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Süleyman Şenyurt
Ordu University

Kebire Hilal Ayvacı
Ordu University

Davut Canlı
Ordu University

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Family of Surfaces with a Common Bertrand D- Curve as Isogeodesic, Isoasymptotic and Line of Curvature

[1]Süleyman Şenyurt, [2]Kebire Hilal Ayvacı and [3]Davut Canlı

[1,2,3]Department of Mathematics
Faculty of Arts and Sciences
Ordu University
52200, Ordu, Turkey
[1]senyurtsuleyman52@gmail.com; [2]kebirehilalayvaci@odu.edu.tr;
[3] davutcanli@odu.edu.tr

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Abstract

In this paper, we establish the necessary and sufficient conditions to parameterize a surface family on which the Bertrand D- partner of any given curve lies as isogeodesic, isoasymptotic or curvature line in $\mathbb{E}^3$. Then, we calculate the fundamental forms of these surfaces and determine the developability and minimality conditions with the Gaussian and mean curvatures. We also extend this idea on ruled surfaces and provide the required conditions for those to be developable. Finally, we present some examples and graph the corresponding surfaces.

Keywords: Bertrand D- curves; Parametric curves; Asymptotic curves; Geodesic curves; Curvature line; Ruled surfaces

MSC 2010: 53A04, 53A05, 53C22

1. Introduction

In differential geometry, two or more curves are simply named to be as associated with the construction of some mathematical relations among the Frenet vectors at their corresponding points.

One of the well-known is the Bertrand curve defined by Joseph Louis François Bertrand in 1850 which states that the principal normal vector of a given curve coincides with the principal normal of its Bertrand partner curve. By utilizing the idea of sharing common vectors at the corresponding points of two curves, Kazaz et al. (2016) extended this notion to surfaces and defined the Bertrand D- partner curves such that the vector g of Darboux frame is common for both curves and provided some characterizations for this new type of associated curves. The relationship between the two Darboux frames of the Bertrand D- pair curves is given in Yıldırım and Kaya (2020). By referring to the Bishop frame, Yerlikaya et al. (2016) defined Bertrand B- partner curves that share the common vector element of B of Type-2 Bishop trihedra. There are similar studies in literature in which researchers apply this time another special curve, namely, Mannheim partner in which the association is based upon the linear dependence of principal and binormal vectors (Liu and Wang (2008); Kazaz et al. (2015); Masal and Azak (2017)).

Apart from associated curves, another typical interest of geometry is to characterize the curves according to an analytically curved surface in E^3 such as geodesic, asymptotic, or curvature line. However, in 2004, Wang et al. approached this problem in reverse and introduced the way to form a surface family possessing a given curve as a spatial geodesic. The generalization of this approach was first studied in Kasap et al. (2008) and the extension of this idea to hypersurfaces was given in Bayram and Kasap (2014b). Another research similar to these was the study of Şaffak and Kasap (2009) on null geodesics. On the other hand, the parametrization for a family of surfaces with a common line of curvature was defined first in Li et al. (2011). The generalization of this approach was comprised in Li et al. (2013). Finally, it was Bayram et al. (2012) in which the construction of the parametric form for the family of surfaces with a common asymptotic curve was introduced. The extension of this to the hypersurfaces was given in Bayram and Kasap (2014a). The family of surfaces with common null asymptotic was examined in Atalay and Kasap (2015). There are other studies that are analogous to these but carried out for different spaces, as well (see Kasap and Akyıldız (2006); Yüzbaşı (2016a); Yüzbaşı and Bektaş (2016b); Atalay et al. (2018); Saad et al. (2021)).

As a combination of these two distinct ideas, researchers begin to question the parametric forms of surfaces possessing a given special curve as geodesic, asymptotic, or curvature line. For example, the surface family with a common Smarandache asymptotic curve was studied in Atalay and Kasap (2016), while the geodesic one was discussed in Atalay and Kasap (2017). In addition to this, the surfaces with a common Mannheim asymptotic and geodesic curve were outlined in Atalay (2018a) and Atalay (2018b), respectively. Ayvacı (2019) studied the surfaces with a common Mannheim-B isogeodesic and isoasymptotic curves and extended the idea to the ruled surfaces. The necessary and sufficient conditions for those ruled surfaces to be developable were provided in Ayvacı (2019), as well. In another study of Ayvacı and Şaffak (2020), the family of the surfaces with a common Bertrand B- isogeodesic curve was defined. The surface family with a common Mannheim D- curve was comprised as isogeodesic in Şenyurt et al. (2020b) and as isoasymptotic in Şenyurt et al. (2020a). More recently, Bilici and Bayram (2021) provided a parameterization to construct a surface family with a common involute line of curvature.

Motivated by all, it is of interest for us to form the family of surfaces with a common Bertrand

D- curve as isogeodesic, isoasymptotic, or curvature line. To do so, we express the surface with a linear combination of Frenet vectors and provide the necessary conditions for a surface such that the Bertrand D- partner curve lies as isogeodesic, isoasymptotic or curvature line in Section 2. In this section, we also include the fundamental forms, the Gaussian and the mean curvature of the surfaces. In Section 3, we attribute the idea to the ruled surfaces. Finally, we provide the conditions for ruled surfaces to be developable, and present some examples with their graphs.

