University of Texas Rio Grande Valley

ScholarWorks @ UTRGV

Mathematical and Statistical Sciences Faculty
Publications and Presentations

College of Sciences

8-13-2018

Solution of mathematical model for gas solubility using fractional-order Bhatti polynomials

Muhammad I. Bhatti
The University of Texas Rio Grande Valley

Paul Bracken
The University of Texas Rio Grande Valley, paul.bracken@utrgv.edu

Nicholas Dimakis
The University of Texas Rio Grande Valley

Armando Herrera

Follow this and additional works at: https://scholarworks.utrgv.edu/mss_fac

Part of the Mathematics Commons

Recommended Citation

PAPER • OPEN ACCESS

Solution of mathematical model for gas solubility using fractional-order Bhatti polynomials

View the article online for updates and enhancements.

Journal of Physics Communications

PAPER

Solution of mathematical model for gas solubility using fractional-order Bhatti polynomials

Muhammad Bhatti [iD] , Paul Bracken, Nicholas Dimakis [iD] and Armando Herrera

Department of Physics, University of Texas Rio Grande Valley, Edinburg, TX 78540, United States of America

E-mail: muhammad.bhatti@utrgv.edu

Abstract

Solutions of a mathematical model for gas solubility in a liquid are attained employing an algorithm based on the generalized Galerkin B-poly basis technique. The algorithm determines a solution of a fractional differential equation in terms of continuous finite number of generalized fractional-order Bhatti polynomial (B-poly) in a closed interval. The procedure uses Galerkin method to calculate the unknown expansion coefficients for constructing a solution to the fractional-order differential equation. Caputo?s fractional derivative is employed to evaluate the derivatives of the fractional B-polys and each term in the differential equation is converted into a matrix problem which is then inverted to construct the solution. The accuracy and efficiency of the B-poly algorithm rely on the size of the basis set as well as the degree of the B-polys used. The fractional-order B-Poly technique has been applied to the mathematical model for a gas diffusion in a liquid with gas volume functions $f(t) = 1 - t^{1/2}$ and $f(t) = 1 - t^{3/2}$. The solutions of the model were obtained which converged with a small number of B-polys basis set. In case of the power series solution, the solution did not converge due to alternating terms present in the solution. We used a Pade approximant on the power series solutions to extract the useful information which showed the solutions are convergent and those solutions were compared with the solutions obtained from the B-poly approach. Excellent agreement was found between the solutions. A Pade approximant was not used on the B-poly solutions because those were convergent with a smaller number of B-polys.

1. Introduction

The Fractional Bhatti polynomial (B-poly) technique [1, 2] is extremely useful for solving varieties of differential equations. The fractional B-polys [2] are precise, define a basis set, straightforwardly differentiable and exemplify an arbitrary function to a desired accuracy over an interval. In recent years, many authors have predicted solutions of differential equations using analytic and numerical methods with high accuracy [3–19]. In the past paper [2], an algorithm had been provided for solving fractional-order differential equations using a generalized Galerkin method and the B-poly basis of fractional-order. The method used the unitary property of generalized B-polys on the interval [0, R] and converted the differential equation into a matrix equation for constructing the solution of a fractional-order equation that provided agility to impose initial as well as boundary conditions. Several examples to determine the solutions of the fractional Harmonic Oscillator were provided [1–5]. In this paper, we will use a similar technique to solve fractional-order differential equation with variable coefficients. One of the fractional-order equations provides a mathematical model for solubility of the gas in a liquid presented by Yu I Babenko [20]. Pressure effects on the solubility of gasses are as follows: At low pressure, a gas has a low solubility and decreased pressure allows more gas molecules to be present in the air, with little gas being dissolved in a liquid. At high pressure, a gas has high probability of solubility in the liquid and increased pressure forces the gas molecules into the solution, liberating the pressure that is applied on the gas volume. This is known as Henry's Law. The more the pressure the more the gas is mixed in a liquid, and the less pressure the less mixing.

