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Abundant Natural Resources, Ethnic Diversity, and Inclusive Growth in sub-Saharan Africa: A Mathematical Approach

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Abstract

The sub-Saharan African region is blessed with abundant natural resources and diverse ethnic groups, yet the region is dominated by the largest number of poor people worldwide due to inequitable distribution of national income. Existing statistics forecast decay in the quality of lives over the years compared to the continent of Asia that shares similar history with the region. In this paper, a five-dimensional first-order nonlinear ordinary differential equation was formulated to give insight into various factors that shaped dynamics of inclusive growth in sub-Saharan Africa. The validity test was performed based on ample mathematical theorems and the model was found to be valid. The model was then studied qualitatively and quantitatively via stability theory of nonlinear differential equations which depended on the policy success ratio and classical fourth-order Runge-Kutta scheme implemented in Maple, respectively. The results from the analysis showed that inclusive growth from abundant natural resources and ethnic diversity in sub-Saharan Africa was a function of policy reform whereby an increase in both equitable distribution of national income and accessibility of common man to the goods and services provided by the state to narrow inequality gap was accompanied with a low level of nepotism.

Keywords: Sub-Saharan Africa; National income; Policy success ratio; Inclusive growth; Inequality gap; Nepotism

MSC 2010 No.: 91B74, 34C60, 92B05

1. Introduction

Natural resources have been identified as an ideal tool for wealth creation and a source of cheap capital for major investments needed for economic development (Chekouri et al. (2017)). If rents from natural resources are well utilized, it is supposed to boost economic activities as well as drive inclusive growth. However, inclusive growth can only be achieved if rent from natural resources managed by the state is appropriated in an unbiased and objective manner. Inclusion in economic prosperity is marked by a sustainable economic growth which is widespread and involves as many people as possible both in the growth process and in sharing of the benefits from such growth sprees.

Ali and Son (2007) opine that growth is inclusive when it is a combination of sustainable and equitable growth, social inclusion, empowerment, and security. When growth is inclusive, more people become part of the growth process and benefit from the growth. This will translate to improved economic activities as most sectors of the economy will boom and manpower will be maximally engaged. The process, in turn, will increase income per head, improve standards of living, and crowd out chronic poverty.

X-raying the impact of ethnic diversity on the inclusive growth, Alesina et al. (1999) and Miguel and Gugerly (2005) opine that if natural resource rents are well-contained and applied in the economy in such a way that there are no ethnic grievances or perceptions of any ethnic group being neglected or relegated, there is a likelihood that ethnically diverse societies will impact positively and directly on economic activities and indirectly on inclusive growth which has been the case in many Asian countries (e.g., Singapore, Malaysia, India, China). Ethnic diversity is key to growth and development and a good amount of literature has been devoted to diversity's positive effects on per capita real income (Florida (2002); Felbermayr et al. (2010); Akay et al. (2017); Bove and Elia (2017)).

Despite diversity's impacts on growth and development especially in the case of some aforementioned Asian countries, evidence abound that ethnic diversity may negatively affect income distribution, poverty as well as human development. Evidently, sub-Saharan African (SSA) countries are blessed with abundant natural resources and diverse ethnic groups with a reasonable output, yet the majorities of individuals found in these countries are absolutely poor and live at the very margin of human existence with inadequate food, shelter, nutrition, education and health care (Noyoo (2000); Gershman and Rivera (2016); Ajide et al. (2019); Appiah et al. (2018); Easterly and Levine (1997); Mbah and Ojo (2018)). The widespread poverty in the SSA region has been blamed on distributive injustice occasioned by ethnic diversity which is a major constraint to inclusive growth (Appiah et al. (2018)). The inequitable distribution of goods and services has been making the rich get richer and the poor to get poorer in the SSA region (Mbah and Ojo (2018)).

To examine the relationship between abundant natural resources, ethnic diversity and inclusive growth in SSA, an epidemic modeling approach is explored. Epidemic modeling is the branch of mathematical modeling that deals with the use of mathematical concepts and language to study the transmission mechanism and control of diseases in man, animals, and plants. Epidemic modeling has been used to study various diseases in man, animals, and plants such as tuberculosis (Liu et al. (2020); Sulyman et al. (2021); Ullah et al. (2018a); Ullah et al. (2020a); Khan et al. (2018)), HIV/AIDS (Nsuami and Witbooi (2018); Arenas et al. (2021); Attuallah and Sohaib (2020)), Ebola (Rachah and Torres (2015); Rachah (2018); Khan and Atangana (2019a)), Lassa fever (Ibrahim and Denes (2021); Akinpelu and Akinwande (2018)), Measles (Ayoade et al. (2019a); Ameen (2018)), Diabetes (Kouidere et al. (2020); Yadav and Maya (2020)), Hepatitis (Khan et al. (2019b); Ullah (2018b); Khan et al. (2013)); Zika (Bonyah et al. (2017); Khan et al. (2019c)); COVID-19 (Okuonghae and Omame (2020); Yang and Wang (2020); Khan and Atangana (2020a); Khan et al. (2020b); Ullah et al. (2020b)), Listeriosis (Bassey (2020); Osman et al. (2018)) to mention but a few.

The use of epidemic modeling is not limited to the study of plants, animals, and human diseases. It has also been used to study various social diseases such as poverty and prostitution (Akinpelu and Ojo (2017)), corruption (Nathan and Jackob (2019); Lemecha and Feyissa (2018)), rumor (Liu et al. (2019)), terrorism (Gambo and Ibrahim (2020)), kidnapping (Okrinya (2018), Iqbal et al. (2017)), banditry (Ugwuishiwu et al. (2019)), sexual harassment (Mamaru et al. (2015)), examination malpractice (Ayoade and Farayola, (2020a)), and unemployment (Ayoade et al. (2020b); Ayoade et al. (2020c); Pathan and Bhathawala (2017)) to mention but just a few.

Inequality and constraints to inclusive growth are also diseases. They are social diseases that are endemic in SSA despite a vast resources and ethnic diversity in the region. Narrowing the inequality gap and achieving inclusive growth is one of the major concerns of governments at all levels in the region. This study is therefore motivated to use an epidemic modeling approach to guide the policy makers in SSA countries towards the achievement of inclusive growth by bridging the inequality gap in the region. While abundant natural resources and ethnic diversity in SSA have been a subject of intensive study, studies on inequality and inclusive growth in SSA despite the region's vast resources and diverse ethnic groups are relatively new in the literature. Besides, until now, no study has explored an epidemic modeling approach to analyze the dynamics of abundant natural resources, ethnic diversity, and inclusive growth in SSA.

2. Materials and Methods

The model of abundant natural resources, ethnic diversity and inclusive growth in SSA is compartmentalized into five subclasses SPRTG. The expectant class is denoted by S(t). S(t) represents the proportion of the population who has not been involved in the sharing of the benefits from economic growth at time t but which has the tendency of being involved. The inclusive class is represented by P(t). P(t) is the proportion of individuals who are already enjoying the full benefits of growth at time t. Government class is denoted by G(t). G(t) is the proportion of individuals from diverse ethnic groups that manage the resources and affairs of the state. The transformed class is represented by T(t). T(t) is the proportion of individuals who have not been previously included in the growth process but are now enjoying equitable distribution of economic growth at time t due to the programs that are put in place by the government to narrow the inequality gap. If the program that is put in place to get more and more people included in the growth process is sustained and continuous then individuals in the compartment T(t) move to the compartment P(t) at rate ω. Finally, R(t) is the compartment for all the resources at time t and the resources are assumed abundant. Economic growth is also assumed. All the compartments of the model are functions of time meaning that components in each compartment can fluctuate with time. Different ethnic groups are recruited to manage the affairs of the state at a rate ϕ. While the recruitment rate into the inclusive class P(t) is $\alpha\pi$, the recruitment rate into the expectant class S(t) is $(1-\alpha)\pi$. The natural mortality occurs for each human compartment at rate μ while the leakages occur for natural resources at rate δ. β is the rate of inclusion. γ and ϵ are the rates of loss in inclusion for those in the inclusive class P(t) and those in the transformed class T(t) respectively. η is the rate at which the resources are being discovered while τ is the rate at which resources are being utilized by the government. It is assumed that the proportion τR from R(t) does not increase the population of those in governance but their ability to promote the welfare of the populace through increase in the gross national income. The flow of the dynamics between the variables is displayed in Figure 1.

