparsity (see [4]) of the controller (e.g., for reliability purposes of networked control systems (NCSs)), the cost or energy consumption of the controller (which are related to various norm-bounds on the controller), the H_∞-norm, the H_2-norm or the linear-quadratic regulator (LQR) functional of the closed loop. The practical meaning of the NP-hardness of the aforementioned problems is that the problems cannot be formulated as convex problems (e.g., through LMIs or SDPs) and cannot have any efficient algorithms (under the widespread belief that P $\neq$ NP). Thus, one has to compromise the exactness (which might affect the feasibility of the solution) or the optimality of the solution. Therefore, one has to utilize the specific structure of the given problem, in order to describe effectively the set of all feasible solutions, by reducing the number of variables and constraints to the minimum, for the purpose of increasing the efficiency and accuracy of the available algorithms. This is the aim of the proposed method.

Several formulations and related algorithms were introduced in the literature for the constrained SOF and other control hard problems. The iterated linear matrix inequalities (ILMI), bilinear matrix inequalities (BMI), and semi-definite programming (SDP) approaches for the constrained SOF problem, for the simultaneous stabilizing SOF problem, and for the robust control via SOF (with related algorithms) were studied in: [5–11]. The problem of pole placement via SOF and the problem of robust pole placement via Static Feedback were studied in: [12, 13]. In [14, 15], the method of alternating projections was utilized to solve the problems of rank minimization and pole placement via SOFs, respectively. The probabilistic and randomized methods for the constrained SOF problem and robust stabilization via SOFs (among other hard problems) were discussed in [16–19]. In [20], the problem of minimal-gain SOF was solved efficiently by the randomized method. A non-smooth analysis approach for H_∞ synthesis and for the SOF problem is given in [21, 22], respectively. A MATLAB® library for multiobjective robust control problems based on the non-smooth analysis approach was introduced in [23]. All these references (and many more references not brought here) show the significance of the constrained SOF problem to control applications.

Many problems can be reduced to the SOF constrained problem, including the reduction of the minimal-degree dynamic-feedback problem and robust or decentralized stability via static-feedback, reduced-order H_∞ filter problem, global minimization of LQR functional via SOF, and the design problem of optimal PID controllers (see [2, 10, 24–27] respectively). It is worth mentioning [28], where the alternating direction method of multipliers was utilized to alternate between optimizing the sparsity of the state feedback matrix and optimizing the closed-loop H_2-norm, where the sparsity measure was introduced as a penalty term, without any pre-assumed knowledge about the sparsity structure of the controller. The method of augmented Lagrangian for optimal structured static-feedbacks was considered in [29], where it is assumed that the structure is known in advance (otherwise, one should solve a combinatorial problem). The computation overhead of all the aforementioned methods can be reduced significantly, if good parametrization of all the SOFs of the given system could be found, where a parametrization can be called "good" if it takes into account the structure of the given specific system and if it well separates between free and dependent parameters, thus resulting in a minimal set of nonlinear nonconvex inequalities/equations needed to be solved.

In [30], a parametrization of all the SFs and SOFs of Linear-Time-Invariant (LTI) continuous-time systems is achieved by using a characterization of all the (marginally) stable matrices as dissipative Hamiltonian matrices, leading to a highly performance sequential semi-definite programming algorithm for the minimal-gain SOF problem. The proposed method there can be applied also to LTI discrete-time systems by adding semi-definite conditions for placing the closed-loop eigenvalues in the unit disk. A new parametrization for SOF control of linear parameter-varying (LPV) discrete-time systems, with guaranteed ℓ_2-gain performance, is provided in [31]. The parametrization there is given in terms of an infinite set of LMIs that becomes finite, if some structure on the parameter-dependent matrices is assumed (e.g., an affine dependency). The H_2-norm guaranteed-performance SOF control for hidden Markov jump linear systems (HMJLS) is studied in [32], where the SOFs are parameterized via convex optimization with LMI constraints, under the assumptions of full-rank sensor matrices and an efficient and accurate Markov chain state estimator. In [33], an iterative LMI algorithm is proposed for the SOF problem for LTI continuous-time negative-imaginary (NI) systems with given H_∞ norm-bound on the closed loop, based on decoupling the dependencies between the SOF and the Lyapunov certificate matrix.

When solving an optimization problem, it is important to have a convenient parametrization for the set of feasible solutions. Otherwise, one needs to use the probability method (i.e., the "generate and check" method), which is seriously doomed to the "curse of dimensionality" (see [16]). In [13], a closed form of all the stabilizing state feedbacks is proved (up to a set of measure 0), for the purpose of exact pole assignment, when the location errors are optimized by lowering the condition number of the similarity matrix, and the controller performance is optimized by minimizing its Frobenius norm. The parametrization in [13] is based on the assumptions that the input-to-state matrix B has a full rank and at least one real state feedback leading to diagonalizable closed-loop matrix exists, where a necessary condition for the existence of such feedback is that the multiplicity of any assigned eigenvalue is less than or equal to $rank(B)$. In this context, it is worth mentioning [34] in which a parametrization of all the exact pole-assignment state feedbacks is given, under the assumption that the set of needed closed-loop poles should contain sufficient number of real eigenvalues (which make no problem if the problem of pole placement is of concern, where it is generally assumed that the region is symmetric with respect to the real axis and contains a real-axis segment with its neighborhood). The results of [34] and of the current chapter are based on a controllability recursive structure that was discovered in [35].

In this chapter, using the aforementioned controllability recursive structure, we introduce a parametrization of the set of all stabilizing SOFs for continuous-time LTI systems with no other assumptions on the given system (for discrete-time LTI systems, the parametrization is much more involved and will be treated in a future work). As opposed to the notable works [36–38], where for the parametrization one still needs to solve some LMIs in order to get the Lyapunov matrix, here we give an explicit recursive formula for the Lyapunov matrix and for the feedback in the case of SF, and a constrained form for the Lyapunov matrix and for the feedback in the case of SOF.

The rest of the chapter goes as follows:

In Section 2, we set notions and give some basic useful lemmas, and in Section 3, we introduce the parametrization of the set of all stabilizing static-state feedbacks for LTI continuous-time systems. In Section 4, we introduce the constrained parametrization of the set of all stabilizing SOFs for LTI continuous-time systems. The effectiveness of the method is shown on a real-life system. Section 5 is based on [34] and is devoted to the problem of exact pole assignment by SF, for LTI continuous-time or discrete-time systems. The effectiveness of the method is shown on a real-life system. Finally, in Section 6, we conclude with some remarks and intentions for a future work.

2. Preliminaries

By $\mathbb{C}$ we denote the complex field and by $\mathbb{C}_-$ the open left half-plane. For $z \in \mathbb{C}$ we denote by $\Re(z)$ its real part, while by $\Im(z)$ we denote its imaginary part. For a square matrix Z, we denote by $\sigma(Z)$ the spectrum of Z. For a $\mathbb{R}^{p \times q}$ matrix Z, we denote by Z^T its transpose, and by $z_{i,j}$ or by $Z_{i,j}$, its (i,j)'th element or block element. A square matrix Z in the continuous-time context (in the discrete-time context) is said to be (asymptotically) stable, if any eigenvalue $\lambda \in \sigma(Z)$ satisfies $\Re(\lambda) < 0$, i.e., $\lambda \in \mathbb{C}_-$ (satisfies $|\lambda| < 1$, i.e. $\lambda \in \mathbb{D}_-$ where $\mathbb{D}_-$ is the open unit disk).

Consider a continuous-time system in the form:

$$\begin{cases} \frac{d}{dt}x(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{r\times n}$, and x, u, y are the state, the input, and the measurement, respectively. Assuming that the state x is fully accessible and fully available for feedback, we define $u = -K^{(0)}x$ to be the state feedback (SF). When the state is not fully accessible or not fully available for feedback but the measurement y is available for feedback, we define $u = -Ky$ to be the static output feedback (SOF). The problems that we consider here are the following:

- (SF-PAR): How can one parameterize the set of all $K^{(0)} \in \mathbb{R}^{m\times n}$ such that the closed-loop $A - BK^{(0)}$ is stable, and what is the best parametrization (in terms of minimal number of parameters and minimal set of constraints)?

- (SOF-PAR): How can one parameterize the set of all $K \in \mathbb{R}^{m\times r}$ such that the closed-loop $A - BKC$ is stable, and what is the best parametrization?

The parameterizations will be used for achieving other goals and performance keys for the system, other than stability, which is the feasibility defining basic key.

A square matrix Z is said to be non-negative (denoted as $Z \geq 0$) if $Z^T = Z$ and $v^T Z v \geq 0$ for any vector v. A non-negative matrix Z is said to be strictly non-negative (denoted as $Z > 0$) if $v^T Z v > 0$ for any vector $v \neq 0$. For two square matrices Z, W, we would write $Z \geq W$ $(Z > W)$ if $Z - W \geq 0$ (respectively, if $Z - W > 0$). For a matrix $Z \in \mathbb{R}^{p\times q}$, we denote by Z^+ the Moore-Penrose pseudo-inverse (see [39, 40] for definition and properties). By L_Z, R_Z we denote the orthogonal projections $I_q - Z^+Z$ and $I_p - ZZ^+$, respectively, where I_s denotes the identity matrix of size $s \times s$. Note that Z^+Z and ZZ^+ (as well as L_Z and R_Z) are symmetric and orthogonally diagonalizable with eigenvalues from $\{0, 1\}$. By *diag* and *bdiag* we denote diagonal and block-diagonal matrices, respectively.

A system triplet (A, B, C) is SOF stabilizable (or just stabilizable) if and only if there exist K and $P > 0$ such that

$$EP + PE^T = -R \tag{2}$$

for some given $R > 0$ ($R = I$ can always be chosen), where $E = A - BKC$. For the "if" direction, note that (2) implies the negativity of the real part of any eigenvalue of E, implying that the closed-loop E is stable. For the "only-if" direction, under the assumption that $E = A - BKC$ is stable for some given K, one can show that $P := \int_0^\infty \exp(Et)R\exp(E^T t)dt$ is well defined, satisfies $P^T = P$ and $P > 0$, and is the unique solution for (2).

Note that the set of all SOFs is given by $K = B^+XC^+ + L_B S + TR_C$ where S, T are any $m \times r$ matrices and X is any $n \times n$ matrix such that $E = A - BB^+XC^+C$ is stable. Thus, one can optimize K by utilizing the freeness in S, T without changing the closed-loop performance achieved by X. This characterization of the feasibility space shows its effectiveness in proving theorems, as will be seen along the chapter (see also [20, 35]). We also conclude that (A, B, C) is stabilizable if and only if (A, BB^+, C^+C) is stabilizable.

In the sequel, we make use of the following lemma (see [39]):

Lemma 2.1 The matrix equation $AX = B$ has solutions if and only if $AA^+B = B$ (equivalently, $R_A B = 0$). When the condition is satisfied, the set of all solutions is given by

$$X = A^+B + L_A Z, \tag{3}$$

where Z is arbitrary matrix. Moreover, we have: $\|X\|_F^2 = \|A^+B\|_F^2 + \|L_A Z\|_F^2$, implying that the minimal Frobenius-norm solution is $X = A^+B$.

Similarly, the equation $YA = B$ has solutions if and only if $BA^+A = B$ (equivalently, $BL_A = 0$). When the condition is satisfied, the set of all solutions is given by

$$Y = BA^+ + WR_A, \tag{4}$$

where W is arbitrary matrix. Moreover, we have: $\|Y\|_F^2 = \|BA^+\|_F^2 + \|WR_A\|_F^2$, implying that the minimal Frobenius-norm solution is $Y = BA^+$.

3. Parametrization of all the static state feedbacks

We start with the following lemma known as the projection lemma (see [41], Theorem 3.1):

Lemma 3.1 The pair (A, BB^+) is stabilizable if and only if there exists $P > 0$ such that

$$R_B\left(I + AP + PA^T\right)R_B = 0. \tag{5}$$

When (5) is satisfied then, X is a stabilizing SF if and only if X is a solution for

$$BB^+XP + PX^TBB^+ = I + AP + PA^T. \tag{6}$$

Moreover, one specific solution for (6) is given by

$$X_0 = \left(I + AP + PA^T\right)\left(I - \frac{1}{2}BB^+\right)P^{-1}. \tag{7}$$

Similarly, $\left(A^T, C^+C\right)$ is stabilizable if and only if there exists $Q > 0$ such that

$$L_C\left(I + A^TQ + QA\right)L_C = 0. \tag{8}$$

When (8) is satisfied then, Y^T is a stabilizing SF (i.e., $A^T - C^+CY^T$ or, equivalently, $A - YC^+C$ is stable) if and only if Y is a solution for:

$$C^+CY^TQ + QYC^+C = I + A^TQ + QA. \tag{9}$$

One specific solution for (9) is given by

$$Y_0 = Q^{-1}\left(I - \frac{1}{2}C^+C\right)\left(I + A^TQ + QA\right). \tag{10}$$

Remark 3.1 The explicit formulas (7) and (10) are our little contribution to the projection lemma, Lemma 3.1. Unfortunately, we do not have such an explicit formulas for LTI discrete-time systems.

In order to describe the set of all solutions for (6) and (9), we need the following lemma that can be proved easily:

Lemma 3.2 Let $P > 0, Q > 0$. Then, the set of all solutions for:

$$ZP + PZ^T = 0, \tag{11}$$

is given by $Z = WP^{-1}$ where $W^T = -W$.

Similarly, the set of all solutions for:

$$Z^T Q + QZ = 0, \tag{12}$$

is given by $Z = Q^{-1}V$ where $V^T = -V$.

The following theorem describes the set of all solutions for (6) and (9), using the controllers (7) and (10):

Theorem 3.1 Let $P > 0$ satisfy (5) and let X_0 be given by (7). Then, X is a solution for (6) if and only if:

$$X = X_0 + WP^{-1} + R_B L, \tag{13}$$

where W satisfies $W^T = -W, R_B W = 0$ and L is arbitrary.

Similarly, let $Q > 0$ satisfy (8) and let Y_0 be given by (10). Then, Y is a solution for (9) if and only if:

$$Y = Y_0 + Q^{-1}V + ML_C, \tag{14}$$

where V satisfies $V^T = -V, VL_C = 0$ and M is arbitrary.

Proof:

Assume that X is a solution for (6). Since X_0 is also a solution for (6), it follows that $BB^+(X - X_0)P + P(X - X_0)^T BB^+ = 0$. Let $Z = BB^+(X - X_0)$. Then, $R_B Z = 0$ and $ZP + PZ^T = 0$. Lemma 3.2 implies that $Z = WP^{-1}$, where $W^T = -W$ and therefore $R_B W = 0$. We conclude that $X - X_0 = WP^{-1} + R_B L$ for some L (namely, $L = X - X_0$).