Carbonated and ammonia waters are good examples of solutions of a gas in a liquid. Carbonated water is made from carbon dioxide (CO_2) being pressured into water. As we know, the mixture of liquid and gas is not the stable condition, an increase in temperature causes the separation. The lower temperature increases chances for dissolving the gas into the liquid. Increasing the gas pressure and decreasing the temperature dissolves more gas into the fluid. Common examples of pressure effects on gas solubility can be verified with carbonated beverages, such as a bottle of soda. Once the pressure within the unopened bottle is released, $CO_2(g)$ is released from the solution as fizzing. In order for deep-sea divers to breathe under water they must take in extremely compressed air in deep water resulting in more nitrogen mixing in their blood, tissues, and other joints. If the diver returns to the surface too quickly; the nitrogen gas scatters out of the blood too quickly and causes discomfort and probably death. To prevent such a painful situation, one can return to the surface steadily so that the gas will scatter slowly and normalize to incremental (fractional) reductions in pressure. Similarly, with soda, the fractional pressure is essential to keep the gas inside the liquid. As the drink becomes flat, almost all the carbon dioxide has been released from the liquid. Water carries dissolved oxygen from the partial pressure of the air (oxygen) in the atmosphere. In this paper, we will consider examples of the model of fractional differential equations with varying concentrations of gas solubility in a liquid and several solutions of the fractional model will be presented to confirm the validity of Henry's Law in everyday life. The gas partial pressure is directly proportional to the gas solubility at constant temperatures. The solubility of a gas in a fluid also depends on the nature of the solvent and the nature of the gas.

Two cases of the model with gas volume functions $f(t) = 1 - t^{1/2}$ and $1 - t^{3/2}$ are considered. Both cases are solved using the B-poly and analytic methods. It is shown that in the first case, $f(t) = 1 - t^{1/2}$, both results from B-poly and power series solutions agree gave converged solutions for an arbitrary value of lambda. In the second case $f(t) = 1 - t^{3/2}$, the B-poly method returned converged solution whereas, power series method did not converge because alternative terms were present in the solution. We used Pade Approximant to the power series solution to obtain approximate solution for comparison with the B-poly approximation. The paper is organized as follows. In the section 2, we provide basic fractional B-poly algorithm for approximating solutions of a fractional differential equation. Explicit details of the method are also provided in [2]. In this paper, a concise description of the method is provided with explicit analytic formulas are given involving fractional derivatives, inner product of B-polys and approximate solution. We also provided important formulas to generate fractional B-ploy basis sets for approximating solutions. In section 3, a model for gas solubility in liquid is provided [20]. A procedure is laid down for calculating approximate solution of this model using B-polys basis set. Proposed method converts the model fractional differential equation into a matrix which is invertible and therefore, provides unique solution of the matrix equation $MA = b$, where matrix A represents unknown coefficients needed to approximate the solution of the model equation (8). Examples are provided to show how current method works to calculate an approximate solution for both examples. In the last section 4, we have concluded the method works well for approximate solutions of the model equation. The B-poly method provides solution which reasonably agreed with the power series solutions.

The B-poly method has been successfully used in many papers and further detailed examples are provided in [1–5]. We have also provided explicit formulas for approximating the solution to fractional differential equations using fractional B-polys which have been for the first time applied to gas solubility model in the paper. These steps can be traced to approximate solutions, sometimes exact solutions are worked out using the current Galerkin B-poly basis method. Explicit expressions of those formulas are provided in equations (1)–(11). We have added more details about analysis of convergence in Conclusion section 4 by considering specific values of parameter $\lambda = \frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$. For example, $n = 1$ and $n = 2$, the λ values are $\frac{2}{\sqrt{\pi}}$, $\frac{3\sqrt{\pi}}{4}$, respectively. The corresponding convergent solutions from B-poly method are exact $1 + \sqrt{x}$ and $1 + \sqrt{x} + x\left(1 - \frac{3\pi}{8}\right)$, respectively. For other values of λ given in the paper, the convergence is obtained by varying the number of B-polys until converged approximate results are achieved and compared with the power series approximate solutions.

2. Procedure for approximating solutions of a fractional equation

In a fractional-order differential equation, the fractional-order derivatives are usually assessed using Caputo's fractional operator D^γ [2, 21, 22]. Also, we assume Caputo's derivative of a constant is zero and the term x^α is given as

$$D^{\gamma}x^{\alpha} = \begin{cases} 0, & \text{for } \alpha \in N_0 \text{ and } \alpha < |\gamma|; \\ \dfrac{\Gamma(\alpha + 1)x^{\alpha-\gamma}}{\Gamma(\alpha + 1 - \gamma)}, & \text{otherwise.} \end{cases} \tag{1}$$

All the fractional-order derivatives of the polynomial terms are evaluated using the above definition. The linear operator property of Caputo's derivative is used in approximating the solution to the fractional differential equations. Here, we shall briefly include some properties of the generalized fractional-order B-polys. The B-polys are delineated on a finite interval [0, R], for further details see [1, 2].

