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Connections of Zero Curvature and Applications to Nonlinear Partial Differential Equations

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Abstract

A general formulation of zero curvature connections in a principle bundle is presented and some applications are discussed. It is proved that a related connection based on a prolongation in an associated bundle remains zero curvature as well. It is also shown that the connection coefficients can be defined so that the partial differential equation to be studied appears as the curvature term in the structure equations. It is discussed how Lax pairs and Bäcklund transformations can be formulated for such equations that occur as zero curvature terms.

MSCs: 53Z05, 57R99, 55R10, 53B50

Keywords: connection, bundle, curvature, structure equations, Lax pair, Bäcklund

1 Introduction

Connections which determine representations of zero curvature have turned out to be a very useful and innovative approach for studying nonlinear partial differential equations. These connection forms have the capacity to produce results which can be used to obtain Lax pairs as well as Bäcklund transformations in a very direct way provided information concerning the structural differential forms of special fiber bundles can be specified. These types of connection have a special property in that the curvature tensor of such a connection contains a subtensor which is directly proportional to a partial differential equation which is of interest. For the case in which the connection tensor with these components vanishes, as on the corresponding lifts of solutions of a given nonlinear equation, it is said the connection determines a representation of zero curvature.

The main ideas which have led to these developments began several decades ago and can be traced to the work of people such as Estabrook and Wahlquist [1-4] and by R. Hermann [5] as well. Hermann first introduced at one point a particular connection of basically this type. He proposed early on to interpret the Bäcklund transformation as a connection similar in a certain sense to the connection which defines a representation of zero curvature. He first introduced the concept of a Bäcklund connection which is defined by the way the connection form is specified. Hermann then formulates Bäcklund's problem as that of finding a section in a bundle space on whose pull-back the Bäcklund connection is plane. He has presented the basic idea in [6], and an introductory outline can be given based on that.

Let M be a manifold and consider two sorts of object on M . First I will be a differential ideal of differential forms on M , and R a Pfaffian system or submodule of the set of differential one-forms on M . Thus, $F^*(M)$ denotes the exterior algebra of differential forms on M , and R is called a prolongation of I if the following condition is satisfied

$$dR \subset F^*(M) \wedge R + I. \tag{1.1}$$

In the initial approach taken by Estabrook and Wahlquist, they primarily start off with I and then search for R . If $I = 0$, then (1.1) expresses the fact that R is completely integrable. The Frobenius complete integrability theorem [7] then asserts that there are, locally, one-forms $\omega_1, \dots, \omega_n \in R$

forming a basis and such that $d\omega_1 = \cdots = d\omega_n = 0$. Second, if R is generated by a single element, ω , such that $d\omega \in I$, then ω is a conservation law for I . Studying the relation (1.1) in more advanced ways and further generalizations has led to an entire geometric approach to the classic AKNS program [8-9], and the study of the geometric properties of non-linear partial differential equations and their associated solutions. There has been much interest in this approach [10-13], and has led to many insights between integrable evolution equations and pseudo-spherical surfaces as well [14-16].

The objective of this work is to go beyond this more primitive formulation which has just been described by starting with a jet-bundle $J^r E$ of r -jets over a lower dimensional bundle E [17]. For purposes here, r is usually two or three when second or third order equations are involved, however, a formulation which doesn't specify r at first will be given. Structure equations are established for the systems of forms on these bundles. A very novel approach to the formulation of zero curvature connections is presented in detail. Several theorems and different proofs of these are presented as well which establish a general theory of the subject from a specific abstract viewpoint. It is shown how the choice of particular connection coefficients can lead to an expression for the curvature, and an expression for the curvature tensor under the assumed form of the coefficients is found and satisfies a particular relation. It is also shown how prolongations of the connections can be generated, and the resulting connections remain zero curvature. Out of this comes a method for writing Lax pairs and Bäcklund transformations [18] for the equations involved. In fact, one of the remarkable features of these differential systems is that once they have been specified, they can be used to yield Lax pairs very easily as well as Bäcklund transformations for the equations which appear as the zero curvature terms in the structure equations. It is explained in detail how these can be constructed. The difficult part as far as applications are concerned is to be able to write down the specific system of connection one-forms to initialize the process. These same forms contain the relevant information for producing these additional structures. Finally, it will be shown how the formalism can be applied in practice to obtain Bäcklund transformations between the Liouville equation and the wave equation. Differential systems which are the zero curvature representations for these two different nonlinear equations will be written down. They

will be shown to have the right zero curvature structure and moreover how information from these differential forms needed to write down Lax pairs and Bäcklund transformations can be extracted.

2 Geometrical Setting

2.1 Framework

The main purpose in formulating connections which define representations of zero curvature is to study nonlinear partial differential equations in a systematic way. By this it is intended that useful structures relevant to the study of these equations, such as Lax pairs and Bäcklund transformations, can be produced. For definiteness, a general third order equation is of the form

$$F(x^i, u, u_j, u_{jk}, u_{jkl}) = 0. \quad (2.1)$$

By enlarging the manifold which supports (2.1), equations of this type can be written in a more general form as

$$F(x^i, u, \lambda_j, \lambda_{jk}, \lambda_{jkl}) = 0, \quad (2.2)$$

This notation is common and can be found in [19-20]. The $\{x^i, u\}$ are adapted local coordinates in the $(n + 1)$ -dimensional bundle E over the n -dimensional base M , whose local coordinates are given by $\{x^i\}$ where $i, j, k = 1, \dots, n$. This larger manifold called $J^r E$ over which (2.2) is defined is called the space of holonomic r -jets of the local sections of the manifold E . It carries the system of coordinates $\{x^i, u, \lambda_{j_1, \dots, j_k}\}$ with $k = 1, \dots, r$. Thus, there exist the following inclusions, $M \subset E \subset J^r E$. Let $\omega^i, \omega^{n+1}, \omega_j^i, \omega_j^{n+1}, \omega_{n+1}^{n+1}, \omega_{jk}^i, \dots$ be a sequence of structural forms of the holonomic frames of the manifold E , symmetric in the subscripts. The forms $\omega^i, \omega^{n+1}, \omega_{i_1, \dots, i_k}^{n+1}$, for $k = 1, \dots, r$, are referred to as principal forms in the bundle of holonomic r -jets, $J^r E$ [21]. These forms will satisfy systems of structural equations which have the form,

$$\begin{aligned} d\omega^i &= \omega^k \wedge \omega_j^i, \\ d\omega^{n+1} &= \omega^j \wedge \omega_j^{n+1} + \omega^{n+1} \wedge \omega_{n+1}^{n+1}, \end{aligned} \quad (2.3)$$

as well as equations which arise in the process of regular prolongation of these by means of Cartan's lemma. That is to say, taking the exterior derivative of the first equation in (2.3) gives

$$0 = d^2\omega^i = d\omega^k \wedge \omega_k^i - \omega^k \wedge d\omega_k^i = \omega^s \wedge (\omega_s^k \wedge \omega_k^i - d\omega_s^i).$$

By the generalized Cartan lemma, the coefficients in the brackets can be expanded in terms of the forms ω^i

$$d\omega_s^i - \omega_s^k \wedge \omega_k^i = \omega^k \wedge \omega_{sk}^i.$$

This can be differentiated in turn and when the process is repeated, a tower of forms can be constructed [22].

