Applications and Applied Mathematics: An International Journal (AAM)

Volume 16 | Issue 2 Article 28

12-2021

(R1513) The Dynamical Study of Variable Mass Test Particle in Nonlinear Sense of Restricted 3-body Problem with Heterogeneous Primaries

Sada Nand Prasad
Acharya Narendra Dev College, University of Delhi

Kumari Shalini
Zakir Hussain Delhi College, University of Delhi

Abdullah A. Ansari
International Center for Advanced Interdisciplinary Research (ICAIR)

Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 16, Issue 2 (December 2021), pp. 1274 – 1294

The Dynamical Study of Variable Mass Test Particle in Nonlinear Sense of Restricted 3-body Problem with Heterogeneous Primaries

[1]Sada Nand Prasad, [2]Kumari Shalini, and [3] *Abdullah A. Ansari

[1]Department of Mathematics
Acharya Narendra Dev College
University of Delhi
Delhi, India
sadanandprasad@andc.du.ac.in

[2]Department of Mathematics
Zakir Hussain Delhi College
University of Delhi
Delhi, India
thakurshalini2000@gmail.com

[3]International Center for Advanced Interdisciplinary Research (ICAIR)
Sangam Vihar, New Delhi, India
icairndin@gmail.com

* Corresponding Author

Received: July 25, 2021; Accepted: September 15, 2021

Abstract

The main idea of this paper is to study the non-linear stability property of the motion of the test particle which is moving under the influence of heterogeneous primaries having N-layers with different densities as well as varying its mass according to Jeans law. The system is also perturbed by the small perturbations in Coriolis as well as centrifugal forces. We evaluate the equations of motion of the test particle under the influence of the above said perturbations. From this system of equations of motion, we reveal analytically the locations of stationary points as well as the non-linear stability.

Keywords: Nonlinear Stability; Variable mass; Lagrangian; Heterogeneous primaries; Layers; Densities; Perturbations

MSC 2010 No.: 70F05; 70F07

1. Introduction

The problems of three bodies is found to be the most famous problems of classical dynamics. The failure in finding a general solutions of the general three bodies problems compelled mathematician and astronomers to study the problem in a restricted three bodies sense. It was Euler who first of all studied circular three bodies problem and found the three collinear liberation points. Again Lagrange found some particular solutions for the general three bodies problems. Besides these, different aspects of the restricted three bodies problem have been discussed by many mathematician and astronomers and their contributions are found to be valuable. For last few decades, scientists have published so many papers in the restricted three-body problem where the primaries are either point masses or spherical in shape considering many other perturbing forces such as oblatenes and rotation forces, Corolies and Centrifugal forces, etc. Some of them are: Deprit and Deprit (1967), Gozdziewski et al. (1991), Haque and Ishwar (1995), Ishwar and Sharma (2012), Kumar and Choudhary (1990), Subbarao and Sharma (1997), etc.

In recent times, Abouelmagd and Ansari (2021), Abouelmagd and Ansari (2019), Ansari (2021), Ansari (2018), Ansari (2017), Bouaziz and Ansari (2021), Jain and Sinha (2014), Kushvah et al. (2007), Pathak et al. (2019), Riaguas et al. (2001), Singh and Ishwar (1985), Singh (2009), Zhang et al. (2012), etc. have worked on the restricted three-body problem by taking different aspects and give many precise results. Moreover, their studies included variation of the masses of the primaries and of the infinitesimal mass, etc. Bhatnagar and Hallan (1983) have studied the non-linear stability of libration point L_4 in the restricted problem by taking the perturbations ϵ and ϵ' in the Coriolis and the Centrifugal forces respectively. Generally, scientists have taken the primaries as homogeneous point masses or spherical in shapes. But primary as heterogeneous in the restricted three-body problem have been studied by Sahdev and Ansari (2021a) and Sahdev and Ansari (2021b).

In our study, we like to find the non-linear stability by considering both the primaries are heterogeneous triaxial rigid body having N layers with different densities as well as varying the mass of the infinitesimal body according to Jean's law, and also the small perturbations in Coriolis as well as centrifugal forces are affected to our system. For this we will apply Moser's modified version of Arnold's theorem (Arnold (1961)) and procedure as adopted by Bhatnagar and Hallan (1983).

The paper is organized as follows. The literature review is presented in the introduction Section 1. The equations of motion with the various perturbations are presented in Section 2. Then in Section 3, we have determined analytically the locations of triangular stationary points. We have evaluated the Lagrangian for the proposed system in Section 4 while Section 5 deals the first order normalization. We have performed first order normal co-ordinates in Section 6, and Section 7 evaluates the second order normal co-ordinates. Second order coefficients in the frequencies are obtained in Section 8. In Section 9, we have checked the non-linear stability of triangular libration points. At the end, Section 10 contains the discussion and conclusion of the obtained results.

2. Equations of motion

This problem contains three masses out of which two massive bodies m_1 and m_2 are considered as heterogeneous shapes with $N-$layers having different densities. Both are moving in circular orbits around their common center of mass which is taken as origin O with radii ℓ_1 and ℓ_2, respectively. The system is also perturbed by the small perturbations in the Coriolis and centrifugal forces with the parameters ϕ and ψ, respectively. And the third smallest body of mass m which is moving in the same plane of the massive bodies and follow the synodic coordinate system which is rotating with angular velocity n. This third body is varying its mass according to Jean's law.

For the non-dimensional units, we have taken $m_1 + m_2 = 1$, $G = 1$, $\ell_1 + \ell_2 = 1$ and $\mu = m_2/(m_1 + m_2)$. Following the procedure used by Abouelmagd and Mostafa (2015), Ansari and Abouelmagd (2020) and Ansari et al. (2019), we can write the equations of motion of the smallest varying mass body taking on account that the variation of mass of this body originates from one point and have zero momenta as:

$$\begin{aligned} \ddot{x} - 2\,n\,\phi\,\dot{y} + \frac{\dot{m}}{m}(\dot{x} - n\,\phi\,y) &= U_x, \\ \ddot{y} + 2\,n\,\phi\,\dot{x} + \frac{\dot{m}}{m}(\dot{y} + n\,\phi\,x) &= U_y, \end{aligned} \tag{1}$$

with

$$U = \frac{n^2\,\psi}{2}(x^2 + y^2) + \frac{(1-\mu)}{r_1} + \frac{1}{2\,r_1^3}\left[J_1 - \frac{3}{r_1^2}\,J_2\,y^2\right] + \frac{\mu}{r_2} + \frac{1}{2\,r_2^3}\left[J_1' - \frac{3}{r_2^2}\,J_2'\,y^2\right], \tag{2}$$

$$n^2 = 1 + c_1\,J_1 + c_1'\,J_1', \;\; c_1 = \frac{3}{2\,(1-\mu)},\; c_1' = \frac{3}{2\,\mu}, \tag{3}$$

$$r_1^2 = (x + \mu)^2 + y^2, \quad \text{and} \quad r_2^2 = (x + \mu - 1)^2 + y^2. \tag{4}$$

J_i and J_i' $(i = 1, 2)$ are the non-dimensional perturbed paramaraters of heterogeneous bodies.

We will use Jean's law (Jeans (1928)) and Meshcherskii transformations (Meshcherskii (1949)) for preserving the dimensions. These are as follows:

$$\begin{aligned} dt &= d\tau, \qquad m = m_0\,e^{-\alpha_0\,t}, \\ (x, \dot{x}, \ddot{x}) &= \beta_0^{-1/2}\left[\alpha, \left(\dot{\alpha} + \frac{\alpha_0}{2}\,\alpha\right), \left(\ddot{\alpha} + \alpha_0\,\dot{\alpha} + \frac{\alpha_0^2}{4}\,\alpha\right)\right], \\ (y, \dot{y}, \ddot{y}) &= \beta_0^{-1/2}\left[\beta, \left(\dot{\beta} + \frac{\alpha_0}{2}\,\beta\right), \left(\ddot{\beta} + \alpha_0\,\dot{\beta} + \frac{\alpha_0^2}{4}\,\beta\right)\right], \end{aligned} \tag{5}$$

where α_0 is constant coefficient, $\beta_0 = \dfrac{m}{m_0}$ and m_0 is the initial mass of the third smallest body.

After using Equation (5) in Equation (3), we get

$$\ddot{\alpha} - 2\,n\,\phi\,\dot{\beta} = \frac{\partial W}{\partial \alpha}, \qquad \ddot{\beta} + 2\,n\,\phi\,\dot{\alpha} = \frac{\partial W}{\partial \beta}, \tag{6}$$

where,

$$W = W_0 \; + \; \beta_0^{3/2}\, W_1, \tag{7}$$

$$W_0 = \left(\frac{n^2\,\psi}{2} \; + \; \frac{\alpha_0^2}{8}\right)(\alpha^2 + \beta^2),$$

$$W_1 = \frac{(1-\mu)}{\rho_1} + \frac{\beta_0}{2\,\rho_1^3}\left(J_1 - \frac{3}{\rho_1^2}\,J_2\,\beta^2\right) + \frac{\mu}{\rho_2} + \frac{\beta_0}{2\,\rho_2^3}\left(J_1' - \frac{3}{\rho_2^2}\,J_2'\,\beta^2\right),$$

$$\rho_1^2 = (\alpha \, + \, \mu\,\beta_0^{1/2})^2 \, + \, \beta^2, \qquad \rho_2^2 = (\alpha \, + \, (\mu - 1)\,\beta_0^{1/2})^2 \, + \, \beta^2. \tag{8}$$

3. Triangular stationary points

The stationary points can be found by solving the equations

$$W_\alpha = 0, \quad W_\beta = 0. \tag{9}$$

In the restricted 3-body problem when both the massive bodies are point mass as well as the mass of the third body is constant, then $\rho_1 = 1$ and $\rho_2 = 1$ are the solutions and the coordinates of triangular points are ($\alpha = \frac{1}{2} - \mu$ and $\beta = \pm\frac{\sqrt{3}}{2}$) (Szebehely (1967)). Here we have taken the perturbations (such as the shape of both massive bodies as heterogeneous with $N-$layers, the coriolis and centrifugal forces as well as the third infinitesimal body is varying its mass). Then let $\rho_1 = 1 + \epsilon_1$ and $\rho_2 = 1 + \epsilon_2$ be the solutions and therefore from Equation (8), we can get

$$\alpha = (\tfrac{1}{2} \, - \, \mu)\,\beta_0^{1/2} \, + \, (\epsilon_1 - \epsilon_2)\,\beta_0^{-1/2}, \qquad \beta = \pm\,(1 - \tfrac{\beta_0}{4} \, + \, \epsilon_1 + \epsilon_2)^{1/2}, \tag{10}$$

where a positive sign represents L_4 while a negative sign represents L_5, and finally from Equation (9), we found $\epsilon_1 = \epsilon_{11}/\epsilon_{12}$, $\epsilon_2 = \epsilon_{21}/\epsilon_{22}$, where ϵ_{11}, ϵ_{12}, ϵ_{21} and ϵ_{22} are given in Appendix A.

4. Lagrangian for the system

The Lagrangian of the system can be written as

$$\Gamma = \tfrac{1}{2}(\dot{\alpha}^2 + \dot{\beta}^2) + n\,\phi(\alpha\,\dot{\beta} - \beta\,\dot{\alpha}) + \left(\frac{n^2\,\psi}{2} + \frac{\alpha_0^2}{8}\right)(\alpha^2 + \beta^2) \, + \, W. \tag{11}$$

Now shifting the origin to the triangular stationary point L_4 and expanding Γ in power series of α and β, we get

$$\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2 + \Gamma_3 + \, ..., \tag{12}$$

where,

$$\Gamma_0 \, = \, \Gamma_{0\,0} \, + \, \Gamma_{0\,1}, \tag{13}$$

$$\Gamma_1 = \Gamma_{10} \, + \, \Gamma_{11}\,\alpha \, + \, \Gamma_{12}\,\beta, \tag{14}$$

$$\Gamma_2 = \Gamma_{20} \, + \, \Gamma_{21}\,\alpha^2 \, + \, \Gamma_{22}\,\beta^2 \, + \, \Gamma_{23}\,\alpha\,\beta, \tag{15}$$

$$\Gamma_3 = \Gamma_{30}\,\alpha^3 + \Gamma_{31}\,\beta^3 + \Gamma_{32}\,\alpha^2\,\beta + \Gamma_{33}\,\alpha\,\beta^2, \tag{16}$$

$$\Gamma_4 = \Gamma_{40}\,\alpha^4 + \Gamma_{41}\,\beta^4 + \Gamma_{42}\,\alpha^2\,\beta^2 + \Gamma_{43}\,\alpha^3\beta + \Gamma_{44}\,\alpha\,\beta^3. \tag{17}$$

The values of Γ_{ij}'s are given in Appendix A.