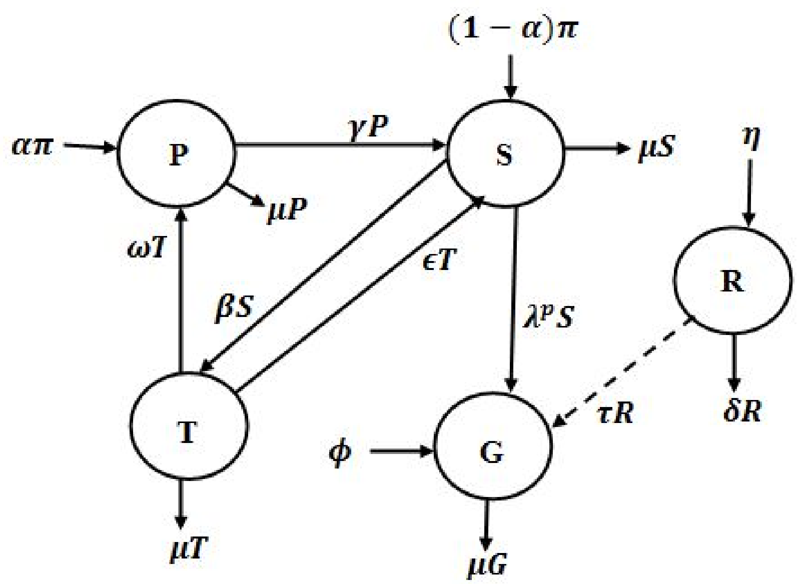

Figure 1. Flow diagram of the model

In Figure 1, λ^p is the force of inclusion and is defined as $\sigma\theta G$ where σ is the per capita probability of involving as many people as possible both in growth process and in sharing of the benefits from such economic growth and θ is the contact rate between individuals who has not been part of the growth (S) and the goods provided by the state to bridge inequality gap. If ρ is the per capita probability at which the ethnic diversity hinders equitable distribution of national cake among the populace then the effective rate of force of inclusion λ^p is $\sigma\theta(1-\rho)G$ (i.e. $\lambda^p = \sigma\theta(1-\rho)G$. Since the model is not a pure epidemic model of human disease, the link between S and G does not imply that individuals are being recruited from S to G but it indicates the extent to which individuals in S are feeling the impact of G quantified in terms of λ^p. The above assumptions, formulations and flow diagram are translated into the following first-order nonlinear ordinary differential equations:

$$\frac{dP}{dt} = \alpha\pi - \gamma p - \mu p + \omega T, \tag{1}$$

$$\frac{dS}{dt} = (1-\alpha)\pi + \epsilon T + \gamma p - \beta S - \sigma\theta(1-\rho)GS - \mu S, \tag{2}$$

$$\frac{dT}{dt} = \beta S - \epsilon T - \omega T - \mu T, \tag{3}$$

$$\frac{dG}{dt} = \phi + \sigma\theta(1-\rho)GS + \tau R - \mu G, \tag{4}$$

$$\frac{dR}{dt} = \eta - \tau R - \delta R. \tag{5}$$

The system of equations (1) - (5) governs the relationship between abundant natural resources, ethnic diversity and inclusive growth in SSA and can be used to guide the policy makers towards achievement of inclusive growth by bridging the inequality gap in the region. The parameters of the model are redefined in Table 1 for ease of reference.

Table 1. Parameters of the model and their interpretations

Parameters	Interpretations
α	Probability of being born into inclusive class
π	Recruitment rate into the general population
γ	Movement rate from the inclusive class to the expectant class
μ	Natural death rate for individuals in each compartment
ω	Sustainability rate of equitable distribution of gross national income
ϵ	Movement rate from the transformed class to the expectant class
β	Movement rate from the expectant class to the transformed class
σ	Per capita probability of equitable distribution of gross national income
θ	Contact rate between the common man and the public goods
ρ	Rate at which ethnic diversity is militating against equitable distribution of gross national income
ϕ	Recruitment rate into the governance
τ	Rate at which the state is tapping the natural resources
η	Discovery rate of natural resources
δ	Rate of loss in the natural resources

The usability of the system of equations (1) - (5) to conduct the study is a function of a number of factors. The solutions of the model must exist and at the same time, be unique. The solutions must also be positive and bounded. If the solutions of the model do not exist or if they exist but are not unique then the model is not valid to conduct the study. Besides, since the model monitors economic phenomena, the solutions to the model are expected to be bounded in the positive region; otherwise, the model is invalid. The existence and uniqueness as well as positivity and boundedness of solutions shall be examined for the model using some ample mathematics theorems.

2.1. Existence and uniqueness of solutions

Suppose the system of equations (1) - (5) is re-rewritten as

$$f_1 = \alpha\pi - \gamma p - \mu p + \omega T, \tag{6}$$
$$f_2 = (1-\alpha)\pi + \epsilon T + \gamma p - \beta S - \sigma\theta(1-\rho)GS - \mu S, \tag{7}$$
$$f_3 = \beta S - \epsilon T - \omega T - \mu T, \tag{8}$$
$$f_4 = \phi + \sigma\theta(1-\rho)GS + \tau R - \mu G, \tag{9}$$
$$f_5 = \eta - \tau R - \delta R. \tag{10}$$

Theorem 2.1.

The solutions for the system (6) - (10) exist and are unique if $\left|\dfrac{\partial f_i}{\partial x_j}\right| < \infty$.

Proof:

(See Ayoade et al. (2019b).) The condition $\left|\dfrac{\partial f_i}{\partial x_j}\right| < \infty$ shall be verified for each equation in the system (6) - (10).

From Equation (6),

$$\left|\frac{\partial f_1}{\partial P}\right| = |-(\mu+\gamma)| = (\mu+\gamma) < \infty; \quad \left|\frac{\partial f_1}{\partial S}\right| = 0 < \infty; \left|\frac{\partial f_1}{\partial T}\right| = |\omega| = \omega < \infty;$$
$$\left|\frac{\partial f_1}{\partial G}\right| = 0 < \infty; \quad \left|\frac{\partial f_1}{\partial R}\right| = 0 < \infty.$$

Also, consider Equation (7),

$$\left|\frac{\partial f_2}{\partial P}\right| = \gamma < \infty; \left|\frac{\partial f_2}{\partial S}\right| = |-(\mu+\beta) - \sigma\theta(1-\rho)G| = (\mu+\beta) + \sigma\theta(1-\rho)G < \infty;$$
$$\left|\frac{\partial f_2}{\partial T}\right| = \epsilon < \infty; \left|\frac{\partial f_2}{\partial G}\right| = |-\sigma\theta(1-\rho)S| = \sigma\theta(1-\rho)S < \infty; \quad \left|\frac{\partial f_2}{\partial R}\right| = 0 < \infty.$$

From Equation (8),

$$\left|\frac{\partial f_3}{\partial P}\right| = 0 < \infty; \left|\frac{\partial f_3}{\partial S}\right| = \beta < \infty; \quad \left|\frac{\partial f_3}{\partial T}\right| \quad = |-(\epsilon+\omega+\mu)| = (\epsilon+\omega+\mu) < \infty;$$
$$\left|\frac{\partial f_3}{\partial G}\right| = 0 < \infty; \quad \left|\frac{\partial f_3}{\partial R}\right| = 0 < \infty.$$

For Equation (9),

$$\left|\frac{\partial f_4}{\partial P}\right| = 0 < \infty; \left|\frac{\partial f_4}{\partial S}\right| = |\sigma\theta(1-\rho)G| = \sigma\theta(1-\rho)G < \infty; \quad \left|\frac{\partial f_4}{\partial T}\right| = 0 < \infty;$$
$$\left|\frac{\partial f_4}{\partial G}\right| = |-\mu + \sigma\theta(1-\rho)S| = \mu + \sigma\theta(1-\rho)S < \infty; \quad \left|\frac{\partial f_4}{\partial R}\right| = \tau < \infty.$$

And lastly, from Equation (10),

$$\left|\frac{\partial f_5}{\partial P}\right| = 0 < \infty; \left|\frac{\partial f_5}{\partial S}\right| = 0 < \infty; \quad \left|\frac{\partial f_5}{\partial T}\right| = 0 < \infty;$$
$$\left|\frac{\partial f_5}{\partial G}\right| = 0 < \infty; \quad \left|\frac{\partial f_5}{\partial R}\right| = |-(\mu+\delta)| = (\mu+\delta) < \infty.$$

Since it is established that each equation in the system (6) - (10) satisfies the condition $\left|\frac{\partial f_i}{\partial x_j}\right| < \infty$, then, there exists a solution for the model and the solution is unique. ∎

2.2. Positivity of solutions

For an economic model, negative solutions are inadmissible. We shall verify whether the solutions for the system of equations (1) - (5) are bounded in the positive region.