It is important in the course of this work to be able to evaluate appropriate sections in these bundles, and it is carried out in the following way. For any section $\Sigma \subset E$ which is defined by the equation $u = u(x^1, \dots, x^n)$, sections in $\Sigma^r \subset J^r E$ are defined by the equations

$$u = u(x^1, \dots, x^n), \quad \lambda_{i_1, \dots, i_k} = u_{i_1, \dots, i_k}, \quad k = 1, \dots, r. \quad (2.4)$$

The subscripts $i+1, \dots, i_k$ on the function u now denote partial derivatives. Consequently, under this process, the equation (2.2) is mapped onto (2.1), the equation of interest. If contact forms are chosen as principal forms on the manifold $J^r E$, then the pull-backs are integral manifolds of the system of Pfaffian equations

$$\omega^{n+1} = \omega_i^{n+1} = \dots = \omega_{i_1 \dots i_k}^{n+1} = 0. \quad (2.5)$$

2.2 Principle Bundle

To begin with, based on this sequence of manifolds, consider the principle bundle $P(J^r E, G)$ over $J^r E$ along with the g parameter structure group G . Let $P(J^r E, G)$ have structural forms ω^A , ($A, B = 1, \dots, g$) which satisfy structure equations of the form

$$d\omega^A = \frac{1}{2}C_{BC}^A \omega^B \wedge \omega^C + \omega^\delta \wedge \omega_\delta^A. \quad (2.6)$$

In (2.6), the C_{BC}^A are the structure constants pertaining to the Lie group G . They are skew-symmetric with respect to the lower indices and satisfy the Jacobi identity

$$C_{BK}^A C_{LM}^B + C_{BL}^A C_{MK}^B + C_{BM}^A C_{KL}^B = 0. \quad (2.7)$$

The forms ω^δ will be principle forms of the base $J^r E$, and will be completely integrable. Thus, their differentials satisfy structure equations of the form

$$d\omega^\delta = \omega^\mu \wedge \omega_\mu^\delta. \quad (2.8)$$

3 General Zero-Curvature Formulation

To show exactly how zero curvature representations can be developed from a rigorous point of view, a connection in the principle bundle $P(J^r E, G)$ has to be defined [19-20]. One way of doing this is to specify the object of connection. This is made precise in the following theorem.

Theorem 3.1 A connection in the principle bundle $P(J^r E, G)$ can be given by the field of a connection object on $J^r E$ which has components Γ_ϵ^A that satisfy the system of differential equations

$$d\Gamma_\epsilon^A + C_{BC}^A \Gamma_\epsilon^B \omega^C - \Gamma_\delta^A \omega_\epsilon^\delta - \omega_\epsilon^A = \Gamma_{\epsilon\delta}^A \omega^\delta, \quad (3.1)$$

The forms ω_ϵ^δ are determined from (2.8). The associated connection forms

$$\tilde{\omega}^A = \omega^A + \Gamma_\epsilon^A \omega^\epsilon \quad (3.2)$$

satisfy the structure equations

$$d\tilde{\omega}^A = \frac{1}{2} C_{BC}^A \tilde{\omega}^B \wedge \tilde{\omega}^C + \Omega^A. \quad (3.3)$$

The Ω^A in (3.3) are curvature forms given by

$$\Omega^A = R_{\epsilon\delta}^A \omega^\epsilon \wedge \omega^\delta. \quad (3.4)$$

Proof: Differentiating the connection forms in (3.2) and requiring the exterior derivative be consistent with (3.3), yields

$$d\omega^A + d(\Gamma_\delta^A \omega^\delta) = \frac{1}{2} C_{BC}^A (\omega^B + \Gamma_\epsilon^B \omega^\epsilon) \wedge (\omega^C + \Gamma_\delta^C \omega^\delta) + \Omega^A.$$

Expanding this out, the following expression results,

$$d\omega^A + d\Gamma_\delta^A \wedge \omega^\delta + \Gamma_\delta^A d\omega^\delta = \frac{1}{2} C_{BC}^A \omega^B \wedge \omega^C + \frac{1}{2} C_{BC}^A \omega^B \wedge \Gamma_\delta^C \omega^\delta + \frac{1}{2} C_{BC}^A \Gamma_\epsilon^B \omega^\epsilon \wedge \omega^C + \frac{1}{2} C_{BC}^A \Gamma_\epsilon^B \Gamma_\delta^C \omega^\epsilon \wedge \omega^\delta + \Omega^A.$$

Substituting (2.8) and (3.1) into this, we obtain,

$$\begin{aligned} d\omega^A - \frac{1}{2}C_{BC}^A \omega^B \wedge \omega^C - \omega^\delta \wedge \omega_\delta^A + (-C_{BC}^A \Gamma_\delta^B \omega^C + \Gamma_\sigma^A \omega_\delta^\sigma + \omega_\delta^A + \Gamma_{\delta\sigma}^A \omega^\sigma) \wedge \omega^\delta + \Gamma_\delta^A \omega^\epsilon \wedge \omega_\epsilon^\delta \\ = -\omega^\delta \wedge \omega_\delta^A + \frac{1}{2}C_{BC}^A \Gamma_\delta^C \omega^B \wedge \omega^\delta + \frac{1}{2}C_{BC}^A \Gamma_\delta^C \omega^\delta \wedge \omega^B + \frac{1}{2}C_{BC}^A \Gamma_\epsilon^B \Gamma_\delta^C \omega^\epsilon \wedge \omega^\delta + \Omega^A. \end{aligned}$$