Lagrange's equations of motion in first order can be written as

$$\ddot{\alpha} - 2\,n\,\phi\,\dot{\beta} = \alpha_1\,\alpha + \alpha_2\,\beta, \qquad \ddot{\beta} + 2\,n\,\phi\,\dot{\alpha} = \beta_1\,\alpha + \beta_2\,\beta. \tag{18}$$

The characteristic polynomial of the above equation will be

$$\lambda^4 + \lambda^2(4\,n^2\,\phi - \alpha_1 - \beta_2) + (\alpha_1\,\beta_2 - \alpha_2\,\beta_1). \tag{19}$$

The values α_1, α_2, β_1 and β_2 are given in Appendix A. Let the four roots of the above polynomial be $\pm\,i\,f_1'$ and $\pm\,i\,f_2'$, where f_1' and f_2' are the perturbed frequencies. Then

$$\begin{aligned} (f_1')^2 + (f_2')^2 &= d_1 + (J_1 + J_1')\,d_2 + (J_2 + J_2')\,d_3, \\ (f_1')^2 * (f_2')^2 &= d_4 + (J_1 + J_1')\,d_5 + (J_2 + J_2')\,d_6. \end{aligned} \tag{20}$$

When we put $J_1 = 0$, $J_2 = 0$, $J_1' = 0$, $J_2' = 0$, then the values of f_1' and f_2' represent the unperturbed basic frequencies. If we denote f_1 and f_2 by the unperturbed frequencies, then

$$(f_1)^2 + (f_2)^2 = 1, \qquad (f_1)^2 * (f_2)^2 = \frac{27}{16}(1 - (1 - 2\,\mu)^2), \qquad (0 < f_2 < \tfrac{1}{\sqrt{2}} < f_1 < 1). \tag{21}$$

We can write

$$f_1' = f_1\,\{1 + p(J_1 + J_1') + p'(J_2 + J_2')\}, \qquad f_2' = f_2\,\{1 + q\,(J_1 + J_1') + q'(J_2 + J_2')\}. \tag{22}$$

Putting the values of f_1', f_2' in (20) and rejecting terms of second and higher degree in J_1, J_1', J_2, J_2', and also equating the coefficients of J_1, J_1', J_2, J_2' on both sides and then solving, we obtain as

$$p = -\frac{d_5 - d_1\,f_1^2}{2\,f_1^2(f_1^2 - f_2^2)},\; p' = -\frac{d_6 - d_3\,f_1^2}{2\,f_1^2(f_1^2 - f_2^2)}, \quad q = -\frac{d_5 - d_1\,f_2^2}{2\,f_2^2(f_2^2 - f_1^2)},\; q' = -\frac{d_6 - d_3\,f_2^2}{2\,f_2^2(f_2^2 - f_1^2)}, \tag{23}$$

The values of d_i (i = 1,..., 6) are given in Appendix A.

5. First Order Normalization

The Hamiltonian of the corresponding Lagrangian can be written as

$$H = -\Gamma + p_\alpha\,\dot{\alpha} + p_\beta\,\dot{\beta}, \tag{24}$$

where p_α and p_β are the generalized momenta which are given by

$$p_\alpha = \dot{\alpha} - n\,\phi\,\beta, \quad p_\beta = \dot{\beta} + n\,\phi\,\alpha, \tag{25}$$

Using these defined values we can get the Hamiltonian in the form

$$H = H_0 + H_1 + H_2 + H_3 + \ldots, \tag{26}$$

where

$$H_0 = -\,\Gamma_0, \quad H_1 = -\,\Gamma_1,$$
$$H_2 = -\,\Gamma_2 + p_\alpha\,\dot{\alpha} + p_\beta\,\dot{\beta}$$
$$= \tfrac{1}{2}(p_\alpha^2 + p_\beta^2) + n\,\phi\,(\beta\,p_\alpha - \alpha\,p_\beta) + \frac{A_1}{2}\,\alpha^2 + A_2\,\alpha\,\beta + \frac{A_3}{2}\,\beta^2. \tag{27}$$

We use the procedure from Whittaker (1965) to investigate the stability of the motion as

$$-\,\lambda\,p_\alpha = \frac{\partial H_2}{\partial \alpha} = A_1\,\alpha + A_2\,\beta - n\,\phi\,p_\beta, \qquad -\,\lambda\,p_\beta = \frac{\partial H_2}{\partial \beta} = A_2\,\alpha + A_3\,\beta + n\,\phi\,p_\alpha,$$
$$\lambda\,\alpha = \frac{\partial H_2}{\partial p_\alpha} = p_\alpha + n\,\phi\,\beta, \qquad \lambda\,\beta = \frac{\partial H_2}{\partial p_\beta} = p_\beta - n\,\phi\,\alpha, \tag{28}$$

where

$$A_1 = 2\,\{\,g_{18} + g_{19}\,(J_1 + J_1') + g_{20}(J_2 + J_2')\}\,,\; A_2 = g_{24} + g_{25}(J_1 + J_1') + g_{26}\,(J_2 + J_2')\,,$$
$$A_3 = 2\,\{\,g_{21} + g_{22}(J_1 + J_1') + g_{23}(J_2 + J_2')\}\,, \tag{29}$$

and the values of g_i's are given in Appendix B.

Now Equation (28) can be written as

$$A\,X = 0,$$

where

$$A = \begin{pmatrix} A_1 & A_2 & \lambda & -\,n\,\phi \\ A_2 & A_3 & n\,\phi & \lambda \\ -\,\lambda & n\,\phi & 1 & 0 \\ -\,n\,\phi & -\,\lambda & 0 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} \alpha \\ \beta \\ p_\alpha \\ p_\beta \end{pmatrix}.$$

The characteristic equation of matrix A is given by

$$\lambda^4 + A_4\,\lambda^2 + A_5 = 0, \tag{30}$$

where

$$A_4 = 2\,n^2\,\phi^2 + A_1 + A_3\,, \qquad A_5 = n^4\,\phi^4 - n^2\,\phi^2\,(A_1 + A_3) + A_1\,A_3 - A_2^2. \tag{31}$$

The discriminant of Equation (30) will be

$$(A_1 - A_3)^2 + 8\,n^2\,\phi^2\,(A_1 + A_3) + 4\,A_2^2. \tag{32}$$

When the discriminant is greater than zero, then the roots $\pm\,i\,f_1'$ and $\pm\,i\,f_2'$ (f_1' and f_2' are the long/short periodic frequencies) are related to each other as $(f_1')^2 + (f_2')^2 = -\,A_4$ and $(f_1')^2 * (f_2')^2 = A_5$, such as $0 \le f_2' \le \frac{1}{\sqrt{2}} \le 1$.

6. First order normal co-ordinates evaluation

To evaluate the normal coordinates, we use the procedure given by Whittaker (1965) with transformation $(\alpha, \beta, p_\alpha, p_\beta) \to (\alpha_1, \alpha_2, p_1, p_2)$ which produces as

$$\chi = M\,Y,$$

where,

$$\chi = \begin{pmatrix} \alpha \\ \beta \\ p_\alpha \\ p_\beta \end{pmatrix}, \; Y = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ p_1 \\ p_2 \end{pmatrix}, \quad M = (M_{ij}), 1 \leq i, j \leq 4, \quad \alpha_s = \left(\frac{2\,I_s}{f'_s}\right)^{\frac{1}{2}} \sin\,\theta_s,$$

$$p_s = (2\,I_s\,f'_s)^{\frac{1}{2}}\;cos\,\theta_s, (s = 1, 2),$$

and $M_{i\,j}$ are given in Appendix B

where, θ_s and I_s are the phase angles and actions, respectively.

After the transformation, we get the Hamiltonian in the normal form as $H_2 = f'_1\,I_1 - f'_2\,I_2$, and hence the general solutions will be $I_i = C_i (i = 1, 2)$, $\theta_1 = f'_1\,t + C_3$, and $\theta_2 = -\,f'_2\,t + C_4$, where C_1, C_2, C_3 and C_4 are constants.

7. Second order normal co-ordinates evaluation

We shall follow the procedure as adopted by Bhatnagar and Hallan (1983) to perform the second order normal coordinates. The first order components $A_1^{1,\,0}$ and $A_1^{0,\,1}$ are the values of α and β. The second order components $A_2^{1,0}$ and $A_2^{0,\,1}$ are the solutions of the partial differential equations:

$$\Delta_1\,\Delta_2\,A_2^{1,0} = \phi_1, \quad \Delta_1\,\Delta_2\,A_2^{0,1} = \phi_2, \tag{33}$$

where

$$\Delta_s = D^2 + (f'_s)^2,\; (s = 1, 2), \text{ and } D = f'_1(\tfrac{\partial}{\partial\theta_1}) - f'_2(\tfrac{\partial}{\partial\theta_2}),$$

$$\phi_1 = [D^2 - 2\{t_{33} + t_{22}(J_1 + J'_1) + t_{25}(J_2 + J'_2)\}]\alpha_2 + \{2n\phi D + t_{35} + t_{23}(J_1 + J'_1) + t_{26}(J_2 + J'_2)\}\beta_2,$$

$$\phi_2 = \{-2n\phi D + t_{34} + t_{23}(J_1 + J'_1) + t_{26}(J_2 + J'_2)\}\alpha_2 + [D^2 - 2\{t_{32} + t_{21}(J_1 + J'_1) + t_{24}(J_2 + J'_2)\}]\beta_2.$$

Here α_2 and β_2 can get from $\dfrac{\partial\,\Gamma_3}{\partial\,\alpha}$ and $\dfrac{\partial\,\Gamma_3}{\partial\,\beta}$ respectively.

8. Second order coefficients in the frequencies

The third order components $A_3^{1,0}$ and $A_3^{0,1}$ in the coordinates α, β and second order polynomials E_2, F_2 in the frequencies θ'_1, θ'_2 satisfy the partial differential equations:

$$\Delta_1\,\Delta_2\,A_3^{1,0} = \phi_3 - 2\,E_2\,U_1 - 2\,F_2\,U_2, \qquad \Delta_1\,\Delta_2\,A_3^{0,1} = \phi_4 - 2\,E_2\,U_3 - 2\,F_2\,U_4, \tag{34}$$

where

$$\phi_3 = [D^2 - 2\{t_{33} + t_{22}(J_1 + J_1') + t_{25}(J_2 + J_2')\}]\alpha_3 + \{2\, n\, \phi\, D + t_{34} + t_{23}\,(J_1 + J_1') + t_{26}\,(J_2 + J_2')\}\beta_3,$$

$$\phi_4 = [D^2 - 2\{t_{32} + t_{21}(J_1 + J_1') + t_{24}(J_2 + J_2')\}]\beta_3 - \{2n\phi D - t_{34} + t_{23}(J_1 + J_1') + t_{26}(J_2 + J_2')\}\,\alpha_3,$$

$$U_1 = \frac{\partial}{\partial\theta_1}\,(T_1\,T_2\, +\, T_3\,T_4), \quad U_2 = \frac{\partial}{\partial\theta_2}\,(T_5\,T_6\, +\, T_7\,T_8),$$
$$U_3 = \frac{\partial}{\partial\theta_1}\,(T_9\,T_{10}\, +\, T_{11}\,T_{12}), U_4 = \frac{\partial}{\partial\theta_2}\,(T_{13}\,T_{14}\, +\, T_{15}\,T_{16}),$$

and T_i's are given in Appendix B. We can get α_3 and β_3 from $\frac{\partial}{\partial\alpha}(\Gamma_3 + \Gamma_4)$ and $\frac{\partial}{\partial\beta}(\Gamma_3 + \Gamma_4)$, respectively. Now we choose suitable coefficients in the polynomial to eliminate the critical terms as:

$$E_2 = E'_{2,0}\,I_1\, +\, E'_{0,2}\,I_2, \;\; F_2\, =\, F'_{2,0}\,I_1\, +\, F'_{0,2}\,I_2,$$

where,

$$E'_{2,0} = \frac{\text{Coefficient of } \cos\theta_1 \text{ in } \phi_3}{2\,\text{Coefficient of } \cos\theta_1 \text{ in } U_1}, \quad E'_{0,2} = F'_{2,0} = \frac{\text{Coefficient of } \cos\theta_2 \text{ in } \phi_3}{2\,\text{Coefficient of } \cos\theta_2 \text{ in } U_2},$$

$$F'_{0,2} = \frac{\text{Coefficient of } \cos\theta_2 \text{ in } \phi_4}{2\,\text{Coefficient of } \cos\theta_1 \text{ in } U_2}.$$

9. Non-linear stability

We will apply KAM theorem to numerically examine the non-linear stability in two parts. In the first part we will consider as variable mass parameters ($\alpha_0 = 0.2, \beta_0 = 0.8$), Coriolis and centrifugal forces parameters ($\phi = 1.2, \psi = 1.2$)) and in the second part we will take as variable mass parameters ($\alpha_0 = 0, \beta_0 = 1$), Coriolis and centrifugal forces parameters ($\phi = 1, \psi = 1$).