Theorem 2.2.

Given the initial values for the state variables as $P_\circ$, $S_\circ$, $T_\circ$, $G_\circ$ and $R_\circ$ then the solutions $P(t)$, $S(t)$, $T(t)$, $G(t)$ and $R(t)$ of the model are positive for all $t \geq 0$.

Proof:

Consider Equation (1),

$$\frac{dP}{dt} = \alpha\pi - (\mu+\gamma)P + \omega T,$$

$$\Rightarrow \frac{dP}{dt} \geq -(\mu+\gamma)P, \tag{11}$$
$$\Rightarrow \frac{dP}{P} \geq -(\mu+\gamma)dt, \tag{12}$$
$$\Rightarrow \int \frac{dP}{dt} \geq -(\mu+\gamma)\int dt, \tag{13}$$
$$\Rightarrow \ln P \geq -(\mu+\gamma) + c, \tag{14}$$
$$\therefore P(t) \geq ke^{-(\mu+\gamma)t}, \tag{15}$$

where

$$k = e^c.$$

At start, $t = 0$ and the number of people who have been enjoying the full benefit of growth $P(t)$ takes the initial value $P_\circ$. Substituting $t = 0$ and $P(t) = P_\circ$ into Equation (15),

$$k = P_\circ. \tag{16}$$

Putting Equation (16) into Inequality (15), the inequality (15) becomes

$$P(t) \geq P_\circ e^{-(\mu+\gamma)t}. \tag{17}$$

Following the same approach for the remaining equations in the model, the following results are derived as

$$S(t) \geq S_\circ e^{-(\mu+\beta)t}, \tag{18}$$

$$T(t) \geq T_\circ e^{-(\mu+\epsilon+\omega)t}, \tag{19}$$

$$G(t) \geq G_\circ e^{-\mu t}, \tag{20}$$

$$R(t) \geq R_\circ e^{-(\mu+\tau)t}. \tag{21}$$

Since $e^q > 0$ for every value of q that is real and $P_\circ$, $S_\circ$, $T_\circ$, $G_\circ$ and $R_\circ$ are generally nonnegative (initial values for physical phenomena), then, it is sufficient to conclude that the solutions P(t), S(t), T(t), G(t) and R(t) of the model are positive for all $t \geq 0$. ■

2.3. Invariant region or boundedness of solutions

Theorem 2.3.

Given the positive initial conditions for the variables P, S, T, G, and R as $P_\circ$, $S_\circ$, $T_\circ$, $G_\circ$ and $R_\circ$, respectively, then the solution for the system (1) - (5) is feasible in the region

$$\Omega = \left\{(P, S, T, G, R) \in \Re^5_+; P, S, T, G \geq 0 : N(t) \leq \frac{\pi+\phi}{\mu}; R \geq 0 : R(t) \leq \frac{\eta}{\tau+\delta}\right\}.$$

Proof:

The rate of change for the sum of human population in Equations (1) - (4) is

$$\begin{aligned}
&\frac{dN}{dt} = \pi + \phi - (P+S+T+G)\mu + \tau R, \\
&\Rightarrow \frac{dN}{dt} \leq \pi + \phi - \mu N, \\
&\Rightarrow \frac{dN}{\pi+\phi-\mu N} \leq dt, \\
&\Rightarrow \pi + \phi - \mu N(t) \geq k_1 e^{-\mu t}.
\end{aligned}$$

As $t = 0$, $N(t) = N_\circ$ and $k_1 = \pi + \phi - \mu N_\circ$.

Hence,

$$\begin{aligned}
&\pi + \phi - \mu N(t) \geq (\pi + \phi - \mu N_\circ)e^{-\mu t}, \\
&\Rightarrow N(t) \leq \frac{\pi+\phi}{\mu}\left(\frac{\pi+\phi-\mu N_\circ}{\mu}\right)e^{-\mu t}.
\end{aligned}$$

As $t \to \infty$, $0 \leq N(t) \leq \dfrac{\pi+\phi}{\mu}$ which indicates that the solution set for human population remains bounded within $\dfrac{\pi+\phi}{\mu}$.

Also, consider the total number of natural resources at time t in Equation (5), i.e.,

$$\frac{dR}{dt} = \eta - (\tau + \delta)R.$$

Assuming

$$\frac{dR}{dt} \leq \eta - (\tau + \delta)R \quad \Rightarrow \quad R(t) \leq \frac{\eta}{\tau + \delta}\left(1 - k_2 e^{-(\tau+\delta)t}\right).$$

As $t \to \infty$ then $R(t) \leq \frac{\eta}{\tau + \delta}$ which shows that the solution for the natural resources deposit remains bounded within the region $\frac{\eta}{\tau + \delta}$. Therefore, as stated in Theorem 2.3, the feasible solutions for the model exist within the region

$$\Omega = \left\{(P, S, T, G, R) \in \Re^5_+; P, S, T, G \geq 0 : N(t) \leq \frac{\pi + \phi}{\mu}; R \geq 0 : R(t) \leq \frac{\eta}{\tau + \delta}\right\}. \quad \blacksquare$$

Since the solutions for the system of Equations (1) - (5) exist and unique, positive and bounded then the model is suitable to conduct the study.

3. Model Analysis

The model shall be subjected to both qualitative and quantitative analysis to examine the relationship between abundant natural resources, ethnic diversity and inclusive growth in sub-Saharan Africa.

3.1. Equilibrium

Given the first-order differential equations

$$\frac{dx}{dt} = f(x, y), \tag{22}$$

$$\frac{dy}{dt} = g(x, y). \tag{23}$$

The equilibrium points can be obtained from

$$f(x, y) = 0, \tag{24}$$

$$g(x, y) = 0. \tag{25}$$

Two scenarios can be taken into consideration. We can imagine a situation when the policies implemented by the government to bring about inclusive growth achieve total success so that every individual in compartment S is moved to compartment T and later to compartment P. In that sense, compartments S and T become empty with time. At the same time, we can also imagine a situation where the policies implemented by the government to enhance inclusive growth fail to achieve the purpose for which they are being implemented so that no compartment in the model is empty.

The former scenario corresponds to disease free equilibrium (DFE) while the latter corresponds to endemic equilibrium (EE) in disease modeling. However, since the present model is a disease model of the society and not of man, we shall denote the former situation policy success equilibrium (PSE) and the latter, policy failure equilibrium (PFE). The two possible situations shall be analyzed one after the other following the approach of epidemic modeling.