2. Surfaces with a common Bertrand D- curve as isogeodesic, isoasymptotic, and line of curvature

Let α and α^* be given on two directed surfaces φ and φ^*, respectively, and $\{T, g, n\}$ and $\{T^*, g^*, n^*\}$ denote the Darboux frame of each. If g and g^* are linearly dependent then, (α, α^*) pair is called to be Bertrand D- partner curves (Kazaz et al. (2016)). According to this definition, we have $\alpha^*(t^*) = \alpha(t) + \lambda(t)g(t)$. The parametric equation of the surface by means of Darboux frame of α^* is given as

$$\begin{aligned} \varphi^*(t, v) = \alpha^*(t) + [x(t, v)T^*(t) + y(t, v)g^*(t) + z(t, v)n^*(t)], \\ L_1 \leq t \leq L_2, \quad K_1 \leq v \leq K_2. \end{aligned} \tag{1}$$

Here $x(t, v)$, $y(t, v)$ and $z(t, v)$ are called differentiable marching-scale functions. By referring the transitional relation between the Frenet and Darboux frame vectors (see Kazaz et al. (2016)), we reform the parametrization of $\varphi^*(t, v)$ as

$$\begin{aligned} \varphi^*(t, v) = \alpha^*(t) + x(t, v)T^*(t) + \big(y(t, v)\cos\theta^* - z(t, v)\sin\theta^*\big)N^*(t) \\ + \big(y(t, v)\sin\theta^* + z(t, v)\cos\theta^*\big)B^*(t). \end{aligned} \tag{2}$$

Theorem 2.1.

α^* is isogeodesic on the surface $\varphi^*(t, v)$ if and only if

$$\begin{cases} x(t, v_0) = & y(t, v_0) = \ z(t, v_0) = 0, \\ y(t, v_0) = & (v - v_0)\beta_1(t)\sin\theta^*, \\ z(t, v_0) = & (v - v_0)\beta_1(t)\cos\theta^*, \end{cases} \tag{3}$$

where $\beta_1(t)$ is a non-zero function ($\beta_1(t) \neq 0$) and $\theta^* = \sphericalangle(N^*, g^*)$.

Proof:

Since the Bertrand D- partner curve is parametric on the surface, for a constant $v = v_0$, we clearly have

$$x(t, v_0) = y(t, v_0) = z(t, v_0) = 0. \tag{4}$$

The normal of the surface n^* is

$$\begin{aligned} n^*(t,v) &= \frac{\partial \varphi^*(t,v)}{\partial t} \times \frac{\partial \varphi^*(t,v)}{\partial v} \\ &= \left[-\frac{\partial y(t,v)}{\partial v} \sin\theta^* - \frac{\partial z(t,v)}{\partial v} \cos\theta^* \right] N^*(t) \\ &\quad + \left[\frac{\partial y(t,v)}{\partial v} \cos\theta^* - \frac{\partial z(t,v)}{\partial v} \sin\theta^* \right] B^*(t). \end{aligned} \tag{5}$$

For $v = v_0$, we rewrite n^* as

$$n^*(t,v_0) = \phi_1(t,v_0)T^*(t) + \phi_2(t,v_0)N^*(t) + \phi_3(t,v_0)B^*(t), \tag{6}$$

where

$$\begin{cases} \phi_1(t,v_0) = & 0, \\ \phi_2(t,v_0) = & -\dfrac{\partial y(t,v_0)}{\partial v} \sin\theta^* - \dfrac{\partial z(t,v_0)}{\partial v} \cos\theta^*, \\ \phi_3(t,v_0) = & \dfrac{\partial y(t,v_0)}{\partial v} \cos\theta^* - \dfrac{\partial z(t,v_0)}{\partial v} \sin\theta^*. \end{cases} \tag{7}$$

Now, recall the geodesicity condition that is $n^* \parallel N^*$ (see O'Neill (1966)). We write

$$\begin{aligned} -\frac{\partial y(t,v)}{\partial v}\Big|_{v_0} \sin\theta^*(t) - \frac{\partial z(t,v)}{\partial v}\Big|_{v_0} \cos\theta^* \neq 0, \\ \frac{\partial y(t,v)}{\partial v}\Big|_{v_0} \cos\theta^*(t) - \frac{\partial z(t,v)}{\partial v}\Big|_{v_0} \sin\theta^* = 0. \end{aligned} \tag{8}$$

This last expression can be restated with a given non-zero function $\beta_1(t) \neq 0$, as

$$\begin{cases} y(t,v_0) = & (v-v_0)\beta_1(t)\sin\theta^*, \\ z(t,v_0) = & (v-v_0)\beta_1(t)\cos\theta^*, \end{cases} \tag{9}$$

which completes the proof. ■

Theorem 2.2.

α^* is isoasymptotic on the surface $\varphi^*(t,v)$ if and only if

$$\begin{cases} x(t,v_0) = & y(t,v_0) = z(t,v_0) = 0, \\ y(t,v_0) = & (v-v_0)\beta_2(t)\cos\theta^*, \\ z(t,v_0) = & -(v-v_0)\beta_2(t)\sin\theta^*, \end{cases} \tag{10}$$

where $\beta_2(t)$ is a non-zero function.

Proof:

As the Bertrand D- partner curve is parametric, the relations (4), (5), (6), and (7) hold. By referring, this time, the asymptoticity condition (see Bayram et al. (2012)), we have

$$\begin{aligned} \Big\langle \frac{\partial n^*}{\partial t}(t,v), T^*(t) \Big\rangle = 0 \iff & \frac{\partial(\phi_1(t,v))}{\partial t} - \kappa^*\phi_2(t,v) = 0, \qquad \kappa^* \neq 0 \\ \iff & \phi_2(t,v) = 0. \end{aligned} \tag{11}$$

From the relation (7), $\alpha^*(t)$ is isoasymptotic on the surface $\varphi^*(t,v)$, if and only if

$$-\frac{\partial y(t,v_0)}{\partial v}\sin\theta^* - \frac{\partial z(t,v_0)}{\partial v}\cos\theta^* = 0. \tag{12}$$

The latter can be rewritten by considering a non zero function $\beta_2(t) \neq 0$ as

$$\begin{cases} y(t,v_0) = & (v-v_0)\beta_2(t)\cos\theta^*, \\ z(t,v_0) = & -(v-v_0)\beta_2(t)\sin\theta^*, \end{cases} \tag{13}$$

which completes the proof. ■

Theorem 2.3.