9.1. First part

The first condition of KAM theorem fails when $f_1' = 2\, f_2'$ and $f_1' = 3\, f_2'$. Hence, we will discuss both cases separately.

Case-I: Eliminating f_1' and f_2' from Equation (20) by using the condition $f_1' = 2\, f_2'$ and solving for $\mu = \mu_1'$ (say), we obtain

$$\mu_1' = 0.521863...\, + 5.36405...\, J_1 + 36.6305...\, J_1' - 96.3674...\, J_2 - 185.524...\, J_2'.$$

Case-II: Eliminating f_1' and f_2' from Equation (20) by using the condition $f_1' = 3\, f_2'$ and solving for $\mu = \mu_2'$ (say), we get

$$\mu_2' = 0.539145...\, + 4.9072...\, J_1 - 31.5729...\, J_1' - 111.489...\, J_2 - 191.392...\, J_2',$$

Again, the normalized Hamiltonian up to fourth order is given as:

$$H = f_1' I_1 - f_2' I_2 + \frac{1}{2}(B_1 I_1^2 + 2 B_2 I_1 I_2 + B_3 I_2^2), \tag{35}$$

the determinant $D = - B_1 (f_2')^2 - 2 B_2 f_1' f_2' - B_3 (f_1')^2$. Now following the procedure adopted by Bhatnagar and Hallan (1983) and using well-known software Mathematica, we find that

$$\mu_3' = 61.6192... + 6.66068...\, J_1 + 63.5839...\, J_1' - 45.0279...\, J_2 - 117.29...\, J_2'.$$

9.2. Second part

Following the same procedures as in the first part, here we obtain as

$$\mu_1' = (0.0242939...) + (1.107438...)(J_1 + J_1') + (-77.366...)(J_2 + J_2'),$$

and

$$\mu_2' = (0.0135160...) + (199.786...)(J_1 + J_1') + (-76.6674...)(J_2 + J_2').$$

From the second condition of KAM theorem, we obtain

$$\mu_3' = (0.01091366...) + (0.62007...)(J_1 + J_1') + (-587.294...)(J_2 + J_2').$$

When $J_1 = J_1' = J_2 = J_2' = 0$, we get

$$\mu_1' = \mu_1 = 0.024293897,\ \mu_2' = \mu_2 = 0.013516016,\ \mu_3' = \mu_3 = 0.0109137.$$

These values agree with those founded by Deprit and Deprit (1967).

10. Conclusion

The heterogeneous primaries having N-layers with different densities and mass variation of test particle in the frame of CR3BP with the assumptions, the effects of small perturbations in Coriolis and centrifugal forces are studied. We have evaluated the equations of motion of the test particle under the influence of the above said perturbations by using the Jean's law and Meshcherskii space-time transformations. Also we revealed analytically the locations of perturbed triangular stationary points and then the relations between the perturbed basic frequencies and unperturbed basic frequencies are obtained. If we eliminate the perturbation parameters used in our case, we will get the unperturbed case, that is, the classical case. We have computed the Lagrangian for the perturbed system and performed the first order normalized Hamiltonian. The canonical transformation from the phase space into the phase space of the angle coordinates as well as the action momenta, to the first order is revealed so that the second order part of the Hamiltonian is transformed to the normal form. Here the Birkhoff's normalization up-to only second order is performed. Also, the critical mass ratios, for which Birkhoff's normalization can not be satisfied, are found.

The examination for the Non-linear stability is done in two parts by using KAM theorem. In the first part we have taken as variable mass parameters, Coriolis and centrifugal forces parameters. It is observed that triangular stationary points are stable for less than critical mass except for three mass ratios at which Moser's theorem does not apply. To show the difference between first and second part, we also determined the values of mass ratio in second part. Again, if we choose the values of densities parameters to zero, then the values of mass ratios, agree with classical case. Hence, the perturbations considered by us have great impact on the dynamical behaviour of this system.

REFERENCES

Abouelmagd, E. I. and Ansari, A. A. (2019). The motion properties of the infinitesimal body in the framework of bi-circular Sun-perturbed Earth-Moon system, New Astronomy, Vol. 73, pp. 101282. https://doi.org/10.1016/j.newast.2019.101282

Abouelmagd, E. I. and Ansari, A. A. (2021). Variable mass motion in the Hénon–Heiles system, Modern Physics Letters A, Vol. 36, No. 21, pp. 2105150.

Abouelmagd, E. I. and Mostafa, A. (2015). Out of plane equilibrium points locations and the forbidden movement regions in the restricted three-body problem with variable mass, Astrophys. Space Sci., Vol. 357, No. 58. doi:10.1007/s10509-015-2294-7

Ansari, A. A. (2017). Effect of Albedo on the motion of the infinitesimal body in circular restricted three-body problem with variable masses, Italian Journal Of Pure and Applied Mathematics, Vol. 38, pp. 581-600.

Ansari, A. A. (2018). The circular restricted four- body problem with triaxial primaries and variable infinitesimal mass, Applications and Applied Mathematics: An International Journal, Vol. 13, No. 2, pp. 818-838.

Ansari, A. A. (2021). Kind of Robe's restricted problem with heterogeneous irregular primary of N-layers when outer most layer has viscous fluid, New Astronomy, Vol. 83. https://doi.org/10.1016/j.newast.2020.101496

Ansari, A. A. and Abouelmagd, E. I. (2020). Gravitational potential formulae between two bodies with finite dimensions, Astronomical Notes. DOI: 10.1002/asna.202013726

Ansari, A. A., Singh, J., Alhussain, Z. and Belmabrouk, H. (2019). Perturbed Robe's CR3BP with Viscous Force, Astrophys. Space Sci., Vol. 364, No. 95. https://doi.org/10.1007/s10509-019-3586-0

Arnold, V. I. (1961). Soviet Math. Dolk., Vol. 2, pp. 247.

Bhatnagar, K. B. and Hallan, P. P. (1983). The effect of perturbation in the restricted problem of three bodies, Celest. Mech. Dyn. Astron., Vol. 30, pp. 97.

Bouaziz, F. and Ansari, A. A. (2021). Perturbed Hill's problem with variable mass, Astronomical Notes, Vol. 346, No. 1, pp. 1-9. https://doi.org/10.1002/asna.202113870

Deprit, A. and Deprit, B. (1967). Stability of the triangular Lagrangian points, Astronomy and Astrophysics, Vol. 72, pp. 173-179.

Gozdziewski, K., Maciejewski, A. J. and Niedzielska Z. (1991). About the libration points in the restricted photogravitational three-body problem, Celest. Mech. Dyn. Astron., Vol. 52, pp. 195–201.

Haque, M. N. and Ishwar, B. (1995). Non-linear stability in the perturbed photo-gravitational restricted three-body problem, Bull. Astr. Soc. India, Vol. 23, pp. 195–210.

Ishwar, B. and Sharma, J.P. (2012). Non-linear stability in photo-gravitational non-planer R3BP with oblate smaller primary, Astrophys. Space Sci., Vol. 337, pp. 563–571.

Jain, R. and Sinha, D. (2014). Non-linear stability of L_4 in the restricted problem when the primaries are finite straight segments under resonances, Astrophys. Space Sci., Vol. 353, pp. 73-88. doi:10.1007/s10509-014-1996-6

Jeans, J. H. (1928). *Astronomy and Cosmogony*, Cambridge University Press, Cambridge.

Kumar, V. and Choudhary, R. K. (1990). Non-linear stability of the triangular libration points for the photo-gravitational elliptic restricted problem of three bodies, Celest. Mech. Dyn. Astron., Vol. 48, pp. 299-317.

Kushvah, B.S., Sharma, J.P. and Ishwar, B. (2007). Non-linear stability in the generalized photo-gravitational restricted three-body problem with Poynting-Robertson drag, Astrophys. Space Sci., Vol. 312, No. 3-4, pp. 279-293.

Meshcherskii, I. V. (1949). *Studies on the Mechanics of Bodies of Variable Mass*, GITTL, Moscow.

Pathak, N., Thomas, V. O. and Abouelmagd, E. I. (2019). The perturbed photo-gravitational restricted three-body problem: Analysis of resonant periodic orbits, Discrete and Continuous Dynamical Systems - Series S, Vol. 12, No. 4-5, pp. 849-875.

Riaguas, A., Elipe, A. and Lopez, M. T. (2001). Non-linear stability of the equilibria in the gravity field of a finite straight segment, Celest. Mech. Dyn. Astron., Vol. 81, No. 3, pp. 235-248.

Sahdev, S. K. and Ansari, A. A. (2021a). Generalized Robe's problem having oblate heterogeneous primary containing viscous fluid inside the outer most layer and radiating spherical secondary with modified Newtonian potential, Science International, Lahore, Vol. 33, No. 2, pp. 147–151.

Sahdev, S. K. and Ansari, A. A. (2021b). Motion of test particle in the outer layer of heterogeneous body, Gedrag & Organisatie Review, Vol. 34, No. 1, pp. 1–12.

Singh, J. and Ishwar, B. (1985). Effect of perturbations on the stability of triangular points in the restricted problem of three bodies with variable mass, Celest. Mech., Vol. 35, pp. 201–207.

Singh, J. (2009). Effect of perturbations on the non-linear stability of triangular points in the restricted three-body problem with variable mass, Astrophys. Space Sci., Vol. 321, No. 2, pp. 127-135.

SubbaRao, P. V. and Sharma, R.K. (1997). Effect of oblateness on the non-linear stability of L_4 in the restricted three-body problem, Celest. Mech. Dyn. Astron., Vol. 65, pp. 291–312.

Szebehely, V. (1967). *Theory of Orbits: The Restricted Problem of Three Bodies*, Academic Press Inc. Ltd., London.

Whittaker, E. T. (2013). *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, London.