Now replace $d\omega^A$ using (2.6) to obtain

$$\begin{aligned} -C_{BC}^A \Gamma_\delta^B \omega^C \wedge \omega^\delta + \Gamma_\sigma^A \omega_\delta^\sigma \wedge \omega^\delta + \omega_\delta^A \wedge \omega^\delta + \Gamma_{\delta\sigma}^A \omega^\sigma \wedge \omega^\delta + \Gamma_\delta^A \omega^\epsilon \wedge \omega_\epsilon^\delta \\ = -\omega^\delta \wedge \omega_\delta^A + C_{BC}^A \Gamma_\delta^C \omega^B \wedge \omega^\delta + \frac{1}{2}C_{BC}^A \Gamma_\epsilon^B \Gamma_\delta^C \omega^\epsilon \wedge \omega^\delta + \Omega^A. \end{aligned}$$

The fact that the C_{BC}^A are antisymmetric in the lower indices simplifies this result to the form,

$$\Omega^A = \Gamma_{\delta\sigma}^A \omega^\sigma \wedge \omega^\delta - \frac{1}{2}C_{BC}^A \Gamma_\epsilon^B \Gamma_\delta^C \omega^\epsilon \wedge \omega^\delta.$$

Factoring the one-forms in the first part of Ω^A , it is found that

$$\Omega^A = -\frac{1}{2}(\Gamma_{\epsilon\delta}^A - \Gamma_{\delta\epsilon}^A + C_{BC}^A \Gamma_\epsilon^B \Gamma_\delta^C) \omega^\epsilon \wedge \omega^\delta. \quad (3.5)$$

This gives Ω^A explicitly and finishes the proof.

The coefficients of Ω^A in (3.5) give the components of $R_{\epsilon\delta}^A$ and the theorem allows us to identify the components of the curvature tensor as

$$R_{\epsilon\delta}^A = -\frac{1}{2}(\Gamma_{\epsilon\delta}^A - \Gamma_{\delta\epsilon}^A + C_{BC}^A \Gamma_\epsilon^B \Gamma_\delta^C). \quad (3.6)$$

Theorem 3.2 The curvature tensor satisfies the following relation

$$dR_{\lambda\mu}^A + R_{\lambda\mu}^B C_{BC}^A \omega^C - R_{\sigma\mu}^A \omega_\lambda^\sigma - R_{\lambda\sigma}^A \omega_\mu^\sigma = 0, \quad \text{mod } \omega^\Delta, \quad (3.7)$$

where ω^Δ are principle forms of the jet manifold.

Proof: Differentiating both sides of (3.3) exteriorly, it is found that

$$\begin{aligned} 0 &= \frac{1}{2}C_{BC}^A d\tilde{\omega}^B \wedge \tilde{\omega}^C - \frac{1}{2}C_{BC}^A \tilde{\omega}^B \wedge d\tilde{\omega}^C + dR_{\lambda\mu}^A \wedge \omega^\lambda \wedge \omega^\mu + R_{\lambda\mu}^A d\omega^\lambda \wedge \omega^\mu - R_{\lambda\mu}^A \omega^\lambda \wedge d\omega^\mu \\ &= C_{BC}^A \left(\frac{1}{2}C_{DQ}^B \tilde{\omega}^D \wedge \tilde{\omega}^Q + R_{\lambda\mu}^B \omega^\lambda \wedge \omega^\mu \right) \wedge \tilde{\omega}^C + dR_{\lambda\mu}^A \wedge \omega^\lambda \wedge \omega^\mu + R_{\lambda\mu}^A \omega^\sigma \wedge \omega_\sigma^\lambda \wedge \omega^\mu - R_{\lambda\mu}^A \omega^\lambda \wedge \omega^\sigma \wedge \omega_\sigma^\mu \end{aligned}$$

$$= \frac{1}{2} C_{TC}^A C_{DB}^T \tilde{\omega}^D \wedge \tilde{\omega}^B \wedge \tilde{\omega}^C + C_{BC}^A R_{\lambda\mu}^B \tilde{\omega}^C \wedge \omega^\lambda \wedge \omega^\mu + dR_{\lambda\mu}^A \wedge \omega^\lambda \wedge \omega^\mu - R_{\lambda\mu}^A \omega_\sigma^\lambda \wedge \omega^\sigma \wedge \omega^\mu - R_{\lambda\mu}^A \omega_\sigma^\mu \wedge \omega^\lambda \wedge \omega^\sigma.$$

Invoking the Jacobi identity (2.7), this result reduces to the following form

$$(dR_{\lambda\mu}^A + R_{\lambda\mu}^B C_{BC}^A \tilde{\omega}^C - R_{\sigma\mu}^A \omega_\lambda^\sigma - R_{\lambda\sigma}^A \omega_\mu^\sigma) \wedge \omega^\lambda \wedge \omega^\mu = 0.$$

This implies that the coefficient of $\omega^\lambda \wedge \omega^\mu$ is zero mod ω^Δ , the principle forms of the jet manifold, so that $\tilde{\omega}^C = \omega^C$. The result in (3.7) then follows.

Thus, the curvature tensor components include, in particular, the components R_{kl}^A . As a consequence of these theorems, the following result is very important as far as the application of the zero-curvature idea to specific nonlinear differential equations is concerned.

Theorem 3.3 For the connection given in the principle bundle $P(J^r E, G)$ to define the representation of zero curvature which corresponds to an equation $F(x^i, u, \lambda_j, \lambda_{jk}, \dots) = 0$, it is necessary and sufficient that the components R_{kl}^A of the curvature vanish on the pull-backs of the solutions to the equation.

Proof: Since the vanishing of the forms of curvature $\Omega^A = R_{\lambda\mu}^A \omega^\lambda \wedge \omega^\mu$ on the pull-backs of solutions is invariant, it suffices to show the statement for some special choice of the principle forms. The statement then becomes obvious if contact forms are taken as principle forms since, in this case, the relations $\Omega^A = R_{kl}^A \omega^k \wedge \omega^l$ hold on the pull-back of any section $\Sigma \subset E$.

In practical terms, the curvature tensor will be, or will have a subtensor, which is proportional to the equation under consideration, and will clearly vanish identically on solutions of that equation. Thus, a connection is called a connection determining a representation of zero curvature for a differential equation if the curvature form vanishes on the solutions, or on the corresponding lifts of solutions, and only on solutions.