$$B_{i,n}(\alpha, x) = \sum_{k=i}^{n} \beta_{i,k} \left(\frac{x}{R} \right)^{\alpha k}. \tag{2}$$

here $\beta_{i,k}$ are given,

$$\beta_{i,k} = (-1)^{i-k} \binom{n}{k} \binom{k}{i}.$$

For convenience, one can also generate fractional B-polys using the recursive formula given in [2]. The generalized B-polys present a basis set and the sum of all the B-polys of fractional-order α is unity over the entire interval [0, R]. An illustration is provided in [2] to show the properties of the fractional B-polys. Any unknown function may be articulated in terms of generalized fractional-order polynomials, such as [2],

$$y(x) = \sum_{i=0}^{n} a_i B_{i,n}(\alpha, x). \tag{3}$$

Here a_i are unknown constants and n is the number of the polynomials in the above equation. The unknown constants are calculated after imposing the initial/boundary conditions. In fact, as mentioned earlier, a generalized basis set and the Galerkin method [2] are used to approximate the solution to the fractional-order equations. Furthermore, an approximation and sometimes exact solution to the fractional-order differential equation is derived by converting the differential equation into a matrix equation. The initial or boundary conditions are applied on to the matrix and its inverse matrix is multiplied on the right-hand side of the equation to find the unknown constants of the expansion equaion (3). Finally, equation (3) is used to express the solution to the fractional-order differential equation. The inner products in terms of fractional-order B-polys and their derivatives are provided in Reference [2]. The explicit formulas are briefly presented here,

$$b_{i,j} = (B_{i,n}(\alpha, x), B_{j,n}(\alpha, x)) = \sum_{k=i}^{n} \beta_{i,k} \left(\frac{x}{R} \right)^{\alpha k} \sum_{l=j}^{n} \alpha_{i,k} \left(\frac{x}{R} \right)^{\alpha l} \frac{R}{(k+l)\alpha}. \tag{4}$$

Caputo's derivative of the B-Polys is given by,

$$D^{\gamma}B_{i,n}(\alpha, x) = \sum_{k=i}^{n} \alpha_{i,k} D^{\gamma} \left(\frac{x}{R} \right)^{\alpha k} = \sum_{k=i}^{n} \frac{\beta_{i,k}}{R^{\alpha k}} \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha k + 1 - \gamma)} x^{\alpha k - \gamma} \tag{5}$$

$$d_{i,j}^{(\gamma)} = (D^{\gamma}B_{i,n}(\alpha, x), B_{j,n}(\alpha, x)) = \sum_{k=i, l=j}^{n} \beta_{i,k}\beta_{j,k} \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha k + 1 - \gamma)} \frac{R^{1-\gamma}}{((k+l)\alpha + 1 - \gamma)}, \tag{6}$$

and

$$f_i = (f(x), B_{i,n}(\alpha, x)) = \sum_{k=i}^{n} \frac{\beta_{i,k}}{R^{\alpha k}} \int_0^R f(x) x^{\alpha k} \, dx. \tag{7}$$

we shall consider the physical example of the fractional-order differential equation for modeling the solubility of a gas in a liquid [20]. In the following, we show how current algorithm provides results when applied to the physical examples of the fractional-order differential equation for modeling the solubility of a gas in liquid [20].

3. Model for gas solubility into a liquid

Babenko [20] offers an example of a mathematical model of a gas solubility in a liquid. His model designates the change of the mass of a gas volume due to diffusion through the contact surface. The model assumes the gas temperature (T) remains constant as the compression of the gas into the liquid varies slowly. Also, the mass increase depends on the concentration near the contact surface. This model is presented in terms of initial value fractional problems for determining the dimensionless gas pressure p(t) near the surface [20, 22].

$$f'(t)y(t) + f(t)y'(t) + \lambda \, _0D_t^{1/2} y(t) = -f'(t), \qquad y(0) = 0, \qquad p(t) = 1 + y(t). \tag{8}$$

The above model is an inhomogeneous fractional differential equation with variable coefficients where the function f(t) is known. For a particular choice of $f(t)$ that represents the change of the gas volume due to

diffusion through the contact surface. The model (8) allows an analytical solution using the power series method. The constant depends on several variables such as gas concentration, gas diffusion constant, mass, volume, temperature, etc [22]. We would like to find numerical and if possible analytic solutions to equation (8) in the region [0, 1] using the fractional B-polys technique and imposing the initial condition $y(0) = 0$ and with known function $f(t)$ of change in gas volume as pressure changes with time. An approximate solution to equation (8) may be written as represented in (3). Notice that at $t = 0$, (3) does not satisfy the initial condition, $y(0) = 0$ because all of the B-poly are not zero at $t = 0$. To satisfy the initial condition automatically, we need to start summation from $i = 1$ instead of $i = 0$ in (3). The unknown constants are calculated using generalized Galerkin and fractional-order B-poly methods [2]. After linearizing the equation (8) in terms of B-polys basis set, we obtain