Conversely, let X be given by (13) and let $Z = WP^{-1}$. Then, $BB^+(X - X_0) = BB^+Z = Z$ since $BB^+R_B = 0$ and since $R_B Z = 0$. Now, $ZP + PZ^T = 0$ implies that

$$BB^+(X - X_0)P + P(X - X_0)^T BB^+ = 0,$$

from which we conclude that X satisfies (6), since X_0 satisfies (6). The second claim is proved similarly. ■

In the following we describe the set $\mathcal{P}$ of all matrices $P > 0$ satisfying (5). Note that in Theorem 3.1 the existence of $P > 0$ satisfying (5) is guaranteed by the assumption that (A, BB^+) is stabilizable and as a result of Lemma 3.1. Let $P \in \mathcal{P}$ and let

$$\begin{cases} X_0 = \left(I + AP + PA^T\right) \cdot \left(I - \frac{1}{2}BB^+\right)P^{-1} \\ W \text{ arbitrary such that } W^T = -W, R_B W = 0 \\ X = X_0 + WP^{-1} + R_B L \text{ where } L \text{ is arbitrary} \\ K = B^+X + L_B F \text{ where } F \text{ is arbitrary.} \end{cases} \tag{15}$$

Let $\mathcal{X}(P)$ denote the set of all matrices X satisfying (15) for a fixed $P \in \mathcal{P}$, and let $\mathcal{K}(P)$ denote the set of all matrices K satisfying (15) for a fixed $P \in \mathcal{P}$. Note that for a fixed $P \in \mathcal{P}$, the set $\mathcal{X}(P)$ is convex (actually affine) and $\cup_{P \in \mathcal{P}} \mathcal{X}(P)$ contains all the stabilizing X parameters of the stabilizable pair (A, BB^+). Finally, $\cup_{P \in \mathcal{P}} \mathcal{K}(P)$ contains all the stabilizing SF's K of the stabilizable pair (A, B).

For a stabilizable pair $\left(A^T, C^T\right)$, let $\mathcal{Q}$ be the set of all matrices $Q > 0$ satisfying (8), and let

$$\begin{cases} Y_0 = Q^{-1} \cdot \left(I - \frac{1}{2} C^+ C\right)\left(I + A^T Q + QA\right) \\ V \text{ arbitrary such that } V^T = -V, VL_C = 0 \\ Y = Y_0 + Q^{-1} V + ML_C \text{ where } M \text{ is arbitrary} \\ K = YC^+ + GR_C \text{ where } G \text{ is arbitrary.} \end{cases} \tag{16}$$

Let $\mathcal{Y}(Q)$ denote the set of all matrices Y satisfying (16) for a fixed $Q \in \mathcal{Q}$, and let $\mathcal{K}(Q)$ denote the set of all matrices K satisfying (16) for a fixed $Q \in \mathcal{Q}$. Then, $\cup_{Q \in \mathcal{Q}} K(Q)$ contains all the stabilizing SFs K of the stabilizable pair $\left(A^T, C^T\right)$.

In the following we assume (without loss of generality, see Remark 4.2) that (A, BB^+) is controllable. Under this assumption, we recursively (go downwards and) define a sequence of sub-systems of the given system (A, BB^+). Since BB^+ is symmetric matrix (with simple eigenvalues from the set $\{0, 1\}$), it is diagonalizable by an orthogonal matrix. Let U denote an orthogonal matrix such that

$$\widehat{B} = U^T BB^+ U = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} = bdiag(I_k, 0) \tag{17}$$

(where $k = rank(B) = rank(BB^+) \geq 1$ since (A, B) is controllable). Let $\widehat{A} = U^T AU = \begin{bmatrix} \widehat{A}_{1,1} & \widehat{A}_{1,2} \\ \widehat{A}_{2,1} & \widehat{A}_{2,2} \end{bmatrix}$ be partitioned accordingly. Let $U^{(0)} = U$ and let $A^{(0)} = A, B^{(0)} = B, n_0 = n, k_0 = rank\left(B^{(0)}\right)$. Similarly, let $U^{(1)}$ be an orthogonal matrix such that $U^{(1)T} B^{(1)} B^{(1)+} U^{(1)} = bdiag(I_{k_1}, 0)$, where $B^{(1)} = \widehat{A}_{2,1}$. Let $A^{(1)} = \widehat{A}_{2,2}, n_1 = n_0 - k_0, k_1 = rank\left(B^{(1)}\right)$. Then, $\left(A^{(1)}, B^{(1)}\right)$ is controllable since $\left(A^{(0)}, B^{(0)}\right)$ is controllable (see [35] and see Lemma 5.1 in the following).

Recursively, assume that the pair $\left(A^{(i)}, B^{(i)}\right)$ was defined and is controllable. Let $U^{(i)}$ be an orthogonal matrix such that $\widehat{B^{(i)}} = U^{(i)T} B^{(i)} B^{(i)+} U^{(i)} = bdiag(I_{k_i}, 0)$, where $k_i \geq 1$ (since $\left(A^{(i)}, B^{(i)}\right)$ is controllable). Let $\widehat{A^{(i)}} = U^{(i)T} A^{(i)} U^{(i)} = \begin{bmatrix} \widehat{A^{(i)}}_{1,1} & \widehat{A^{(i)}}_{1,2} \\ \widehat{A^{(i)}}_{2,1} & \widehat{A^{(i)}}_{2,2} \end{bmatrix}$ be partitioned accordingly, with sizes $k_i \times k_i$ and $(n_i - k_i) \times (n_i - k_i)$ of the main diagonal blocks. Let $A^{(i+1)} = \widehat{A^{(i)}}_{2,2}, B^{(i+1)} = \widehat{A^{(i)}}_{2,1}, n_{i+1} = n_i - k_i, k_i = rank\left(B^{(i)}\right)$. Then, $\left(A^{(i+1)}, B^{(i+1)}\right)$ is controllable. We stop the recursion when $B^{(i)} B^{(i)+} = I_{k_i}$ for some $i = b$ (i.e. the base case, in which also $k_b = n_b$).

Now, we go upward and define the Lyapunov matrices and the related SFs of the sub-systems. For the base case $i = b$, let $P^{(b)} > 0$ be arbitrary (note that it is a free parameter!). Let

$$\begin{cases} X_0^{(b)} = \frac{1}{2}\left(I_{n_b} + A^{(b)}P^{(b)} + P^{(b)}A^{(b)T}\right)\left(P^{(b)}\right)^{-1} \\ W^{(b)} \text{ arbitrary such that } W^{(b)T} = -W^{(b)} \\ X^{(b)} = X_0^{(b)} + W^{(b)}\left(P^{(b)}\right)^{-1} \\ K^{(b)} = B^{(b)+}X^{(b)} + L_{B^{(b)}}F^{(b)} \text{ where } F^{(b)} \text{ is arbitrary,} \end{cases} \tag{18}$$

and note that $R_{B^{(b)}} = 0$ in the base case. Now, it can be checked that $E^{(b)} = A^{(b)} - B^{(b)}K^{(b)} = A^{(b)} - X^{(b)}$ is stable. We therefore have a parametrization of $\mathcal{K}^{(b)}\left(P^{(b)}\right)$ through arbitrary $P^{(b)} > 0$.

Let $\mathcal{P}^{(i+1)}$ denote the set of all $P^{(i+1)} > 0$ satisfying:

$$R_{B^{(i+1)}}\left(I_{n_{i+1}} + A^{(i+1)}P^{(i+1)} + P^{(i+1)}A^{(i+1)T}\right)R_{B^{(i+1)}} = 0, \tag{19}$$

and assume that $\mathcal{K}^{(i+1)}\left(P^{(i+1)}\right)$ was parameterized through $P^{(i+1)} > 0$ ranging in the set $\mathcal{P}^{(i+1)}$, as is defined by (19). Similarly, let $\mathcal{P}^{(i)}$ denote the set of all $P^{(i)} > 0$ satisfying:

$$R_{B^{(i)}}\left(I_{n_i} + A^{(i)}P^{(i)} + P^{(i)}A^{(i)T}\right)R_{B^{(i)}} = 0, \tag{20}$$

and assume that $\mathcal{K}^{(i)}\left(P^{(i)}\right)$ was parameterized through $P^{(i)} > 0$ ranging in the set $\mathcal{P}^{(i)}$, as is defined by (20).

Now, we need to characterize the matrices $P^{(i)} > 0$ belonging to the set $\mathcal{P}^{(i)}$. Multiplying (20) from the left by $U^{(i)T}$ and from the right by $U^{(i)}$ we get:

$$\widehat{R_{B^{(i)}}}\left(I_{n_i} + \widehat{A^{(i)}}\widehat{P^{(i)}} + \widehat{P^{(i)}}\widehat{A^{(i)T}}\right)\widehat{R_{B^{(i)}}} = 0,$$

where $\widehat{R_{B^{(i)}}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n_i-k_i} \end{bmatrix}$ and $\widehat{P^{(i)}} = U^{(i)T}P^{(i)}U^{(i)} = \begin{bmatrix} \widehat{P^{(i)}}_{1,1} & \widehat{P^{(i)}}_{1,2} \\ \widehat{P^{(i)T}}_{1,2} & \widehat{P^{(i)}}_{2,2} \end{bmatrix}$ is partitioned accordingly. The condition (20) is therefore equivalent to:

$$\begin{aligned} & I_{n_i-k_i} + \left(\widehat{A^{(i)}}_{2,2} + \widehat{A^{(i)}}_{2,1}\widehat{P^{(i)}}_{1,2}\left(\widehat{P^{(i)}}_{2,2}\right)^{-1}\right)\widehat{P^{(i)}}_{2,2} + \\ & + \widehat{P^{(i)}}_{2,2}\left(\widehat{A^{(i)}}_{2,2} + \widehat{A^{(i)}}_{2,1}\widehat{P^{(i)}}_{1,2}\left(\widehat{P^{(i)}}_{2,2}\right)^{-1}\right)^T = 0, \end{aligned} \tag{21}$$

which is equivalent to:

$$\begin{aligned} & I_{n_i-k_i} + \left(A^{(i+1)} + B^{(i+1)}\widehat{P^{(i)}}_{1,2}\left(\widehat{P^{(i)}}_{2,2}\right)^{-1}\right)\widehat{P^{(i)}}_{2,2} + \\ & + \widehat{P^{(i)}}_{2,2}\left(A^{(i+1)} + B^{(i+1)}\widehat{P^{(i)}}_{1,2}\left(\widehat{P^{(i)}}_{2,2}\right)^{-1}\right)^T = 0. \end{aligned} \tag{22}$$

Let $P^{(i+1)} \in \mathcal{P}^{(i+1)}$ and let $K^{(i+1)} \in \mathcal{K}^{(i+1)}\left(P^{(i+1)}\right)$. Set $\widehat{P^{(i)}}_{2,2} := P^{(i+1)}$ and set $\widehat{P^{(i)}}_{1,2} := -K^{(i+1)}P^{(i+1)}$. Now, since $\widehat{P^{(i)}}_{2,2} := P^{(i+1)} > 0$, (22) implies that the system:

$$A^{(i+1)} + B^{(i+1)}\widehat{P^{(i)}}_{1,2}\left(\widehat{P^{(i)}}_{2,2}\right)^{-1} := A^{(i+1)} - B^{(i+1)}K^{(i+1)}, \tag{23}$$

is stable. Now,

$$\widehat{P^{(i)}} = \begin{bmatrix} \widehat{P^{(i)}}_{1,1} & -K^{(i+1)}P^{(i+1)} \\ -P^{(i+1)}K^{(i+1)T} & P^{(i+1)} \end{bmatrix} \tag{24}$$

and we need to define $\widehat{P^{(i)}}_{1,1}$ in order to complete $\widehat{P^{(i)}}$ to a strictly non-negative matrix. Since:

$$\begin{bmatrix} \widehat{P^{(i)}}_{1,1} & -K^{(i+1)}P^{(i+1)} \\ -P^{(i+1)}K^{(i+1)T} & P^{(i+1)} \end{bmatrix} =$$

$$= \begin{bmatrix} I_{k_i} & -K^{(i+1)} \\ 0 & I_{n_i-k_i} \end{bmatrix}\cdot$$

$$\cdot\begin{bmatrix} \widehat{P^{(i)}}_{1,1} - K^{(i+1)}P^{(i+1)}K^{(i+1)T} & 0 \\ 0 & P^{(i+1)} \end{bmatrix}\cdot$$

$$\cdot\begin{bmatrix} I_{k_i} & 0 \\ -K^{(i+1)T} & I_{n_i-k_i} \end{bmatrix},$$

it follows that $\widehat{P^{(i)}} > 0$ if and only if $\widehat{P^{(i)}}_{1,1} - K^{(i+1)}P^{(i+1)}K^{(i+1)T} > 0$ or equivalently if and only if $\widehat{P^{(i)}}_{1,1} = \Delta\widehat{P^{(i)}}_{1,1} + K^{(i+1)}P^{(i+1)}K^{(i+1)T}$, where $\Delta\widehat{P^{(i)}}_{1,1}$ is arbitrary strictly non-negative matrix (a free parameter!).

Conversely, if $P^{(i)} > 0$ satisfies (20) then (23) is stable and thus:

$$K^{(i+1)} = -\widehat{P^{(i)}}_{1,2}\left(\widehat{P^{(i)}}_{2,2}\right)^{-1} \in \mathcal{K}^{(i+1)}\left(R^{(i+1)}\right),$$

for some $R^{(i+1)} \in \mathcal{P}^{(i+1)}$. But since $K^{(i+1)} \in \mathcal{K}^{(i+1)}\left(R^{(i+1)}\right)$ if and only if:

$$I_{n_i-k_i} + \left(A^{(i+1)} - B^{(i+1)}K^{(i+1)}\right)R^{(i+1)} + R^{(i+1)}\left(A^{(i+1)} - B^{(i+1)}K^{(i+1)}\right)^T = 0, \tag{25}$$

since the last equation has unique strictly non-negative solution and since $\widehat{P^{(i)}}_{2,2}$ satisfies this equation, it follows that $R^{(i+1)} = \widehat{P^{(i)}}_{2,2}$. Let $P^{(i+1)} = \widehat{P^{(i)}}_{2,2}$. Then, $K^{(i+1)} \in \mathcal{K}^{(i+1)}\left(P^{(i+1)}\right)$ and since $\widehat{P^{(i)}}_{1,2} = -K^{(i+1)}P^{(i+1)}$, it follows that $\widehat{P^{(i)}}$ has the form (24). Thus, $\widehat{P^{(i)}}_{1,1} = \Delta\widehat{P^{(i)}}_{1,1} + K^{(i+1)}P^{(i+1)}K^{(i+1)T}$ where $\Delta\widehat{P^{(i)}}_{1,1} > 0$ is arbitrary (a free parameter!) and

$$\widehat{P^{(i)}} = U^{(i)T}P^{(i)}U^{(i)} = \begin{bmatrix} \Delta\widehat{P^{(i)}}_{1,1} + K^{(i+1)}P^{(i+1)}K^{(i+1)T} & -K^{(i+1)}P^{(i+1)} \\ -P^{(i+1)}K^{(i+1)T} & P^{(i+1)} \end{bmatrix}. \tag{26}$$

Therefore, $\mathcal{P}^{(i)}$ is the set of all $P^{(i)} > 0$ such that $\widehat{P^{(i)}} = U^{(i)T}P^{(i)}U^{(i)}$ is given by (26). We thus have a parametrization of all $P^{(i)} > 0$ satisfying (20). Specifically, $\mathcal{P}^{(0)}$ is the set of all $P^{(0)} > 0$ satisfying (5).