Zhang, M. J., Zhao, C. Y. and Xiong, Y.Q. (2012). On the triangular libration points in photo-gravitational restricted three-body problem with variable mass, Astrophys. Space Sci., Vol. 337, pp. 107-113. doi:10.1007/s10509-011-0821-8

Appendix A

$$\Gamma_{00} = \left(n^2 \psi + \frac{\alpha_0^2}{4}\right)\left(\frac{1}{2} + \frac{\beta_0}{8} - \frac{\beta_0 \mu}{2} + \epsilon_1 (1 - \mu) + \epsilon_2 \mu\right) + \beta_0^{3/2},$$

$$\Gamma_{01} = +\frac{1}{2}(J_1 + J_1')\beta_0^{5/2} - (J_2 + J_2')\frac{3\beta_0^{5/2}}{2}(1 - \frac{\beta_0}{2}) - \epsilon_1 \beta_0^{3/2}(1 - \mu) - \epsilon_2 \beta_0^{3/2}\mu,$$

$$\Gamma_{10} = t_1 \dot{\alpha} + t_2 \dot{\beta},\ \Gamma_{11} = t_{15} + (J_1 + J_1') t_7 + (J_2 + J_2') t_9,$$

$$\Gamma_{12} = t_{16} + (J_1 + J_1') t_8 + (J_2 + J_2') t_{10},$$

$$\Gamma_{20} = \tfrac{1}{2}(\dot{\alpha}^2 + \dot{\beta}^2) + n\,\phi\,(\alpha\,\dot{\beta} - \beta\,\dot{\alpha},)\ \Gamma_{21} = t_{32} + (J_1 + J_1') t_{21} + (J_2 + J_2') t_{24},$$

$$\Gamma_{22} = t_{33} + (J_1 + J_1') t_{22} + (J_2 + J_2') t_{25}, \Gamma_{23} = t_{34} + (J_1 + J_1') t_{23} + (J_2 + J_2') t_{26},$$

$$\Gamma_{30} = t_{35} + (J_1 + J_1') t_{39} + (J_2 + J_2') t_{43},\ \Gamma_{31} = t_{36} + (J_1 + J_1') t_{40} + (J_2 + J_2') t_{44},$$

$$\Gamma_{32} = t_{37} + (J_1 + J_1') t_{41} + (J_2 + J_2') t_{45},\ \Gamma_{33} = t_{39} + (J_1 + J_1') t_{42} + (J_2 + J_2') t_{46},$$

$$\Gamma_{40} = t_{47} + (J_1 + J_1') t_{52} + (J_2 + J_2') t_{57},\ \Gamma_{41} = t_{48} + (J_1 + J_1') t_{53} + (J_2 + J_2') t_{58},$$

$$\Gamma_{42} = t_{49} + (J_1 + J_1') t_{54} + (J_2 + J_2') t_{59},\ \Gamma_{43} = t_{50} + (J_1 + J_1') t_{55} + (J_2 + J_2') t_{60},$$

$$\Gamma_{44} = t_{51} + (J_1 + J_1') t_{56} + (J_2 + J_2') t_{61},$$

$$\alpha_1 = t_{17} - t_{19} + 2\,t_{29} + \tfrac{1}{2}\,t_{21}(J_1 + J_1') + 2\,t_{24}(J_2 + J_2'),$$

$$\alpha_2 = \beta_1 = t_{20} - t_{31} + t_{23}(J_1 + J_1') + (\beta_0^{3/2}\,t_{26} + \beta_0^{5/2}\,t_{28})(J_2 + J_2'),$$

$$\beta_2 = t_{17} + 2\,t_{19} + t_{30} + 2\,t_{22}(J_1 + J_1') + (t_{25} + 3\,t_{27})(J_2 + J_2'),$$

$$d_1 = 4n^2\phi^2 - 2t_{32} - 2t_{33},\ d_2 = -2t_{21} + t_{22}),\ d_3 = -2(t_{24} + t_{25} + t_{27}),\ d_4 = 4t_{32}t_{33} - t_{34}^2,$$

$$d_5 = 4\,t_{22}\,t_{32} + 4\,t_{21}\,t_{33} - 2\,t_{23}\,t_{34},\ d_6 = 4\,t_{25}\,t_{32} + 4\,t_{27}\,t_{32} + 4\,t_{24}\,t_{33} - 2\,t_{26}\,t_{34} - 2\,t_{28}\,t_{34},$$

$$\epsilon_{11} = (-2\,\alpha_0^2 + \beta_0^{3/2}(8 + 3\,\beta_0(4(J_1 - 3\,J_2 + J_1' - J_2') + (5\,J_2 + 3\,J_2')\,\beta_0)) - 8\,n^2\,\psi)$$

$$(4\,\alpha_0^2 + \beta_0^{3/2}(-16 + 3\,\beta_0(-8\,J_1 + J_2(40 - 30\,\beta_0) + 4\,J_1'(-2 + 5\,\beta_0) + 5\,J_2'(8$$

$$+\beta_0(-26 + 7\,\beta_0)) + 8\,\mu)) + 16\,n^2\,\psi) + 3\,\beta_0^{5/2}(20\,J_2\,\beta_0 + 20\,J_1'\,\beta_0 + J_2'\,\beta_0(-32$$

$$+21\,\beta_0) + 8\,\mu)(\beta_0^{3/2}(8 + 3(4\,J_1 - 4\,J_1' + 5\,J_2(-4 + \beta_0) - 5\,J_2'(-4 + \beta_0))\,\beta_0$$

$$-16\,\mu) + \alpha_0^2(-2 + 4\,\mu) + 8\,n^2(-1 + 2\,\mu)\,\psi),$$

$$\epsilon_{12}=\epsilon_{22}=3\,\beta_0^{3/2}\,((20\,J_2\,\beta_0\,+\,20\,J_1'\,\beta_0\,+\,J_2'\,\beta_0\,(-\,32\,+\,21\,\beta_0)\,+\,8\,\mu)\,(4\,\alpha_0^2\,+\,\beta_0^{3/2}\,(-\,16$$
$$+\,30\,(2\,J_1\,-\,13\,J_2\,-\,3\,J_2')\,\beta_0^2\,+\,105\,J_2\,\beta_0^3\,-\,24\,\beta_0\,(-\,1\,+\,J_1\,-\,5\,J_2\,+\,J_1'$$
$$-\,5\,J_2'\,+\,\mu))\,+\,16\,n^2\,\psi)\,+\,(8\,+\,20\,J_1\,\beta_0\,+\,12\,J_2'\,\beta_0\,+\,5\,J_2\,\beta_0\,(-\,16\,+\,7\,\beta_0)$$
$$-\,8\,\mu)\,(4\,\alpha_0^2\,+\,\beta_0^{3/2}\,(-\,16\,+\,3\,\beta_0\,(-\,8\,J_1\,+\,J_2\,(40\,-\,30\,\beta_0)\,+\,4\,J_1'\,(-\,2$$
$$+\,5\,\beta_0)\,+\,5\,J_2'\,(8\,+\,\beta_0\,(-\,26\,+\,7\,\beta_0))\,+\,8\,\mu))\,+\,16\,n^2\,\psi)),$$

$$\epsilon_{21}=(-\,2\,\alpha_0^2\,+\,\beta_0^{3/2}\,(8\,+\,3\,\beta_0\,(4\,(J_1\,-\,3\,J_2\,+\,J_1'\,-\,J_2')\,+\,(5\,J_2\,+\,3\,J_2')\,\beta_0))\,-\,8\,n^2\,\psi)$$
$$(4\,\alpha_0^2+\,\beta_0^{3/2}\,(\,-\,16\,+\,30\,(2\,J_1\,-\,13\,J_2\,-\,3\,J_2')\,\beta_0^2\,+\,105\,J_2\,\beta_0^3\,-\,24\,\beta_0\,(-\,1\,+\,J_1$$
$$-5\,J_2\,+\,J_1'-\,5\,J_2'\,+\,\mu))\,+\,16\,n^2\,\psi)\,-\,3\,\beta_0^{5/2}\,(8\,+\,20\,J_1\,\beta_0\,+\,12\,J_2'\,\beta_0$$
$$+\,5\,J_2\,\beta_0\,(-\,16\,+\,7\,\beta_0)\,-\,8\,\mu)\,(\beta_0^{3/2}\,(8\,+\,3\,(4\,J_1-\,4\,J_1'\,+\,5\,J_2\,(\,-\,4\,+\,\beta_0)$$
$$-\,5\,J_2'\,(\,-\,4\,+\,\beta_0))\,\beta_0\,-\,16\,\mu)\,+\,\alpha_0^2\,(-\,2\,+\,4\,\mu)\,+\,8\,n^2\,(-\,1\,+\,2\,\mu)\,\psi),$$

$$t_1=-\,\frac{n\,\phi}{2\,\sqrt{\beta_0}}\,\sqrt{(4-\beta_0+4\epsilon_1+4\epsilon_2)\beta_0},\qquad t_2=\frac{n\,\phi}{2\,\sqrt{\beta_0}}(\beta_0+2\,\epsilon_1-\,2\,\epsilon_2-2\,\beta_0\,\mu),$$

$$t_3=\left(n^2\,\psi\,+\,\frac{\alpha_0^2}{4}\right)(\tfrac{1}{2}\,-\,\mu\,+\,(\epsilon_1\,-\,\epsilon_2)\,\beta_0^{-1/2}),\,t_4=\left(n^2\,\psi\,+\,\frac{\alpha_0^2}{4}\right)(1\,-\,\tfrac{\beta_0}{4}\,+\,\epsilon_1\,+\,\epsilon_2)^{1/2},$$

$$t_5=\frac{\beta_0^{5/2}}{2}(-\,1\,+\,\mu\,+\,\beta_0^{-1/2}\,\mu),\quad t_6=-\,\frac{\beta_0^{3/2}}{2}\sqrt{4\,-\,\beta_0},\quad t_7=-\,\frac{3\,\beta_0^{7/2}}{4},\quad t_8=-\,\frac{3\,\beta_0\,t_6}{2},$$

$$t_9=\frac{15\,\beta_0^{7/2}}{8}(4\,-\,\beta_0),\quad t_{11}=\epsilon_1\,(-\,\beta_0\,+\,\tfrac{3}{4}\,\beta_0^{5/2}(1\,-\,\mu)),\quad t_{12}=\,\tfrac{3}{2}\,\epsilon_1\,\beta_0^{3/2}\sqrt{4\,-\,\beta_0}(1\,-\,\mu),$$

$$t_{13}=\epsilon_2\,\beta_0\,(1\,-\,\tfrac{3}{2}\,\beta_0\,\mu),\qquad t_{10}=\frac{3\,\beta_0^{5/2}}{4}\{5(4\,-\,\beta_0)^{3/2}+4\,\sqrt{-\,\beta_0\,+\,4\,(\,1\,+\,\epsilon_1\,+\,\epsilon_2)}\},$$

$$t_{14}=\epsilon_2\,\beta_0^{3/2}(4-\beta_0)^{-1/2}(1+\tfrac{3}{2}\mu(-4+\beta_0)),\,t_{15}=t_3+t_5+t_{11}+t_{13},\,t_{16}=t_4+t_6+t_{12}$$
$$+\,t_{14},\quad t_{17}=\frac{n^2\,\psi}{2}\,+\,\frac{\alpha_0^2}{8},\,t_{18}=\frac{3\,\beta_0^{3/2}}{8}((-\,\tfrac{4}{3}\,+\,\beta_0^2)\,+\,\beta_0\,\mu\,(1\,-\,\beta_0)),\,t_{19}=\tfrac{3}{8}\,\beta_0^{3/2}\,(\tfrac{8}{3}\,-\,\beta_0),$$

$$t_{20}=\tfrac{3}{8}\,\beta_0^{3/2}\,(-\,2\,\beta_0\,\sqrt{4\,-\,\beta_0}(-\,1\,+\,\mu\,+\,\beta_0^{-1/2}\,\mu),\,t_{21}=\frac{3\,\beta_0^{5/2}}{16}(-\,4\,+\,5\,\beta_0^2),\,t_{23}=3\,\beta_0^{7/2}$$
$$\sqrt{4\,-\,\beta_0},\quad t_{24}=\frac{5\,\beta_0^{5/2}}{32}(24\,-\,54\,\beta_0\,+\,21\,\beta_0^2),\,t_{25}=\frac{\beta_0^{5/2}}{32}(-\,768\,+\,570\,\beta_0\,-\,105\,\beta_0^2),$$

$$t_{26}=\frac{210(\beta_0(4-\beta_0))^{\frac{3}{2}}}{32}, t_{27}=120\sqrt{(-\beta_0+4(1+\epsilon_1+\epsilon_2))(4-\beta_0)}, t_{28}=-\frac{\beta_0^{1/2}t_{27}}{\sqrt{-\beta_0+4}},$$

$$t_{29}=\frac{3\,(\epsilon_1\,-\epsilon_2)\,\beta_0^{3/2}}{8}(4\,\beta_0^{1/2}(-1+\mu)+\mu\,(8-5\,\beta_0)),\, t_{30}=\frac{(4-16\mu+5\beta_0\mu)\,t_{29}}{(4\beta_0^{1/2}(-1+\mu)+\mu(8-5\beta_0))},$$

$$t_{31}=\frac{3\,(\epsilon_1\,-\,\epsilon_2)\,\beta_0^{3/2}}{8}\,\beta_0\,((\,4\,-\,\beta_0)^{1/2}\,(4\,-\,10\,\beta_0\,\mu)\beta_0^{-3/2}\,+\,4\,(1\,-\,\mu)\,(\,4\,-\,\beta_0)^{-1/2}),$$