$$\sum_{i=1}^{n} a_i \left(\int_0^R [f'(t)B_{i,n}(\alpha, t)B_{j,n}(\alpha, t) + f(t)B'_{i,n}(\alpha, t)B_{j,n}(\alpha, t) + \lambda D^{1/2}B_{i,n}(\alpha, t)B_{j,n}(\alpha, t)] \, dt \right)$$
$$+ \int_0^R f'(t)B_{j,n}(\alpha, t) \, dt = 0, \tag{9}$$

$$B_{i,j} = \int_0^R [f'(t)B_{i,n}(\alpha, t)B_{j,n}(\alpha, t) + f(t)B'_{i,n}(\alpha, t)B_{j,n}(\alpha, t) + \lambda D^{1/2}B_{i,n}(\alpha, t)B_{j,n}(\alpha, t)] \, dt \tag{10}$$

and

$$b_j = \int_0^R f'(t)B_{j,n}(\alpha, t) \, dt = 0.$$

The symbol $D^{1/2}$ is Caputo's derivative in these equations and equations (9), (10) determine an $n \times n$ system of equations in a matrix form, $BA = b$, in variables $a_1, a_2, \cdots, a_n$. The integrals in equations (9), (10) are evaluated using formulas provided in (4)–(6). We also obtained analytic solutions of the model with $f(t) = 1 - t^{1/2}$ and using various B-poly basis sets for n = 1 and 2 which agreed with the solution provided in [22].

The B-Poly basic Technique has been used in many of the papers published in various journals and some references are provided. The method applied to various cases had returned approximate convergent results which have been reported here. The algorithm hinges over the inverse of the matrix which is successfully obtained using Wolfram Mathematica 11.2 symbolic programs in all of our calculations. The convergence or stability analysis is improved step by step by increasing the number of B-polys for each case in hand. For more details of the convergence analysis [2]. In the B-poly techniques, we are dealing with $n \times n$ matrices. An $n \times n$ matrix A is invertible if there exists an $n \times n$ matrix A^{-1} such that $A^{-1}A = AA^{-1} = I$, where I is the $n \times n$ identity matrix. We also checked that the Matrix $n \times n$ is invertible if and only if its determinant is nonzero. According to the linear algebra theorem, if matrix A is invertible then the equation $Ax = b$ has unique solution $x = A^{-1}b$. Our calculation showed after that the final matrix is invertible and has unique solution which is then compared with exact solution of the solubility model equation (11).

To demonstrate the usefulness of the numerical procedure described above for approximating solutions of the fractional-order differential equations, we will consider two examples below:

Example 1. Let us consider as a first example, the function $f(t) = 1 - t^{1/2}$ describing the change of gas volume due to partial pressure applied on the gas. A typical solution of the mathematical model (8) with $n = 8$ degree-B-poly basis set, initial condition $y(0) = 0$ and various gas concentration values of λ are obtained using the fractional B-poly technique. The solutions converge with a small number of basis set of B-polys of degree $n = 8$. The solutions are given for four particular values of $\lambda = 1.0, 1.5, 2.0$ and 2.5, respectively.

$$y(t) = \quad 0.999\,965\,9t^{1/2} + 0.114\,450\,8t + 0.022\,522\,1t^{3/2} + 0.033\,802\,4t^2 - 0.054\,794\,9t^{5/2}$$
$$+0.081\,733\,9t^3 - 0.057\,939\,4t^{7/2} + 0.018\,897\,7t^4$$
$$y(t) = 0.999\,999\,9\,t^{1/2} - 0.329\,338\,3\,t + 0.042\,265\,6\,t^{3/2} + 0.000\,180\,6\,t^2 - 0.000\,096\,9\,t^{5/2}$$
$$+0.000\,126\,2\,t^3 - 0.000\,073\,9\,t^{7/2} + 0.000\,018\,5\,t^4$$
$$y(t) = 0.999\,999\,9\,t^{1/2} - 0.772\,451\,4\,t + 0.389\,690\,5\,t^{3/2} - 0.128\,286\,3\,t^2 + 0.026\,011\,6\,t^{5/2}$$
$$-0.002\,680\,5\,t^3 + 7.651\,069\,2 \times 10^{-6}\,t^{7/2} + 0.000\,023\,2\,t^4$$
$$y(t) = 0.999\,997\,9t^{1/2} - 1.215\,523\,6t + 1.070\,081\,5t^{3/2} - 0.706\,501\,0t^2 + 0.352\,453\,6t^{5/2}$$
$$-0.129\,459\,5t^3 + 0.031\,890\,4t^{7/2} - 0.003\,968\,0t^4$$