Now, let $P^{(i)} \in \mathcal{P}^{(i)}$ and let

$$\begin{cases} X_0^{(i)} = \left(I_{n_i} + A^{(i)}P^{(i)} + P^{(i)}A^{(i)T}\right)\cdot \\ \qquad \cdot\left(I_{n_i} - \frac{1}{2}B^{(i)}B^{(i)+}\right)\left(P^{(i)}\right)^{-1} \\ W^{(i)} \text{ arbitrary such that } W^{(i)T} = -W^{(i)}, R_{B^{(i)}}W^{(i)} = 0 \\ X^{(i)} = X_0^{(i)} + W^{(i)}\left(P^{(i)}\right)^{-1} + R_{B^{(i)}}L^{(i)} \text{ where } L^{(i)} \text{ is arbitrary} \\ K^{(i)} = B^{(i)+}X^{(i)} + L_{B^{(i)}}F^{(i)} \text{ where } F^{(i)} \text{ is arbitrary.} \end{cases} \tag{27}$$

Then, it can be checked that $E^{(i)} = A^{(i)} - B^{(i)}K^{(i)} = A^{(i)} - B^{(i)}B^{(i)+}X^{(i)}$ is stable. We therefore have a parametrization of $\mathcal{K}^{(i)}\left(P^{(i)}\right)$ through $P^{(i)} \in \mathcal{P}^{(i)}$. We conclude the discussion above with the following:

Theorem 3.2 Let (A, B) be a controllable pair. Then, in the above notations, for $i = b-1, \dots, 0$, $P^{(i)} > 0$ satisfies (20) if and only if $\widehat{P^{(i)}} = U^{(i)T}P^{(i)}U^{(i)}$ has the structure (26) where $\Delta\widehat{P^{(i)}}_{1,1} > 0$ is arbitrary (free parameter), where $K^{(i+1)} \in \mathcal{K}^{(i+1)}\left(P^{(i+1)}\right)$, where $P^{(b)} > 0$ is arbitrary (free parameter) and $\mathcal{K}^{(b)}\left(P^{(b)}\right)$ is given by (18). Moreover, $\mathcal{K}^{(i)}\left(P^{(i)}\right)$ for $i = b-1, \dots, 0$ is given by (27).

Similarly to the discussion above, relating to $\left(A^T, C^+C\right)$ and defining sub-systems for $j = 0, \dots, c$, we have a parametrization of all $Q^{(j)} > 0$ satisfying (8) for the related sub-system and specifically, $\mathcal{Q}^{(0)}$ is the set of all $Q^{(0)} > 0$ satisfying (8). The parametrizations of all the stabilizing SF's of (A, BB^+) and $\left(A^T, C^+C\right)$ are given in the following:

Corollary 3.1 Let (A, BB^+) be a given controllable pair. Then, the set of all stabilizing SF's of (A, BB^+) is given by $X = X_0 + WP^{-1} + R_BL$ where

$$X_0 = \left(I + AP + PA^T\right)\left(I - \frac{1}{2}BB^+\right)P^{-1},$$

where L is arbitrary, W satisfies $W^T = -W, R_BW = 0$, and $P > 0$ satisfies

$$R_B\left(I + AP + PA^T\right)R_B = 0,$$

i.e. $P \in \mathcal{P}^{(0)}$.

Similarly, let $\left(A^T, C^+C\right)$ be a given controllable pair. Then, the set of all stabilizing SF's of $\left(A^T, C^+C\right)$ is given by $Y = Y_0 + Q^{-1}V + ML_C$ where

$$Y_0 = Q^{-1}\left(I - \frac{1}{2}C^+C\right)\left(I + A^TQ + QA\right),$$

where M is arbitrary, V satisfies $V^T = -V, VL_C = 0$, and $Q > 0$ satisfies

$$L_C\left(I + A^TQ + QA\right)L_C = 0,$$

i.e. $Q \in \mathcal{Q}^{(0)}$.

4. Parametrizations of all the static output feedbacks

In this section, we give two parametrizations for the set of all the stabilizing SOFs. We start with the following lemma, which was extensively used in [20]:

Lemma 4.1 A system (A, B, C) is stabilizable if and only if (A, B) and (A^T, C^T) are stabilizable and there exist matrices $X, Y \in \mathbb{R}^{n\times n}$ such that $A - BB^+X$ and $A - YC^+C$ are stable and $BB^+X = YC^+C$. When the conditions hold, the set of all stabilizing SOFs related to the chosen matrices X, Y is given by $K_X = B^+XC^+ + L_BS + TR_C$ or by $K_Y = B^+YC^+ + L_BF + HR_C$ respectively, where S, T, F, H are any $m \times r$ matrices. The closed-loop matrix is given by $E = A - BK_XC = A - BB^+X = A - YC^+C = A - BK_YC$.

Remark 4.1 Under the hypotheses of Corollary 3.1, note that $X = X_0 + WP^{-1} + R_BL$ and $Y = Y_0 + Q^{-1}V + ML_C$ satisfies $BB^+X = YC^+C$ if and only if $BB^+X_0 + WP^{-1} = Y_0C^+C + Q^{-1}V$, since $BB^+W = W, BB^+R_B = 0, VC^+C = V, L_CC^+C = 0$. Moreover, this condition can be simplified to (meaning that it does not include matrix inverses):

$$\begin{aligned} &QBB^+\left(I + AP + PA^T\right)\left(I - \frac{1}{2}BB^+\right) + QW = \\ &= \left(I - \frac{1}{2}C^+C\right)\left(I + A^TQ + QA\right)C^+CP + VP. \end{aligned} \tag{28}$$

We can state now the first parametrization for the set of all the stabilizing SOFs:

Corollary 4.1 Let (A, B, C) be a given system triplet. Assume that (A, B), (A^T, C^T) are controllable. Then, the system has a stabilizing static output feedback if and only if there exist $P, Q > 0$ and W, V such that

$$\begin{cases} R_B\left(I + AP + PA^T\right)R_B = 0 \ \left(i.e. P \in \mathcal{P}^{(0)}\right) \\ L_C\left(I + A^TQ + QA\right)L_C = 0 \ \left(i.e. Q \in \mathcal{Q}^{(0)}\right) \\ W^T = -W, R_BW = 0 \\ V^T = -V, VL_C = 0 \\ QBB^+\left(I + AP + PA^T\right)\left(I - \frac{1}{2}BB^+\right) + QW = \\ = \left(I - \frac{1}{2}C^+C\right)\left(I + A^TQ + QA\right)C^+CP + VP. \end{cases}$$

In this case, $A - BKC$ is stable if and only if

$$\begin{cases} K = K_X = B^+XC^+ + L_BS + TR_C \\ X = X_0 + WP^{-1} + R_BL \\ X_0 = \left(I + AP + PA^T\right)\left(I - \frac{1}{2}BB^+\right)P^{-1}, \end{cases}$$

where S, T, L are arbitrary.

Similarly, $A - BKC$ is stable if and only if

$$\begin{cases} K = K_Y = B^+YC^+ + L_BF + HR_C \\ Y = Y_0 + Q^{-1}V + ML_C \\ Y_0 = Q^{-1}\left(I - \frac{1}{2}C^+C\right)\left(I + A^TQ + QA\right), \end{cases}$$

where F, H, M are arbitrary.

We conclude this section with a second SOF parametrization:

Corollary 4.2 Let (A, B) and $\left(A^T, C^T\right)$ be controllable pairs. Then, $A - BKC$ is stable if and only if there exists $K^{(0)} \in \mathcal{K}^{(0)}\left(P^{(0)}\right)$ for some $P^{(0)} \in \mathcal{P}^{(0)}$, such that $K^{(0)}L_C = 0$. In this case, the set of all K's such that $A - BKC$ is stable, is given by $K = K^{(0)}C^+ + GR_C$ where $K^{(0)} \in \mathcal{K}^{(0)}\left(P^{(0)}\right)$, and G is arbitrary.

Proof: If there exists $K^{(0)} \in \mathcal{K}^{(0)}\left(P^{(0)}\right)$ such that $K^{(0)}L_C = 0$ for some $P^{(0)} \in \mathcal{P}^{(0)}$ then $K^{(0)} = K^{(0)}C^+C$. Since $K^{(0)} \in \mathcal{K}^{(0)}\left(P^{(0)}\right)$ it follows that $A - BK^{(0)}$ is stable. Thus, for $K = K^{(0)}C^+$ we get that $A - BKC$ is stable.

Conversely, if $A - BKC$ is stable for some K then, for $K^{(0)} = KC$ we have $K^{(0)}L_C = 0$ and since $A - BK^{(0)}$ is stable, Theorem 3.2 implies that there exists $P^{(0)} \in \mathcal{P}^{(0)}$ such that $K^{(0)} \in \mathcal{K}^{(0)}\left(P^{(0)}\right)$. ■.

Remark 4.2 Note that when (A, BB^+) is stabilizable, there exists an orthogonal matrix V such that

$$V^TAV = \begin{bmatrix} \widehat{A}_{1,1} & 0 \\ \widehat{A}_{2,1} & \widehat{A}_{2,2} \end{bmatrix}, V^TBB^+V = \begin{bmatrix} 0 & 0 \\ 0 & \widehat{B}_{2,2}\widehat{B}_{2,2}^+ \end{bmatrix},$$

where $\widehat{A}_{1,1}$ is stable and $\left(\widehat{A}_{2,2}, \widehat{B}_{2,2}\widehat{B}_{2,2}^+\right)$ is controllable (see [35], Lemma 3.1, p. 536). Thus, we may assume without loss of generality that the given pair is controllable. If (A, B, C) is a system triplet such that (A, BB^+) and $\left(A^T, C^+C\right)$ are stabilizable then, there exists an orthogonal matrix V such that $\left(\widehat{A}_{2,2}, \widehat{B}_{2,2}\widehat{B}_{2,2}^+\right)$ and $\left(\widehat{A}_{2,2}^T, \widehat{C}_{2,2}^+\widehat{C}_{2,2}\right)$ are controllable, where $\widehat{C} = V^TC^+CV$ is partitioned accordingly (see [35], Theorem 4.1 and Remark 4.1, p. 539). Thus, the assumption in Corollary 4.2 that the given pairs are controllable does not make any loss of generality of the results.

The effectiveness of the method is shown in the following example, but first, for the convenience of the reader, we summarize the whole method in Algorithm 1 (with its continuation in Algorithm 2). Let $f(K)$ denote a target function of the SOF K, to be minimized (e.g., $\|K\|_F$, the LQR functional, the H_∞-norm or the H_2-norm of the closed loop, the pole-placement errors of the closed loop, or any other key performance that depends on K).

Regarding the LQR problem, let the LQR functional be defined by:

$$J(x_0, u) = \int_0^\infty \left(x(t)^TQx(t) + u(t)^TRu(t)\right)dt, \tag{29}$$

where $Q > 0$ and $R \geq 0$ are given. We need to find $u(t)$ that minimizes the functional value for any initial disturbance x_0 from the equilibrium point 0. Assuming that $u(t)$ is realized by a stabilizing SOF, let $u(t) = -Ky(t) = -KCx(t)$. Then, by substitution of the last into (29), we get:

$$J(x_0, K) = \int_0^\infty x(t)^T\left(Q + C^TK^TRKC\right)x(t)dt. \tag{30}$$

Now, since $Q + C^T K^T RKC > 0$ and since $E := A - BKC$ is stable, the Lyapunov equation:

$$E^T P + PE = -\left(Q + C^T K^T RKC\right), \tag{31}$$

has unique solution $P_{LQR}(K) > 0$ given by:

$$\begin{aligned} P_{LQR}(K) &= \int_0^\infty \exp\left(E^T t\right)\left(Q + C^T K^T RKC\right)\exp\left(Et\right)dt = \\ &= -mat\left(\left(I \otimes E^T + E^T \otimes I\right)^{-1}\right) vec\left(Q + C^T K^T RKC\right). \end{aligned} \tag{32}$$

By substitution of (31) into (30), we get:

$$J(x_0, K) = x_0^T P_{LQR}(K) x_0 = \left\| P_{LQR}(K)^{\frac{1}{2}} x_0 \right\|_2^2. \tag{33}$$

Algorithm 1. An Algorithm For Optimal SOF's.

Require: *An algorithm for optimizing $f(K)$ under LMI and linear constraints, an algorithm for computing the Moore-Penrose pseudo-inverse and an algorithm for orthogonal diagonalization.*

Input: *System triplet (A, B, C) such that (A, B), $\left(A^T, C^T\right)$ are controllable.*

Output: *SOF K such that $A - BKC$ is stable minimizing $f(K)$— if exists*

1. $A^{(0)} \leftarrow A$
2. $B^{(0)} \leftarrow B$
3. $i \leftarrow 0$
4. $k_0 \leftarrow rank\left(B^{(0)}\right)$
5. **while** $B^{(i)} B^{(i)+} \neq I_{k_i}$ **do**
6. *compute orthogonal matrix* $U^{(i)}$ *such that* $U^{(i)T} B^{(i)} B^{(i)+} U^{(i)} = bdiag(I_{k_i}, 0)$
7. $\widehat{A^{(i)}} \leftarrow U^{(i)T} A^{(i)} U^{(i)}$
8. *partition* $\widehat{A^{(i)}} = \begin{bmatrix} \widehat{A^{(i)}}_{1,1} & \widehat{A^{(i)}}_{1,2} \\ \widehat{A^{(i)}}_{2,1} & \widehat{A^{(i)}}_{2,2} \end{bmatrix}$
9. $A^{(i+1)} \leftarrow \widehat{A^{(i)}}_{2,2}$
10. $B^{(i+1)} \leftarrow \widehat{A^{(i)}}_{2,1}$
11. $i \leftarrow i + 1$
12. $k_i \leftarrow rank\left(B^{(i)}\right)$
13. **end while**
14. $b \leftarrow i$
15. *let* $P^{(b)}$ *be a symbol for* $P^{(b)} > 0$
16. $X_0^{(b)} \leftarrow \frac{1}{2}\left(I_{n_b} + A^{(b)} P^{(b)} + P^{(b)} A^{(b)T}\right)\left(P^{(b)}\right)^{-1}$
17. *let* $W^{(b)}$ *be a symbol for a matrix satisfying* $W^{(b)T} = -W^{(b)}$
18. $X^{(b)} \leftarrow X_0^{(b)} + W^{(b)}\left(P^{(b)}\right)^{-1}$
19. *let* $F^{(b)}$ *be a symbol for arbitrary matrix*
20. $K^{(b)} \leftarrow B^{(b)+} X^{(b)} + L_{B^{(b)}} F^{(b)}$

Thus,

$$\begin{aligned}
J(x_0, K) &= \left\| P_{LQR}(K)^{\frac{1}{2}} x_0 \right\|_2^2 \leq \\
&\leq \left\| P_{LQR}(K)^{\frac{1}{2}} \right\|^2 \|x_0\|_2^2 = \\
&= \left\| P_{LQR}(K) \right\| \|x_0\|_2^2 = \\
&= \sigma_{max}\left(P_{LQR}(K)\right) \|x_0\|_2^2,
\end{aligned}$$

Algorithm 2. An Algorithm For Optimal SOF's, Continued.