$$t_{22}=\frac{3\,\beta_0^{5/2}}{16}(16\,-\,5\,\beta_0),\, t_{32}=t_{17}+t_{18}+t_{29},\, t_{33}=t_{17}+t_{19}+t_{30},\, t_{34}=t_{20}+t_{31},$$

$$t_{35}=\frac{\beta_0}{16}\,\{\beta_0(12-24\mu-5\beta_0+5\,\beta_0\,\mu)+\epsilon_1(24-90\,\beta_0+35\beta_0^2+60\beta_0\,\mu-25\,\beta_0^2\,\mu)$$

$$+\,\epsilon_2(-24+30\,\beta_0+60\beta_0\,\mu-25\beta_0^2\mu)\},\;\; t_{36}=\frac{5\,\beta_0^{3/2}}{16}\,\sqrt{-\beta_0+4(1+\epsilon_1+\epsilon_2)}\{(-2$$

$$+\beta_0)+\epsilon_1(12-7\,\beta_0-16\,\mu+7\,\beta_0\,\mu)+\epsilon_2(-4\,+16\,\mu-7\,\beta_0\,\mu)\},$$

$$t_{37}=\frac{15\,\beta_0^{3/2}}{16}\,\sqrt{-\beta_0+4(1+\epsilon_1+\epsilon_2)}\,\{(1-\beta_0)+\epsilon_1(-8+7\,\beta_0+12\,\mu-7\,\beta_0\,\mu)$$

$$+\,\epsilon_2(4-12\,\mu\,+\,7\,\beta_0\,\mu)\},\quad t_{38}=\frac{15\,\beta_0}{16}\,\{\beta_0(-\,4+\beta_0+8\,\mu-2\,\beta_0\,\mu)+\epsilon_1(-\,8+\,22\,\beta_0$$

$$-7\,\beta_0^2-\,16\,\beta_0\,\mu+7\,\beta_0^2\,\mu)\,+\,\epsilon_2(8-6\,\beta_0-\,16\,\beta_0\,\mu+7\,\beta_0^2\,\mu)\},\;\; t_{39}=\frac{\beta_0^{5/2}}{8}\beta_0\,(6\,-5\beta_0^2),$$

$$t_{40}=-\frac{\beta_0^{5/2}}{8}(56-34\beta_0+5\beta_0^2),\, t_{41}=\frac{6\beta_0^{5/2}}{8}\sqrt{4-\beta_0}(1-4\beta_0^2),\;\; t_{42}=\frac{6\beta_0^{7/2}}{8}(-15+4\beta_0)$$

$$(-\,15\,+\,4\,\beta_0),\, t_{43}=\frac{15\,\beta_0^3}{4}(\beta_0^2\,-\,3\,\beta_0\,+\,2),\;\; t_{44}=\frac{15\,\beta_0^{5/2}}{16}\{+\,4(\,28\,-\,27\,\beta_0\,+\,9\,\beta_0^2-\,\beta_0^3)$$

$$(4\,-\,\beta_0)^{-1/2}+(-\,24\,+\,7\,\beta_0)\,\sqrt{-\,\beta_0\,+\,4(1\,+\,\epsilon_1\,+\,\epsilon_2)}\},\;\; t_{45}=\frac{15\,\beta_0^{5/2}}{16}\{4(-\,8\,+\,30\,\beta_0$$

$$-\,19\,\beta_0^2\,+\,3\,\beta_0^3)\,(4\,-\,\beta_0)^{-1/2}\,+\,\sqrt{-\,\beta_0\,\,+\,4\,(1\,+\,\epsilon_1\,+\,\epsilon_2)}(7\,-\,5\beta_0)\},\, t_{46}=\frac{15\,\beta_0^3}{8}\,\{2$$

$$(17\,\beta_0\,-\,3\,\beta_0^2\,-\,23)\,+\,7\,\sqrt{4\,-\,\beta_0}\,\sqrt{-\,\beta_0\,+\,4\,(1\,+\,\epsilon_1\,+\,\epsilon_2)}\},\, t_{47}=\frac{15\,\beta_0^{3/2}}{128}\{(3-8\,\beta_0$$

$$+35\beta_0^2)+\epsilon_1(-\,48+96\,\beta_0-315\,\beta_0^2\,+80\,\mu\,-96\,\beta_0\,\mu+315\,\beta_0^2\,\mu)+\epsilon_2(32\,-\,24\,\beta_0-\,48\,\mu$$

$$+96\,\beta_0\,\mu-21\,\beta_0^2\,\mu)\},\; t_{48}=\frac{15\,\beta_0^{3/2}}{128}\{(26-32\,\beta_0+35\,\beta_0^2)+\epsilon_1(-256\,+\,280\,\beta_0\,-63\,\beta_0^2$$

$$+ 384\,\mu - 336\,\beta_0\,\mu + 63\,\beta_0^2\,\mu) + \epsilon_2(128 - 56\,\beta_0 - 384\,\mu + 346\,\beta_0\,\mu - 315\,\beta_0^2\,\mu)\},$$

$$t_{49} = \frac{3\,\beta_0^{3/2}}{128}\{(-128 + 280\,\beta_0 - 70\,\beta_0^2) + \epsilon_1(1760 - 2520\,\beta_0 + 630\,\beta_0^2 - 1728\,\mu + 560\beta_0\,\mu)$$

$$+ \epsilon_2(-1120 + 560\,\beta_0 + 1728\mu - 560\,\beta_0\,\mu)\},$$

$$t_{50} = \frac{15\beta_0}{32}\sqrt{-\beta_0 + 4(1+\epsilon_1+\epsilon_2)}\{\tfrac{\beta_0}{3}(-12 + 7\beta_0 + 24\mu - 14\beta_0\mu) + \epsilon_1(-8 + 42\beta_0 - 21\beta_0^2$$

$$- 28\,\beta_0\,\mu + 21\,\beta_0^2\,\mu) + \epsilon_2(8 - 14\beta_0 - 28\beta_0\,\mu + 21\,\beta_0^2\,\mu)\},$$

$$t_{51} = \frac{5\,\beta_0}{32}\sqrt{-\beta_0 + 4(1+\epsilon_1+\epsilon_2)}\{\beta_0(16 - 7\,\beta_0 - 32\,\mu + 14\,\beta_0\,\mu) + \epsilon_1(32 - 154\,\beta_0 + 63\,\beta_0^2$$

$$+ 256\,\beta_0\mu - 126\,\beta_0^2\,\mu) + \epsilon_2\,(-32 + 42\,\beta_0 - 32\,\beta_0\,\mu)\},\;\; t_{52} = \frac{3\,\beta_0^{5/2}}{8}\sqrt{4-\beta_0}(1-\beta_0^2)^2,$$

$$t_{53} = \frac{3\,\beta_0^{5/2}}{8}\sqrt{4-\beta_0}(8 - 5\,\beta_0 + \beta_0^2), t_{54} = \frac{3\,\beta_0^{5/2}}{4}\sqrt{4-\beta_0}(-3 + \beta_0 + 27\beta_0^2 - 7\beta_0^3),$$

$$t_{55} = \frac{21\,\beta_0^{7/2}}{8}(4-\beta_0), t_{56} = \frac{3\,\beta_0^{7/2}}{8}(11 - 7\beta_0 + \beta_0^2), t_{57} = \frac{15\,\beta_0^{5/2}}{4}(-1 + 4\beta_0 - 3\beta_0^2 + \beta_0^3),$$

$$t_{58} = 120\,\beta_0^{5/2}\,(-12 + 11\beta_0 - 4\beta_0^2 + \beta_0^3), t_{60} = \frac{15\,\beta_0^3}{4\,\sqrt{4-\beta_0}}(-24 + 47\,\beta_0 - 25\,\beta_0^2 + 3\,\beta_0^3),$$

$$t_{59} = \frac{75\,\beta_0^{5/2}}{4}(2 - 9\,\beta_0 + 6\,\beta_0^2 - \beta_0^3), t_{62} = \frac{15\,\beta_0^{5/2}}{2\sqrt{4-\beta_0}}\sqrt{-\beta_0 + 4(1+\epsilon_1+\epsilon_2)}(12 - 7\beta_0$$

$$+\beta_0^2), t_{61} = \frac{45\beta_0^3}{\sqrt{4-\beta_0}}(9 - \beta_0 + 3\beta_0^2 - \beta_0^3), t_{63} = \frac{15\beta_0^{5/2}}{2\sqrt{4-\beta_0}}\sqrt{-\beta_0 + 4(1+\epsilon_1+\epsilon_2)}(-4 + 13\beta_0$$

$$-3\,\beta_0^2), t_{64} = \frac{15\beta_0^3}{2}\sqrt{-\beta_0 + 4(1+\epsilon_1+\epsilon_2)}(1-\beta_0), t_{65} = \frac{45\beta_0^3}{2}\sqrt{-\beta_0 + 4(1+\epsilon_1+\epsilon_2)}(\beta_0$$

$$-4), t_{69} = 2\,g_{18}\,g_{30}\,n\,\phi\,f_1^2 - 2\,g_{30}\,n^3\,\phi^3\,f_1^2 - 2\,g_{30}\,n\,\phi\,f_1^4,\;\; t_{66} = -\,g_{21}^2\,g_{36} - 4\,g_{36}\,n^2\,\phi^2\,f_2^2,$$

$$t_{67} = -\,2\,g_{21}\,g_{22}\,g_{36} - g_{21}^2\,g_{37} - 4\,g_{37}\,n^2\,\phi^2\,f_2^2 - 8\,g_{36}\,n^2\,p_1\,\phi^2\,f_2^2, \quad t_{68} = -\,2\,g_{21}\,g_{23}\,g_{36}$$

$$-\,g_{21}^2\,g_{38} - 4(g_{38} + 2\,g_{36}\,q_1)n^2\,\phi^2\,f_2^2, \qquad t_{70} = 2(g_{19}\,g_{30} + g_{18}\,g_{31} + 2\,g_{18}\,g_{30}\,p)\,n\,\phi\,f_1^2$$

$$-\,2\,g_{31}\,n^3\,\phi^3\,f_1^2 - 4\,g_{30}\,n^3\,p\,\phi^3\,f_1^2 - 2\,g_{31}\,n\,\phi\,f_1^4 - 8\,g_{30}\,n\,p\,\phi\,f_1^4,$$

$$t_{71}=2\,g_{20}\,g_{30}\,n\,\phi\,f_1^2+2\,g_{18}\,g_{32}\,n\,\phi\,f_1^2+4\,g_{18}\,g_{30}\,n\,q\,\phi\,f_1^2-2\,g_{32}\,n^3\,\phi^3\,f_1^2-4\,g_{30}\,n^3\,q\,\phi^3\,f_1^2$$
$$-2\,g_{32}\,n\,\phi\,f_1^4-8\,g_{30}\,n\,q\,\phi\,f_1^4,\qquad t_{72}=2\,g_{18}\,g_{36}\,n\,\phi\,f_2^2-2\,g_{36}\,n^3\,\phi^3\,f_2^2-2\,g_{36}\,n\,\phi\,f_2^4,$$
$$t_{73}=2\,g_{19}\,g_{36}\,n\,\phi\,f_2+2\,g_{18}\,g_{37}\,n\,\phi\,f_2^2+4\,g_{18}\,g_{36}\,n\,p_1\,\phi\,f_2^2-2\,g_{37}\,n^3\,\phi^3\,f_2^2-4\,g_{36}\,n^3\,p_1\,\phi^3\,f_2^2$$
$$-2\,g_{37}\,n\,\phi\,f_2^4-8\,g_{36}\,n\,p_1\,\phi\,f_2^4,$$
$$t_{74}=2\,g_{20}\,g_{36}\,n\,\phi\,f_2^2+2\,g_{18}\,g_{38}\,n\,\phi\,f_2^2+4\,g_{18}\,g_{36}\,n\,q_1\,\phi\,f_2^2-2\,g_{38}\,n^3\,\phi^3\,f_2^2-2\,g_{38}\,n\,\phi\,f_2^4$$
$$-4\,g_{36}\,n^3\,q_1\,\phi^3\,f_2^2-8\,g_{36}\,n\,q_1\,\phi\,f_2^4,\;\; t_{75}=-\,g_{18}\,g_{21}\,g_{30}\;+\;g_{21}\,g_{30}\,n^2\,\phi^2+g_{21}\,g_{30}\,f_1^2,$$