These approximate solutions given above are also compared with exact solutions obtained from power series expansions and recursive relationships provided in [22]. The errors between the solutions were acceptable for considered accuracy. Various graphs of the approximate solutions of equation (8) are presented in figure 1. It is noted from the solutions that Henry's Law is obeyed which states the solubility of a gas in a liquid is directly
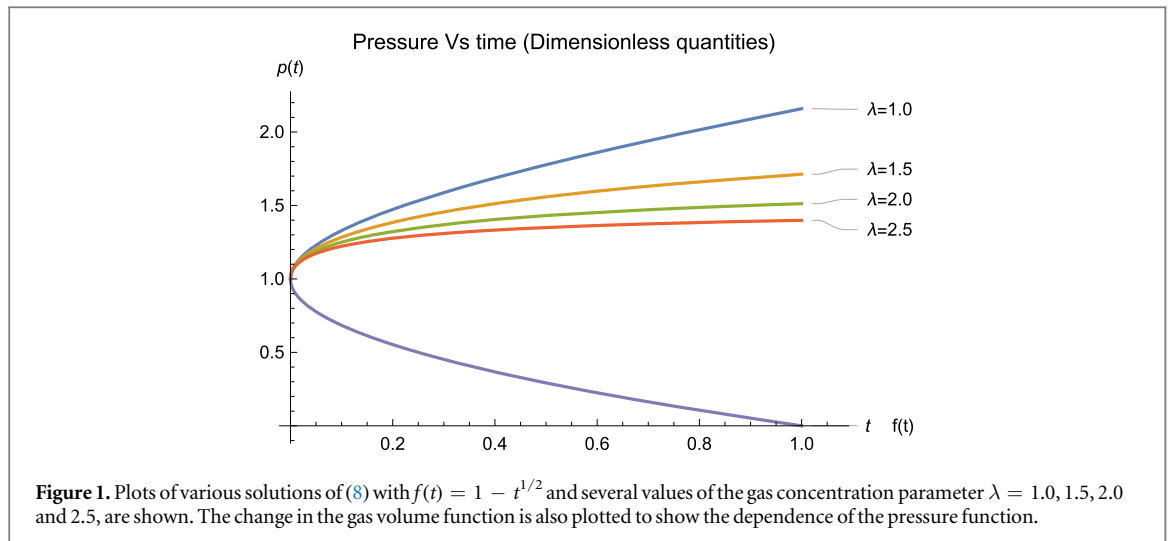
Figure 1. Plots of various solutions of (8) with $f(t) = 1 - t^{1/2}$ and several values of the gas concentration parameter $\lambda = 1.0, 1.5, 2.0$ and 2.5, are shown. The change in the gas volume function is also plotted to show the dependence of the pressure function.

proportional to the pressure on that gas above the surface of the liquid. All the quantities shown in the graphs are dimensionless as mentioned above in equation (8). The pressure function $p(t) = 1 + y(t)$ and the change in the gas volume $f(t)$ are also plotted in the graphs in figure 1. The remaining gas volume above the surface of a liquid decreases as the pressure increases. Also, both the pressure and the change of the gas volume are normalized to unity at t = 0. From the graphs in figure 1., it is obvious to see that as the gas concentration parameter is increased, reduced pressure is needed to dissolve the gas in a liquid.

Example 2. Consider a second example of the mathematical model for gas solubility (8), inserting the function for gas volume as $f(t) = 1 - t^{3/2}$ and setting the three initial conditions to $y(0) = 0, a_1 = 0$, and $a_2 = 0$. Furthermore, we seek the solution of (8) over the closed interval [0, 1]. Applying the generalized Galerkin-B-poly techniques to this problem as described in the previous example 1, we convert the problem again into matrix form and approximate the solution using equations (9), (10). Equation (9) may be reconstructed in the form of a generalized matrix equation, $BA = b$, in unknown variables, $a_1, a_2, \cdots, a_n$. The matrices **B** and **b** are constructed from the elements of (10). The integrals are evaluated according to the prescription provided in equations (4)–(7). In addition, we impose initial conditions by neglecting the first three B-polys (B_0, B_1, B_2) in the summation of (3) providing the appropriate solution to the solubility model equation (8). To provide the details of the procedure, for example, the results of the fractional-order model equation equation (8) with $\alpha = 1/2, \lambda = 1$ and $n = 4$ in the interval [0, 1] are given below. After neglecting the first three B-polys for imposing initial conditions on the matrix **B**, we get the basis set,

$$(4t^{3/2} - 4t^2, t^2), \qquad \mathbf{B} = \begin{pmatrix} 0.042\,469 & 0.198\,336 \\ -0.148\,258 & 0.197\,971 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 0.285\,714 \\ 0.428\,571 \end{pmatrix}.$$

We can now calculate expansion coefficients by multiplying the inverse of matrix **B** with the column matrix **b**. The matrix of coefficients $A = (-0.752\,081, 1.601\,596)$ is multiplied by the basis set according to equation (3) to obtain an approximate solution which is $y(t) = -3.008\,326t^{3/2} + 4.6\,09923t^2$. Unfortunately, this is not a convergent solution to the model problem.