The Bertrand D- partner curve α^* is a line of curvature on the surface $\varphi^*(t,v)$ if and only if

$$\begin{cases} x(t,v_0) = y(t,v_0) = z(t,v_0) = 0, \\ \gamma(t) = \int \tau^*(t)dt, \\ \left.\frac{\partial z(t,v)}{\partial v}\right|_{v_0} = -\mu(t)\cos(\gamma-\theta^*)(t), \end{cases} \tag{14}$$

where $\mu(t) \neq 0, \quad \sphericalangle(N^*,\eta) = \gamma$ and $\eta(t)$ denotes the orthogonal vector field of the surface.

Proof:

Since $\eta(t)$ is the orthogonal vector field of the surface on which the Bertrand D- partner curve lies, we may write

$$\eta(t) = \cos\gamma(t)N^*(t) + \sin\gamma(t)B^*(t).$$

The condition for α^* to be a line of curvature on the surface is twofold (O'Neill (1966)). First is that $\eta(t) \parallel n(t,v_0)$, which results in the following:

$$\frac{-\frac{\partial y(t,v)}{\partial v}\sin\theta^*(t) - \frac{\partial z(t,v)}{\partial v}\cos\theta^*(t)}{\cos\gamma(t)} = \frac{-\frac{\partial y(t,v)}{\partial v}\cos\theta^*(t) - \frac{\partial z(t,v)}{\partial v}\sin\theta^*(t)}{\sin\gamma(t)} = \mu(t),$$

and therefore,

$$\left.\frac{\partial z(t,v)}{\partial v}\right|_{v_0} = -\mu(t)\cos(\gamma-\theta^*)(t).$$

The second condition is that the surface accepting α^* as the base curve and $\eta(t)$ as the director, $\Phi(t,v) = \alpha^*(t) + v\eta(t)$ should be developable, which corresponds to that $det(\alpha^{*\prime},\eta,\eta') = 0$ (Ravani and Ku (1991)). By this condition, we obtain

$$\gamma'(t) + \tau^*(t) = 0 \quad \Longrightarrow \quad \gamma(t) = -\int \tau^*(t)dt,$$

which completes the proof. ■

Theorem 2.4.

The first and the second fundamental forms, together with the Gaussian and the mean curvatures of the surface with Bertrand D- partner curve α^*, are given by the following:

$$I = dt^2 + 2\frac{\partial x}{\partial v}dsdv + \left(\frac{\partial^2 x}{\partial^2 v} + \frac{\partial^2 y}{\partial^2 v} + \frac{\partial^2 z}{\partial^2 v}\right)dv^2,$$

$$II = -A\kappa dt^2 + 2Cdtdv + \Bigg((\kappa+\tau)(A+B) + \frac{\partial^2 z}{\partial t \partial v}(A\sin\theta^*(t) + B\cos\theta^*(t)) + \frac{\partial^2 y}{\partial t \partial v}(B\sin\theta^*(t) - A\cos\theta^*(t)) - \kappa A\frac{\partial z}{\partial v}\Bigg)dv^2,$$

$$K = \frac{1}{\frac{\partial^2 y}{\partial^2 v} + \frac{\partial^2 z}{\partial^2 v}}\Bigg((\kappa+\tau)(A+B) + \frac{\partial^2 z}{\partial t \partial v}(A\sin\theta^*(t) + B\cos\theta^*(t)) + \frac{\partial^2 y}{\partial t \partial v}(B\sin\theta^*(t) - A\cos\theta^*(t)) - \kappa A\frac{\partial z}{\partial v} - C^2\Bigg),$$

$$H = \frac{1}{2\left(\frac{\partial^2 y}{\partial^2 v} + \frac{\partial^2 z}{\partial^2 v}\right)}\Bigg((\kappa+\tau)(A+B) + \frac{\partial^2 z}{\partial t \partial v}(A\sin\theta^*(t) + B\cos\theta^*(t)) + \frac{\partial^2 y}{\partial t \partial v}(B\sin\theta^*(t) - A\cos\theta^*(t)) - 2C\frac{\partial x}{\partial v} - \kappa A\left(\frac{\partial z}{\partial v} + \frac{\partial^2 x}{\partial^2 v} + \frac{\partial^2 y}{\partial^2 v} + \frac{\partial^2 z}{\partial^2 v}\right)\Bigg),$$

respectively, where

$$A = \frac{\partial y}{\partial v}\sin\theta^*(t) + \frac{\partial z}{\partial v}\cos\theta^*(t),\ B = \frac{\partial y}{\partial v}\cos\theta^*(t) + \frac{\partial z}{\partial v}\sin\theta^*(t),\ C = \frac{\partial^2 z}{\partial^2 v}\frac{\partial y}{\partial v} - \frac{\partial^2 y}{\partial^2 v}\frac{\partial z}{\partial v}.$$

Proof:

The coefficients of the fundamental forms of the surface $\varphi^*(t,v)$ are

$$E = \langle \varphi_t, \varphi_t \rangle = 1, \qquad F = \langle \varphi_t, \varphi_v \rangle = \frac{\partial x}{\partial v}, \qquad G = \langle \varphi_v, \varphi_v \rangle = \frac{\partial^2 x}{\partial^2 v} + \frac{\partial^2 y}{\partial^2 v} + \frac{\partial^2 z}{\partial^2 v},$$

and

$$L = \langle \varphi_{tt}, \varphi_t \times \varphi_v \rangle = -\kappa\left(\frac{\partial y}{\partial v}\sin\theta^*(t) + \frac{\partial z}{\partial v}\cos\theta^*(t)\right),$$

$$M = \langle \varphi_{tv}, \varphi_t \times \varphi_v \rangle = \frac{\partial^2 z}{\partial^2 v}\frac{\partial y}{\partial v} - \frac{\partial^2 y}{\partial^2 v}\frac{\partial z}{\partial v},$$

$$N = \langle \varphi_{vv}, \varphi_t \times \varphi_v \rangle = (\kappa+\tau)(A+B) - \kappa A\frac{\partial z}{\partial v} + \frac{\partial^2 z}{\partial t \partial v}(A\sin\theta^*(t) + B\cos\theta^*(t)) + \frac{\partial^2 y}{\partial t \partial v}(B\sin\theta^*(t) - A\cos\theta^*(t)).$$

By utilizing these coefficients and recalling the definitions given in O'Neill (1966), the proof is complete. ∎

Corollary 2.1.

In order for the surface $\varphi^*(t,v)$ with a common Bertrand D- isogeodesic curve to be developable, there exists the following relation:

$$A\left(\kappa+\tau+\frac{\partial^2 z}{\partial t\partial v}\sin\theta^*(t)-\frac{\partial^2 y}{\partial t\partial v}\cos\theta^*(t)-\kappa\frac{\partial z}{\partial v}\right)=C^2.$$

Proof:

As a result of Theorem 2.1, the coefficient B of Theorem 2.4 vanishes. On the other hand, a surface is said to be developable if and only if the Gaussian curvature vanishes, that is, $K=0$ (O'Neill (1966)). Hence, by utilizing these, the proof is complete. ∎

Corollary 2.2.

In order for the surface $\varphi^*(t,v)$ with a common Bertrand D- isoasymptotic curve to be developable, there exists the following relation:

$$B\left(\kappa+\tau+\frac{\partial^2 z}{\partial t\partial v}\sin\theta^*(t)-\frac{\partial^2 y}{\partial t\partial v}\cos\theta^*(t)\right)=C^2.$$

Proof:

By referring this time to Theorem 2.2, the coefficient A of Theorem 2.4 vanishes. By the developability condition, that is, $K=0$, we complete the proof. ∎

Corollary 2.3.

In order for the surface $\varphi^*(t,v)$ with a common Bertrand D- isogeodesic curve to be minimal, there exists the following relation:

$$A\left(\kappa+\tau+\frac{\partial^2 z}{\partial t\partial v}\sin\theta^*(t)-\frac{\partial^2 y}{\partial t\partial v}\cos\theta^*(t)-\kappa\frac{\partial z}{\partial v}\right)=2C\frac{\partial x}{\partial v}+\kappa A+\frac{\partial z}{\partial v}+\frac{\partial^2 x}{\partial^2 v}+\frac{\partial^2 y}{\partial^2 v}+\frac{\partial^2 z}{\partial^2 v}.$$

Proof:

By Theorem 2.1, $B=0$. Moreover, a surface is said to be minimal if and only if the mean curvature vanishes, that is, $H=0$, (O'Neill (1966)). Hence, by using these together, the proof is complete. ∎

Corollary 2.4.

In order for the surface $\varphi^*(t,v)$ with a common Bertrand D- isoasymptotic curve to be minimal, there exists the following relation:

$$B\left(\kappa+\tau+\frac{\partial^2 z}{\partial t\partial v}\sin\theta^*(t)-\frac{\partial^2 y}{\partial t\partial v}\cos\theta^*(t)-\kappa\frac{\partial z}{\partial v}\right)=2C\frac{\partial x}{\partial v}.$$

Proof:

Similarly, $A=0$ as a result of Theorem 2.2 and $H=0$ as a condition of minimality. Therefore, the proof is clear. ∎

3. Ruled surfaces with a common Bertrand D- isogeodesic and isoasymptotic curve

The parametric equation of a ruled surface having α^* as the base vector is given as

$$\varphi^*(t,v) = \varphi^*(t,v_0) + (v-v_0)R(t). \tag{15}$$

For $v = v_0$ and from the relation (2), we have

$$\begin{aligned}\varphi^*(t,v_0) = \alpha^*(t) &+ x(t,v_0)T^*(t)\\ &+\big(y(t,v_0)\cos\theta^* - z(t,v_0)\sin\theta^*\big)N^*(t)\\ &+\big(y(t,v_0)\sin\theta^* + z(t,v_0)\cos\theta^*\big)B^*(t).\end{aligned} \tag{16}$$

Now by considering the parametricity condition given in (4) and using it in the latter expression, we get

$$\varphi^*(t,v_0) = \alpha^*(t). \tag{17}$$

Upon substitution of (2) and (17) into (15), we form the following relation:

$$\begin{aligned}(v-v_0)R(t) = x(t,v_0)T^*(t) &+ \big(y(t,v_0)\cos\theta^* - z(t,v_0)\sin\theta^*\big)N^*(t)\\ &+ \big(y(t,v_0)\sin\theta^* + z(t,v_0)\cos\theta^*\big)B^*(t).\end{aligned} \tag{18}$$