3.1.1. *Existence of policy success equilibrium, E°*

If government programs are implemented in such a way that a total inclusiveness is achieved then compartments S and T will be empty with time (i.e. $S = 0$ and $T = 0$). Solving the system of equations (1) - (5) at equilibrium based on the condition $S = 0$ and $T = 0$ then the policy success (PSE) for the model is derived as

$$E^\circ = (P^\circ, S^\circ, T^\circ, G^\circ, R^\circ) = \left(\frac{\alpha\pi}{(\mu+\gamma)}, 0, 0, \frac{\phi(\tau+\delta)+\tau\eta}{\mu(\tau+\delta)}, \frac{\eta}{(\tau+\delta)}\right). \tag{26}$$

3.1.2. *Existence of policy failure equilibrium, E^**

When the policies to drive inclusiveness are poorly implemented, the society is taken over by inequality and each compartment of the model exists. There exists resources but while few individuals are enjoying the benefits of growth from the abundant natural resources and ethnic diversity, the majority is wallowing in abject poverty despite growth in the economy. This has been the case for many countries in sub-Saharan Africa. To determine the population of individuals in each compartment and the available resources when the growth is not inclusive, the right-hand side of the system of equations (1) - (5) is equated to zero and the reduced system is solved simultaneously but due to nonlinear nature of the system, the system is solved in terms T^* as in Nsuami and Witbooi (2018). Thus,

$$\alpha\pi - \gamma P^* - \mu P^* + \omega T^* = 0, \tag{27}$$

$$(1-\alpha)\pi + \epsilon T^* + \gamma P^* - \beta S^* - \sigma\theta(1-\rho)G^*S^* - \mu S^* = 0, \tag{28}$$

$$\beta S^* - \epsilon T^* - \omega T^* - \mu T^* = 0, \tag{29}$$

$$\phi + \sigma\theta(1-\rho)G^*S^* + \tau R^* - \mu G^* = 0, \tag{30}$$

$$\eta - \tau R^* - \delta R^* = 0. \tag{31}$$

From Equation (31),

$$R^* = \frac{\eta}{(\tau+\delta)}. \tag{32}$$

From Equation (27),

$$P^* = \frac{\alpha\pi + \omega T^*}{(\mu+\gamma)}. \tag{33}$$

From Equation (29),

$$S^* = \frac{(\mu+\epsilon+\omega)T^*}{\beta}. \tag{34}$$

Putting Equations (33) and (34) into Equation (28),

$$G^* = \frac{\pi\beta(1-\alpha)(\mu+\gamma)+\beta(\mu+\gamma)\epsilon T^*+\alpha\pi\beta\gamma+\beta\gamma\omega T^*-\beta(\mu+\gamma)(\mu+\epsilon+\omega)}{\sigma\theta(\mu+\gamma)(1-\rho)(\mu+\epsilon+\omega)T^*+\mu(\mu+\gamma)(\mu+\epsilon+\omega)}. \tag{35}$$

From Equation (29), the value of T^* can be determined in terms of $S^*$and it is greater than zero.

3.2. The policy success ratio P_G

In epidemic modeling, there exists an important non-dimensional quantity known as the basic reproductive ratio. The quantity measures the number of infections an infected individual is likely to generate in a total susceptible population (Diekmann et al. (1990)). The quantity is derived from the infectious compartments in the model around the disease-free equilibrium. Inequality and constraints to inclusive growth when there are abundant natural resources and diverse ethnic groups are social diseases in a society. Therefore, an epidemic modeling approach shall be employed to derive a policy success ratio P_G, a threshold for successful implementation of policies to drive inclusiveness in the society. If $P_G > 1$, government policies to drive inclusiveness are successful so that abundant natural resources and ethnic diversity turn around distributive injustice and inequality in the society. However, if $P_G < 1$, policies to facilitate inclusive growth are poorly implemented and fail to achieve their purpose. The compartments S(t) and G(t) are employed in deriving the quantity P_G just as in epidemic modeling where infectious compartments are employed. Emphasis is placed on these compartments because policies to drive inclusiveness are usually designed and implemented by the government. Besides, individuals in the compartments S(t) are those who have not, in any way, been part of the growth process. Following the Next Generational Matrix approach which was originally formulated by Diekmann et al. (1990) but developed by Driessche and Wathmough (2002), $n \times n$ matrices are constructed from the model by considering only G(t) and S(t) classes. The transmission and transition terms are partitioned into matrices $\mathbf{F_i}$ and $\mathbf{V_i}$ from the compartments G(t) and S(t) as follows:

$$\mathbf{F_i} = \begin{pmatrix} \sigma\theta(1-\rho)GS \\ 0 \end{pmatrix}, \tag{36}$$

$$\mathbf{V_i} = \begin{pmatrix} \mu G - \tau R \\ (\mu+\beta)S - \epsilon T - \gamma P \end{pmatrix}. \tag{37}$$

$\sigma\theta(1-\rho)GS$ is considered in $\mathbf{F_i}$ because it is the term that measures the rate of success of government policies in bridging the inequality gap in the society which has a direct link with individuals who have not been part of the growth process. It has been assumed and stated below Figure 1 that the link between S and G does not imply that individuals are being recruited from S to G but it indicates the extent to which individuals in S are feeling the impact of the policies implemented by the government to include them in the growth process quantified in terms of $\sigma\theta(1-\rho)GS$.

Also, since the model is not a pure epidemic model of human disease, $\mathbf{F_i}$ is differentiated partially with respect to S and then partially with respect to G while $\mathbf{V_i}$ is differentiated partially first with respect to G and then with respect to S to determine matrices $\mathbf{F}$ and $\mathbf{V}$ thus,

$$\mathbf{F} = \begin{pmatrix} \sigma\theta(1-\rho)G & \sigma\theta(1-\rho)S \\ 0 & 0 \end{pmatrix}, \tag{38}$$

$$\mathbf{V} = \begin{pmatrix} \mu & 0 \\ 0 & (\mu+\beta) \end{pmatrix}. \tag{39}$$

The inverse of $\mathbf{V}$ is obtained as

$$\mathbf{V^{-1}} = \begin{pmatrix} \frac{1}{\mu} & 0 \\ 0 & \frac{1}{(\mu+\beta)} \end{pmatrix}. \tag{40}$$

The product of $\mathbf{F}$ and $\mathbf{V^{-1}}$ is

$$\mathbf{FV^{-1}} = \begin{pmatrix} \frac{\sigma\theta(1-\rho)G}{\mu} & \frac{\sigma\theta(1-\rho)S}{(\mu+\beta)} \\ 0 & 0 \end{pmatrix}. \tag{41}$$

As in epidemic modeling where $R_\circ$ is evaluated at the disease-free equilibrium, P_G is evaluated here at the policy success equilibrium point given in Equation (26). If the value of G(t) and S(t) in Equation (26) is used to evaluate Equation (41) then

$$\mathbf{FV^{-1}} = \begin{pmatrix} \frac{\sigma\theta(1-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu^2(\tau+\delta)} & 0 \\ 0 & 0 \end{pmatrix}. \tag{42}$$

The policy success ratio is therefore the spectral radius (largest eigenvalues) of Equation (42), i.e.,

$$P_G = \frac{\sigma\theta(1-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu^2(\tau+\delta)}. \tag{43}$$

Equation (43) sets a threshold for successful implementation of policies to drive inclusiveness in terms of abundant natural resources and ethnic diversity. If $P_G > 1$, it means that the program that is put in place by the government to get more people included in the growth process is sustained and continuous thereby making individuals in S(t) to be moving to T(t) at rate β and the individuals in the transformed class T(t) to be moving into the inclusive class P(t) at rate ω (see Figure 1). However, if $P_G < 1$, the program that is put in place by the government to get more people involved in the growth process is not sustained thereby making individuals who have just left expectant class S(t) to the transformed class T(t) to be moving back to their former compartment at rate ϵ (see Figure 1). A quick check of Equation (43) shows that an increase in both σ and θ accompanied with a low value of ρ can activate inclusiveness as $P_G > 1$ is more feasible under the condition.