In order to get a convergent solution to equation (8), we need to increase the size of the basis set by increasing the number of B-polys of higher degree (n) to reach the desired degree of accuracy. A few convergent solutions of the mathematical model (8), with $n = 14$ B-poly basis set, three initial conditions $y(0) = 0, a_1 = 0, a_2 = 0$ and various gas concentration values are obtained using the generalized Galerkin fractional B-poly technique. The solutions are given for various values of $\lambda = 1.0, 1.5, 2.0$ and 2.5, respectively.
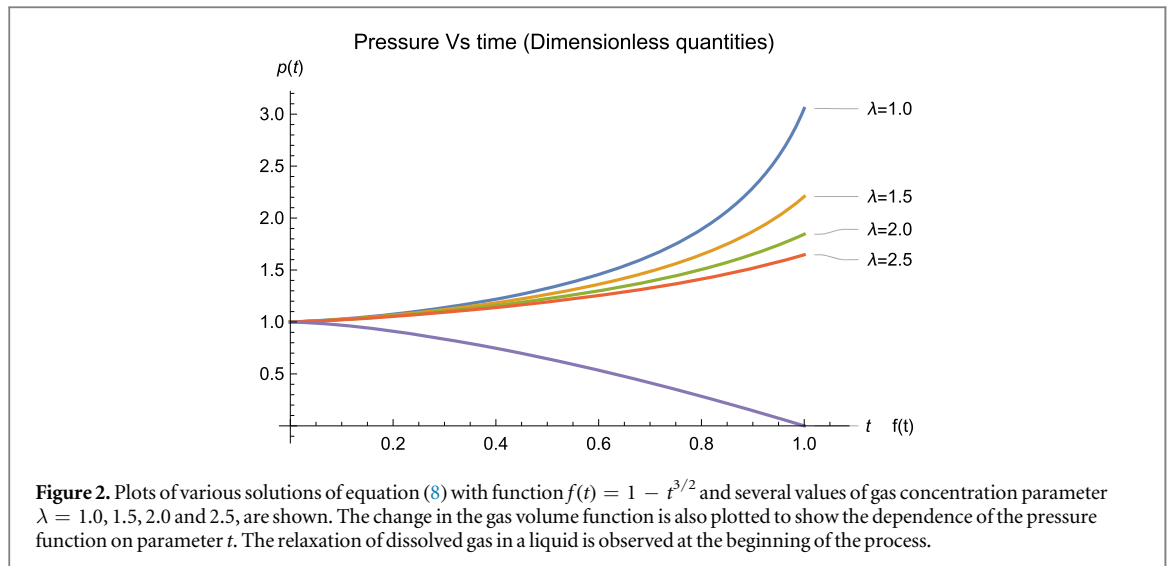
Figure 2. Plots of various solutions of equation (8) with function $f(t) = 1 - t^{3/2}$ and several values of gas concentration parameter $\lambda = 1.0, 1.5, 2.0$ and 2.5, are shown. The change in the gas volume function is also plotted to show the dependence of the pressure function on parameter t. The relaxation of dissolved gas in a liquid is observed at the beginning of the process.

$$y(t) = t^{3/2}(17.847\,351 - 343.449\,864t^{1/2} + 3\,048.672\,646t - 15\,286.367\,924t^{3/2} + 46\,922.570\,769t^2$$
$$- 88\,831.409\,292t^{5/2} + 94\,735.244\,609t^3 - 29\,330.832\,71t^{7/2} - 57\,802.566\,541t^4 + 82\,001.722\,239t^{9/2}$$
$$- 44\,321.405\,847t^5 + 9\,192.029\,656t^{11/2})$$

$$y(t) = t^{3/2}(3.678\,865\,7 - 52.039\,069\,7t^{1/2} + 426.654\,000\,0t - 2\,013.319\,804\,3t^{3/2} + 5\,864.879\,237\,6t^2$$
$$- 10\,593.196\,683\,5t^{5/2} + 10\,767.808\,419\,3t^3 - 2\,912.666\,287\,5t^{7/2} - 6\,822.741\,938\,7t^4$$
$$+ 9\,205.783\,343\,5t^{9/2} - 4\,871.800\,014\,2t^5 + 998.167\,114\,8t^{11/2})$$