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1. **for** $i = b-1$ *downto* 0 **do**
2. *let* $\Delta\widehat{P^{(i)}}_{1,1}$ *be a symbol for* $\Delta\widehat{P^{(i)}}_{1,1} > 0$
3. $\widehat{P^{(i)}} \leftarrow \begin{bmatrix} \Delta\widehat{P^{(i)}}_{1,1} + K^{(i+1)}P^{(i+1)}K^{(i+1)T} & -K^{(i+1)}P^{(i+1)} \\ -P^{(i+1)}K^{(i+1)T} & P^{(i+1)} \end{bmatrix}$
4. $\left(\widehat{P^{(i)}}\right)^{-1} \leftarrow \begin{bmatrix} \left(\Delta\widehat{P^{(i)}}_{1,1}\right)^{-1} & \left(\Delta\widehat{P^{(i)}}_{1,1}\right)^{-1} K^{(i+1)} \\ K^{(i+1)T}\left(\Delta\widehat{P^{(i)}}_{1,1}\right)^{-1} & \left(P^{(i+1)}\right)^{-1} + K^{(i+1)T}\left(\Delta\widehat{P^{(i)}}_{1,1}\right)^{-1} K^{(i+1)} \end{bmatrix}$
5. $P^{(i)} \leftarrow U^{(i)}\widehat{P^{(i)}}U^{(i)T}$
6. $\left(P^{(i)}\right)^{-1} \leftarrow U^{(i)}\left(\widehat{P^{(i)}}\right)^{-1}U^{(i)T}$
7. $X_0^{(i)} \leftarrow \left(I_{n_i} + A^{(i)}P^{(i)} + P^{(i)}A^{(i)T}\right)\left(I_{n_i} - \frac{1}{2}B^{(i)}B^{(i)+}\right)\left(P^{(i)}\right)^{-1}$
8. *let* $W^{(i)}$ *be a symbol for a matrix satisfying* $W^{(i)T} = -W^{(i)}$ *and* $R_{B^{(i)}}W^{(i)} = 0$
9. *let* $L^{(i)}$ *be a symbol for arbitrary matrix*
10. *let* $F^{(i)}$ *be a symbol for arbitrary matrix*
11. $X^{(i)} \leftarrow X_0^{(i)} + W^{(i)}\left(P^{(i)}\right)^{-1} + R_{B^{(i)}}L^{(i)}$
12. $K^{(i)} \leftarrow B^{(i)+}X^{(i)} + L_{B^{(i)}}F^{(i)}$
13. **end for**
14. *optimize* $f(K)$ *under the matrix equation* $K^{(0)}L_C = 0$
    and the constraints $\Delta\widehat{P^{(0)}}_{1,1} > 0, \ldots, \Delta\widehat{P^{(b-1)}}_{1,1} > 0, P^{(b)} > 0$
    with respect to $F^{(0)}, \ldots, F^{(b)}$ *to* $W^{(0)}, \ldots, W^{(b)}$
    and to $\Delta\widehat{P^{(0)}}_{1,1}, \ldots, \Delta\widehat{P^{(b-1)}}_{1,1}, P^{(b)}$ *as variables*
15. **if** *a solution was found* **then**
16. **return** K
17. **else**
18. **return** "*no solution was found*"
19. **end if**

---

where $\sigma_{max}\left(P_{LQR}(K)\right)$ is the largest eigenvalue of $P_{LQR}(K)$. Therefore,

$$\frac{J(x_0, K)}{\|x_0\|_2^2} \leq \sigma_{max}\left(P_{LQR}(K)\right). \tag{34}$$

Now, if x_0 is known then we can minimize $J(x_0, K)$ by minimizing $x_0^T P_{LQR}(K) x_0$. Otherwise, and if we design for the worst-case, we need to minimize $\sigma_{max}\left(P_{LQR}(K)\right)$.

In the following examples, we have executed the algorithm on Processor: Intel (R) Core(TM) i5-2400 CPU @ 3.10GHz 3.10 GHz, RAM: 8.00 GB, Operating System: Windows 10, System Type: 64-bit Operating System, x64-based processor, Platform: MATLAB®, Version: R2018b, Function: fmincon.

Example 4.1 A system of Boeing B-747 aircraft (the "AC5" system in [42] and see also [43]) is given by the general model (given here with slight changes):

$$\begin{cases} \dfrac{d}{dt}x(t) = Ax(t) + B_1w(t) + Bu(t) \\ z(t) = C_1x(t) + D_{1,1}w(t) + D_{1,2}u(t) \\ y(t) = Cx(t) + D_{2,1}w(t), \end{cases}$$

where x is the state, w is the noise, u is the control input, z is the regulated output, and y is the measurement, where:

$$A = \begin{bmatrix} 0.980100000000000 & 0.000300000000000 & -0.098000000000000 & 0.003800000000000 \\ -0.386800000000000 & 0.907100000000000 & 0.047100000000000 & -0.000800000000000 \\ 0.159100000000000 & -0.001500000000000 & 0.969100000000000 & 0.000300000000000 \\ -0.019800000000000 & 0.095800000000000 & 0.002100000000000 & 1.000000000000000 \end{bmatrix}$$

$$B = \begin{bmatrix} -0.000100000000000 & 0.005800000000000 \\ 0.029600000000000 & 0.015300000000000 \\ 0.001200000000000 & -0.090800000000000 \\ 0.001500000000000 & 0.000800000000000 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$B_1 = B, C_1 = C, D_{1,1} = 0_4, D_{2,1} = 0_{2\times 4}, D_{1,2} = 0_{4\times 2}.$

with

$$\sigma(A) = \begin{Bmatrix} 0.978871342065923 \pm 0.128159143146289i \\ 0.899614120838404 \\ 0.998943195029751 \end{Bmatrix}.$$

Note that (A, B) and $\left(A^T, C^T\right)$ here are controllable. Let $u = u_r - Ky$, where u_r is a reference input. Then, $u = u_r - KCx - KD_{2,1}w$ and substitution the last into the system yields the closed-loop system:

$$\begin{cases} \dfrac{d}{dt}x(t) = (A - BKC)x(t) + (B_1 - BKD_{2,1})w(t) + Bu_r(t) \\ z(t) = (C_1 - D_{1,2}KC)x(t) + (D_{1,1} - D_{1,2}KD_{2,1})w(t) + D_{1,2}u_r(t), \end{cases}$$

where the behavior of z is of our interest. Note that we actually have:

$$\begin{cases} \dfrac{d}{dt}x(t) = (A - BKC)x(t) + Bw(t) + Bu_r(t) \\ z(t) = Cx(t) = y(t). \end{cases}$$

For the stabilization via SOF with minimal Frobenius-norm, we need to minimize $f(K) = \|K\|_F$. For the LQR problem we need to minimize $f(K) = x_0^T P_{LQR}(K)x_0$ when x_0 is known and to minimize $f(K) = \sigma_{max}\left(P_{LQR}(K)\right)$ when x_0 is unknown, where $P_{LQR}(K)$ is given by (32). For the H_∞ and the H_2 problems, we need to minimize $f(K) = \|T_{w,z}(s)\|_{H_\infty}$ and $f(K) = \|T_{w,z}(s)\|_{H_2}$, resp. where:

$$T_{w,z}(s) = (D_{1,1} - D_{1,2}KD_{2,1}) + (C_1 - D_{1,2}KC)(sI - A + BKC)^{-1}(B_1 - BKD_{2,1}) = \\ = C(sI - A + BKC)^{-1}B.$$

These problems needed to be solved under the constraint that $A - BKC$ stable, i.e., that $K = K^{(0)}C^{+} + GR_C$, where $K^{(0)} \in K^{(0)}\left(P^{(0)}\right)$ for some $P^{(0)} \in P^{(0)}$, such that $K^{(0)}L_C = 0$.

Applying the algorithm we had:

$$U^{(0)} = \begin{bmatrix} -0.063882699439918 & 0 & -0.997957414277919 & 0 \\ 0.012195875662698 & 0.998643130545343 & -0.000780700106257 & -0.050621625208610 \\ 0.997882828566422 & -0.012222870716218 & -0.063877924951025 & 0.000269421014890 \\ 0.000348971870720 & 0.050621134381318 & -0.000022338894237 & 0.998717867304654 \end{bmatrix}$$

$$A^{(1)} = \begin{bmatrix} 0.983650838535766 & -0.003772277911855 \\ 0.000091490905229 & 0.994959072276129 \end{bmatrix}, B^{(1)} = \begin{bmatrix} 0.098883972659475 & -0.001544978964136 \\ 0.000939225045070 & 0.100248978115558 \end{bmatrix}.$$

The “while-loop” stops because $B^{(1)}B^{(1)+} = I_2$. We have

$$B^{(0)+} = \begin{bmatrix} -0.386571795892900 & 33.468356299440529 & 5.629731696802776 & 1.694878880538571 \\ 0.696955535886478 & 0.442712098375399 & -10.893875111014371 & 0.025378383262705 \end{bmatrix}$$

$$B^{(1)+} = \begin{bmatrix} 10.111382188357680 & 0.155830743345230 \\ -0.094732770050116 & 9.973704058215121 \end{bmatrix}$$

$$L_{B^{(0)}} = 0_2, R_{C^{(0)}} = 0_2, L_{B^{(1)}} = 0_2,$$

$$R_{B^{(0)}} = \begin{bmatrix} 0.995919000712269 & 0.000779105459367 & 0.063747448813564 & 0.000022293265130 \\ 0.000779105459367 & 0.002563158431417 & 0.000036230973158 & -0.050556704127861 \\ 0.063747448813564 & 0.000036230973158 & 0.004080461883732 & 0.000270502543607 \\ 0.000022293265130 & -0.050556704127861 & 0.000270502543607 & 0.997437378972582 \end{bmatrix}, R_{B^{(1)}} = 0_2$$

$$I_4 - \frac{1}{2}B^{(0)}B^{(0)+} = \begin{bmatrix} 0.997959500356135 & 0.000389552729683 & 0.031873724406782 & 0.000011146632565 \\ 0.000389552729683 & 0.501281579215708 & 0.000018115486579 & -0.025278352063931 \\ 0.031873724406782 & 0.000018115486579 & 0.502040230941866 & 0.000135251271804 \\ 0.000011146632565 & -0.025278352063931 & 0.000135251271804 & 0.998718689486291 \end{bmatrix}$$

$$I_2 - \frac{1}{2}B^{(1)}B^{(1)+} = \frac{1}{2}I_2.$$

Now, we parameterize all the matrices $K^{(0)}$ such that $A^{(0)} - B^{(0)}K^{(0)}$ is stable. Let $P^{(1)} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix}$, where $p_1, d_1 := p_1p_3 - p_2^2 > 0$. Let w_1 be arbitrary and let $W^{(1)} = \begin{bmatrix} 0 & w_1 \\ -w_1 & 0 \end{bmatrix}$. Let

$$S^{(1)} = B^{(1)+}\left(\frac{1}{2}\left(I_2 + A^{(1)}P^{(1)} + P^{(1)}A^{(1)T}\right) + W^{(1)}\right).$$

Then

$$K^{(1)} = S^{(1)}\left(P^{(1)}\right)^{-1}.$$

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Let $\Delta\widehat{P^{(0)}}_{1,1} = \begin{bmatrix} p_4 & p_5 \\ p_5 & p_6 \end{bmatrix}$, where $p_4, d_2 := p_4 p_6 - p_5^2 > 0$. Then,

$$P^{(0)} = U^{(0)} \begin{bmatrix} \Delta\widehat{P^{(0)}}_{1,1} + K^{(1)} P^{(1)} K^{(1)T} & -K^{(1)} P^{(1)} \\ -P^{(1)} K^{(1)T} & P^{(1)} \end{bmatrix} U^{(0)T} =$$

$$= U^{(0)} \begin{bmatrix} \Delta\widehat{P^{(0)}}_{1,1} + S^{(1)} K^{(1)T} & -S^{(1)} \\ -S^{(1)T} & P^{(1)} \end{bmatrix} U^{(0)T},$$

and

$$\left(P^{(0)}\right)^{-1} = U^{(0)} \begin{bmatrix} \left(\Delta\widehat{P^{(0)}}_{1,1}\right)^{-1} & \left(\Delta\widehat{P^{(0)}}_{1,1}\right)^{-1} K^{(1)} \\ K^{(1)T}\left(\Delta\widehat{P^{(0)}}_{1,1}\right)^{-1} & \left(P^{(1)}\right)^{-1} + K^{(1)T}\left(\Delta\widehat{P^{(0)}}_{1,1}\right)^{-1} K^{(1)} \end{bmatrix} U^{(0)T}.$$

Let

$$S^{(0)} = B^{(0)+}\left(I_4 + A^{(0)} P^{(0)} + P^{(0)} A^{(0)T}\right)\left(I_4 - \frac{1}{2} B^{(0)} B^{(0)+}\right).$$

Then

$$K^{(0)} = S^{(0)}\left(P^{(0)}\right)^{-1}.$$

Note that $W^{(0)T} = -W^{(0)}, R_{B^{(0)}} W^{(0)} = 0$ implies that $W^{(0)} = 0_4$. We have completed the parametrization of all the SFs of the system, where the parameters $W^{(1)}$, $P^{(1)} > 0$, and $\Delta\widehat{P^{(0)}}_{1,1} > 0$ are free.