3.3. Local stability analysis of policy success equilibrium E°

The local stability analysis of the policy success equilibrium shall be investigated following the linearization approach (Nathan and Jackob (2019); Osman et al. (2018)). The Jacobian matrix is computed by finding the partial derivative of the system (1) - (5) with respect to each state variable P, S, T, G, R. Hence, the Jacobian matrix of the system is given by

$$\mathbf{J}(\mathbf{E}^{\circ}) = \begin{pmatrix} -(\mu+\gamma) & 0 & \omega & 0 & 0 \\ \gamma & -(\beta+\sigma\theta(1-\rho)G+\mu) & \epsilon & -\sigma\theta(1-\rho)S & 0 \\ 0 & \beta & -(\mu+\omega+\epsilon) & 0 & 0 \\ 0 & \sigma\theta(1-\rho)G & 0 & (\sigma\theta(1-\rho)S-\mu) & \tau \\ 0 & 0 & 0 & 0 & -(\tau+\delta) \end{pmatrix}. \tag{44}$$

Since the stability analysis is to be carried out at the policy success equilibrium then Equation (26) is used to evaluate Equation (44) and Equation (44) reduces to

$$\mathbf{J}(\mathbf{E}^{\circ}) = \begin{pmatrix} -(\mu+\gamma) & 0 & \omega & 0 & 0 \\ \gamma & -\left(\beta+\dfrac{\sigma\theta(1-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu(\tau+\delta)}+\mu\right) & \epsilon & 0 & 0 \\ 0 & \beta & -(\mu+\omega+\epsilon) & 0 & 0 \\ 0 & \left(\dfrac{\sigma\theta(1-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu(\tau+\delta)}\right) & 0 & -\mu & \tau \\ 0 & 0 & 0 & 0 & -(\tau+\delta) \end{pmatrix}. \tag{45}$$

It is known in disease modeling that an outbreak will not take off or will die out if $R_{\circ} < 1$ which is a DFE stable condition and a desirable situation. However, since the present model monitors a societal disease, government efforts to drive inclusiveness will only be effective if $P_G > 1$ which is a PSE stable condition and a desirable situation in the present analysis. Government efforts to achieve a purpose using an epidemic modeling approach are successful if the threshold quantity that is similar to $R_{\circ}$ is greater than one (Ayoade and Farayola (2021)). Disease free equilibrium (DFE) in human disease models means policy success equilibrium in the present model. In the same vein, $R_{\circ} < 1$ in human disease models means $P_G > 1$ in the present analysis. Therefore, what $R_{\circ} < 1$ establishes in human disease models using linearization approach is what $P_G > 1$ establishes in the present analysis.

Theorem 3.1.

The policy success equilibrium of the system (1) - (5) is locally asymptotically if $P_G > 1$ but unstable if $P_G < 1$.

Proof:

The policy success equilibrium of the system (1) - (5) is locally asymptotically if all the eigenvalues of $\mathbf{J}(\mathbf{E}^{\circ})$ in Equation (45) are negative or if the trace of $\mathbf{J}(\mathbf{E}^{\circ})$ is negative whenever the determinant of $\mathbf{J}(\mathbf{E}^{\circ})$ is positive.

The first two eigenvalues in Equation (45) are already negative, i.e., $\lambda_1 = -(\tau+\delta)$ and $\lambda_2 = -\mu$.

The remaining eigenvalues can be obtained from submatrix $\mathbf{B}$ given as

$$\mathbf{B} = \begin{pmatrix} -(\mu+\gamma) & 0 & \omega \\ \gamma & -\left(\beta + \dfrac{\sigma\theta(1-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu(\tau+\delta)} + \mu\right) & \epsilon \\ 0 & \beta & -(\mu+\omega+\epsilon) \end{pmatrix}. \tag{46}$$

Elementary row operation reduces Equation (46) to

$$\mathbf{B} = \begin{pmatrix} -(\mu+\gamma) & 0 & \omega \\ 0 & -\dfrac{(\mu+\gamma)}{\gamma}\left(\beta + \dfrac{\sigma\theta(1-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu(\tau+\delta)} + \mu\right) & \epsilon\dfrac{(\mu+\gamma)}{\gamma} + \omega \\ 0 & \beta & -(\mu+\omega+\epsilon) \end{pmatrix}. \tag{47}$$

Equation (47) has one of its eigenvalues to be $\lambda_3 = -(\mu+\gamma)$ and the remaining eigenvalues can be determined from submatrix $\mathbf{C}$ given as

$$\mathbf{C} = \begin{pmatrix} -\dfrac{(\mu+\gamma)}{\gamma}\left(\beta + \dfrac{\sigma\theta(1-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu(\tau+\delta)} + \mu\right) & \epsilon\dfrac{(\mu+\gamma)}{\gamma} + \omega \\ \beta & -(\mu+\omega+\epsilon) \end{pmatrix}. \tag{48}$$

The trace and determinant of Equation (48) are, respectively,

$$\mathcal{T}r(\mathbf{C}) = -\left[\frac{(\mu+\gamma)}{\gamma}\left(\beta + \frac{\sigma\theta(\mathbf{1}-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu(\tau+\delta)} + \mu\right) + (\mu+\omega+\epsilon)\right], \tag{49}$$

$$\mathcal{D}et(\mathbf{C}) = \frac{(\mu+\gamma)(\mu+\omega+\epsilon)}{\gamma}\left(\beta + \frac{\sigma\theta(\mathbf{1}-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu(\tau+\delta)} + \mu\right) - \beta\left[\epsilon\frac{(\mu+\gamma)}{\gamma} + \omega\right]. \tag{50}$$

Generally the two eigenvalues of Equation (48) are negative if $\mathcal{T}r(\mathbf{C}) < \mathbf{0}$ and $\mathcal{D}et(\mathbf{C}) > \mathbf{0}$. $\mathcal{T}r(\mathbf{C}) < \mathbf{0}$ is already true while $\mathcal{D}et(\mathbf{C}) > \mathbf{0}$ if

$$\frac{(\mu+\gamma)(\mu+\omega+\epsilon)}{\gamma}\left(\beta + \frac{\sigma\theta(1-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu(\tau+\delta)} + \mu\right) > \beta\left[\epsilon\frac{(\mu+\gamma)}{\gamma} + \omega\right].$$

Hence, the policy success equilibrium, E° is locally asymptotically stable if

$$\frac{(\mu+\gamma)(\mu+\omega+\epsilon)}{\gamma}\left(\beta + \frac{\sigma\theta(1-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu(\tau+\delta)} + \mu\right) > \beta\left[\epsilon\frac{(\mu+\gamma)}{\gamma} + \omega\right]. \tag{51}$$

The relation between the policy success P_G and the expression (51) is that $P_G > 1$ only if expression (51) is true.

This ends the proof. ∎

The implication of the result for the proof of Theorem 3.1 is that if inequality (51) is true then the policies implemented by the government to narrow the inequality gap is successful hence abundant natural resources and ethnic diversity translate into inclusive growth. On the other hand, if inequality (51) is not true then the policies implemented by the government to bridge the inequality gap defeated the purpose for which they were being implemented and growth was not inclusive despite abundant natural resources and ethnic diversity.

3.4. Global stability of policy success equilibrium E°

The global stability analysis of the policy success equilibrium of the model shall be examined following comparison theorem (Lakshmikantham et al. (1989)).

Theorem 3.2.

The policy success equilibrium of the model is globally asymptotically stable if $P_G > 1$ such that the eigenvalues of $(\mathbf{F} - \mathbf{V})$ at the policy success equilibrium are all negative.