The inner product of this last expression with T^*, N^*, and B^* results in

$$\begin{aligned} x(t,v) &= (v-v_0) < R, T^* >,\\ \big(y(t,v_0)\cos\theta^* - z(t,v_0)\sin\theta^*\big) &= (v-v_0) < R, N^* >,\\ \big(y(t,v_0)\sin\theta^* + z(t,v_0)\cos\theta^*\big) &= (v-v_0) < R, B^* > .\end{aligned} \tag{19}$$

By taking into account (3), we write

$$\begin{aligned} x(t,v) &= (v-v_0)f(t),\\ y(t,v) &= (v-v_0)\beta_1(t)\sin\theta^*,\\ z(t,v) &= (v-v_0)\beta_1(t)\cos\theta^*,\end{aligned} \tag{20}$$

where $\beta_1(t) \neq 0$ and $f(t)$ are any real valued functions. Substituting this in (15), we define the parametric representation of the ruled surfaces with a common Bertrand D- isogeodesic curve as

$$\varphi^*(t,v) = \alpha^*(t) + (v-v_0)\big(f(t)T^*(t) + \beta_1(t)B^*(t)\big). \tag{21}$$

In order for φ^* to be developable,

$$\begin{aligned} det(\alpha', R, R') = 0 &\Rightarrow \beta_1(t)(\tau^*(t)\beta_1(t) - \kappa^*(t)f(t)) = 0,\\ &\Rightarrow f(t) = \frac{\tau^*(t)}{\kappa^*(t)}\beta_1(t), \qquad \beta_1(t)\neq 0.\end{aligned}$$

When considering both (10) and (19), we write

$$\begin{aligned} x(t,v) &= (v-v_0)h(t),\\ y(t,v) &= (v-v_0)\beta_2(t)\cos\theta^*,\\ z(t,v) &= -(v-v_0)\beta_2(t)\sin\theta^*,\end{aligned} \tag{22}$$

where $\beta_2(t) \neq 0$ and $h(t)$ are any real valued functions. Substituting this in (15), the parametrization of the ruled surfaces with a common Bertrand D- isoasymptotic curve is

$$\varphi^*(t,v) = \alpha^*(t) + (v - v_0)\big(h(t)T^*(t) + \beta_2(t)N^*(t)\big). \tag{23}$$

In order for φ^* to be developable,

$$\begin{aligned} det(\alpha', R, R') = 0 &\Rightarrow \beta_2(t)\tau^*(t) = 0, \\ &\Rightarrow \tau^*(t) = 0, \qquad \beta_2(t) \neq 0. \end{aligned}$$

Example 3.1.

Let us consider the unit speed curve, $\alpha(t) = \left(\frac{3}{5}\cos(t), \frac{3}{5}\sin(t), \frac{4}{5}t\right)$ and the surface $\varphi(t,v) = \left(\frac{3}{5}\cos(t), \frac{3}{5}\sin(t), \frac{4}{5}t + v\right)$. The Darboux frame and the curvatures of α are given as

$$\begin{cases} T(t) = & \left(-\frac{3}{5}\sin(t), \frac{3}{5}\cos(t), \frac{4}{5}\right) \\ g(t) = & \left(\frac{4}{5}\sin(t), -\frac{4}{5}\cos(t), \frac{3}{5}\right), \qquad \kappa = \frac{3}{25}, \quad \tau = \frac{4}{25}. \\ n(t) = & (\cos(t), \sin(t), 0) \end{cases}$$

For $\lambda = 1$, the Bertrand D- mate $\alpha^*(t)$ of the curve $\alpha(t)$ is found by

$$\begin{aligned} \alpha^*(t) =& \alpha(t) + \lambda g(t) \\ =& \left(\frac{3}{5}\cos(t) + \frac{4}{5}\sin(t), \frac{3}{5}\sin(t) - \frac{4}{5}\cos(t), \frac{4}{5}t + \frac{3}{5}\right), \end{aligned}$$

and its curvatures are $\kappa^* = \frac{5}{\sqrt{41}}$ and $\tau^* = \frac{4}{\sqrt{41}}$. The Frenet vectors of the curve $\alpha^*(t)$ are

$$\begin{cases} T^*(t) = & \left(-\frac{3}{\sqrt{41}}\sin(t) + \frac{4}{\sqrt{41}}\cos(t), \frac{3}{\sqrt{41}}\cos(t) + \frac{4}{\sqrt{41}}\sin(t), \frac{4}{\sqrt{41}}\right), \\ N^*(t) = & \left(-\frac{3}{5}\cos(t) - \frac{4}{5}\sin(t), -\frac{3}{5}\sin(t) + \frac{4}{5}\cos(t), 0\right), \\ B^*(t) = & \left(\frac{12}{5\sqrt{41}}\sin(t) - \frac{16}{5\sqrt{41}}\cos(t), -\frac{12}{5\sqrt{41}}\cos(t) - \frac{16}{5\sqrt{41}}\sin(t), \frac{5}{\sqrt{41}}\right). \end{cases}$$

From this line of the paper, in order to form surfaces, we consider manipulating marching scale functions such that the geodesicity, asymptoticity, and line of curvature conditions defined, respectively, in Equation 3, Equation 10 and Equation 14 are satisfied. Thus, we present four distinct cases of scale functions meeting each of the three predefined conditions to construct the family of regular and the ruled surfaces with a common Bertrand D- curve.