4 Prolongations on These Spaces

An additional bundle associated with the principle bundle $P(J^r E, E)$, which is called $F(P(J^r E, G))$, can now be constructed. A larger space is now being associated with P . The typical fiber of this new bundle is a space F which is an N -dimensional space of the representation of the Lie group

G . The representation of the group G as a group of transformations of the space F can be defined by the specification of the system of Pfaffian equations

$$dX^I - \xi_A^I(X)w^a = 0. \quad (4.1)$$

In (4.1), the w^A are invariant forms of the group G which satisfy the structural equations

$$dw^A = \frac{1}{2} C_{BC}^A w^B \wedge w^C. \quad (4.2)$$

Indeed, it is worth recalling that if G is connected, any diffeomorphism $f : G \rightarrow G$ which preserves left-invariant forms, θ^α , so that $f^*\theta^\alpha = \theta^\alpha$ is left translation. If N is a smooth manifold and w^α linearly independent forms on N satisfying (4.2), then for any point in N , there exists a neighborhood U and a diffeomorphism $f : U \rightarrow G$ such that $\theta^\alpha = f^*(w^\alpha)$.

The following theorem will produce a condition that, when satisfied, will guarantee that system (4.1) is completely integrable.

Theorem 4.1 Pfaffian system (4.1) is completely integrable provided the set of $\xi_A^I(X)$ satisfy the following constraint,

$$\xi_B^K \frac{\partial \xi_C^I}{\partial X^K} - \xi_C^K \frac{\partial \xi_B^I}{\partial X^K} + \xi_A^I C_{BC}^A = 0. \quad (4.3)$$

Proof: Differentiate both sides of system (4.1) to obtain,

$$\frac{\partial \xi_A^I}{\partial X^K} \xi_C^K(X) w^C \wedge w^A + \frac{1}{2} \xi_B^I(X) C_{BC}^A w^B \wedge w^C = 0.$$

The first term in this equation can be put in the form

$$\frac{1}{2} \{ \xi_B^K(X) \frac{\partial \xi_C^I}{\partial X^K} w^B \wedge w^C + \xi_C^K(X) \frac{\partial \xi_B^I}{\partial X^K} w^C \wedge w^B \} + \frac{1}{2} \xi_A^I(X) C_{BC}^A w^B \wedge w^C = 0.$$

Equating the coefficient of $w^B \wedge w^C$ to zero, the condition (4.3) for complete integrability is obtained. These conditions are often referred to as the Lie identities.

If there exists a connection in $P(J^r E, G)$ which determines a representation of zero curvature, it is remarkable that the same property holds in the associated bundle $F(P(J^r E, G))$. The N -dimensional space F is coordinatized by means of coordinates $\{X^i\}_1^N$ and carries a representation of the group. Moreover, the curvature forms of $F(P(J^r E, G))$ are defined by

$$\theta^I = dX^I - \xi_A^I(X^1, \dots, X^N) \omega^A, \quad I, J, K = 1, \dots, N. \quad (4.4)$$

In (4.4), the ω^A are structural forms of the principle bundle.

If a connection with the connection forms

$$\tilde{\omega}^A = \omega^A + \Gamma_{\lambda}^A \omega^\lambda, \quad (4.5)$$

is defined in the principle bundle, then along with this connection in the principle bundle, a connection is induced in the associated bundle $F(P(J^r E, G))$ and it has connection forms

$$\tilde{\theta}^I = dX^I - \xi_A^I(X) \tilde{\omega}^A. \quad (4.6)$$

Proposition 4.1 The Pfaffian system $\tilde{\theta}^I$ satisfies the system of structural equations

$$d\tilde{\theta}^I = \tilde{\theta}^K \wedge \tilde{\theta}_K^I - \xi_A^I(X) R_{\lambda\mu}^A \omega^\lambda \wedge \omega^\mu. \quad (4.7)$$

The $\xi_A^I(X)$ satisfy the Lie identities (4.3) and the $\tilde{\theta}_K^I$ are given by

$$\tilde{\theta}_K^I = -\frac{\partial \xi_A^I}{\partial X^K} \tilde{\omega}^A. \quad (4.8)$$

The $R_{\lambda\mu}^A$ are the components of the curvature tensor defined in $P(J^r E, G)$.

Proof: Differentiating the set of forms in (4.6), it is found that

$$\begin{aligned} d\tilde{\theta}^I &= -\frac{\partial \xi_A^I}{\partial X^K} dX^K \wedge \tilde{\omega}^A - \xi_A^I(X) d\tilde{\omega}^A \\ &= -\frac{\partial \xi_A^I}{\partial X^K} (\tilde{\theta}^K + \xi_C^K(X) \tilde{\omega}^C) \wedge \tilde{\omega}^A - \xi_A^I(X) d\tilde{\omega}^A \\ &= -\frac{\partial \xi_A^I}{\partial X^K} \tilde{\theta}^K \wedge \tilde{\omega}^A - \xi_C^K(X) \frac{\partial \xi_A^I}{\partial X^K} \tilde{\omega}^C \wedge \tilde{\omega}^A - \xi_A^I(X) d\tilde{\omega}^A \\ &= \tilde{\theta}^K \wedge \left(-\frac{\partial \xi_A^I}{\partial X^K}\right) \tilde{\omega}^A - \xi_B^K(X) \frac{\partial \xi_C^I}{\partial X^K} \tilde{\omega}^B \wedge \tilde{\omega}^C - \frac{1}{2} C_{BC}^A \xi_A^I(X) \tilde{\omega}^B \wedge \tilde{\omega}^C - \xi_A^I(X) R_{\lambda\mu}^A \omega^\lambda \wedge \omega^\mu. \end{aligned}$$

Assuming that the Lie identities (4.3) hold and $\tilde{\theta}_K^I$ are defined by (4.8), the desired result (4.7) appears directly,

$$d\tilde{\theta}^I = \tilde{\theta}^K \wedge \tilde{\theta}_K^I - \xi_A^I(X) R_{\lambda\mu}^A \omega^\lambda \wedge \omega^\mu.$$

Therefore, if the connection defined in the principle bundle specifies a representation of zero curvature for an equation, then the related connection just defined in the associated bundle generated by it will define a representation of zero curvature as well. Its curvature tensor $\xi_A^I R_{\lambda\mu}^A$ vanishes on

sections $\Sigma \subset E$ if and only if the sections are solutions of the equations. This has established the following.

Corollary 4.1 The system of forms $\tilde{\theta}^I$ defined by (4.6) is completely integrable on the pull-backs of solutions to the associated equation and only on these solutions.