Proof:

The matrices $\mathbf{F}$ and $\mathbf{V}$ are defined in Equations (38) and (39), respectively. Therefore, $\mathbf{F} - \mathbf{V}$ at the policy success equilibrium is evaluated thus,

$$\mathbf{F} - \mathbf{V} = \begin{pmatrix} \dfrac{\sigma\theta(1-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu(\tau+\delta)} & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \mu & 0 \\ 0 & (\mu+\beta) \end{pmatrix}, \tag{52}$$

$$\mathbf{F} - \mathbf{V} = \begin{pmatrix} \dfrac{\sigma\theta(1-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu(\tau+\delta)} - \mu & 0 \\ 0 & -(\mu+\beta) \end{pmatrix}. \tag{53}$$

Therefore, $|(\mathbf{F} - \mathbf{V}) - \lambda| = \mathbf{0}$

$$\Rightarrow \begin{vmatrix} \dfrac{\sigma\theta(1-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu(\tau+\delta)} - \mu - \lambda & 0 \\ 0 & -(\mu+\beta+\lambda) \end{vmatrix} = 0. \tag{54}$$

Therefore,

$$(\mu+\beta+\lambda)\left(\frac{\sigma\theta(1-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu(\tau+\delta)} - \mu - \lambda\right) = 0. \tag{55}$$

$$\begin{aligned} \Rightarrow \lambda_1 &= -(\mu+\beta), \\ \lambda_2 &= \left(\frac{\sigma\theta(1-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu(\tau+\delta)} - \mu\right) \end{aligned} \tag{56}$$

From

$$\lambda_2 = \left(\frac{\sigma\theta(1-\rho)[\phi(\tau+\delta)+\tau\eta]}{\mu(\tau+\delta)} - \mu\right),$$

$$\lambda_2 = \mu(P_G - 1). \tag{57}$$

As expounded below Equation (45), $R_{\circ} < 1$ is technically $P_G > 1$. $R_{\circ} < 1$ in disease modeling means $P_G > 1$ in this analysis so what $R_{\circ} < 1$ guarantees in disease modeling is what $P_G > 1$ guarantees here. Based on this argument, in Equation (57), λ_2 can only be regarded negative if $P_G > 1$ which is what $R_{\circ} < 1$ guarantees in human disease models.

This ends the proof. ■

The implication of Theorem 3.2 is that the stability of policy success equilibrium is a function of the value of P_G whether in the short-run (local stability) or in the long-run (global stability). That is growth would be inclusive in the society only if $P_G > 1$.

3.5. Local stability of policy failure equilibrium, E^*

The stability analyzes (both local and global) of the policy failure equilibrium have to be examined having investigated the stability of the policy success equilibrium. As in the stability of the policy success equilibrium, the linearization procedure shall be employed to analyze the local stability of the policy failure equilibrium. Consider the Jacobian matrix of the system in Equation (44) bearing in mind that at the policy failure equilibrium, inequality exists in the society and no compartment is empty (no variable is zero). Therefore, the Jacobian matrix at the policy failure equilibrium is computed thus,

$$\mathbf{J}(\mathbf{E}^*) = \begin{pmatrix} -(\mu+\gamma) & 0 & \omega & 0 & 0 \\ \gamma & -(\beta+\sigma\theta(1-\rho)G^*+\mu) & \epsilon & -\sigma\theta(1-\rho)S^* & 0 \\ 0 & \beta & -(\mu+\omega+\epsilon) & 0 & 0 \\ 0 & \sigma\theta(1-\rho)G^* & 0 & (\sigma\theta(1-\rho)S^*-\mu) & \tau \\ 0 & 0 & 0 & 0 & -(\tau+\delta) \end{pmatrix}, \tag{58}$$

where G^* and S^* represent the values of G and S at the policy failure equilibrium.

Theorem 3.3.

The policy failure equilibrium is locally asymptotically stable and $P_G < 1$ if all the eigenvalues of the Jacobian matrix in Equation (58) are less than zero.

Proof:

One of the eigenvalues in Equation (58) is $\lambda_1 = -(\tau+\delta)$. The remaining eigenvalues can be obtained from matrix $\mathbf{J_1}(\mathbf{E}^*)$ given in Equation (59) which has been subjected to elementary row operation from Equation (58), i.e.,

$$\mathbf{J_1}(\mathbf{E}^*) = \begin{pmatrix} -(\mu+\gamma) & 0 & \omega & 0 \\ 0 & -\dfrac{(\mu+\gamma)}{\gamma}(\beta+\sigma\theta(1-\rho)G^*+\mu) & \dfrac{\epsilon(\mu+\gamma)}{\gamma}+\omega & -\dfrac{(\mu+\gamma)\sigma\theta(1-\rho)S^*}{\gamma} \\ 0 & \beta & -(\mu+\omega+\epsilon) & 0 \\ 0 & \sigma\theta(1-\rho)G^* & 0 & (\sigma\theta(1-\rho)S^*-\mu) \end{pmatrix}. \tag{59}$$

One of the eigenvalues in Equation (59) is $\lambda_2 = -(\mu+\gamma)$. The remaining eigenvalues can be obtained from matrix $\mathbf{J_2}(\mathbf{E}^*)$ given in Equation (60) which has been subjected to elementary row operation from Equation (59), i.e.,

$$\mathbf{J_2}(\mathbf{E}^*) = \begin{pmatrix} -m_1+\sigma\theta(1-\rho)G^* & m_2 & 0 \\ \beta & -m_3 & 0 \\ \sigma\theta(1-\rho)G^* & 0 & (\sigma\theta(1-\rho)S^*-\mu) \end{pmatrix}, \tag{60}$$

where

$$
\begin{aligned}
m_1 &= \frac{(\mu+\gamma)(\sigma\theta(1-\rho)S^*-\mu)(\mu+\beta+\sigma\theta(1-\rho)G^*)}{(\mu+\gamma)\sigma\theta(1-\rho)S^*},\\
m_2 &= \frac{\gamma(\sigma\theta(1-\rho)S^*-\mu)}{(\mu+\gamma)\sigma\theta(1-\rho)S^*}\left[\frac{\epsilon(\mu+\gamma)}{\gamma}+\omega\right],\\
m_3 &= (\mu+\omega+\epsilon).
\end{aligned}
$$

One of the eigenvalues in Equation (60) is $\lambda_3 = (\sigma\theta(1-\rho)S^* - \mu)$. The remaining eigenvalues can be obtained from matrix $\mathbf{J_3}(\mathbf{E}^*)$ given in Equation (61) i.e.,

$$
\mathbf{J_3}(\mathbf{E}^*) = \begin{pmatrix} -m_1+\sigma\theta(1-\rho)G^* & m_2 \\ \beta & -m_3 \end{pmatrix}. \tag{61}
$$

The trace and determinant of $\mathbf{J_3}(\mathbf{E}^*)$ are respectively

$$
\mathcal{T}r(\mathbf{J_3}(\mathbf{E}^*)) = -(m_1+m_2-\sigma\theta(1-\rho)G^*), \tag{62}
$$

$$
\mathcal{D}et(\mathbf{J_3}(\mathbf{E}^*)) = m_1m_3-(m_2\beta+m_3\sigma\theta(1-\rho)G^*). \tag{63}
$$

The result for the proof of Theorem 3.3 shows that all the eigenvalues for the Jacobian in equation (61) are negative if $(m_1+m_2-\sigma\theta(1-\rho)G^*)>0$ for $\mathcal{T}r(\mathbf{J_3}(\mathbf{E}^*))$ and $m_1m_3-(m_2\beta+m_3\sigma\theta(1-\rho)G^*)>0$ for $\mathcal{D}et(\mathbf{J_3}(\mathbf{E}^*))$. So, If $\lambda_3<0$, $\mathcal{T}r(\mathbf{J_3}(\mathbf{E}^*))<\mathbf{0}$ and $\mathcal{D}et(\mathbf{J_3}(\mathbf{E}^*))>\mathbf{0}$ then $P_G<1$. The policy failure equilibrium is stable under this condition meaning that the policies designed by the government to enhance inclusiveness in the society fail to achieve the purpose. The interpretation therefore is that policy reform is required to turn the situation around and make $P_G>1$ to drive inclusiveness with abundant natural resources and ethnic diversity. This is feasible if a policy is driven at making any of the inequalities $\lambda_3<0$, $\mathcal{T}r(\mathbf{J_3}(\mathbf{E}^*))<\mathbf{0}$ or $\mathcal{D}et(\mathbf{J_3}(\mathbf{E}^*))>\mathbf{0}$ untrue.■

3.6. Global stability of policy failure equilibrium, E^*

The global stability of policy failure equilibrium shall be investigated by Lyapunov principle of Goh Volterra type (Hugo and Simanjilo, (2019); Nsuami, and Witbooi, (2018)).