Case 1.
a) By choosing $x(t,v)=0$, $y(t,v)=vt\sin(t)$, $z(t,v)=vt\cos(t)$ and $\beta_1(t)=t$, $v_0=0$, one of the surfaces with a common Bertrand D- isogeodesic curve is given as follows:

$$\begin{aligned}\varphi^1_{ig}(t,v)=\Big(&\frac{3}{5}\ \cos(t)+\frac{4}{5}\sin(t)+\frac{12}{5\sqrt{41}}vt\sin(t)-\frac{16}{5\sqrt{41}}vt\cos(t),\\ &\frac{3}{5}\sin(t)-\frac{4}{5}\ \cos(t)-\frac{12}{5\sqrt{41}}vt\cos(t)-\frac{16}{5\sqrt{41}}vt\sin(t),\\ &\frac{4}{5}t+\frac{3}{5}+\frac{5}{\sqrt{41}}vt\Big).\end{aligned}$$

b) On the other hand, by taking $x(t,v)=0$, $y(t,v)=vt\cos(t)$, $z(t,v)=-vt\sin(t)$ and $\beta_2(t)=t$, $v_0=0$, we get the following parametrization for the surface with a common Bertrand D- isoasymptotic curve as:

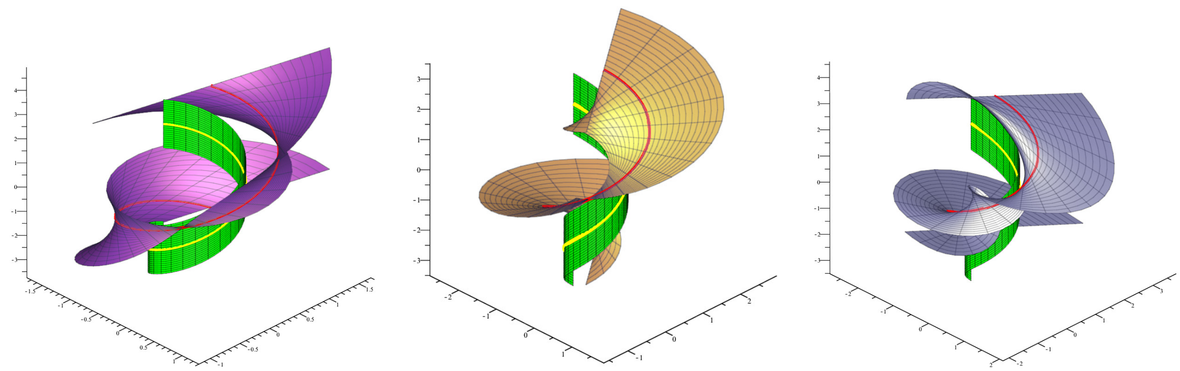
$$\begin{aligned}\varphi^1_{ia}(t,v)=\Big(&\frac{3}{5}\cos(t)+\frac{4}{5}\sin(t)-\frac{3}{5}vt\cos(t)-\frac{4}{5}vt\sin(t),\\ &\frac{3}{5}\sin(t)-\frac{4}{5}\cos(t)-\frac{3}{5}vt\sin(t)+\frac{4}{5}vt\cos(t),\\ &\frac{4}{5}t+\frac{3}{5}\Big).\end{aligned}$$

c) Now, if marching scales are chosen such that $x(t,v)=vt$, $y(t,v)=vt$, $z(t,v)=-v\cos(\gamma-\theta^*)$ with $\theta^*=t$, $\mu(t)=1$ and $\gamma=-\int\tau^*dt=-\frac{20}{41}t$, we obtain the following parametrization for the surface on which α^* lies as a line of curvature:

$$\begin{aligned}\varphi^1_{lc}(t,v)=\Bigg(&(1-vt)\left(\frac{3}{5}\cos(t)+\frac{4}{5}\sin(t)\right)-\left(\frac{1}{\sqrt{41}}vt+\frac{4}{5\sqrt{41}}v\cos\left(\frac{61}{41}t\right)\right)(3\sin(t)-4\cos(t))\,,\\ &(1-vt)\left(\frac{3}{5}\sin(t)-\frac{4}{5}\cos(t)\right)+\left(\frac{1}{\sqrt{41}}vt+\frac{4}{5\sqrt{41}}vt\cos\left(\frac{61}{41}t\right)\right)(3\cos(t)+4\sin(t))\,,\\ &\frac{4}{5}t+\frac{3}{5}+\frac{4}{\sqrt{41}}vt-\frac{5}{\sqrt{41}}v\cos\left(\frac{61}{41}t\right)\Bigg).\end{aligned}$$

In the following figures, we illustrate the main surface φ with green color and the surfaces where Bertrand D- partner curve lies as isogeodesic (a), isoasymptotic (b) , and line of curvature (c) with colors purple, orange, and gray, respectively.

Figure 1 corresponds to the surfaces formed in the example.



(a) The surface φ_{ig}^1 with isogeodesic curve

(b) The surface φ_{ia}^1 with isoasymptotic curve

(c) The surface φ_{lc}^1 with line of curvature

Figure 1. Bertrand D- surface mates for the 1^{st} example

Case 2.

a) Another series of functions $x(t,v) = v$, $y(t,v) = vt^2\sin(t)$, $z(t,v) = vt^2\cos(t)$ and $\beta_1(t) = t^2$, $v_0 = 0$ satisfying the geodesicity condition given by (3), produce the following parametrization for the surface as

$$\begin{aligned}\varphi_{ig}^2(t,v) = \Bigg(&\frac{1}{5}(3\cos(t)+4\sin(t)) + \frac{v}{\sqrt{41}}(-3\sin(t)+4\cos(t)) - \frac{4vt^2}{5\sqrt{41}}(-3\sin(t)+4\cos(t)),\\ &\frac{1}{5}(3\ \sin(t)-4\ \cos(t)) + \frac{v}{\sqrt{41}}(3\ \cos(t)+4\ \sin(t)) - \frac{4vt^2}{5\sqrt{41}}(3\ \cos(t)+4\ \sin(t)),\\ &\frac{4}{5}t+\frac{3}{5}+\frac{4}{\sqrt{41}}v+\frac{5}{\sqrt{41}}vt^2\Bigg).\end{aligned}$$

b) Similarly, the asymptoticity condition given in (10) is valid for the functions $x(t,v) = v$, $y(t,v) = vt^2\cos(t)$, $z(t,v) = -vt^2\sin(t)$ and $\beta_2(t) = t^2$, $v_0 = 0$. This results the following parametric equation:

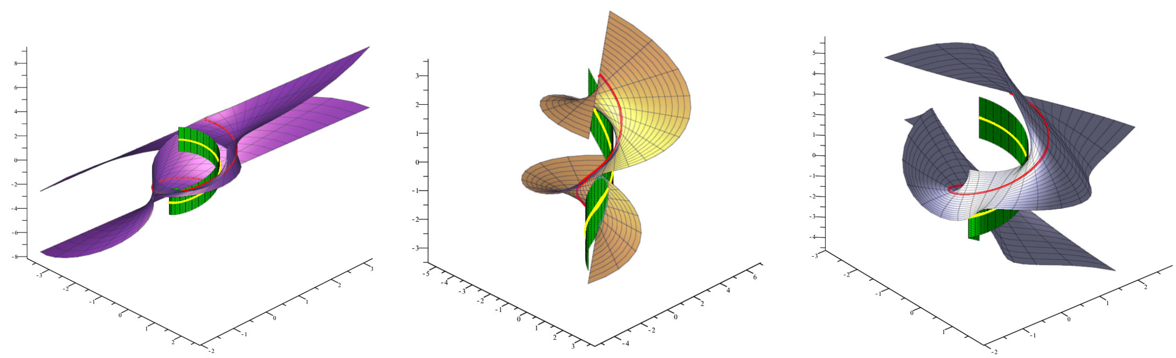
$$\begin{aligned}{\varphi_{ia}}^2(t,v) = \Bigg(&\frac{1}{5}(3\cos(t)+4\sin(t)) + \frac{v}{\sqrt{41}}(-3\sin(t)+4\cos(t)) + \frac{vt^2}{5}(-3\cos(t)-4\sin(t)),\\ &\frac{1}{5}(3\ \sin(t)-4\ \cos(t)) + \frac{v}{\sqrt{41}}(3\ \cos(t)+4\ \sin(t)) + \frac{vt^2}{5}(-3\ \sin(t)+4\ \cos(t)),\\ &\frac{4}{5}t+\frac{3}{5}+\frac{4}{\sqrt{41}}v\Bigg).\end{aligned}$$

c) Finally, we form another surface with α^* as a line of curvature by choosing $x(t,v) = vt$,

$y(t,v) = \sin(v)$, $z(t,v) = -vt\cos(\gamma - \theta^*)$ with $\theta^* = t$, $\mu(t) = t$ and $\gamma = -\int \tau^* dt = -\frac{20}{41}t$,

$$\begin{aligned}\varphi_{lc}^2(t,v) = \Bigg(& (1-\sin(v))\left(\frac{3}{5}\cos(t) + \frac{4}{5}\sin(t)\right) \\ & - \left(\frac{1}{\sqrt{41}}vt + \frac{4}{5\sqrt{41}}vt\cos\left(\frac{61}{41}t\right)\right)(3\sin(t) - 4\cos(t)), \\ & (1-\sin(v))\left(\frac{3}{5}\sin(t) - \frac{4}{5}\cos(t)\right) \\ & + \left(\frac{1}{\sqrt{41}}vt + \frac{4}{5\sqrt{41}}vt\cos\left(\frac{61}{41}t\right)\right)(3\cos(t) + 4\sin(t)), \\ & \frac{4}{5}t + \frac{3}{5} + \frac{4}{\sqrt{41}}vt - \frac{5}{\sqrt{41}}vt\cos\left(\frac{61}{41}t\right)\Bigg).\end{aligned}$$

For this example of Case 2, the corresponding pictures of these surfaces are given in Figure 2.



(a) The surface φ_{ig}^2 with isogeodesic curve

(b) The surface φ_{ia}^2 with isoasymptotic curve

(c) The surface φ_{lc}^2 with line of curvature

Figure 2. Bertrand D- surface mates for the second example

Case 3.

a) For a ruled surface with a common Bertrand D- isogeodesic curve, we choose the functions as $f(t) = 0$ and $\beta_1(t) = 1$. The corresponding parametrization for this surface is

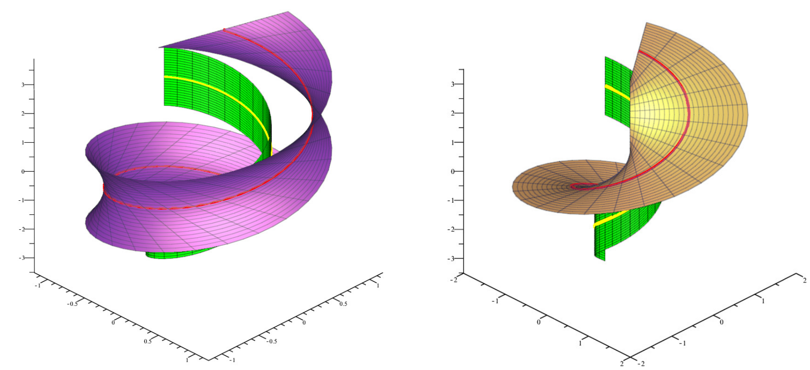
$$\begin{aligned}\varphi_{igr}^1(t,v) = \Bigg(& \frac{12}{5\sqrt{41}}v\sin(t) - \frac{16}{5\sqrt{41}}v\cos(t) + \frac{4}{5}\sin(t) + \frac{3}{5}\cos(t), \\ & -\frac{16}{5\sqrt{41}}v\sin(t) - \frac{12}{5\sqrt{41}}v\cos(t) + \frac{3}{5}\sin(t) - \frac{4}{5}\cos(t), \\ & \frac{4}{5}t + \frac{3}{5} + \frac{5}{\sqrt{41}}v\Bigg).\end{aligned}$$