The theoretical advantage then in introducing the general formalism is that the $R_{\lambda\mu}^A$ can be interpreted as curvature forms with respect to this larger manifold. This also suggests an application for these results. It is possible that a system of forms $\tilde{\theta}^K$ can be found such that a set of equations of the form (4.7) obtain. The curvature terms may automatically vanish or be proportional to some nonlinear partial differential equation of interest which vanishes on some transverse integral manifold of solutions. Along with Bäcklund connections on bundles having one-dimensional fibers, Bäcklund connections on bundles with two-dimensional fibers can be studied; for example, on a bundle associated to a two-dimensional vector space of the representation of the group $Sl(2)$. This connection is often referred to as a Lax connection as it can be made to lead directly to formulation of Lax pairs for the equation. In this event, the specific forms can then be used to generate both Lax pairs and Bäcklund transformations. This will be illustrated clearly in the following general theorem below [23].

Hermann used a one-form with the structure (4.6) for the KdV equation and realized that it could be written in a particular way [5]. He inferred that the Wahlquist-Estabrook prolongation structure could be interpreted as a type of connection. As for the form $\tilde{\theta}$, it is a form of connection in a bundle with a one-dimensional typical fiber associated with the principal bundle $P(J^r E, Sl(2))$. This connection is also a connection defining a representation of zero curvature. Note that a one-form is a connection form in a bundle with a one-dimensional typical fiber associated with the principal bundle $P(J^r E, Sl(2))$ if and only if it takes the form

$$dy - \xi(y)\tilde{\theta}_0 - \xi_1^2(y)\tilde{\theta}_1 - \xi_2^1(y)\tilde{\theta}_2.$$

The Lie identities satisfied by these coefficients are obtained from the system

$$\frac{\partial \xi_B^I}{\partial y^K} \xi_C^K - \frac{\partial \xi_C^I}{\partial y^K} \xi_B^K = \xi_A^I C_{BC}^A.$$

Consider a Bäcklund mapping in the one-dimensional case. In this case the system of Pfaff equations that define the Bäcklund mapping consist of a single equation

$$dy - \xi(y)\tilde{\omega} - \xi_1^2(y)\tilde{\omega}_2^1 - \xi_2^1(y)\tilde{\omega}_1^2 = 0. \quad (4.9)$$

The Lie identities satisfied by the coefficients in (4.9) are of the following form

$$\begin{aligned} \xi \frac{\partial \xi_1^2}{\partial y} - \xi_1^2 \frac{\partial \xi}{\partial y} &= \xi_1^2, \\ \xi \frac{\partial \xi_2^1}{\partial y} - \xi_2^1 \frac{\partial \xi}{\partial y} &= -\xi_2^1, \\ \xi_1^2 \frac{\partial \xi_2^1}{\partial y} - \xi_2^1 \frac{\partial \xi_1^2}{\partial y} &= 2\xi. \end{aligned} \quad (4.10)$$

Theorem 4.2. The Pfaff equation (4.9) which defines the Bäcklund mapping with the associated space of the structure group G of dimension one can be represented in either of the two forms,

$$d\varphi - \tilde{\omega}_1^2 - \varphi\tilde{\omega} + \varphi^2\tilde{\omega}_2^1 = 0, \quad (4.11)$$

$$d\psi - \tilde{\omega}_2^1 - \psi\tilde{\omega} - \psi^2\tilde{\omega}_1^2 = 0.$$

Proof: Take the second equation in (4.10) and divide it by $(\xi_2^1)^2$ to obtain

$$-\frac{\xi}{(\xi_2^1)^2} d\xi_2^1 + \frac{d\xi}{\xi_2^1} = \frac{dy}{\xi_2^1}.$$

This is equivalent to

$$d\left(\frac{\xi}{\xi_2^1}\right) = \frac{dy}{\xi_2^1}.$$

Define the variable $\varphi = \xi/\xi_2^1$ and use it in this result to give,

$$d\varphi = \frac{dy}{\xi_2^1}. \quad (4.12)$$

Dividing by $(\xi_2^1)^2$, the third equation becomes

$$-\frac{\xi_1^2}{(\xi_2^1)^2} d\xi_2^1 + \frac{d\xi_1^2}{\xi_2^1} = -2\frac{\xi}{(\xi_2^1)^2} dy.$$

Consequently, using (4.12),

$$d\left(\frac{\xi_1^2}{\xi_2^1}\right) = -2\frac{\xi}{\xi_2^1} \frac{dy}{\xi_2^1} = -d\varphi^2.$$

Thus, we can identify $-\varphi^2 = \xi_1^2/\xi_2^1$. Since the form (4.9) can be written in the following way,

$$\frac{dy}{\xi_2^1} - \tilde{\omega}_1^2 - \frac{\xi(y)}{\xi_2^1(y)} \tilde{\omega} - \frac{\xi_1^2(y)}{\xi_2^1(y)} \tilde{\omega}_2^1 = 0, \quad (4.13)$$

the required first equation in (4.11) follows by substituting these results for φ and φ^2 into (4.13). The second equation in (4.11) follows in a similar fashion.

An example which shows how the results in these last two sections can be combined and made into something useful will be presented. Here M will be the two-dimensional base manifold which is coordinatized by the coordinates $(x^1, x^2) = (x, t)$. Now consider the following application which starts with Theorem 3.1. A system of structural forms $\tilde{\omega}^A$ is required to satisfy the structure equations (3.3) expressed as

$$d\tilde{\omega}^1 = 2\tilde{\omega}^2 \wedge \tilde{\omega}^3 + R_{12} dx^1 \wedge dx^2, \quad d\tilde{\omega}^2 = \tilde{\omega}^1 \wedge \tilde{\omega}^2 + R_{112}^2 dx^1 \wedge dx^2, \quad d\tilde{\omega}^3 = \tilde{\omega}^3 \wedge \tilde{\omega}^1 + R_{212}^1 dx^1 \wedge dx^2. \quad (4.14)$$

The last terms in these are the curvature terms which are required to vanish when they are considered on the lifting of a section. This will result in producing a particular equation in the end. In the notation of (2.2), take for the forms $\tilde{\omega}^A$

$$\tilde{\omega}^1 = 2\lambda_1 dx^2, \quad \tilde{\omega}^2 = \frac{1}{2}\lambda_1 dx^1 + (u\lambda_1 - \lambda_{11}) dx^2, \quad \tilde{\omega}^3 = dx^1 + 2u dx^2. \quad (4.15)$$

It is easily verified that these forms satisfy system (4.14). The curvature term in the first and third is zero. The second is satisfied provided that considered on the lifting of a section in which the notation reverts to that of (2.1), u satisfies the following Burgers-type equation $-\frac{1}{2}u_{12} + \frac{1}{2}(u^2)_{11} - u_{111} + (u_1)^2 = 0$. Replacing $(x^1, x^2) = (x, t)$ in this, the following form for the equation is obtained,

$$u_{xt} = (u^2)_{xx} - 2u_{xxx} + 2(u_x)^2. \quad (4.16)$$

Following along the lines of Theorem 4.2, there should be a Bäcklund transformation of the form $dy + \tilde{\omega}^2 - y\tilde{\omega}^1 - y^2\tilde{\omega}^3 = 0$. Substituting the forms (4.15) into this, the following differential system is obtained

$$y_x = -\frac{1}{2}u_x + y^2, \quad y_t = u_{xx} - uu_x + 2u_x y + 2uy^2. \quad (4.17)$$

Evaluating the derivatives y_{xt} and y_{tx} , and subtracting, all higher order terms in the expression above y^0 are found to cancel. Only the y^0 term remains and it is precisely the equation (4.16).