Theorem 3.4.

The policy failure equilibrium point of the model's Equations (1) - (5) is globally asymptotically stable if $P_G<1$.

Proof:

The method of Lyapunov functional shall be employed to prove the theorem. Suppose there exists a positive definite function V such that the time derivative of the function is less than or equal to zero i.e. $\frac{dV}{dt}\leq 0$ then the function V is called the Lyapunov function for the system and the system is globally asymptotically stable. Now, construct the Lyapunov function V as

$$\begin{aligned} V(P,S,T,G,R) = &\left(P - P^* - P^* \log \frac{P}{P^*}\right) + \left(S - S^* - S^* \log \frac{S}{S^*}\right) \\ &+ \left(T - T^* - T^* \log \frac{T}{T^*}\right) + \left(G - G^* - G^* \log \frac{G}{G*}\right) \\ &+ \left(R - R^* - R^* \log \frac{R}{R^*}\right), \end{aligned} \tag{64}$$

$$\begin{aligned} \frac{dV}{dt} = &\left(\frac{dP}{dt} - \frac{P^*}{P}\frac{dP}{dt}\right) + \left(\frac{dS}{dt} - \frac{S^*}{S}\frac{dS}{dt}\right) + \left(\frac{dT}{dt} - \frac{T^*}{T}\frac{dT}{dt}\right) + \left(\frac{dG}{dt} - \frac{G^*}{G}\frac{dG}{dt}\right) \\ &+ \left(\frac{dR}{dt} - \frac{R^*}{R}\frac{dR}{dt}\right), \end{aligned} \tag{65}$$

$$\begin{aligned} \frac{dV}{dt} = &\left(1 - \frac{P^*}{P}\right)\frac{dP}{dt} + \left(1 - \frac{S^*}{S}\right)\frac{dS}{dt} + \left(1 - \frac{T^*}{T}\right)\frac{dT}{dt} + \left(1 - \frac{G^*}{G}\right)\frac{dG}{dt} \\ &+ \left(1 - \frac{R^*}{R}\right)\frac{dR}{dt}, \end{aligned} \tag{66}$$

$$\begin{aligned} \frac{dV}{dt} = &\left(\frac{P - P^*}{P}\right)\frac{dP}{dt} + \left(\frac{S - S^*}{S}\right)\frac{dS}{dt} + \left(\frac{T - T^*}{T}\right)\frac{dT}{dt} + \left(\frac{G - G^*}{G}\right)\frac{dG}{dt} \\ &+ \left(\frac{R - R^*}{R}\right)\frac{dR}{dt}, \end{aligned} \tag{67}$$

$$\begin{aligned} \frac{dV}{dt} = &\left(\frac{P - P^*}{P}\right)[\alpha\pi - \gamma p - \mu p + \omega T] \\ &+ \left(\frac{S - S^*}{S}\right)[(1-\alpha)\pi + \epsilon T + \gamma p - \beta S - \sigma\theta(1-\rho)GS - \mu S] \\ &+ \left(\frac{T - T^*}{T}\right)[\beta S - \epsilon T - \omega T - \mu T] \\ &+ \left(\frac{G - G^*}{G}\right)[\phi + \sigma\theta(1-\rho)GS + \tau R - \mu G] \\ &+ \left(\frac{R - R^*}{R}\right)[\eta - \tau R - \delta R], \end{aligned} \tag{68}$$

$$\begin{aligned}\frac{dV}{dt} &= \left(\frac{P-P^*}{P}\right)[(\mu+\gamma)P^* - (\mu+\gamma)P] \\ &+ \left(\frac{S-S^*}{S}\right)[(\mu+\beta+\sigma\theta(1-\rho)G)S^* - (\mu+\beta+\sigma\theta(1-\rho)G)S] \\ &+ \left(\frac{T-T^*}{T}\right)[(\epsilon+\omega+\mu)T^* - (\epsilon+\omega+\mu)T] \\ &+ \left(\frac{G-G^*}{G}\right)[(-\sigma\theta(1-\rho)S+\mu)G^* - (-\sigma\theta(1-\rho)S+\mu)G] \\ &+ \left(\frac{R-R^*}{R}\right)[(\tau+\delta)R^* - (\eta+\delta)R],\end{aligned} \tag{69}$$

$$\begin{aligned}\frac{dV}{dt} &= \left(\frac{P-P^*}{P}\right)(\mu+\gamma)[P^* - P] \\ &+ \left(\frac{S-S^*}{S}\right)(\mu+\beta+\sigma\theta(1-\rho)G)[S^* - S] \\ &+ \left(\frac{T-T^*}{T}\right)(\epsilon+\omega+\mu)[T^* - T] \\ &+ \left(\frac{G-G^*}{G}\right)(-\sigma\theta(1-\rho)S+\mu)[G^* - G] \\ &+ \left(\frac{R-R^*}{R}\right)(\tau+\delta)[R^* - R],\end{aligned} \tag{70}$$

$$\begin{aligned}\frac{dV}{dt} &= -\left(\frac{P-P^*}{P}\right)(\mu+\gamma)[P - P^*] \\ &- \left(\frac{S-S^*}{S}\right)(\mu+\beta+\sigma\theta(1-\rho)G)[S - S^*] \\ &- \left(\frac{T-T^*}{T}\right)(\epsilon+\omega+\mu)[T - T^*] \\ &- \left(\frac{G-G^*}{G}\right)(-\sigma\theta(1-\rho)S+\mu)[G - G^*] \\ &- \left(\frac{R-R^*}{R}\right)(\tau+\delta)[R - R^*].\end{aligned} \tag{71}$$

Since $X^* = (P^*, S^*, T^*, G^*, R^*)$ is a point inside $X = (P, S, T, G, R)$, then $P \geq P^*, S \geq S^*, T \geq T^*, G \geq G^*$ and $R \geq R^*$. Hence, from Equation (71), $\frac{dV}{dt} \leq 0$. Also, $\frac{dV}{dt} = 0$ if and only if $P = P^*, S = S^*, T = T^*, G = G^*$ and $R = R^*$. Therefore, the policy failure equilibrium of the model is globally asymptotically stable if $P_G < 1$ (which is the same as $R_\circ > 1$ in disease modeling) following LaSalle's invariant principle (LaSalle and Lefschetz (1961); LaSalle (1968)). ■

As in the stability of policy success equilibrium, the stability of policy failure equilibrium is also a function of the value of P_G whether in the short-run (local stability) or in the long-run (global stability). The implication of the result of the proof of Theorem 3.4 is that government efforts to promote inclusive growth would fail as long as $P_G < 1$.

4. Results and Discussion

The model shall be studied numerically using the software Maple to confirm the theoretical results in Section 3 and to visualize the effect of parameter perturbations on the structure of the model. Since the model parameters are rates and probabilities, their values are rightly chosen between zero and one. Some of the parameters are estimated while others are assumed. The SSA rate of fertility in 2019 was 4.62 children per woman (O'Neill (2021a)). The recruitment rate into the population is therefore estimated as the inverse of SSA fertility rate, i.e., $\pi = \dfrac{1}{4.62} = 0.22$. Also, the SSA life expectancy in 2019 was 61.63 years (O'Neill (2021b)). Generally, natural mortality rate is the inverse of life expectancy. Therefore, the SSA natural mortality rate $\mu = \dfrac{1}{61.63} = 0.016$. Again, the data for ethnic diversity for countries in SSA region in 2015 are supplied by the World Development Indicator (WDI) (WDI (2019)). Following WDI (2019), the rate at which ethnic diversity hinders equitable distribution of gross national income $\rho = 0.85$. We assume a reasonable set of values for other parameters since their values are not readily available as displayed in Table 2. The definition for each parameter has been stated in Table 1.