$$t_{76}=-\,g_{19}\,g_{21}\,g_{30}-g_{18}\,g_{22}\,g_{30}-g_{18}\,g_{21}\,g_{31}+\;g_{22}\,g_{30}\,n^2\,\phi^2+g_{21}\,g_{31}\,n^2\,\phi^2+g_{22}\,g_{30}\,f_1^2$$
$$+g_{21}\,g_{31}\,f_1^2+2\,g_{21}\,g_{30}\,p\,f_1^2,\;\; t_{77}=-\,g_{20}\,g_{21}\,g_{30}-g_{18}\,g_{23}\,g_{30}-g_{18}\,g_{21}\,g_{32}+g_{23}\,g_{30}\,n^2\,\phi^2$$
$$+g_{21}\,g_{32}\,n^2\,\phi^2+g_{23}\,g_{30}\,f_1^2+g_{21}\,g_{32}\,f_1^2+2\,g_{21}\,g_{30}\,q\,f_1^2,\;\; t_{78}=g_{18}\,g_{21}\,g_{36}-g_{21}\,g_{36}\,n^2\,\phi^2$$
$$-g_{21}\,g_{36}\,f_2^2,\quad t_{79}=g_{19}\,g_{21}\,g_{36}+g_{18}\,g_{22}\,g_{36}\;+g_{18}\,g_{21}\,g_{37}-g_{22}\,g_{36}\,n^2\,\phi^2-g_{21}\,g_{37}\,n^2\,\phi^2$$
$$-g_{22}\,g_{36}\,f_2^2-g_{21}\,g_{37}\,f_2^2-2\,g_{21}\,g_{36}\,p_1\,f_2^2,\quad t_{80}=g_{20}\,g_{21}\,g_{36}\;+g_{18}\,g_{23}\,g_{36}+g_{18}\,g_{21}\,g_{38}$$
$$-g_{23}\,g_{36}\,n^2\,\phi^2-g_{21}\,g_{38}\,n^2\,\phi^2-g_{23}\,g_{36}\,f_2^2-g_{21}\,g_{38}\,f_2^2-2\,g_{21}\,g_{36}\,q_1\,f_2^2,$$
$$t_{81}=-\,g_{21}^2\,g_{30}\,f_1^2-2\,g_{18}\,g_{30}\,n^2\,\phi^2\,f_1^2+2\,g_{30}\,n^4\,\phi^4\,f_1^2-2\,g_{30}\,n^3\,\phi^3\,f_1^4,$$
$$t_{82}=-\,2\,g_{21}\,g_{22}\,g_{30}\,f_1^2-g_{21}^2\,g_{31}\,f_1^2-2\,g_{21}^2\,g_{30}\,p\,f_1^2-2\,g_{19}\,g_{30}\,n^2\,\phi^2\,f_1^2-2\,g_{18}\,g_{31}\,n^2\,\phi^2\,f_1^2$$
$$-4\,g_{18}\,g_{30}\,n^2\,p\,\phi^2\,f_1^2+2\,g_{31}\,n^4\,\phi^4\,f_1^2+4\,g_{30}\,n^4\,p\,\phi^4\,f_1^2-2\,g_{31}\,n^3\,\phi^3\,f_1^4-8\,g_{30}\,n^3\,p\,\phi^3\,f_1^4,$$

$$t_{83}=-\,2\,g_{21}\,g_{23}\,g_{30}\,f_1^2-g_{21}^2\,g_{32}\,f_1^2-2\,g_{21}^2\,g_{30}\,q\,f_1^2-2\,g_{20}\,g_{30}\,n^2\,\phi^2\,f_1^2-2\,g_{18}\,g_{32}\,n^2\,\phi^2\,f_1^2$$
$$-4\,g_{18}\,g_{30}\,n^2\,q\,\phi^2\,f_1^2+2\,g_{32}\,n^4\,\phi^4\,f_1^2+4\,g_{30}n^4q\,\phi^4\,f_1^2-2\,g_{32}n^3\,\phi^3\,f_1^4-8\,g_{30}n^3q\,\phi^3\,f_1^4,$$

$$t_{84}=i\,(g_{18}\,g_{21}\,g_{36}\,n\,\phi\,f_2-g_{21}\,g_{36}\,n^3\,\phi^3\,f_2-2\,g_{21}\,g_{36}\,n\,\phi\,f_2^3+g_{21}\,g_{36}\,n^2\,\phi^2\,f_2^3),$$
$$t_{85}=i(g_{19}g_{21}g_{36}n\phi f_2+g_{18}g_{22}g_{36}n\phi f_2+g_{18}g_{21}g_{37}n\phi f_2+g_{18}g_{21}g_{36}np_1\phi f_2-g_{22}g_{36}n^3\,\phi^3\,f_2$$
$$-g_{21}g_{37}n^3\phi^3 f_2-g_{21}g_{36}n^3p_1\phi^3 f_2-2g_{22}g_{36}n\phi f_2^3-2g_{21}g_{37}n\phi f_2^3-6g_{21}g_{36}np_1\phi f_2^3$$
$$+g_{22}\,g_{36}\,n^2\,\phi^2\,f_2^3+g_{21}\,g_{37}\,n^2\,\phi^2\,f_2^3+3\,g_{21}\,g_{36}\,n^2\,p_1\,\phi^2\,f_2^3),$$

$$t_{86}=i\,(g_{20}\,g_{21}\,g_{36}\,n\,\phi\,f_2+g_{18}\,g_{23}\,g_{36}\,n\,\phi\,f_2+g_{18}\,g_{21}\,g_{38}\,n\,\phi\,f_2+g_{18}\,g_{21}\,g_{36}\,n\,q_1\,\phi\,f_2$$

$$-g_{23}\,g_{36}\,n^3\,\phi^3\,f_2-g_{21}\,g_{38}\,n^3\,\phi^3\,f_2-g_{21}\,g_{36}\,n^3\,q_1\,\phi^3 f_2-2g_{23}g_{36}n\phi f_2^3-2g_{21}g_{38}n\phi f_2^3$$

$$-6\,g_{21}\,g_{36}\,n\,q_1\,\phi\,f_2^3+g_{23}\,g_{36}\,n^2\,\phi^2\,f_2^3+g_{21}\,g_{38}\,n^2\,\phi^2\,f_2^3+3\,g_{21}\,g_{36}\,n^2\,q_1\,\phi^2\,f_2^3),$$

$$t_{87}=g_{18}\,g_{21}\,g_{30}\,n\,\phi-g_{21}\,g_{30}\,n^3\,\phi^3-2\,g_{21}\,g_{30}\,n\,\phi\,f_1^2+g_{21}\,g_{30}\,n^2\,\phi^2\,f_1^2,\quad t_{88}=g_{19}\,g_{21}\,g_{30}\,n\,\phi$$

$$+\,g_{18}\,g_{22}\,g_{30}\,n\,\phi+g_{18}\,g_{21}\,g_{31}\,n\,\phi-g_{22}\,g_{30}\,n^3\,\phi^3-g_{21}\,g_{31}\,n^3\,\phi^3-2\,g_{22}\,g_{30}\,n\,\phi\,f_1^2$$

$$-2\,g_{21}\,g_{31}\,n\,\phi\,f_1^2\,-4\,g_{21}\,g_{30}\,n\,p\,\phi\,f_1^2+g_{22}\,g_{30}\,n^2\,\phi^2\,f_1^2+g_{21}\,g_{31}\,n^2\,\phi^2\,f_1^2+2\,g_{21}\,g_{30}\,n^2\,p\,\phi^2\,f_1^2,$$

$$t_{89}=g_{20}\,g_{21}\,g_{30}\,n\,\phi+g_{18}\,g_{23}\,g_{30}\,n\,\phi+g_{18}\,g_{21}\,g_{32}\,n\,\phi-g_{23}\,g_{30}\,n^3\,\phi^3-g_{21}\,g_{32}\,n^3\,\phi^3$$

$$-2\,g_{23}\,g_{30}\,n\,\phi\,f_1^2-2\,g_{21}\,g_{32}\,n\,\phi\,f_1^2-4\,g_{21}\,g_{30}\,n\,q\,\phi\,f_1^2+g_{23}\,g_{30}\,n^2\,\phi^2\,f_1^2+g_{21}\,g_{32}\,n^2\,\phi^2\,f_1^2$$

$$+2\,g_{21}\,g_{30}\,n^2\,q\,\phi^2\,f_1^2,\quad t_{90}=i(g_{21}^2\,g_{36}\,f_2+2\,g_{18}\,g_{36}\,n^2\,\phi^2\,f_2-2\,g_{36}\,n^4\,\phi^4\,f_2+2\,g_{36}\,n^3\,\phi^3\,f_2^3),$$

$$t_{91}=i(2\,g_{21}\,g_{22}\,g_{36}\,f_2+g_{21}^2\,g_{37}\,f_2+g_{21}^2\,g_{36}\,p_1\,f_2+2\,g_{19}\,g_{36}\,n^2\,\phi^2\,f_2+2\,g_{18}\,g_{37}\,n^2\,\phi^2\,f_2$$

$$+2\,g_{18}\,g_{36}\,n^2\,p_1\,\phi^2\,f_2-2\,g_{37}\,n^4\,\phi^4\,f_2-2\,g_{36}\,n^4\,p_1\,\phi^4\,f_2+2\,g_{37}\,n^3\,\phi^3\,f_2^3+6\,g_{36}\,n^3\,p_1\,\phi^3\,f_2^3),$$

$$t_{92}=i(2\,g_{21}\,g_{23}\,g_{36}\,f_2+g_{21}^2\,g_{38}\,f_2+g_{21}^2\,g_{36}\,q_1\,f_2+2\,g_{20}\,g_{36}\,n^2\,\phi^2\,f_2+2\,g_{18}\,g_{38}\,n^2\,\phi^2\,f_2$$

$$+2\,g_{18}\,g_{36}\,n^2\,q_1\,\phi^2\,f_2-2\,g_{38}\,n^4\,\phi^4\,f_2-2\,g_{36}\,n^4\,q_1\,\phi^4\,f_2+2\,g_{38}\,n^3\,\phi^3\,f_2^3+6\,g_{36}\,n^3\,q_1\,\phi^3\,f_2^3),$$

$$t_{93}=g_{18}g_{21}g_{30}f_1^2-g_{21}g_{30}n^2\phi^2 f_1^2-g_{21}g_{30}f_1^4,\;\; t_{96}=g_{18}g_{21}g_{36}f_2^2-g_{21}g_{36}n^2\phi^2 f_2^2-g_{21}g_{36}f_2^4,$$

$$t_{94}=g_{19}\,g_{21}\,g_{30}\,f_1^2+g_{18}\,g_{22}\,g_{30}\,f_1^2+g_{18}\,g_{21}\,g_{31}\,f_1^2+2\,g_{18}\,g_{21}\,g_{30}\,p\,f_1^2-g_{22}\,g_{30}\,n^2\,\phi^2\,f_1^2$$

$$-g_{21}\,g_{31}\,n^2\,\phi^2\,f_1^2-2\,g_{21}\,g_{30}\,n^2\,p\,\phi^2\,f_1^2-g_{22}\,g_{30}\,f_1^4-g_{21}\,g_{31}\,f_1^4-4\,g_{21}\,g_{30}\,p\,f_1^4),$$

$$t_{95}=g_{20}\,g_{21}\,g_{30}\,f_1^2+g_{18}\,g_{23}\,g_{30}\,f_1^2+g_{18}\,g_{21}\,g_{32}\,f_1^2+2\,g_{18}\,g_{21}\,g_{30}\,q\,f_1^2-g_{23}\,g_{30}\,n^2\,\phi^2\,f_1^2$$

$$-g_{21}\,g_{32}\,n^2\,\phi^2\,f_1^2-2\,g_{21}\,g_{30}\,n^2\,q\,\phi^2\,f_1^2-g_{23}\,g_{30}\,f_1^4-g_{21}\,g_{32}\,f_1^4-4\,g_{21}\,g_{30}\,q\,f_1^4,$$