Another approach to Lax and Bäcklund systems will be presented in the next section.

5 Lax and Bäcklund Systems

Perhaps the most interesting aspect of the theoretical development presented so far is that there exists a clear relationship between connections which define a representation of zero curvature and specific Lax and Bäcklund systems for the equation. Let the group be $G = Gl(2)$, so that r is selected to suit the system under consideration. In fact, for the example given here, we take $r = 2$, and the following theorem holds.

Theorem 5.1 Given a connection in $P(J^r E, G)$, where $G = Gl(2)$ or a subgroup, which defines a representation of zero curvature corresponding to an equation of the form (2.1), a Lax system exists which can be defined in terms of the connection coefficients.

Proof: Let

$$\tilde{\omega}_j^i = \omega_j^i + \Gamma_{j\lambda}^i \omega^\lambda \quad (5.1)$$

be connection forms in the principle bundle $P(J^r E, Gl(2))$ which define the representation of zero curvature for the $\tilde{\omega}^A$. This connection which is defined in the principle bundle generates a connection in the associated bundle whose typical fiber is a two-dimensional linear space. The connection forms in the associated bundle corresponding to the connection in P can be written in the form (4.6)

$$\tilde{\theta}^i = dX^i + X^j \tilde{\omega}_j^i.$$

As for the connection in P , the connection in the associated bundle is also a connection which defines a representation of zero curvature for the equation. Consequently, the restriction of the $\tilde{\theta}^i$ to the corresponding pull-back of the section $\Sigma \subset E$ defined by $u = u(x, y)$ is completely integrable if and only if the section $\Sigma \subset E$ is a solution of the equation. ♣

In practical terms, if contact forms are taken as principle forms then ω_j^i will be equal to zero

and the forms $\tilde{\theta}^i$ take the form

$$\tilde{\theta}^i = dX^i + X^j \Gamma_{j\lambda}^i \omega^\lambda. \quad (5.2)$$

In this case, with $(x^1, x^2) = (x, y)$, the system of equations $\tilde{\theta}^i|_\Sigma = 0$ have the form

$$dX^i + X^j \Gamma_{j1}^i(x, y, u, u_k, u_{kl}) dx + X^j \Gamma_{j2}^i(x, y, u, u_k, u_{kl}) dy = 0. \quad (5.3)$$

Of course, this is equivalent to the following system of partial differential equations

$$X_x^i = -\Gamma_{j1}^i(x, y, u, u_k, u_{kl}) X^j, \quad X_y^i = -\Gamma_{j2}^i(x, y, u, u_k, u_{kl}) X^j. \quad (5.4)$$

In matrix form for a two-dimensional representation of G , (5.4) can be written as

$$\begin{pmatrix} X^1 \\ X^2 \end{pmatrix}_x = \begin{pmatrix} -\Gamma_{11}^1 & -\Gamma_{21}^1 \\ -\Gamma_{11}^2 & -\Gamma_{21}^2 \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}, \quad \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}_y = \begin{pmatrix} -\Gamma_{12}^1 & -\Gamma_{22}^1 \\ -\Gamma_{12}^2 & -\Gamma_{22}^2 \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}. \quad (5.5)$$

This system is completely integrable and has solutions satisfying any initial conditions if and only if $u = u(x, y)$ is a solution of the associated nonlinear equation.

There are relationships between Bäcklund transformations and the connections defining representations of zero curvature, as Hermann pointed out [3]. Consider restricting the problem to investigate how to write Bäcklund transformations between two second order equations. Suppose x, y, u and x, y, v are adapted local coordinates in bundles E_1 and E_2 respectively which share a common base manifold M with local coordinates x, y . The variables $x, y, u, \lambda_i, \lambda_{jk}$ and x, y, v, μ_i, μ_{jk} are local coordinates in the bundles of second order jets $J^2 E_1$ and $J^2 E_2$. In this case, x, y, u, λ_i and x, y, v, μ_i are local coordinates in the corresponding bundles of first order jets $J^1 E_1$ and $J^1 E_2$. In this event, the equations then take the form

$$F_1(x, y, u, \lambda_i, \lambda_{jk}) = 0, \quad (5.6)$$

and,

$$F_2(x, y, v, \mu_i, \mu_{jk}) = 0. \quad (5.7)$$

A Bäcklund transformation between these two equations can be defined as a system of equations

$$\Phi(x, y, u, v, u_i, v_j) = 0. \quad (5.8)$$

Equation (5.8) will be integrable over u if and only if $v = v(x, y)$ is a solution of (5.7) and integrable over v if and only if $u = u(x, y)$ is a solution of (5.6). For any specified solution u of (5.6), or v of (5.7), (5.8) makes it possible to determine a certain solution v of (5.7), or of (5.6), respectively.

It is said that a Bäcklund transformation is established between (5.6) and (5.7) if connections have been defined in the two principle bundles $P(J^1 E_1, G_1)$ and $P(J^1 E_2, G_2)$ which define representations of zero curvature for each equation. In each of the manifolds E_1 and E_2 a structure of the bundle is defined with a one-dimensional fiber associated. In the case of E_2 , it is with the principle bundle $P(J^1 E_1, G_1)$ and in the case of E_1 with $P(J^1 E_2, G_2)$. Therefore, the connections which are defined in the principle bundles and specify representations of zero curvature generate corresponding representations of zero curvature in the associated bundles. The forms for these two connection forms are written θ and ϑ .