$$y(t) = t^{3/2}(2.531\,088\,9 - 28.649\,469\,6t^{1/2} + 212.890\,992\,5t - 920.913\,528\,9t^{3/2} + 2\,453.390\,740\,1t^2$$
$$- 4\,018.942\,784\,7t^{5/2} + 3\,593.024\,847\,3t^3 - 494.140\,546\,8t^{7/2} - 2\,643.633\,268\,7t^4 + 3\,036.571\,868\,9t^{9/2}$$
$$- 1\,473.699\,592\,0t^5 + 282.414\,319\,9t^{11/2})$$

$$y(t) = t^{3/2}(2.295\,452\,2 - 24.328\,598\,9t^{1/2} + 173.476\,615\,0t - 725.020\,384\,4t^{3/2} + 1\,861.252\,775\,9t^2$$
$$- 2\,917.386\,055\,8t^{5/2} + 2\,445.909\,645\,7t^3 - 179.736\,204\,2t^{7/2} - 1\,882.058\,255\,4t^4 + 1\,993.911\,665\,6t^{9/2}$$
$$- 913.676\,202\,8t^5 + 166.006\,236\,2t^{11/2}).$$

The approximate solutions given above are also compared with a Pade approximant to the exact solutions obtained from the power series recursive formula [22]. The solution obtained from the recursive formula were not apparently convergent because of the alternating terms present in the approximant to the solution obtained from the recursive power series formula with $n = 100$ terms in the power series, an appropriate rational function of a given order rendered a convergent approximate solution. Both solutions, the Pade approximant solution and the solution using B-poly technique, were matched and the errors between the solutions were acceptable. Various graphs of the approximate solutions presented above of equation (8) are presented in Figure 2. It is noted that Henry's law is still obeyed which states the solubility of a gas in a liquid is directly proportional to the pressure on that gas above surface of the liquid. But, it is also observed that in the model in which $f(t) = 1 - t^{3/2}$ was chosen, there was, initially, a relaxation in the solubility of gas in a liquid. First the gas is dissolved very slowly and after some time, the gas is dissolved much faster as pressure is constantly increased. In other words, slow dissolution of gas and then at a later time much faster dissolution of gas has taken place. This phenomenon is new predicted by this type of gas model we have not seen in the literature. All the quantities shown in the graphs are dimensionless as mentioned earlier in equation (8). The pressure function $p(t) = 1 + y(t)$ and changes in the gas volume $f(t)$ are also plotted in the same graphs in figure 2. The remaining gas volume above the surface of a liquid decreases as the pressure increases. Also, both the pressure and the change of gas volume are normalized to unity at $t = 0$. From the graphs in figure 2, it is obvious to see that as the gas concentration parameter λ is increased, reduced pressure is needed to dissolve the gas in the liquid.

4. Conclusion

In this paper, a generalized Galerkin-fractional B-poly basis method has been employed to solve a mathematical model for gas solubility in a liquid. Two examples of the model were considered with gas volume functions $f(t) = 1 - t^{1/2}$ and $f(t) = 1 - t^{3/2}$. It is interesting that in example 1, the results obtained from the recursive formula [22],

$$y(t) = \sum_{n=0}^{\infty} a_n t^{n/2}, \qquad f(t) = \sum_{n=0}^{\infty} b_n t^{n/2}$$

where,

$$\sum_{k=0}^{n} a_{n+1-k} b_k + \lambda a_n \frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)} = -b_{n+1}, \tag{11}$$

provided convergent solutions to the model in equation (8) for arbitrary values of λ, in the power series solutions for several values of λ. Actually, if special vales of parameter $\lambda = \frac{\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$ are used to terminate the series in equation (11), B-polys method provided exact solutions to the model in example 1. For example, when n $=$ 4 number of B-polys are taken, explicitly, we showed that method does not provide convergent solution. We had to use n $=$ 14 fractional B-polys to obtain convergent solution which could be compared with n $=$ 100 terms solution obtained from power series solution in equation (11). Power series solution did not provide convergent solution even after using 100 terms, we had to use Pade Approximation on the solution to compare with the approximate solution obtained from B-poly method.