b) By taking $h(t) = 0$ and $\beta_2(t) = 1$, we form the following equation for the ruled surface with a

Bertrand D- isoasymptotic curve as:

$$\varphi^1_{iar}(t,v) = \Big(-\frac{4}{5}\,v\sin(t) - \frac{3}{5}\,v\cos(t) + \frac{4}{5}\,\sin(t) + \frac{3}{5}\,\cos(t),\\ -\frac{3}{5}\,v\sin(t) + \frac{4}{5}\,v\cos(t) + \frac{3}{5}\,\sin(t) - \frac{4}{5}\,\cos(t),\\ \frac{4}{5}\,t + \frac{3}{5}\Big).$$

The pictures of these ruled surfaces are presented in Figure 3.



(a) The ruled surface φ^1_{igr} with isogeodesic curve

(b) The ruled surface φ^1_{iar} with isoasymptotic curve

Figure 3. Bertrand D- ruled surface mates for the third example

Case 4.

a) To illustrate more, a final example for an isogeodesic Bertrand D- curve included ruled surface could be given as

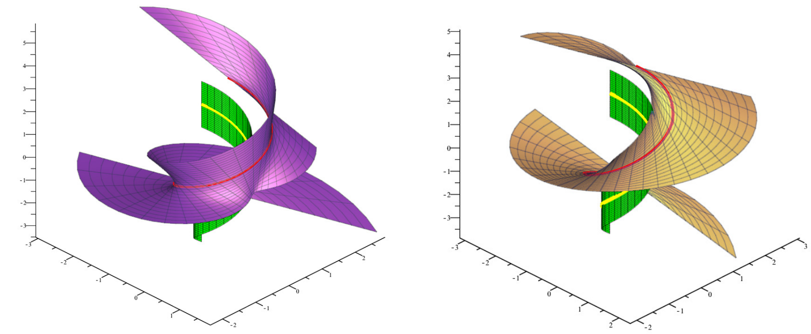
$$\varphi^2_{igr}(t,v) = \Big(\frac{4}{\sqrt{41}}\,vt\ \cos(t) - \frac{3}{\sqrt{41}}vt\sin(t) - \frac{16}{5\sqrt{41}}v\ \cos(t) + \frac{12}{5\sqrt{41}}v\ \sin(t) + \frac{3}{5}\ \cos(t) + \frac{4}{5}\ \sin(t),\\ \frac{3}{\sqrt{41}}vt\cos(t) + \frac{4}{\sqrt{41}}vt\sin(t) - \frac{12}{5\sqrt{41}}\,v\cos(t) - \frac{16}{5\sqrt{41}}\,v\sin(t) - \frac{4}{5}\ \cos(t)\frac{3}{5}\ \sin(t),\\ \frac{4}{5}t + \frac{3}{5} + \frac{4}{\sqrt{41}}vt + \frac{5}{\sqrt{41}}v\Big),$$

by taking $f(t) = t$ and $\beta_1(t) = 1$.

b) For an isoasymptotic Bertrand D- curve included ruled surface, we have the parametrization:

$$\varphi^2_{iar}(t,v) = \Big(\frac{4}{\sqrt{41}}\,vt\cos(t) - \frac{3}{\sqrt{41}}\,vt\sin(t) - \frac{3}{5}v\cos(t) - \frac{4}{5}\,v\sin(t) + \frac{3}{5}\ \cos(t) + \frac{4}{5}\ \sin(t),\\ \frac{3}{\sqrt{41}}\,vt\cos(t) + \frac{4}{\sqrt{41}}\,vt\sin(t) + \frac{4}{5}\,v\cos(t) - \frac{3}{5}\,v\sin(t) - \frac{4}{5}\ \cos(t) + \frac{3}{5}\ \sin(t),\\ \frac{4}{5}\,t + \frac{3}{5} + \frac{4}{\sqrt{41}}\,vt\Big),$$

with the functions $h(t) = t$ and $\beta_2(t) = 1$. Figure 4 shows these final set of surfaces.



(a) The ruled surface φ^2_{igr} with isogeodesic curve

(b) The ruled surface φ^2_{iar} with isoasymptotic curve

Figure 4. Bertrand D- ruled surface mates for the fourth example

4. Conclusion

The idea of associated curves by a moving frame is a popular concept in the theory of curves. Recent studies show that this association has seemingly been extended to different frames like Darboux. On the other hand, the construction of the new types of surfaces has the potential to lead to new perspectives for related areas such as industrial engineering, computer-aided designs, and so on. By this study, we have provided the mathematical formulations to model a new surface family such that the Bertrand D- partner of any given curve lies on as isogeodesic, isoasymptotic or curvature line. We have also examined the forms of these surfaces and provide some characterizations. By following the same manner, we have introduced the family of ruled surfaces and discussed their developability conditions in the context. Finally, we have presented some examples and pictured the graph of relevant surfaces.

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