Table 2. Numerical values for the model parameters

Parameters	Values	Source
α	0.01	Assumed
π	0.22	O'Neill (2021a)
γ	0.0001	Assumed
μ	0.016	O'Neill (2021b)
ω	0.0001	Assumed
ϵ	0.001	Assumed
β	0.0001	Assumed
σ	0.03	Assumed
θ	0.01	Assumed
ρ	0.85	WDI (2019)
ϕ	0.01	Assumed
τ	0.25	Assumed
η	0.11	Assumed
δ	0.001	Assumed

In Section 3, the issue of whether government policy will activate inclusive growth or not given abundant natural resources and ethnic diversity is resolved by a threshold parameter termed the policy success ratio P_G in Equation (43). Using the values in Table 2 as the base, the numerical values for P_G at various values of parameters perturbations are displayed in Table 3.

Table 3. Effectiveness of government policy in driving inclusive growth

S/No.	σ	θ	ρ	ϕ	τ	δ	η	μ	P_G	Remark
1	0.03	0.01	0.85	0.01	0.25	0.001	0.11	0.016	0.020	Unstable
2	0.02	0.009	0.86	0.01	0.26	0.001	0.11	0.016	0.012	Unstable
3	0.01	0.008	0.87	0.01	0.27	0.001	0.11	0.016	0.005	Unstable
4	0.009	0.007	0.88	0.01	0.28	0.001	0.11	0.016	0.004	Unstable
5	0.008	0.006	0.89	0.01	0.29	0.001	0.11	0.016	0.003	Unstable
6	0.06	0.02	0.425	0.01	0.25	0.001	0.11	0.016	0.322	Unstable
7	0.12	0.04	0.213	0.01	0.26	0.001	0.11	0.016	1.765	Stable
8	0.18	0.06	0.142	0.01	0.27	0.001	0.11	0.016	4.33	Stable
9	0.24	0.08	0.106	0.01	0.28	0.001	0.11	0.016	8.02	Stable
10	0.3	0.1	0.085	0.01	0.29	0.001	0.11	0.016	12.827	Stable

Generally, income distribution, in most cases, is determined by the government so if every attempt is made by the government to distribute income evenly and to ensure that goods and services provided by the state to bridge inequality gap get to the common man, the probability of ethnic diversity hindering inclusive growth would be low. The initial values for the parameters in Table 2 are used to compute the initial value for P_G which is unstable (see S/No. 1 in Table 3). Some of these values are then varied to study the behavior of policy reform threshold (P_G) to changes in the values of the key model parameters.

Policy success equilibrium is unstable (i.e., $P_G < 1$) when a decrease in σ and θ, per capita probability of equitable distribution of national income and accessibility rate between the common man and the goods provided by the state to narrow inequality gap respectively, is accompanied with a high level of ρ, the rate at which ethnic diversity hinders equitable distribution of national income (see S/No. 2 to S/No.5). On the other hand, when an increase in σ and θ is accompanied with a low level of ρ, policy success equilibrium is at first unstable but later becomes stable (see S/No. 6 to S/No.10). The values for the parameters σ, θ, ρ and τ are varied while others are held constant because they have direct influence on the government policy reform. The results in Table 3 are supported graphically in Figure 2 - Figure 3. Figure 2 - Figure 3 display the interplay between abundant natural resources and ethnic diversity when policies are implemented to drive inclusive growth.

Figure 2 is plotted with parameters values in S/No. 1 in Table 2 and it depicts the situation in the region within S/No.1 - S/No. 5 in Table 2 when a decrease in σ and θ is accompanied with an increase in ρ such that $P_G < 1$. The interpretation is that if a policy is implemented to drive inclusiveness but if the government fails to set a good example by ensuring equitable distribution of national income and at the same time, ensuring that goods and services provided to narrow the inequality gap get to the common man then ethnic diversity would jeopardize growth inclusiveness. Under this condition, the policy would make the rich get richer and the poor to get poorer.

However, the situation of the rich getting richer and the poor getting poorer is a disequilibrium that could not persist as at a point, certain forces would act to restore the equilibrium. Certain

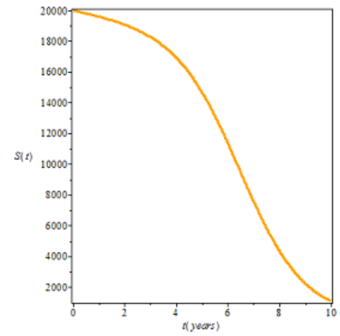

Figure 2. Policy effectiveness in driving inclusiveness when $P_G < 1$

forces would provoke resource redistribution from surplus hands to deficit hands and these forces have been attributed to the rise of separatist movements in virtually all countries in SSA - Nigeria, Cameroon, Ethiopia, Angola, Central African Republic, Democratic Republic of the Congo, Republic of the Congo, Equatorial Guinea, Ghana, Mali to mention but a few (Boukari (2018)). According to Boukari (2018), some ethnic groups in SSA countries, as a result of distributive injustice, felt marginalized and began secessionist agitations. Secessionist agitation is inevitable where abundant natural resources and ethnic diversity are not accompanied with inclusive growth. Figure 2 indicates that if the government is not serious about parameters σ and θ, after some time (a decade in Figure 2), inclusiveness would be provoked through various agitations when the population of individuals who are being marginalized tends to zero. The failure of the policy to address distributive injustice on time pushed the marginalized individuals to agitations in Figure 2.

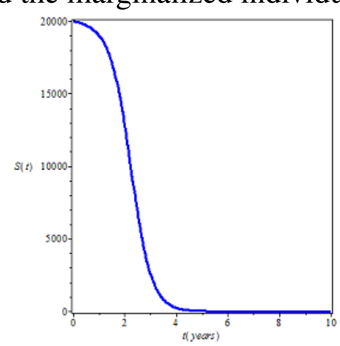

Figure 3. Policy effectiveness in driving inclusiveness when $P_G > 1$ with ($\sigma = 0.0035$,$\theta = 0.0015$,$\rho = 0.75$)

On the other hand, in Figure 3, it is observed that a small perturbation in σ, θ and ρ (i.e., a small increase in σ and θ accompanied with a low level of ρ) prevents distributive injustice and the population of individuals who are excluded from growth process tends to zero after four years. The policy implemented to drive inclusiveness is effective and able to achieve the purpose on time without allowing marginalization to push any group to agitations unlike in Figure 2. Secessionist agitation could be prevented where inclusive growth is facilitated by abundant natural resources and ethnic diversity. Figure 3 depicts the situation in the region within S/No. 7 - S/No. 10 in Table 2 when $P_G > 1$. The implication of Figure 3 is that if equitable distribution of national income and supply of goods and services to common man to bridge inequality gap are handled with all seriousness, the tendency of abundant natural resources and ethnic diversity translating to inclusive growth in SSA is feasible in no time.

5. Conclusion

In this work, the dynamics of inclusive growth given abundant natural resources and ethnic diversity in the SSA region had been quantified via the approach of mathematical modeling. A-five dimensional first-order nonlinear differential equations had been designed and the validity test had been carried out to establish the validity of the model. The equilibria of the model had been obtained and the threshold for the attainment of inclusiveness had been derived. The stability of the inclusiveness (policy success equilibrium) and non-inclusiveness (policy failure equilibrium) had been conducted based on inclusiveness threshold (policy success ratio). Numerical simulation had been performed to justify the analytical results and the outcome of the simulation indicated that constraints to inclusive growth in SSA were instigated by government inability to ensure equitable distribution of national income and at the same time, ensuring that goods and services provided to narrow the inequality gap get to the common man which had been a cause of rising separatist movements in the region.

It was also discovered from the simulation that inclusive growth through abundant natural resources and ethnic diversity was possible in SSA if policy reform to facilitate equitable distribution of resources and to increase accessibility of common man to goods and services provided to bridge inequality gap was inaugurated. However, while the study had expounded an elegant framework to study the relationship between abundant natural resources, ethnic diversity and inclusive growth in SSA, the accuracy of the outcome could be affected by the theoretical values employed for some parameters in simulation. Besides, since the model is the disease model of the society and not of man, it does not agree totally with what is obtainable in epidemic modeling especially in terms of the reproduction number. Future research may focus on a more accurate result by sourcing for primary data for those parameters that their values were imagined in this work.

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