Regarding the optimization stage, we had the following results: starting from the point (feasible for SF but not feasible for SOF):

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & w_1 \end{bmatrix} =$$
$$= \begin{bmatrix} 750 & 0 & 750 & 750 & 0 & 750 & 0 \end{bmatrix},$$

in $CPU - Time = 0.59375[sec]$ the fmincon function (with the interior-point option and the default optimization parameters) has converged to the optimal point

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & w_1 \end{bmatrix} =$$
$$= 10^7 \cdot \begin{bmatrix} 0.009103654227197 & 0.000105735816664 & 0.006486122495094 & 1.912355216342858 & 0.057053301125719 & 1.792930229298237 & -0.000647092932030 \end{bmatrix},$$

resulting with the following optimal Frobenius-norm SF and SOF

$$K^{(0)} = 10^3 \cdot \begin{bmatrix} -0.157533999747776 & -0.000000000000000 & -0.000000000000000 & 1.273776653891793 \\ 0.332650954180600 & -0.000000000000000 & -0.000000000000000 & 0.000655866128882 \end{bmatrix}$$

$$K = 10^3 \cdot \begin{bmatrix} -0.157533999747776 & 1.273776653891793 \\ 0.332650954180600 & 0.000655866128882 \end{bmatrix},$$

with $\|K\|_F = 1.325888763265586 \cdot 10^3$. The resulting closed-loop eigenvalues are:

$$\sigma\left(A^{(0)} - B^{(0)}KC^{(0)}\right) = \left\{ \begin{array}{c} -0.000004083911866 \pm 1.423412274467895i \\ -0.000005220069661 \pm 1.681989978268793i \end{array} \right\}.$$

For a comparison, in this (small) example we had seven scalar indeterminate, four scalar equations and four scalar inequalities while by the BMI method $\left(A^{(0)} - B^{(0)}KC^{(0)}\right)P + P\left(A^{(0)} - B^{(0)}KC^{(0)}\right)^T < 0, P > 0$ we would have 14 scalar indeterminate and eight scalar inequalities. This shows the potential of the method in reducing the number of variables and inequalities/equations, thus enabling to deal efficiently with larger problems. Moreover, the method removes the decoupling of P and K, in the sense that now K depends on P and the dependence of P on K has removed, thus making the problem more relaxed.

Figures 1 and **2** show the impulse response and the step response of the closed-loop system, in terms of the regulated output $z = y$, where $w = 0$ and u_r is the delta Dirac function or the unit-step function, respectively. While the amplitudes seem to be reasonable, the settling time of order 10^5 seems unreasonable. This happens because lowering the SOF-norm results in pushing the closed-loop eigenvalues toward the imaginary axes, as can be seen from the dense oscillations. We therefore must set a barrier on the abscissa of the closed-loop eigenvalues as a constraint. Note however that as a starting point for other optimization keys where we need any stabilizing SOF that we can get, the above SOF might be sufficient.

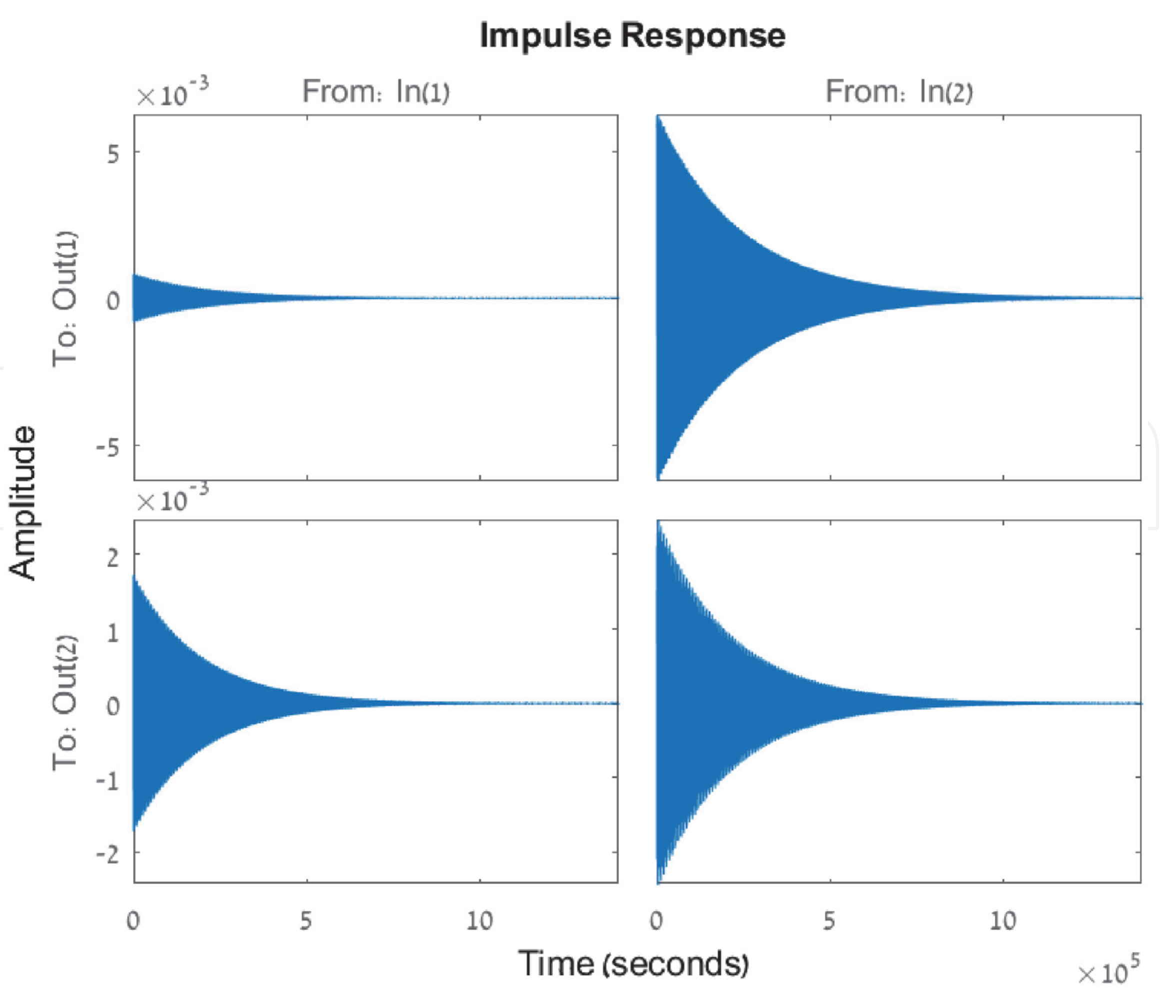

Figure 1.
Impulse response of the closed loop with the minimal-norm SOF.

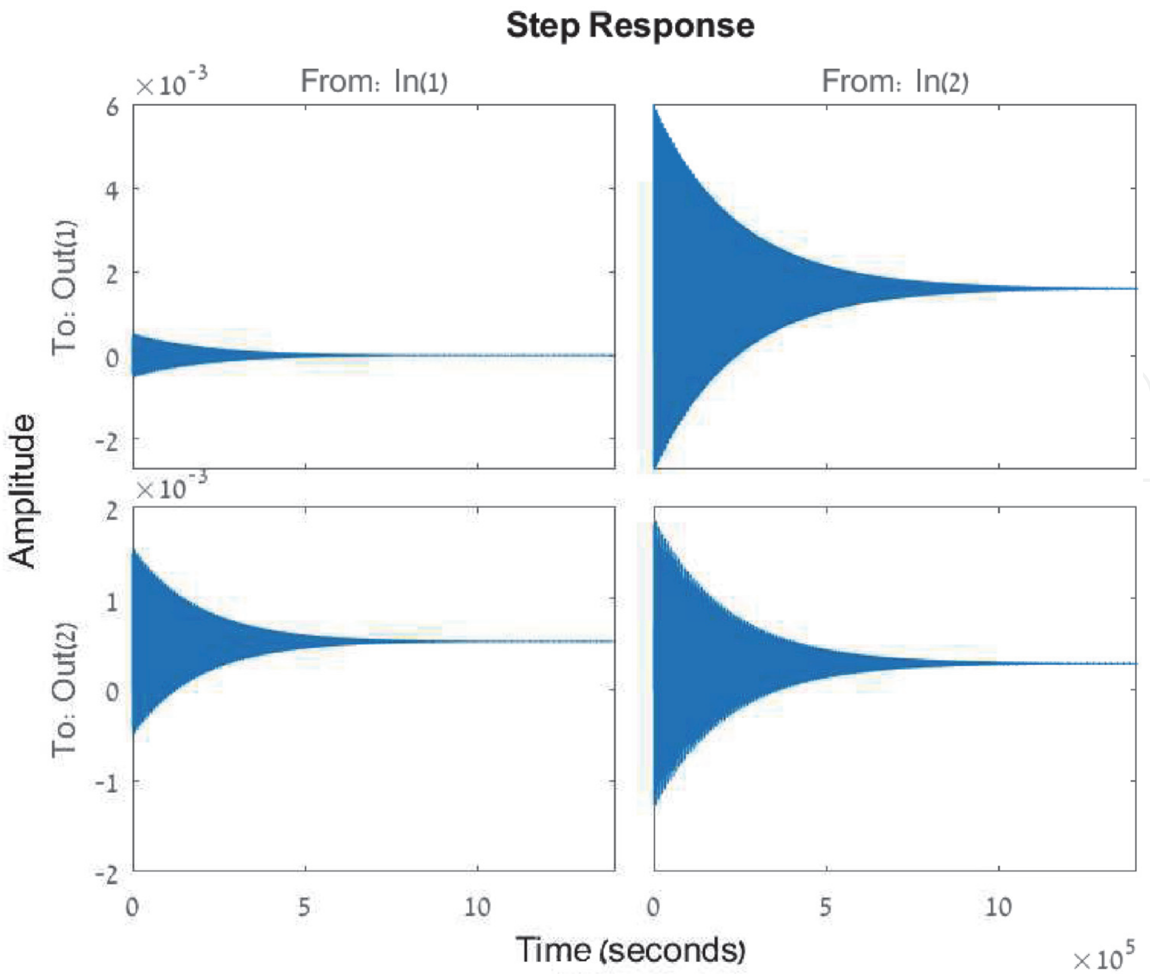

Figure 2.
Step response of the closed loop with the minimal-norm SOF.

Regarding the LQR functional with $Q = I, R = I$, starting from:

$$\begin{aligned}&\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & w_1 \end{bmatrix} = \\ &= \begin{bmatrix} 300 & 0 & 300 & 300 & 0 & 300 & 15 \end{bmatrix},\end{aligned}$$

in $CPU - Time = 0.90625[sec]$ the fmincon function has converged to the optimal point

$$\begin{aligned}&\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & w_1 \end{bmatrix} = \\ &= 10^2 \cdot \begin{bmatrix} 0.010603084813420 & -0.002595549083521 & 0.009614240830002 & 1.009076872432389 & -0.832493933321482 & 1.812148701422345 & 0.001750170924431 \end{bmatrix},\end{aligned}$$

resulting with the following optimal SF and SOF

$$K^{(0)} = 10^3 \cdot \begin{bmatrix} -1.466094499085196 & 0.000000000000002 & -0.000000000000000 & 2.731682559443639 \\ 0.703352598722306 & 0.000000000000000 & 0.000000000000001 & -0.149101634953946 \end{bmatrix}$$

$$K = 10^3 \cdot \begin{bmatrix} -1.466094499085196 & 2.731682559443639 \\ 0.703352598722306 & -0.149101634953946 \end{bmatrix},$$

with $\|K\|_F = 3.182523976577676 \cdot 10^3$ and LQR wort-case functional-value $\sigma_{max}(P_{LQR}(K)) = 1.981249586261248 \cdot 10^6$. The resulting closed-loop eigenvalues are:

$$\sigma\left(A^{(0)} - B^{(0)}KC^{(0)}\right) = \left\{ \begin{array}{c} -1.723892066022943 \pm 1.346849871126735i \\ -0.450106460827155 \pm 1.711908728695912i \end{array} \right\}.$$

The entries of $z = y$ under $w = 0$ and $u_r = 0$, when the closed-loop system is derived by the initial condition $x_0 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}^T$ are depicted in **Figure 3**. The results might not be satisfactory regarding the amplitudes or the settling time; however, as a starting point for other optimization keys where we need any stabilizing SOF that we can get, the above SOF might be sufficient.

For the problem of pole placement via SOF, assume that the target is to place the closed-loop eigenvalue as close as possible to $-10 \pm i, \; -1 \pm 0.1i$. Then, starting from:

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & w_1 \end{bmatrix} = \\ = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix},$$

in $CPU - Time = 1.828125[sec]$ the fmincon function has converged to the optimal point

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & w_1 \end{bmatrix} = \\ = 10^3 \cdot \begin{bmatrix} 0.017543717092354 & -0.025265281638022 & 0.040984285298812 & 1.855250747170489 & -3.397079720955738 & 6.251476924192442 & 0.013791203700439 \end{bmatrix},$$

resulting with the following optimal SF and SOF

$$K^{(0)} = 10^5 \cdot \begin{bmatrix} -1.014246236498231 & 0.000000000000000 & 0.000000000000000 & 0.708417204523523 \\ -0.177349433397692 & 0.000000000000000 & 0.000000000000000 & 0.124922842398310 \end{bmatrix}$$

$$K = 10^5 \cdot \begin{bmatrix} -1.014246236498231 & 0.708417204523523 \\ -0.177349433397692 & 0.124922842398310 \end{bmatrix},$$