For the case in which $G_1 = G_2 = Gl(2)$, the forms θ and ϑ take the form

$$\theta = dv - \xi_j^i(v) \tilde{\omega}_i^j, \quad (5.9)$$

and,

$$\vartheta = du - \eta_j^i(u) \tilde{\pi}_i^j. \quad (5.10)$$

The structure forms on the right of (5.9) and (5.10) are given by

$$\tilde{\omega}_j^i = \omega_j^i + \Gamma_{jk}^i(x, y, u, \lambda_l) \omega^k, \quad \tilde{\pi}_j^i = \pi_j^i + \Phi_{jk}^i(x, y, v, \mu_l) \omega^k, \quad i, j = 1, 2. \quad (5.11)$$

These will be connection forms in $P(J^1 E_1, Gl(2))$ and $P(J^1 E_2, Gl(2))$, respectively. If contact forms are selected as principle forms in the bundle of jets, then $\omega_j^i = 0$ and $\pi_j^i = 0$ hold. The forms in (5.11) simplify to

$$\tilde{\omega}_j^i = \Gamma_{jk}^i(x, y, u, \lambda_l) \omega^k, \quad \tilde{\pi}_j^i = \Phi_{jk}^i(x, y, v, \mu_l) \omega^k. \quad (5.12)$$

In this case, the equations $\theta = 0$ and $\vartheta = 0$ considered on pull-backs of solutions of the equations (5.6) and (5.7), respectively, are written as

$$\begin{aligned} dv - \xi_j^i(v) \Gamma_{i1}^j(x, y, u, u_k) dx - \xi_j^i(v) \Gamma_{i2}^j(x, y, u, u_k) dy &= 0, \\ du - \eta_j^i(u) \Phi_{i1}^j(x, y, v, v_k) dx - \eta_j^i(u) \Phi_{i2}^j(x, y, v, v_k) dy &= 0. \end{aligned} \quad (5.13)$$

Of course, (5.13) are equivalent to the following systems of partial differential equations

$$v_x = \xi_j^i(v)\Gamma_{i1}^j(x, y, u, u_k), \quad v_y = \xi_j^i(v)\Gamma_{i2}^j(x, y, u, u_k), \quad (5.14)$$

and

$$u_x = \eta_j^i(u)\Phi_{i1}^j(x, y, v, v_k), \quad u_y = \eta_j^i(u)\Phi_{i2}^j(x, y, v, v_k). \quad (5.15)$$

6 An Application of the Theory

This formalism is now applied to obtain Bäcklund transformations between the Liouville equation $u_{xy} = e^u$ and the wave equation $v_{xy} = 0$. These can now be defined by specifying the connections in two principle bundles which define representations of zero curvature, and the corresponding connections in the associated bundles. In this case, the connection forms in the principle bundles are defined as in (5.12).

A system of forms which will accomplish the task can be specified as follows

$$\begin{aligned} \tilde{\omega}_1^1 &= -\frac{\lambda_1}{4}dx + \frac{\lambda_2}{4}dy, & \tilde{\omega}_2^2 &= \frac{\lambda_1}{4}dx - \frac{\lambda_2}{4}dy, & \tilde{\omega}_1^2 &= \frac{1}{\sqrt{2}}e^{u/2}dx, & \tilde{\omega}_2^1 &= \frac{1}{\sqrt{2}}e^{u/2}dy \\ \tilde{\pi}_1^1 &= \frac{\mu_1}{4}dx - \frac{\mu_2}{4}dy, & \tilde{\pi}_2^2 &= -\frac{\mu_1}{4}dx + \frac{\mu_2}{4}dy, & \tilde{\pi}_1^2 &= \sqrt{2}(e^{-v/2}dx + e^{v/2}dy), & \tilde{\pi}_2^1 &= 0. \end{aligned} \quad (6.1)$$

Based on this collection of definitions, the required coefficients Γ_{ik}^j and Φ_{ik}^j can be read off

$$\Gamma_{11}^1 = -\frac{\lambda_1}{4}, \quad \Gamma_{21}^2 = \frac{\lambda_1}{4}, \quad \Gamma_{11}^2 = \frac{1}{\sqrt{2}}e^{u/2}, \quad \Gamma_{21}^1 = 0, \quad (6.2)$$

$$\Gamma_{12}^1 = \frac{\lambda_2}{4}, \quad \Gamma_{22}^2 = -\frac{\lambda_2}{4}, \quad \Gamma_{12}^2 = 0, \quad \Gamma_{22}^1 = \frac{1}{\sqrt{2}}e^{u/2}.$$

and as well,

$$\Phi_{11}^1 = \frac{\mu_1}{4}, \quad \Phi_{21}^2 = -\frac{\mu_1}{4}, \quad \Phi_{11}^2 = \sqrt{2}e^{-v/2}, \quad \Phi_{21}^1 = 0, \quad (6.3)$$

$$\Phi_{12}^1 = -\frac{\mu_2}{4}, \quad \Phi_{22}^2 = \frac{\mu_2}{4}, \quad \Phi_{12}^2 = \sqrt{2}e^{v/2}, \quad \Phi_{22}^1 = 0.$$

Now the corresponding sets of forms θ^i and ϑ^i are defined in terms of the structural forms (6.1),

$$\theta^1 = 2(\tilde{\omega}_1^1 - \tilde{\omega}_2^2), \quad \theta^2 = 2\tilde{\omega}_1^2, \quad \theta^3 = -2\tilde{\omega}_2^1, \quad (6.4)$$

$$\vartheta^1 = 2(\tilde{\pi}_1^1 - \tilde{\pi}_2^2), \quad \vartheta^2 = \tilde{\pi}_1^2, \quad \vartheta^3 = 0.$$

It will be shown explicitly that these forms define representations of zero curvature for the two equations above. The first three structure equations in the θ^i are given by

$$\begin{aligned} d\theta^1 + \theta^2 \wedge \theta^3 &= -d\lambda_1 \wedge dx + d\lambda_2 \wedge dy - 2e^u dx \wedge dy, \\ d\theta^2 - \frac{1}{2}\theta^1 \wedge \theta^2 &= \frac{1}{\sqrt{2}}e^{u/2}(du - \lambda_2 dy) \wedge dx, \\ d\theta^3 + \frac{1}{2}\theta^1 \wedge \theta^3 &= \frac{1}{\sqrt{2}}e^{u/2}(-du + \lambda_1 dx) \wedge dy. \end{aligned} \tag{6.5}$$

On a section $\Sigma_1 \subset E_1$, using (2.4) it follows that $\lambda_1 = u_x$, $\lambda_2 = u_y$ and all three of these equations vanish provided that u satisfies $u_{xy} = e^u$.