After the application of a Pade approximant to the solution yielded convergent results. At about $n = 14$ terms, the solutions of the model problem seem to converge with various values using the recursive formula in equation (11). Comparing the solutions calculated using the currently employed fractional B-poly technique with $n = 8$ degree-B-polys, imposing the initial conditions as mentioned in example 1, the solutions agree perfectly. Furthermore, the generalized Galerkin B-poly technique shows superior convergence over the exact formula in (11), with a smaller number of B-polys. The four plots of the pressure function $p(t) = 1 + y(t)$ has been depicted in figure 1 for various values of the parameter $\lambda = 1.0, 1.5, 2.0$ and 2.5. The solutions obtained obviously obey Henry's law which states the solubility of a gas in a liquid is directly proportional to the pressure on that gas above the surface of a liquid. Example 2 of the fractional-order model is based on the gas volume function $f(t) = 1 - t^{3/2}$. The solutions obtained from a power series expansion and the recursive formula (11) weren't apparently convergent because of the alternating terms present in the series solution. We had to use a Pade approximant on the solution that yielded the results that were convergent. This resulted in the form of a rational function of given order converged and was compared with the results of the fractional B-Poly technique. Both methods gave us results which had errors within desired accuracy when $n = 14$ degree B-polys were utilized for better convergence. The four graphs of the pressure function $p(t) = 1 + y(t)$ along with changes of gas volume function $f(t)$ has been depicted in figure 2 for various values of the gas concentration parameter $\lambda = 1.0, 1.5, 2.0$ and 2.5. The solutions obtained obey Henry's law, the solubility of a gas in a liquid is directly proportional to the pressure on that gas above the surface of a liquid. We also observed in those solutions that the gas diffusion is extremely slow in the beginning of the process and then progressed at much faster rate. This is new phenomenon predicted by the model with the unknown function taken as $f(t) = 1 - t^{3/2}$ and seems to be closer to a real situation in nature. From the graphs in figure 2, it is also obvious that as the gas concentration is increased the lower pressure is required to increase diffusion of the gas in the liquid.

Explicit formulas [2] are used concerning the inner products of the B-polys and their derivatives expressed in (4)–(7) via equations (1), (2). The algorithm for predicting the solution of the mathematical model for gas solubility is successfully applied and solutions of the model are compared with power series solutions in equation (11). The solutions obtained from the current algorithm shows that the B-poly approach is somewhat robust to solve the fractional-order differential equations with a smaller number of B-polys. The method can be easily extended in other research disciplines, such as fractional quantum mechanics as well as in atomic physics.

ORCID iDs

Muhammad Bhatti ⓘ https://orcid.org/0000-0002-0162-2624
Nicholas Dimakis ⓘ https://orcid.org/0000-0003-2478-237X

References

[1] Bhatti M I 2014 *Physics and Technical Sciences* **2** 8–13
[2] Bhatti M I 2013 *Adv. Studies Theor. Phys.* **7** 11–3
[3] Bhatti M I 2017 *Current Trends in Technology and Sciences* **VI, II** 707
[4] Bhatti M I 2009 *Adv. Studies Theor. Phys.* **3** 451–60
[5] Bhatti M I and Bracken P 2007 *J. Comp. and Applied Math.* **205** 272–80
[6] Gejji V D and Jafari H 2007 *Appl. Math. Comput.* **189** 541–8
[7] Ray S S, Chaudhuri K S and Bera R K 2006 *Appl. Math. Comput.* **182** 544–52
[8] Cenesiz Y, Keskin Y and Kurnaz A 2010 *J. Franklin Inst.* **347** 452–66

[9] Doha E H, Bhrawy A H and Ezz-Eldien S S 2012 *Applied Mathematical Modelling* **36** 4931–43
[10] Sweilam N H, Khader M M and Al-Bar R F 2007 *Phys. Lett.* A **371** 26–33
[11] Hashim I, Abdulaziz O and Momani S 2009 *Commun. Nonlinear Sci. Numer. Simulat.* **14** 674–84
[12] Kumar P and Agrawal O P 2006 *Signal Process* **86** 2602–10
[13] Wang Y and Fan Q 2012 *Appl. Math. Comput.* **218** 8592–601
[14] Vanani S K and Aminataei A 2011 *Comput. Math. Appl.* **62** 1075–83
[15] Isik O R, Güney Z and Sezer M 2012 *J. Differ. Equ. Appl.* **18** 357–74
[16] Saadatmandi A and Dehghan M 2010 *Compt. Math. Appl.* **59** 1326–36
[17] Yüzbasi S 2013 *Appl. Math. Comput.* **219** 6328–43
[18] Yousefi S A and Behroozifar M 2010 *Int. J. Systems Sci.* **41** 709–16
[19] Isik O R, Sezer M and Güney Z 2011 *Appl. Math. Comput.* **217** 9438–50
[20] Babenko Y I 1986 *Heat and Mass Tranfer* (Leningrad: Khimiya) (in Russian)
[21] Caputo M 1967 *J. Ry. Aust. Soc.* **13** 529–39
[22] Polubny I 1999 *Fractional Differential equations* (New York: Academic)