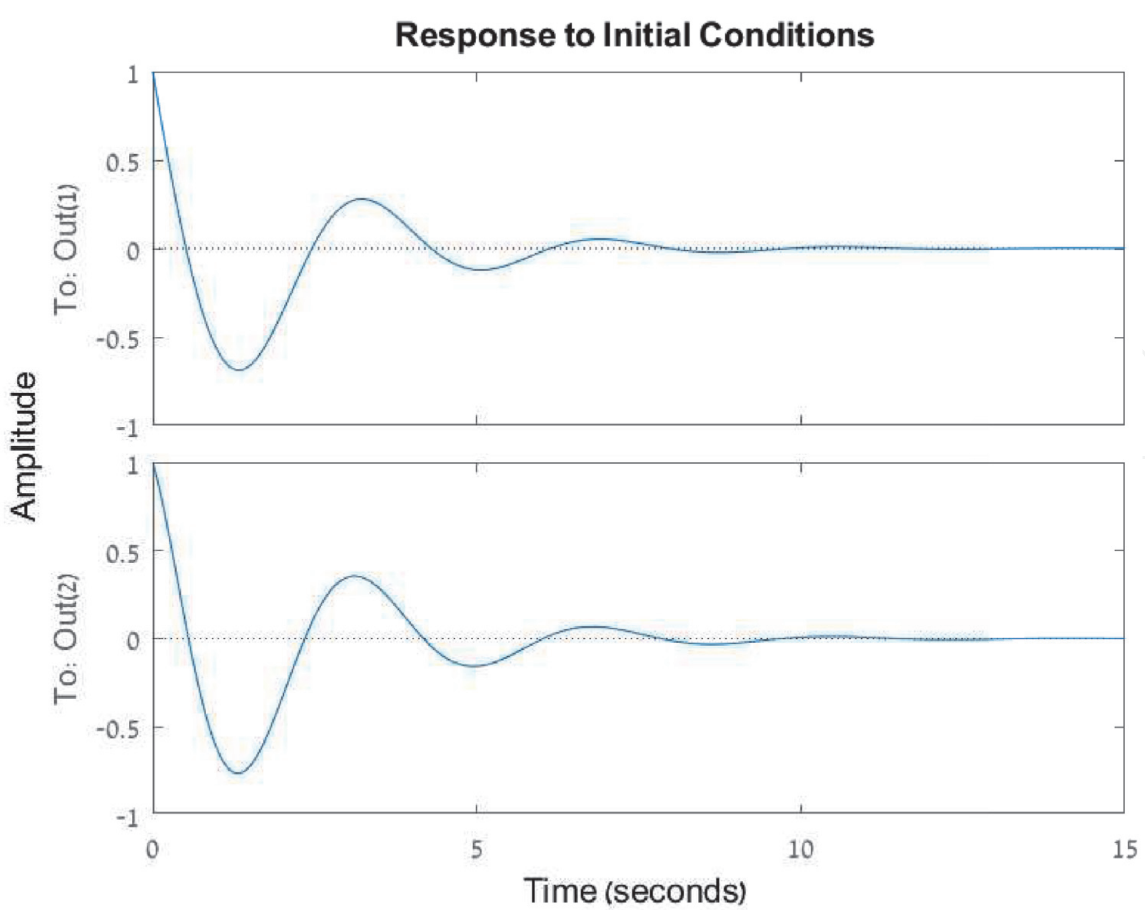

Figure 3.
Response of initial condition of the closed loop with the LQR SOF.

with $\|K\|_F = 1.256029021159584 \cdot 10^5$. The resulting closed-loop eigenvalues are:

$$\sigma\left(A^{(0)} - B^{(0)}KC^{(0)}\right) = \left\{ \begin{array}{c} -9.682859336407926 \pm 0.940019732471932i \\ -0.157090195949127 \pm 1.083963060760387i \end{array} \right\}.$$

Figures 4 and **5** depict the impulse response and the step response of the closed loop with the pole-placement SOF. The amplitudes look reasonable but the settling time might be unsatisfactory.

Regarding the H_∞-norm of the closed loop, starting from:

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & w_1 \end{bmatrix} = \\ = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 0 \end{bmatrix},$$

in $CPU - Time = 0.703125[sec]$ the fmincon function has converged to the optimal point

$$\begin{bmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & w_1 \end{bmatrix} = \\ = \begin{bmatrix} 0.888221316790683 & 0.005450463395221 & 0.509688006534611 & 0.351367770700493 & 0.108479534948988 & 2.135683618863295 & -0.023286321711901 \end{bmatrix},$$

resulting with the following optimal SF and SOF

$$K^{(0)} = 10^4 \cdot \begin{bmatrix} -0.434126348764817 & -0.000000000000000 & 0.000000000000000 & 6.230425140286776 \\ 6.139781209081431 & 0.000000000000000 & -0.000000000000000 & 0.417994561595739 \end{bmatrix}$$

$$K = 10^4 \cdot \begin{bmatrix} -0.434126348764817 & 6.230425140286776 \\ 6.139781209081431 & 0.417994561595739 \end{bmatrix},$$

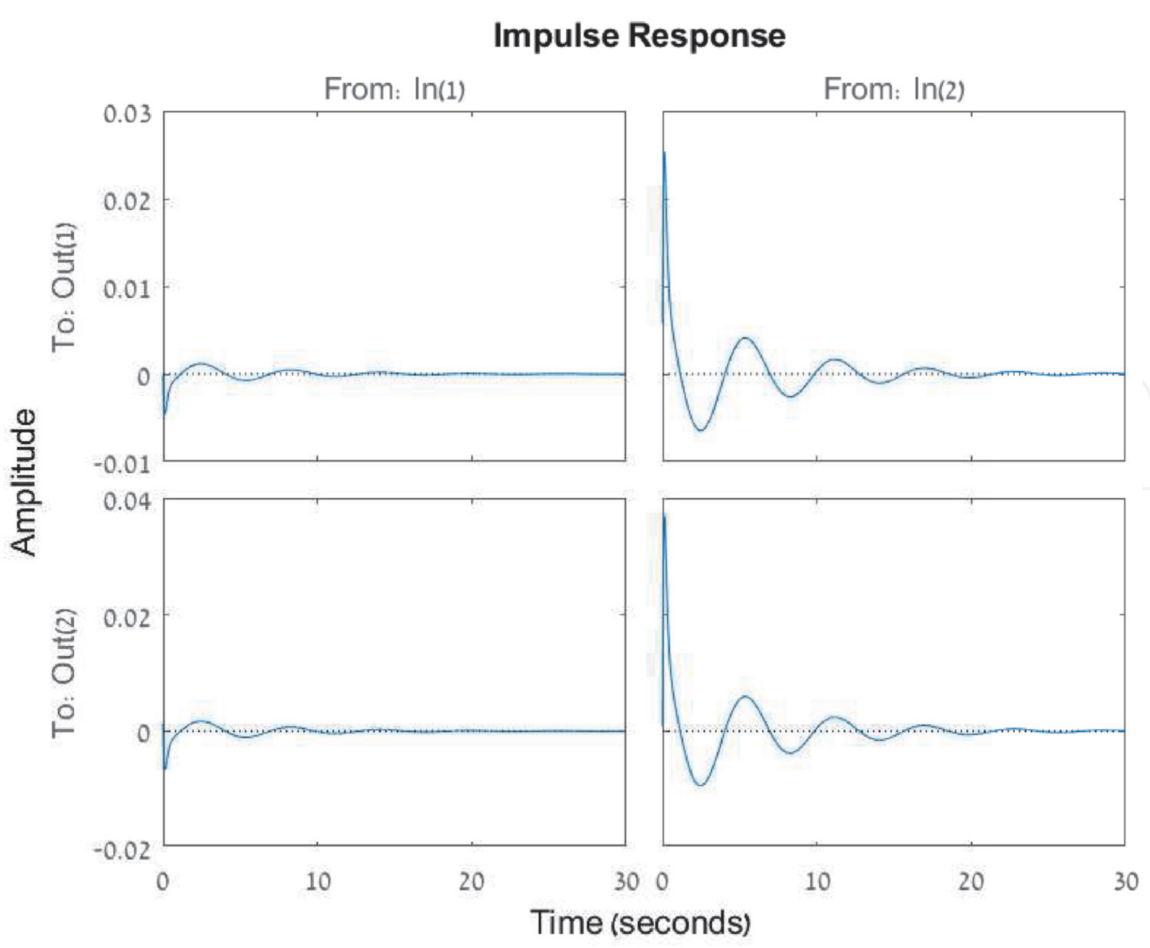

Figure 4.
Impulse response of the closed loop with the pole-placement SOF.

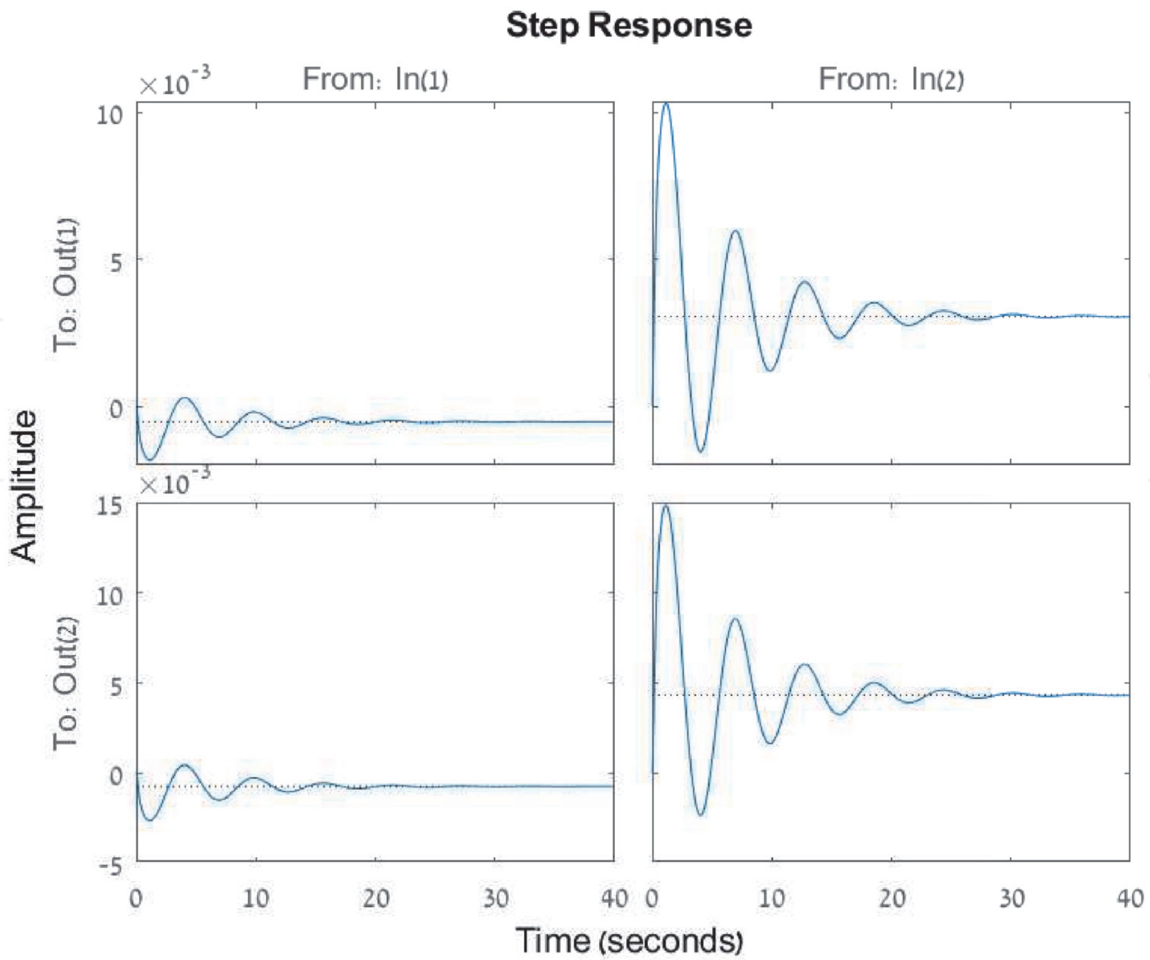

Figure 5.
Step response of the closed loop with the pole-placement SOF.

with $\|K\|_F = 8.768026908280017 \cdot 10^4$ and $\|T_{w,z}(s)\|_{H_\infty} = 1.631954397074613 \cdot 10^{-5}$. The resulting closed-loop eigenvalues are:

$$\sigma\left(A^{(0)} - B^{(0)}KC^{(0)}\right) = \left\{\begin{matrix} -356.9401845964764 \\ -90.9530886951882 \\ -1.0236129938218 \\ -0.5685837870690 \end{matrix}\right\}.$$

The simulation results of the closed-loop system are given in **Figure 6**, where w is normally distributed random disturbance, where each entry is $\mathcal{N}\left(0, 10^6\right)$ distributed. The maximum absolute values of the entries of $z = y$ are

$$[\,0.034719714201842 \quad 0.014588756724050\,]^T,$$

and the maximum absolute values of the entries of x are

$$[\,0.034719714201842 \quad 0.279876853192629 \quad 0.549124101316666 \quad 0.014588756724050\,]^T.$$

The results here are good.

Regarding the H_2-norm of the closed loop, starting from:

$$\begin{aligned} &[p_1 \quad p_2 \quad p_3 \quad p_4 \quad p_5 \quad p_6 \quad w_1] = \\ &= [1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0], \end{aligned}$$

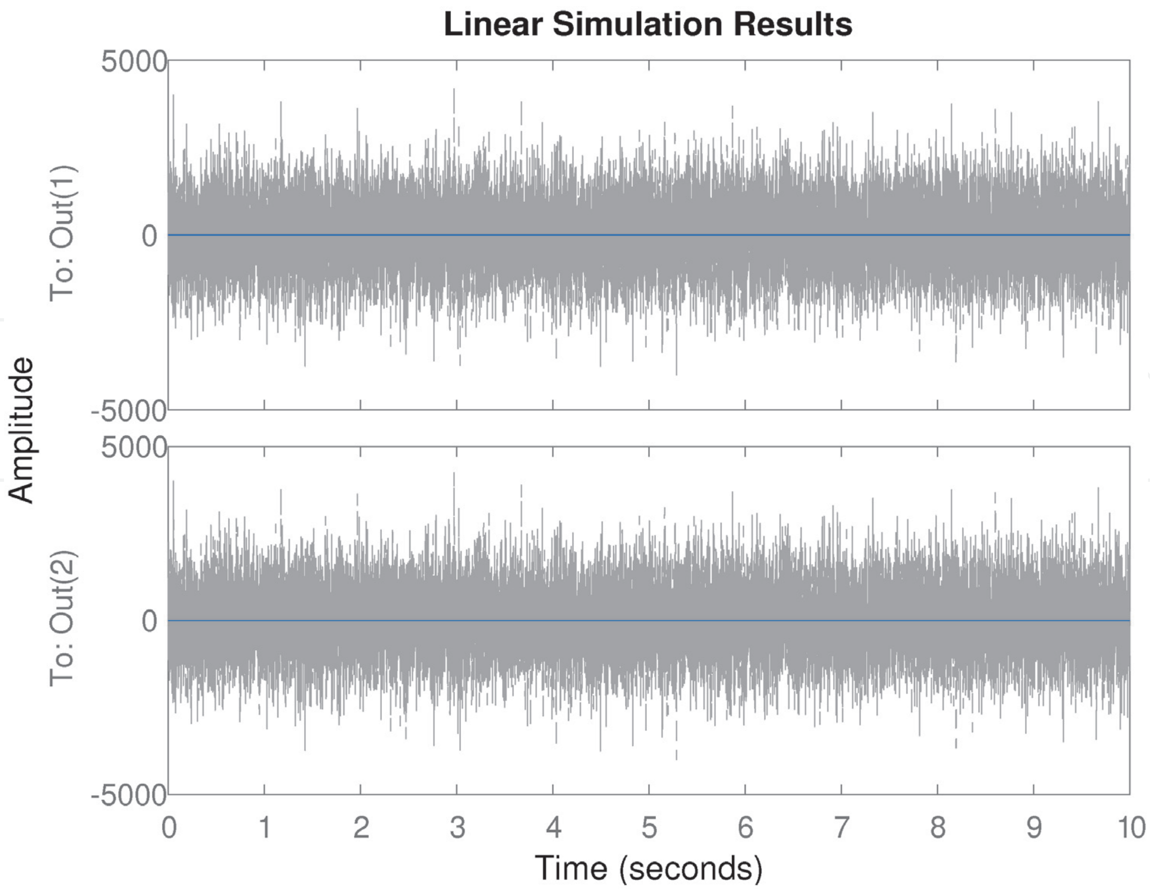

Figure 6.
Response of the closed loop with the optimal H_∞-norm SOF to a 10^6 variance zero-mean normally distributed random disturbance.