$$t_{97}=g_{19}\,g_{21}\,g_{36}\,f_2^2+g_{18}\,g_{22}\,g_{36}\,f_2^2+g_{18}\,g_{21}\,g_{37}\,f_2^2+2\,g_{18}\,g_{21}\,g_{36}\,p_1\,f_2^2-g_{22}\,g_{36}\,n^2\,\phi^2\,f_2^2$$

$$-g_{21}\,g_{37}\,n^2\,\phi^2\,f_2^2-2\,g_{21}\,g_{36}\,n^2\,p_1\,\phi^2\,f_2^2-g_{22}\,g_{36}\,f_2^4-g_{21}\,g_{37}\,f_2^4-4\,g_{21}\,g_{36}\,p_1\,f_2^4,$$

$$t_{98}=g_{20}\,g_{21}\,g_{36}\,f_2^2+g_{18}\,g_{23}\,g_{36}\,f_2^2+g_{18}\,g_{21}\,g_{38}\,f_2^2+2\,g_{18}\,g_{21}\,g_{36}\,q_1\,f_2^2-g_{23}\,g_{36}\,n^2\,\phi^2\,f_2^2$$

$$-g_{21}\,g_{38}\,n^2\,\phi^2\,f_2^2-2\,g_{21}\,g_{36}\,n^2\,q_1\,\phi^2\,f_2^2-g_{23}\,g_{36}\,f_2^4-g_{21}\,g_{38}\,f_2^4-4\,g_{21}\,g_{36}\,q_1\,f_2^4,$$

$$t_{99} = g_{21}^2\, g_{30}\, n\, \phi + 2\, g_{18}\, g_{30}\, n\, \phi\, f_1^2 + 2\, g_{30}\, n^3\, \phi^3\, f_1^2 - 2\, g_{30}\, n\, \phi\, f_1^4,$$

$$t_{100} = 2\, g_{21}\, g_{22}\, g_{30}\, n\, \phi + g_{21}^2\, g_{31}\, n\, \phi + 2\, g_{19}\, g_{30}\, n\, \phi\, f_1^2 + 2\, g_{18}\, g_{31}\, n\, \phi\, f_1^2 + 4\, g_{18}\, g_{30}\, n\, p\, \phi\, f_1^2 + 2\, g_{31}\, n^3\, \phi^3\, f_1^2 + 4\, g_{30}\, n^3\, p\, \phi^3\, f_1^2 - 2\, g_{31}\, n\, \phi\, f_1^4 - 8\, g_{30}\, n\, p\, \phi\, f_1^4,$$

$$t_{101} = 2\, g_{21}\, g_{23}\, g_{30}\, n\, \phi + g_{21}^2\, g_{32}\, n\, \phi + 2\, g_{20}\, g_{30}\, n\, \phi\, f_1^2 + 2\, g_{18}\, g_{32}\, n\, \phi\, f_1^2 + 4\, g_{18}\, g_{30}\, n\, q\, \phi\, f_1^2 + 2\, g_{32}\, n^3\, \phi^3\, f_1^2 + 4\, g_{30}\, n^3\, q\, \phi^3\, f_1^2 - 2\, g_{32}\, n\, \phi\, f_1^4 - 8\, g_{30}\, n\, q\, \phi\, f_1^4,$$

$$t_{102} = -\, g_{21}^2\, g_{36}\, n\, \phi - 2\, g_{18}\, g_{36}\, n\, \phi\, f_2^2 - 2\, g_{36}\, n^3\, \phi^3\, f_2^2 + 2\, g_{36}\, n\, \phi\, f_2^4,$$

$$t_{103} = -\, 2\, g_{21}\, g_{22}\, g_{36}\, n\, \phi - g_{21}^2\, g_{37}\, n\, \phi - 2\, g_{19}\, g_{36}\, n\, \phi\, f_2^2 - 2\, g_{18}\, g_{37}\, n\, \phi\, f_2^2 - 4\, g_{18}\, g_{36}\, n\, p_1\, \phi\, f_2^2 - 2\, g_{37}\, n^3\, \phi^3\, f_2^2 - 4\, g_{36}\, n^3\, p_1\, \phi^3\, f_2^2 + 2\, g_{37}\, n\, \phi\, f_2^4 + 8\, g_{36}\, n\, p_1\, \phi\, f_2^4,$$

$$t_{104} = -\, 2\, g_{21}\, g_{23}\, g_{36}\, n\, \phi - g_{21}^2\, g_{38}\, n\, \phi - 2\, g_{20}\, g_{36}\, n\, \phi\, f_2^2 - 2\, g_{18}\, g_{38}\, n\, \phi\, f_2^2 - 4\, g_{18}\, g_{36}\, n\, q_1\, \phi\, f_2^2 - 2\, g_{38}\, n^3\, \phi^3\, f_2^2 - 4\, g_{36}\, n^3\, q_1\, \phi^3\, f_2^2 + 2\, g_{38}\, n\, \phi\, f_2^4 + 8\, g_{36}\, n\, q_1\, \phi\, f_2^4,$$

$$t_{105} = -\, 6\, t_{22}\, t_{35} + t_{23}\, t_{37} - 6\, t_{33}\, t_{39} + t_{34}\, t_{41}, \quad t_{106} = -\, 6\, t_{25}\, t_{35} + t_{26}\, t_{37} - 6\, t_{33}\, t_{43} + t_{34}\, t_{45},$$

$$t_{107} = 3\, t_{23}\, t_{36} - 2\, t_{22}\, t_{38} + 3\, t_{34}\, t_{40} - 2\, t_{33}\, t_{42}, \; t_{108} = 3\, t_{26}\, t_{36} - 2\, t_{25}\, t_{38} + 3\, t_{34}\, t_{44} - 2\, t_{33}\, t_{46},$$

$$t_{109} = -\, 2\, t_{22}\, t_{37} + t_{23}\, t_{38} - 2\, t_{33}\, t_{41} + t_{34}\, t_{42}, \; t_{110} = -\, 2\, t_{25}\, t_{37} + t_{26}\, t_{38} - 2\, t_{33}\, t_{45} + t_{34}\, t_{46},$$

$$t_{111} = 3\, t_{23}\, t_{35} - 2\, t_{21}\, t_{37} + 3\, t_{34}\, t_{39} - 2\, t_{32}\, t_{41}, \; t_{112} = 3\, t_{26}\, t_{35} - 2\, t_{24}\, t_{37} + 3\, t_{34}\, t_{43} - 2\, t_{32}\, t_{45},$$

$$t_{113} = -\, 6\, t_{21}\, t_{36} + t_{23}\, t_{38} - 6\, t_{32}\, t_{40} + t_{34}\, t_{42}, \quad t_{114} = -\, 6\, t_{24}\, t_{36} + t_{26}\, t_{38} - 6\, t_{32}\, t_{44} + t_{34}\, t_{46},$$

$$t_{115} = t_{23}\, t_{37} - 2\, t_{21}\, t_{38} + t_{34}\, t_{41} - 2\, t_{32}\, t_{42}, \qquad t_{116} = t_{26}\, t_{37} - 2\, t_{24}\, t_{38} + t_{34}\, t_{45} - 2\, t_{32}\, t_{46}.$$

Appendix B

$$g_1 = \tfrac{3}{8}\, \beta_0^{3/2}, \;\; g_2 = (-\tfrac{4}{3} + \beta_0^2) + \beta_0\, \mu\, (1 - \beta_0), \; g_3 = \tfrac{8}{3} - \beta_0, \; g_4 = 2\, \sqrt{\beta_0\, (4 - \beta_0)}\, \mu(1 - \mu),$$

$$g_5 = -\, \tfrac{3\beta_0^{5/2}}{16}, \; g_6 = -4 + 5\, \beta_0^2, \;\; g_7 = 16 - 5\, \beta_0, \; g_8 = 16\, \beta_0\, \sqrt{4 - \beta_0}, \;\; g_{11} = -\tfrac{525\, \beta_0^3}{16} (4 - \beta_0)^{3/2},$$

$$g_9 = -\, \tfrac{5\beta_0^{5/2}}{32}\, (24 - 54\, \beta_0 + 21\, \beta_0^2), \quad g_{10} = \tfrac{1}{32}\, \beta_0^{5/2} (-\, 768 + 570\, \beta_0 - 105\, \beta_0^2), \; g_{19} = 2\, g_5\, g_6,$$

$$g_{12} = 120\sqrt{-\beta_0 + 4(1+\epsilon_1+\epsilon_2)(4-\beta_0)},\ g_{13} = -\frac{\beta_0^{1/2}\, g_{12}}{\sqrt{-\beta_0+4}},\ \ g_{14} = \frac{3(\epsilon_1-\epsilon_2)\,\beta_0^{3/2}}{8},$$

$$g_{15} = 4\,\beta_0^{1/2}(-1+\mu) + \mu\,(8-5\,\beta_0),\, g_{16} = 4 - 16\,\mu + 5\,\beta_0\,\mu,\ \ g_{22} = g_5\,g_7,\ \ g_{23} = g_{10} + g_{12},$$

$$g_{17} = \beta_0^{-1/2}\sqrt{4-\beta_0}(4 - 10\,\beta_0\mu) + 4\,\beta_0\mu(1-\mu)(4-\beta_0)^{-1/2},\ g_{25} = g_5\,g_8,\ g_{26} = g_{11} + g_{13},$$

$$g_{18} = \tfrac{1}{2}\,n^2\,(\phi^2 - \psi) - \tfrac{\alpha_0^2}{8} - \ g_1\,g_2\ +\ g_{14}\,g_{15},\ \ g_{21} = \tfrac{1}{2}\,n^2\,(\phi^2-\psi) - \tfrac{\alpha_0^2}{8} - 2\,g_1\,g_3\ + 2\,g_{14}\,g_{16},$$

$$g_{20} = 2\,g_5\,g_9,\quad g_{24} = g_{14}\,g_{17} - g_1\,g_4,\quad g_{30} = \frac{1}{\sqrt{g_{27}}},\quad g_{31} = \frac{-g_{28}}{2\,g_{27}\sqrt{g_{27}}},\quad g_{32} = \frac{-g_{29}}{2\,g_{27}\sqrt{g_{27}}},$$

$$g_{27} = f_1^2(g_{21}^2 + 4\,n^2\,\phi^2\,f_1^2)(g_{18}^2 + g_{21}^2 + 2\,g_{18}\,n^2\,\phi^2 - 3\,n^4\,\phi^4 - 2\,g_{18}\,f_1^2 + 2\,n^3\,\phi^3\,f_1^2 + f_1^4),$$

$$\begin{aligned} g_{28} = 2\,f_1^2(&g_{18}\,g_{19}\,g_{21}^2 + g_{18}^2\,g_{21}\,g_{22} + 2\,g_{21}^3\,g_{22} +\ g_{18}^2\,g_{21}^2\,p +\ g_{21}^4\,p + g_{19}\,g_{21}^2\,n^2\,\phi^2 - g_{19}\,g_{21}^2\,f_1^2 \\ &+2g_{18}\,g_{21}g_{22}\,n^2\,\phi^2 + 2g_{18}\,g_{21}^2\,n^2\,\phi^2\,p\ - 3\,g_{21}\,g_{22}\,n^4\,\phi^4 - 3\,g_{21}^2\,n^4\,\phi^4\,p - 2\,g_{18}\,g_{21}\,g_{22}\,f_1^2 \\ &-4\,g_{18}\,g_{21}^2\,p\,f_1^2 + 4\,g_{18}\,g_{19}\,n^2\,\phi^2\,f_1^2 + 4\,g_{21}\,g_{22}n^2\phi^2f_1^2 + 8g_{18}^2n^2\phi^2f_1^2p + 8g_{21}^2n^2\phi^2f_1^2p \\ &+2g_{21}\,g_{22}\,n^3\,\phi^3\,f_1^2 + 4\,g_{21}^2\,n^3\,\phi^3\,f_1^2\,p + 4\,g_{19}\,n^4\,\phi^4\,f_1^2 + 16\,g_{18}\,n^4\,\phi^4\,f_1^2\,p - 24\,n^6\,\phi^6\,f_1^2\,p \\ &+g_{21}g_{22}f_1^4 + 3g_{21}^2pf_1^4 - 4g_{19}n^2\phi^2f_1^4 - 24g_{18}n^2p\phi^2\,f_1^4 + 24\,n^5\,p\,\phi^5\,f_1^4 + 16n^2\,p\,\phi^2\,f_1^6), \end{aligned}$$