Similarly, for the forms ϑ^i , it is found that

$$\begin{aligned} d\vartheta^1 + \vartheta^2 \wedge \vartheta^3 &= d\mu_1 \wedge dx - d\mu_2 \wedge dy, \\ d\vartheta^2 - \frac{1}{2}\vartheta^1 \wedge \vartheta^2 &= \frac{1}{\sqrt{2}}[-e^{v/2}(dv \wedge dx + \mu_2 dx \wedge dy) + e^{v/2}(dv \wedge dy - \mu_1 dx \wedge dy)]. \end{aligned} \tag{6.6}$$

The third vanishes identically since $\vartheta^3 = 0$. On a section $\Sigma_2 \subset E_2$, by applying (2.4), it follows that $\mu_1 = v_x$, $\mu_2 = v_y$ and these equations vanish provided that v satisfies the equation $v_{xy} = 0$.

Using (6.2) and the definitions in (6.4), then under the assignment

$$\xi_1^1 = 2, \quad \xi_2^2 = -2, \quad \xi_2^1 = 2e^{-v/2}, \quad \xi_1^2 = -2e^{v/2}. \tag{6.7}$$

the equation for dv in (5.13) is written as

$$dv - \theta^1 - e^{-v/2}\theta^2 - e^{v/2}\theta^3 = 0. \tag{6.8}$$

Similarly, using (6.3) and identifying

$$\eta_1^1 = -2, \quad \eta_2^2 = 2, \quad \eta_2^1 = e^{u/2}, \tag{6.9}$$

the equation for du in (5.13) becomes

$$du + \vartheta^1 - e^{u/2}\vartheta^2 = 0. \tag{6.10}$$

It can be observed that the one-form of (6.8) is a closed form, whereas (6.10) is not closed, but leads to a consistent result. Now all of the required information is at hand to write down

Bäcklund transformations between these two equations. Substituting (6.2) and (6.7) into (5.14), there results the system

$$u_x + v_x = \sqrt{2}e^{(u-v)/2}, \quad u_y - v_y = \sqrt{2}e^{(u+v)/2}. \quad (6.11)$$

Substituting (6.3) and (6.9) into (5.15), it is found that the same pair appears,

$$u_x + v_x = \sqrt{2}e^{(u-v)/2}, \quad u_y - v_y = \sqrt{2}e^{(u+v)/2}. \quad (6.12)$$

Theorem 6.1. The exterior derivatives of the one-forms in (6.8) and (6.10) vanish modulo the sets of forms $\{dv, d\theta^i\}$ and $\{du, d\theta^i\}$, respectively.

Proof: Let τ denote the one-form on the left-hand side of (6.8). Differentiate τ exteriorly and there results,

$$d\tau = -d\theta^1 + \frac{1}{2}e^{-v/2}dv \wedge \theta^2 - e^{-v/2}d\theta^2 - \frac{1}{2}e^{v/2}dv \wedge \theta^3 - e^{v/2}d\theta^3.$$

Replacing the known forms $d\theta^i$ from (6.5) and dv (6.8), we obtain that

$$d\tau = \theta^2 \wedge \theta^3 + \frac{1}{2}e^{-v/2}\theta^1 \wedge \theta^2 + \frac{1}{2}\theta^3 \wedge \theta^2 - \frac{1}{2}e^{-v/2}\theta^1 \wedge \theta^2 - \frac{1}{2}e^{v/2}\theta^1 \wedge \theta^3 - \frac{1}{2}\theta^2 \wedge \theta^3 + \frac{1}{2}e^{v/2}\theta^1 \wedge \theta^3 = 0,$$

as required.

In the same way, differentiate (6.10) and substitute $d\theta^i$ from (6.6) and du from (6.10).

This provides another way to get the ξ_j^i and η_j^i which appear in (5.14) and (5.15).

Theorem 6.2. Equations (6.11) and (6.12) form a system of Bäcklund transformations which connect the equations $u_{xy} = e^u$ and $v_{xy} = 0$ respectively.

Proof: Differentiating the pair of equations in (6.11) and replacing the first derivatives on the right-hand side, it is found that

$$(u+v)_{xy} = \frac{1}{\sqrt{2}}(u-v)_y e^{(u-v)/2} = e^u, \quad (u-v)_{yx} = \frac{1}{\sqrt{2}}(u+v)_x e^{(u+v)/2} = e^u.$$

Adding these two second derivatives, the Liouville equation $u_{xy} = e^u$ results. Upon subtracting this pair, the wave equation $v_{xy} = 0$ is obtained.

7 Outlook and Summary

A very general and useful formalism has been examined which makes use of connections of zero curvature. The first few sections present one way of giving an abstract formulation to this subject, and the latter part transfers this to the more concrete aspect of actually calculating some differential systems for a pair of specific equations. If the forms are selected in the right way, it should be possible to create auto-Bäcklund transformations, that is transformations between solutions of the same equation. It has been shown that these types of connection have the potential to produce Lax pairs and Bäcklund transformations for nonlinear partial differential equations. In fact the results of the previous section can be used to write Lax pairs for the respective equations. Using coefficients (6.2) for the equation $u_{xy} = e^u$, the following Lax pair is obtained

$$\begin{pmatrix} X^1 \\ X^2 \end{pmatrix}_x = \begin{pmatrix} \frac{u_x}{4} & 0 \\ -\frac{1}{\sqrt{2}}e^{u/2} & -\frac{u_x}{4} \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}, \quad \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}_y = \begin{pmatrix} -\frac{u_y}{4} & -\frac{1}{\sqrt{2}}e^{u/2} \\ 0 & \frac{u_y}{4} \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}.$$

The compatibility condition for this pair can be calculated by differentiating the first matrix equation with respect to y and the second with respect to x . It is seen to hold provided that u satisfies the equation $u_{xy} = e^u$. Similarly, using the results in (6.3) for the equation $v_{xy} = 0$, the following Lax pair results

$$\begin{pmatrix} X^1 \\ X^2 \end{pmatrix}_x = \begin{pmatrix} -\frac{v_x}{4} & 0 \\ -\sqrt{2}e^{-v/2} & \frac{v_x}{4} \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}, \quad \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}_y = \begin{pmatrix} \frac{v_y}{4} & 0 \\ -\sqrt{2}e^{-v/2} & -\frac{v_y}{4} \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \end{pmatrix}.$$

The compatibility condition is again found to hold provided that v satisfies $v_{xy} = 0$.

It might be conjectured as a further application of this work that if Lax pairs of the form (5.5) can be produced by some means, their matrix elements might be used to generate connections of zero curvature as discussed here. If they are found to have zero curvature structure, the results obtained here would be of use in generating Bäcklund transformations for the equations involved.

8 References

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