in $CPU - Time = 8.390625[sec]$ the fmincon function has converged to the optimal point

$$[p_1 \;\; p_2 \;\; p_3 \;\; p_4 \;\; p_5 \;\; p_6 \;\; w_1] =$$
$$= [0.891178477642138 \;\; 0.006639774876451 \;\; 0.508684007482598 \;\; 0.000038288661546 \;\; 0.000053652014908 \;\; 0.000140441315775 \;\; -0.021795268637180],$$

resulting with the following optimal SF and SOF

$$K^{(0)} = 10^9 \cdot \begin{bmatrix} 1.835262537587536 & 0.000000000003900 & 0.000000000012416 & 1.804218059606446 \\ 1.194244874676116 & -0.000000000002080 & 0.000000000007286 & 0.578593785387393 \end{bmatrix}$$
$$K = 10^9 \cdot \begin{bmatrix} 1.835262537587536 & 1.804218059606446 \\ 1.194244874676116 & 0.578593785387393 \end{bmatrix},$$

with $\|K\|_F = 2.895579903518702 \cdot 10^9$ and $\|T_{w,z}(s)\|_{H_2} = 2.289352128445973 \cdot 10^{-6}$. The resulting closed-loop eigenvalues are:

$$\sigma\left(A^{(0)} - B^{(0)}KC^{(0)}\right) = 10^6 \cdot \begin{Bmatrix} -8.825067937292802 \\ -1.087222799352728 \\ -0.000000561491544 \\ -0.000000982645227 \end{Bmatrix}.$$

The simulation results of the closed-loop system is given in **Figure 6**, where w is normally distributed random disturbance, where each entry is $\mathcal{N}(0, 10^6)$ distributed. The maximum absolute values of the entries of $z = y$ are

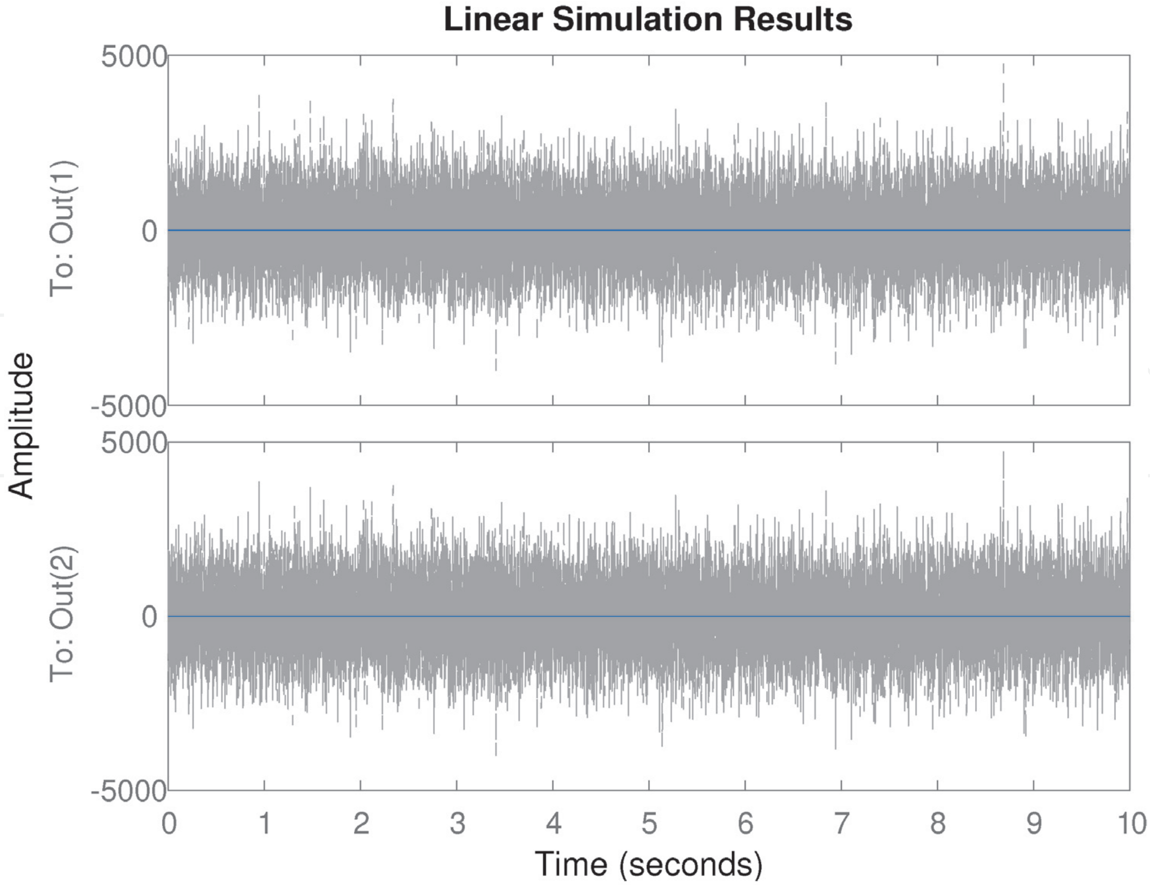

Figure 7.
Response of the closed loop with the optimal H_2-norm SOF to a 10^6 variance zero-mean normally distributed random disturbance.

$$10^{-5} \cdot [\,0.793706028985933 \quad 0.829879751812045\,]^T,$$

and the maximum absolute values of the entries of x are

$$10^{-5} \cdot [\,0.7937060289859 \quad 16.3502413073901 \quad 12.3720896708621 \quad 0.8298797518120\,]^T.$$

The results here are excellent.

We conclude that the best performance of the closed-loop system is achieved with the optimal H_2-norm SOF; however, since the Frobenius-norm of the SOF controller is high, the cost of construction and of operation of the SOF controller might be high, and there are no "free meals." Note also that by minimizing the SOF Frobenius-norm, the eigenvalues of the closed loop tend to get closer to the imaginary axes (to the region of lower degree of stability), while by minimizing the H_2-norm, the eigenvalues of the closed loop tend to escape from the imaginary axes (to the region of higher degree of stability). These are conflicting demands, and therefore, one should use some combination of the related key functions or to use some multiobjective optimization algorithm in order to get the best SOF in some or all of the needed key performance measures (**Figure** 7).

The following counterintuitive example shows that the SOF problem can be unsolvable (or hard to solve) even for small systems. In the example we show how nonexistence of SOF can be detected by the method:

Example 4.2 Let

$$A^{(0)} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, B^{(0)} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, C^{(0)} = [1 \quad 1].$$

Applying the algorithm we have

$$U^{(0)} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, A^{(1)} = \frac{1}{2}, B^{(1)} = -\frac{1}{2}.$$

The "while-loop" stops because $B^{(1)}B^{(1)+} = 1$. Let $P^{(1)} = p_1$ where $p_1 > 0$. Then,

$$K^{(1)} = B^{(1)+}\left(\frac{1}{2}\left(1 + A^{(1)}P^{(1)} + P^{(1)}A^{(1)T}\right)\right)\left(P^{(1)}\right)^{-1} = -\frac{(1+p_1)}{p_1}.$$

Let $\Delta\widehat{P^{(0)}}_{1,1} = p_2$, where $p_2 > 0$. Then

$$P^{(0)} = U^{(0)}\begin{bmatrix} \Delta\widehat{P^{(0)}}_{1,1} + K^{(1)}P^{(1)}K^{(1)T} & -K^{(1)}P^{(1)} \\ -P^{(1)}K^{(1)T} & P^{(1)} \end{bmatrix} U^{(0)T} =$$

$$= \frac{1}{2p_1}\begin{bmatrix} p_2p_1 + 1 & p_2p_1 + 1 + 2p_1 \\ p_2p_1 + 1 + 2p_1 & p_2p_1 + 1 + 4p_1 + 4p_1^2 \end{bmatrix}.$$

We therefore have:

$$K^{(0)} = B^{(0)+}\left(I_2 + A^{(0)}P^{(0)} + P^{(0)}A^{(0)T}\right)\left(I_2 - \frac{1}{2}B^{(0)}B^{(0)+}\right)\left(P^{(0)}\right)^{-1} =$$

$$= \frac{1}{4p_2p_1^3}\begin{bmatrix} 4p_2p_1^2 + 4p_1 + 6p_1^2 + 4p_1^3 + p_2^2p_1^2 + 2p_2p_1 + 4p_2p_1^3 + 1 \\ -2p_2p_1^2 - 2p_1 - 2p_1^2 - p_2^2p_1^2 - 2p_2p_1 - 1 + 4p_2p_1^3 \end{bmatrix}^T,$$

as the free parametrization of all the state feedbacks for which $A^{(0)} - B^{(0)}K^{(0)}$ is stable—for any choice of $p_1, p_2 > 0$. Now $L_{C^{(0)}} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and the equations $K^{(0)}L_{C^{(0)}} = 0$ are equivalent to the single equation:

$$\frac{1}{4p_2p_1^3}\left(p_2^2p_1^2 + p_2\left(2p_1 + 3p_1^2\right) + \left(2p_1^3 + 4p_1^2 + 3p_1 + 1\right)\right) = 0.$$

Assuming $p_1, p_2 > 0$, the last equation implies that

$$p_2 = \frac{-\left(2p_1 + 3p_1^2\right) \pm \sqrt{\left(2p_1 + 3p_1^2\right)^2 - 4p_1^2\left(2p_1^3 + 4p_1^2 + 3p_1 + 1\right)}}{2p_1^2},$$

leading to a contradiction with p_2 being real positive number.

5. A parametrization for exact pole assignment via SFs

This section is based on the results reported in [34], where the proofs of the following lemma and theorem can be found. The aim of this section is to introduce a parametrization of all the SF's for the exact pole-assignment problem, when the set of eigenvalues can be given as free parameters (under some reasonable assumptions). This is done as part of the research of the problem of parametrization of all the SOFs for pole assignment. Note that the problem of exact pole assignment by SOFs is NP-hard (see [3]), meaning that an efficient algorithm for the problem probably does not exist, and therefore an effective description of the set of all solutions might not exist too. Also note that with SOFs, the feasible set Ω might exclude some open set from being a feasible set for the closed-loop spectrum (see [12]). These make the full aim very hard (if not impossible) to achieve. We therefore focus here on the problem of exact pole assignment via SFs.

Let the control system be given by:

$$\begin{cases} \Sigma(x(t)) = Ax(t) + Bu(t) \\ \quad y(t) = Cx(t) \end{cases} \tag{35}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{r\times n}$, $\Sigma(x(t)) = \frac{d}{dt}x(t)$ in the continuous-time context and $\Sigma(x(t)) = x(t+1)$ in the discrete-time context. We assume without loss of generality that (A, B) is controllable. The problem of exact pole assignment by SF is defined as follows:

- (SF-EPA) Given a set $\Omega \subseteq \mathbb{C}_-$, $|\Omega| = n$ (in the discrete-time context $\Omega \subseteq \mathbb{D}_-$), symmetric with respect to the x-axis, find a state feedback $F \in \mathbb{R}^{m\times n}$ such that the closed-loop state-to-state matrix $E = A - BF$ has Ω as its complete set of eigenvalues, with their given multiplicities.

In [13], a closed form of all the exact pole-placement SFs is proved (up to a set of measure 0), based on Moore's method. In order to minimize the inaccuracy of the eigenvalues final placement and in order to minimize the Frobenius-norm of the feedback, a convex combination of the condition number of the similarity matrix and of the feedback norm was minimized. The parametrization proposed in [13] is based on the assumptions that there exists at least one real state feedback that leads to a diagonalizable state-to-state closed-loop matrix and that B is full rank. A necessary condition for such SF to exist is that the final multiplicity of any eigenvalue is less than or equal to $rank(B)$. Here, we do not assume that B is full rank and we only assume that Ω contains sufficient number of real eigenvalues. A survey of most of the methods for robust pole assignment via SFs or by SOFs and the formulation of these methods as optimization problems with optimality necessary conditions is given in [44]. In [45] a performance comparison of most of the algorithmic methods for robust pole placement is given. A formulation of the general problem of robust exact pole assignment via SFs as an SDP problem and LMI-based linearization is introduced in [46], where the robustness is with respect to the condition number of the similarity matrix, which is made in order to hopefully minimize the inaccuracy of the eigenvalues final placement. Unfortunately, one probably cannot gain a parametric closed form of the SFs from such formulations. Moreover, the following proposed method is exact and therefore enables the use of the parametrization free parameters for other (and maybe more important) optimization purposes. Note that since the proposed method is exact, the closed-loop eigenvalues thyself can be inserted to the problem as parameters.

A completely different notion of robustness with respect to pole placement is considered in the following works:

Robust pole placement in LMI regions and H_∞ design with pole placement in LMI regions are considered in [47, 48], respectively. An algorithm based on alternating projections is introduced in [15], which aims to solve efficiently the problem of pole placement via SOFs. A randomized algorithm for pole placement via SOFs with minimal norm, in nonconvex or unconnected regions, is considered in [20].

Let $\Omega = \left\{ \underbrace{\alpha_1, \overline{\alpha_1}}_{c_1 \text{ times}}, \dots, \underbrace{\alpha_m, \overline{\alpha_m}}_{c_m \text{ times}}, \underbrace{\beta_1}_{r_1 \text{ times}}, \dots, \underbrace{\beta_\ell}_{r_\ell \text{ times}} \right\}$, be the intended closed-loop eigenvalues, where the αs denote the paired complex-conjugate eigenvalues (with nonzero imaginary part), the βs denote the real eigenvalues, and $2c_1, \dots, 2c_m, r_1, \dots, r_\ell$ denote their respective multiplicities, where $2\sum_{i=1}^{m} c_i + \sum_{j=1}^{\ell} r_j = n$. In the following we would say that the size of the set (actually, the multiset) Ω is n (counting multiplicities) and we would write $|\Omega| = n$. Note that (A, B) is controllable if and only if (A, BB^+) is controllable, and also note that BB^+ is a real symmetric matrix with simple eigenvalues in the set $\{0, 1\}$ and thus is orthogonally diagonalizable matrix. Let U denote an orthogonal matrix such that:

$$\widehat{B} = U^T BB^+ U = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} = bdiag(I_k, 0), \tag{36}$$

where $k = rank(B) = rank(BB^+) \geq 1$ since (A, B) is controllable, and let $\widehat{A} = U^T A U = \begin{bmatrix} \widehat{A}_{1,1} & \widehat{A}_{1,2} \\ \widehat{A}_{2,1} & \widehat{A}_{2,2} \end{bmatrix}$ be partitioned accordingly. We cite here the following lemma taken from [34] connecting between the controllability of the given system and the controllability of its sub-system:

Lemma 5.1 In the notations above, (A, BB^+) is controllable if and only if $\left(\widehat{A}_{2,2}, \widehat{A}_{2,1}\widehat{A}_{2,1}^{+}\right)$ is controllable.