$$\begin{aligned} g_{29} = 2f_1^2(&g_{18}\,g_{20}\,g_{21}^2 + g_{18}^2\,g_{21}\,g_{23} + 2\,g_{21}^3\,g_{23} + g_{18}^2\,g_{21}^2\,q\ + g_{21}^4\,q + g_{20}\,g_{21}^2\,n^2\,\phi^2 - g_{20}\,g_{21}^2f_1^2 \\ &+2\,g_{18}\,g_{21}\,g_{23}\,n^2\,\phi^2 + 2\,g_{18}\,g_{21}^2\,n^2\,q\,\phi^2 - 3\,g_{21}\,g_{23}\,n^4\,\phi^4 - 3g_{21}^2\,n^4\,q\,\phi^4 - 2\,g_{18}\,g_{21}\,g_{23}\,f_1^2 \\ &-4g_{18}g_{21}^2\,qf_1^2 + 4\,g_{18}g_{20}n^2\phi^2\,f_1^2 + 4\,g_{21}\,g_{23}\,n^2\,\phi^2\,f_1^2 + 8\,g_{18}^2\,n^2\,q\,\phi^2\,f_1^2 + 8\,g_{21}^2\,n^2\,q\,\phi^2\,f_1^2 \\ &+2\,g_{21}\,g_{23}\,n^3\,\phi^3\,f_1^2 + 4\,g_{21}^2\,n^3\,q\,\phi^3\,f_1^2 + 4\,g_{20}\,n^4\,\phi^4\,f_1^2 + 16\,g_{18}\,n^4\,q\,\phi^4\,f_1^2 - 24\,n^6\,q\,\phi^6\,f_1^2 \\ &+g_{21}\,g_{23}\,f_1^4 + 3\,g_{21}^2\,q\,f_1^4 - 4g_{20}n^2\phi^2f_1^4 - 24g_{18}n^2q\phi^2f_1^4 + 24n^5q\phi^5f_1^4 + 16n^2q\phi^2f_1^6), \end{aligned}$$

$$g_{33} = -f_2^2(g_{21}^2 + 4\,n^2\,\phi^2\,f_2^2)(g_{18}^2 + g_{21}^2 + 2\,g_{18}\,n^2\,\phi^2 - 3\,n^4\,\phi^4 - 2\,g_{18}\,f_2^2 + 2\,n^3\,\phi^3\,f_2^2 + f_2^4),$$

$$g_{34} = 2f_2^2(g_{18}g_{19}g_{21}^2 + g_{18}^2g_{21}g_{22} + 2g_{21}^3g_{22} + g_{18}^2g_{21}^2p_1 + g_{21}^4p_1 + g_{19}g_{21}^2n^2\phi^2 - g_{19}g_{21}^2f_2^2$$
$$+2\,g_{18}g_{21}g_{22}n^2\phi^2 + 2g_{18}\,g_{21}^2n^2\,p_1\phi^2 - 3\,g_{21}\,g_{22}\,n^4\,\phi^4 - 3\,g_{21}^2\,n^4\,p_1\,\phi^4 - 2\,g_{18}\,g_{21}\,g_{22}\,f_2^2$$
$$-4\,g_{18}\,g_{21}^2\,p_1\,f_2^2 + 4\,g_{18}\,g_{19}\,n^2\,\phi^2\,f_2^2 + 4\,g_{21}\,g_{22}\,n^2\,\phi^2\,f_2^2 + 8\,g_{18}^2\,n^2\,p_1\,\phi^2\,f_2^2 + g_{21}\,g_{22}\,f_2^4$$
$$+8\,g_{21}^2\,n^2\,p_1\,\phi^2 f_2^2 + 2g_{21}g_{22}n^3\phi^3f_2^2 + 4g_{21}^2n^3p_1\phi^3f_2^2 + 4g_{19}n^4\,\phi^4\,f_2^2 - \,4g_{19}\,n^2\,\phi^2\,f_2^4$$
$$+16\,g_{18}\,n^4\,p_1\,\phi^4\,f_2^2 - 24\,n^6\,p_1\,\phi^6\,f_2^2 + 3\,g_{21}^2\,p_1\,f_2^4 - 24\,g_{18}\,n^2\,p_1\,\phi^2 f_2^4 + 24\,n^5\,p_1\,\phi^5\,f_2^4$$
$$+16\,n^2\,p_1\,\phi^2\,f_2^6),$$

$$g_{35} = -\,2\,f_2^2(g_{18}\,g_{20}\,g_{21}^2 + g_{18}^2\,g_{21}\,g_{23} + 2\,g_{21}^3\,g_{23} + 2\,g_{18}\,g_{21}\,g_{23}\,n^2\,\phi^2$$
$$+g_{18}^2\,g_{21}^2\,q_1 + g_{21}^4\,q_1 + g_{20}\,g_{21}^2\,n^2\,\phi^2 + 2\,g_{18}\,g_{21}^2\,n^2\,q_1\,\phi^2 - 3\,g_{21}\,g_{23}\,n^4\,\phi^4 - 3\,g_{21}^2\,n^4\,q_1\,\phi^4$$
$$+4\,g_{18}\,g_{20}\,n^2\,\phi^2\,f_2^2 + 4\,g_{21}\,g_{23}\,n^2\,\phi^2\,f_2^2 + 8\,g_{18}^2n^2q_1\phi^2f_2^2 + 8g_{21}^2n^2q_1\phi^2f_2^2 - g_{20}g_{21}^2f_2^2$$
$$+2\,g_{21}g_{23}n^3\phi^3f_2^2 + 4\,g_{21}^2\,n^3\,q_1\,\phi^3\,f_2^2 + 4\,g_{20}\,n^4\phi^4f_2^2 + 16g_{18}n^4q_1\phi^4f_2^2 - 24\,n^6\,q_1\,\phi^6\,f_2^2$$
$$+3g_{21}^2q_1f_2^4 - 4g_{20}n^2\phi^2f_2^4 - 24g_{18}n^2q_1\phi^2f_2^4 + 24n^5q_1\phi^5f_2^4 + 16n^2q_1\phi^2f_2^6 + g_{21}g_{23}f_2^4$$
$$-2\,g_{18}\,g_{21}\,g_{23}\,f_2^2 - 4\,g_{18}\,g_{21}^2\,q_1\,f_2^2), g_{36} = \frac{1}{\sqrt{g_{33}}}, g_{37} = \frac{-\,g_{34}\,g_{36}}{2\,g_{33}}, g_{38} = \frac{-\,g_{35}\,g_{36}}{2\,g_{33}}.$$

$$g_{39} = g_{21}^2\,g_{30} + 4\,g_{30}\,n^2\,\phi^2\,f_1^2,$$
$$g_{40} = 2\,g_{21}\,g_{22}\,g_{30} + \,g_{21}^2\,g_{31} + 4\,g_{31}\,n^2\,\phi^2\,f_1^2 + 8\,g_{30}\,n^2\,p\,\phi^2\,f_1^2,$$
$$g_{41} = 2\,g_{21}\,g_{23}\,g_{30} + \,g_{21}^2\,g_{32} + 4\,g_{32}\,n^2\,\phi^2\,f_1^2 + 8\,g_{30}\,n^2\,q\,\phi^2\,f_1^2,$$

$$M_{11} = 0, \qquad M_{12} = 0,$$
$$M_{13} = k_{40} + k_{41}(J_1 + J_1') + k_{42}(J_2 + J_2'), M_{14} = t_{66} + t_{67}(J_1 + J_1') + t_{68}(J_2 + J_2'),$$
$$M_{21} = t_{69} + t_{70}(J_1 + J_1') + t_{71}(J_2 + J_2'), \quad M_{22} = t_{72} + t_{73}(J_1 + J_1') + t_{74}(J_2 + J_2'),$$
$$M_{23} = t_{75} + t_{76}(J_1 + J_1') + t_{77}(J_2 + J_2'), \quad M_{24} = t_{78} + t_{79}(J_1 + J_1') + t_{80}(J_2 + J_2'),$$
$$M_{31} = t_{81} + t_{82}(J_1 + J_1') + t_{83}(J_2 + J_2'), \quad M_{32} = t_{84} + t_{85}(J_1 + J_1') + t_{86}(J_2 + J_2'),$$
$$M_{33} = t_{87} + t_{88}(J_1 + J_1') + t_{89}(J_2 + J_2'), \quad M_{34} = t_{90} + t_{91}(J_1 + J_1') + t_{92}(J_2 + J_2'),$$
$$M_{41} = t_{93} + t_{94}(J_1 + J_1') + t_{95}(J_2 + J_2'), \quad M_{42} = t_{96} + t_{97}(J_1 + J_1') + t_{98}(J_2 + J_2'),$$
$$M_{43} = t_{99} + t_{100}(J_1 + J_1') + t_{101}(J_2 + J_2'), M_{44} = t_{102} + t_{103}(J_1 + J_1') + t_{104}(J_2 + J_2'),$$

$$T_1=(f_1')^2\,\frac{\partial^2}{\partial\,\theta_1^2}-\{t_{33}\,+\,t_{22}\,(J_1+J_1')\,+\,t_{26}\,(J_2+J_2')\},\qquad T_2=f_1'\,\frac{\partial\,A_1^{1,0}}{\partial\,\theta_1}\,-\,n\,\phi\,A_1^{0,1},$$

$$T_3=2\,n\,\phi\,f_1'\,\frac{\partial}{\partial\,\theta_1}+\{t_{34}\,+\,t_{23}\,(J_1+J_1')\,+\,t_{26}\,(J_2+J_2')\},\qquad T_4=f_1'\,\frac{\partial\,A_1^{0,1}}{\partial\,\theta_1}\,+\,n\,\phi\,A_1^{1,0},$$

$$T_5=f_2'\,\frac{\partial^2}{\partial\,\theta_2}-2\{t_{33}\,+\,t_{22}\,(J_1+J_1')\,+\,t_{25}\,(J_2+J_2')\},\qquad T_6=-\,f_2'\,\frac{\partial\,A_1^{1,0}}{\partial\,\theta_2}\,-\,n\,\phi\,A_1^{0,1},$$

$$T_7=-\,2\,n\,\phi\,f_2'\,\frac{\partial}{\partial\,\theta_2}+\{t_{34}\,+\,t_{23}\,(J_1+J_1')\,+\,t_{26}\,(J_2+J_2')\},\quad T_8=-\,f_2'\,\frac{\partial\,A_1^{0,1}}{\partial\,\theta_2}\,+\,n\,\phi\,A_1^{1,0},$$

$$T_9=(f_1')^2\,\frac{\partial^2}{\partial\,\theta_1^2}-2\{t_{32}\,+\,t_{21}\,(J_1+J_1')\,+\,t_{24}\,(J_2+J_2')\},\qquad T_{10}=\,f_1'\,\frac{\partial\,A_1^{0,1}}{\partial\,\theta_1}\,+\,n\,\phi\,A_1^{1,0},$$

$$T_{11}=-\,2\,n\,\phi\,f_1'\,\frac{\partial}{\partial\,\theta_1}+\{t_{34}\,+\,t_{23}\,(J_1+J_1')\,+\,t_{26}\,(J_2+J_2')\},\,T_{12}=\,f_1'\,\frac{\partial\,A_1^{1,0}}{\partial\,\theta_1}\,-\,n\,\phi\,A_1^{0,1},$$

$$T_{13}=(f_2')^2\,\frac{\partial^2}{\partial\,\theta_2^2}-2\{t_{32}\,+\,t_{21}\,(J_1+J_1')\,+\,t_{24}\,(J_2+J_2')\},\qquad T_{14}=-\,f_2'\,\frac{\partial\,A_1^{0,1}}{\partial\,\theta_2}\,+\,n\,\phi\,A_1^{1,0},$$

$$T_{15}=-\,2\,n\,\phi\,f_2'\,\frac{\partial}{\partial\,\theta_2}-\{t_{34}\,-\,t_{23}\,(J_1+J_1')\,-\,t_{26}\,(J_2+J_2')\},\,T_{16}=\,f_2'\,\frac{\partial\,A_1^{1,0}}{\partial\,\theta_2}\,+\,n\,\phi\,A_1^{0,1}.$$