Again, we use the recursive controllable structure. Let $U^{(0)} = U$ and let $A^{(0)} = A, B^{(0)} = B, n_0 = n, k_0 = rank\left(B^{(0)}\right)$. Similarly, let $U^{(1)}$ be an orthogonal matrix such that $U^{(1)T} B^{(1)} B^{(1)+} U^{(1)} = bdiag(I_{k_1}, 0)$, where $B^{(1)} = \widehat{A}_{2,1}$. Let $A^{(1)} = \widehat{A}_{2,2}, n_1 = n_0 - k_0, k_1 = rank\left(B^{(1)}\right)$. Now, Lemma 5.1 implies that $\left(A^{(1)}, B^{(1)}\right)$ is controllable since $\left(A^{(0)}, B^{(0)}\right)$ is controllable. Recursively, assume that the pair $\left(A^{(i)}, B^{(i)}\right)$ is controllable. Let $U^{(i)}$ be an orthogonal matrix such that $\widehat{B^{(i)}} = U^{(i)T} B^{(i)} B^{(i)+} U^{(i)} = bdiag(I_{k_i}, 0)$, where $k_i \geq 1$ (since $\left(A^{(i)}, B^{(i)}\right)$ is controllable). Let $\widehat{A^{(i)}} = U^{(i)T} A^{(i)} U^{(i)} = \begin{bmatrix} \widehat{A^{(i)}}_{1,1} & \widehat{A^{(i)}}_{1,2} \\ \widehat{A^{(i)}}_{2,1} & \widehat{A^{(i)}}_{2,2} \end{bmatrix}$ be partitioned accordingly, with sizes $k_i \times k_i$ and $(n_i - k_i) \times (n_i - k_i)$ of the main block-diagonal blocks. Let $A^{(i+1)} = \widehat{A^{(i)}}_{2,2}, B^{(i+1)} = \widehat{A^{(i)}}_{2,1}, n_{i+1} = n_i - k_i, k_i = rank\left(B^{(i)}\right)$. Then, Lemma 5.1 implies that $\left(A^{(i+1)}, B^{(i+1)}\right)$ is controllable. The recursion stops when $B^{(i)} B^{(i)+} = I_{k_i}$ for some $i = b$ (which we call the base case). Note that in the worst case, the recursion stops when the rank $k_b = 1$.

Theorem 5.1 In the above notations, assume that $\sum_{j=1}^{\ell} r_j \geq a$, where a is the number of parity alternations in the sequence $\langle n_0, n_1, \dots, n_b \rangle$. Let $\Omega_0 = \Omega$. Then, there exist a sequence $\Omega_0 \supseteq \Omega_1 \supseteq \cdots \supseteq \Omega_b$ of symmetric sets with size $|\Omega_i| = n_i$ (counting multiplicities) and there exist a real state feedback $F_i = F_i(G_{i+1}, F_{i+1})$ such that $\sigma\left(A^{(i)} - B^{(i)} F_i\right) = \Omega_i$. Moreover, an explicit (recursive) formula for $F_i(G_{i+1}, F_{i+1})$ is given by:

$$\begin{cases} F_i = B^{(i)+}W^{(i)} \\ W^{(i)} = U^{(i)}\widehat{W}^{(i)}U^{(i)T} \\ \widehat{W}^{(i)} = \begin{bmatrix} \widehat{W}_{1,1}^{(i)} & \widehat{W}_{1,2}^{(i)} \\ 0 & 0 \end{bmatrix} \\ \widehat{W}_{1,1}^{(i)} = \widehat{A}_{1,1}^{(i)} + F_{i+1}\widehat{A}_{2,1}^{(i)} - G_{i+1} \\ \widehat{W}_{1,2}^{(i)} = \widehat{A}_{1,2}^{(i)} + F_{i+1}\widehat{A}_{2,2}^{(i)} - G_{i+1}F_{i+1}, \end{cases} \tag{37}$$

where $\sigma\left(\widehat{A}_{2,2}^{(i)} - \widehat{A}_{2,1}^{(i)}F_{i+1}\right) = \sigma\left(A^{(i+1)} - B^{(i+1)}F_{i+1}\right) = \Omega_{i+1}$ and G_{i+1} is arbitrary real matrix such that $\sigma(G_{i+1}) = \Omega_i \backslash \Omega_{i+1}$.

Example 5.1 Consider the problem of exact pole assignment via SF for the same system from Example 4.1. We therefore assume here that the full state is available for feedback control. Now, using the calculations from Example 4.1, we have: $\langle n_0, n_1 \rangle = \langle 4, 2 \rangle$ implying that the number of parity alternations is $a = 0$. We therefore can assign by the method any symmetric set of eigenvalues to the closed loop. Let $\Omega_0 = \{\alpha, \overline{\alpha}, \beta, \overline{\beta}\}$ be the eigenvalues to be assigned, and let $\Omega_1 = \{\beta, \overline{\beta}\}$. Now,

$$F_1 = B^{(1)+}\left(A^{(1)} - G_2\right),$$

where

$$G_2 = \begin{bmatrix} \Re(\beta) & \Im(\beta) \\ -\Im(\beta) & \Re(\beta) \end{bmatrix},$$

and

$$F_0 = B^{(0)+}W^{(0)},$$

where

$$\begin{aligned} W^{(0)} &= U^{(0)}\widehat{W^{(0)}}U^{(0)T} \\ \widehat{W^{(0)}} &= \begin{bmatrix} \widehat{W^{(0)}}_{1,1} & \widehat{W^{(0)}}_{1,2} \\ 0_2 & 0_2 \end{bmatrix} \\ \widehat{W^{(0)}}_{1,1} &= \widehat{A^{(0)}}_{1,1} + F_1\widehat{A^{(0)}}_{2,1} - G_1 \\ \widehat{W^{(0)}}_{1,2} &= \widehat{A^{(0)}}_{1,2} + F_1\widehat{A^{(0)}}_{2,2} - G_1F_1 \\ G_1 &= \begin{bmatrix} \Re(\alpha) & \Im(\alpha) \\ -\Im(\alpha) & \Re(\alpha) \end{bmatrix}. \end{aligned}$$

We have completed the pole-assignment SF parametrization. As an application, assume that $\alpha = -10 + i, \beta = -1 + 0.1i$. Then,

$$F_0 = 10^3 \cdot \begin{bmatrix} -2.312944765727539 & 0.063274421635669 & -0.033598570868307 & 7.136847864481691 \\ 2.382051359668675 & -0.002249112006026 & 0.011460219946760 & 0.432788287638212 \end{bmatrix},$$

resulting with the closed-loop eigenvalues:

$$\sigma\left(A^{(0)} - B^{(0)}F_0\right) = \left\{ \begin{array}{c} -10.000000000000000 \pm 1.000000000000004i \\ -1.000000000000003 \pm 0.099999999999998i \end{array} \right\}.$$

In our calculations, we used MATLAB®, which has a general precession of $5 - 7$ significant digits in computing eigenvalues. Thus, we have almost no loss of digits by the method. For a comparison, see the last case in Example 4.1 and note that while exact pole assignment can be achieved by SF, in general, it cannot be achieved by SOF because the last is a NP-hard problem (see the introduction of this section). Even regional pole placement is hard to achieve by SOF because of the nonconvexity of the SOF feasibility domain.

Remark 5.1 Note that the indices $\langle k_0, k_1, \dots, k_b \rangle$ as well as the indices $\langle n_0, n_1, \dots, n_b \rangle$, can be calculated from (A, B) in advance. After calculating these indices and the number a of parity alternations in the sequence $\langle n_0, n_1, \dots, n_b \rangle$, the designer can define Ω as to satisfy the assumption of Theorem 5.1, i.e., being symmetric with at least a real eigenvalues, in a parametric way, and get a parametrization of all the real SF leading to Ω as the set of closed-loop eigenvalues. Next, the designer can play with the specific values of these and of other free parameters, in order to gain the needed closed-loop performance requirements. This is in contrast with other methods where the parametrization is calculated ad-hoc for a specific set of eigenvalues, where any change in the set of eigenvalues necessitates new execution of the method.

Remark 5.2 Note that F_{i+1} can be replaced by $F_{i+1} + \left(I - B^{(i)+}B^{(i)}\right)H_{i+1}$ where H_{i+1} is any real matrix, without changing the closed-loop eigenvalues (if one seek feedbacks with minimal Frobenius-norm, then he should take $H_{i+1} = 0$, otherwise he should leave H_{i+1} as another free parameter). Thus, the freeness in $\langle H_b, \dots, H_1, H_0 \rangle$ and in $\langle G_{b+1}, G_b, \dots, G_1 \rangle$ makes the freeness in F_0 (e.g. in order to globally optimize the H_∞-norm of the closed loop, the H_2-norm or the LQR functional of the closed loop or any other performance key thereof). Note also that the sequences $\langle F_b, \dots, F_1, F_0 \rangle$ and $\langle G_{b+1}, G_b, \dots, G_1 \rangle$ can be calculated for Ω as in Theorem 5.1, where the eigenvalues in Ω are given as free parameters. In that case, it can be easily proved by induction that the state feedbacks $\langle F_b, \dots, F_1, F_0 \rangle$ depend polynomially on the eigenvalues parameters and on the other free parameters mentioned above (for complex eigenvalue α they depend polynomially on $\mathfrak{R}(\alpha), \mathfrak{I}(\alpha)$).

Finally, it is wort mentioning that the complementary theorem of Theorem 5.1 was also proved in [34], meaning that under the assumptions of Theorem 5.1, any SF that solves the problem has the form given in the theorem (up to a factor of the form given in Remark 5.2).

6. Concluding remarks

In this chapter, we have introduced an explicit free parametrization of all the stabilizing SF's of a controllable pair (A, B). This enables global optimization over the set of all the stabilizing SF's of such pair, because the parametrization is free. For a system triplet (A, B, C), we have shown how to get the parametrization of all the SOFs of the system by parameterizing all the SFs of (A, B) and all the SFs of $\left(A^T, C^T\right)$ and then imposing the compatibility constraint (28). We have also shown a parametrization of all the SOFs of the system triplet (A, B, C) by imposing the linear constraint $K^{(0)}L_{C^{(0)}} = 0$ on the SF $K^{(0)}$ of the pair (A, B), where $K^{(0)}$ was defined recursively and parameterizes the set of all SFs of (A, B). This leads to a set of polynomial equations (after multiplying by the l.c.m. of the denominators of the rational entries of $K^{(0)}$) and inequalities that can be brought to polynomial equations. The resulting polynomial set of equations can be solved (parametrically) by using the Gröbner basis method (see e.g., [49–52]). By applying the Gröbner basis method, one would get an indication to the existence of solutions and in case that solutions do exist, it would tell what are the free parameters and how other parameters depend on the free parameters. It seems that the proposed method makes the Gröbner basis computations overhead (or other methods thereof) reduced significantly, thus enabling SOF global optimization for larger systems.

In view of Theorem 5.1 (with its complementary theorem proved in [34]), we have introduced a sound and complete parametrization of all the state feedbacks F which make the matrix $\widehat{E} = U^T(A - BF)U$ k-complementary $(n - k)$-invariant with respect to Ω_1 (see [34] for the definition and properties), where U is orthogonal such that $U^T BB^+ U = bdiag(I_k, 0), k = rank(B)$, where Ω is symmetric and has at least a (being the parity alternations in the sequence $\langle n_0, \ldots, n_b \rangle$) real eigenvalues, where $\Omega_1 \subseteq \Omega$ is symmetric with maximum real eigenvalues with size $|\Omega_1| = n - k$. Assuming Ω as above, we have generalized the results of [13] in the sense that we do not assume the existence of real state feedback F that brings the closed-loop $E = A - BF$ to diagonalizable matrix, which actually means that the geometric and algebraic multiplicity coincides for any eigenvalue of the closed loop, and we do not assume the restriction on the multiplicity of each eigenvalue to be less than or equal to $rank(B)$. However, in cases where the number of real eigenvalues in Ω is less than a, one should use the parametrizations given in [13], in [45] or in the references there. Note that in communication systems, where complex SFs and SOFs are sought, the introduced method is complete (with no restrictions) since the number of parity alternations and the restriction on Ω to contain as much real eigenvalues, were needed only to guarantee that F_i for $i = b, \ldots, 0$ is real in each stage, which is needless in communication systems.

In view of Example 5.1, one can see that the accuracy of the final location of the closed-loop eigenvalues given by the proposed method depends only on the accuracy of computing $B^{(i)+}$ and $U^{(i)}$ for $i = 0, \ldots, b$ and in the algorithm that we have to compute the closed-loop eigenvalues (see [53], for example) in order to validate their final location, and it has nothing to do with the specific values of the specific eigenvalues given in Ω. Therefore, by the proposed method, once that $B^{(i)+}$ and $U^{(i)}$ for $i = 0, \ldots, b$ were computed as accurate as possible, the location of the closed-loop eigenvalues will be accurate accordingly. Thus, by the proposed method the designer can save time since he can do it parametrically only once, and afterward he only needs to play with the specific values of the eigenvalues until he gets a satisfactory closed-loop performance, where he can be sure that the accuracy of the final placement will be the same for all of his trials independently on the specific values of the chosen eigenvalues. Also, the given parametrization of F_0 is polynomially dependent on the free parameters and thus is very convenient for applying automatic differentiation and optimization methods.

To conclude, we have introduced parametrizations of SFs and SOFs that are based on the recursive controllable structure that was discovered in [35]. The results has powerful implications for real-life systems, and we expect for more results in this direction. Unfortunately, for uncertain systems, the method cannot work directly because of the dependencies of (A, B, C) in uncertain parameters, for which we cannot compute $U^{(i)}$ for $i = 0, \ldots, b$. However, if a nominal system $(\tilde{A}, \tilde{B}, \tilde{C})$ is known accurately then, the method can be applied to that system and the free parameters of the parametrization can be used to “catch” the uncertainty of the whole system, together with the closed-loop performance requirements. The research of this method will be left for a future work.

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