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# A New Finite Difference Scheme for High-Dimensional Heat Equation

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## Abstract

In this research, a new second-order finite difference scheme is proposed to solve two and three-dimensional heat equation. Finite difference equations are determined via a discretization approach in which spatial second order partial derivatives in x and y directions are approximated simultaneously while in the classic method, each spatial partial derivative is replaced by a central finite difference approximation, separately. By this new discretization scheme and also using the forward difference to the first-order time derivative, a finite difference equation is obtained for the parabolic equation. This approach is explicit and similar to other explicit approaches, an interval for the Courant number, $r$ is determined. This region for $r$ is obtained through Fourier stability analysis. The advantage of this approach is that its stability interval is larger than the interval for traditional methods. Numerical experiments are presented to confirm the theoretical results. It is shown that more accurate approximations can be obtained by the new scheme.

**Keywords**: Finite difference method; parabolic equations; central differences; discretization; stability; Taylor series; Courant number

**MSC 2020 No.**: 65M06, 65M12

## 1. Introduction

For many years, finite difference method (FDM) has been known as the most practical and dominant numerical scheme for solving differential equations, appeared in various scientific fields.

By this numerical technique, both spatial and time derivatives can be approximated with difference quotients which are originally found by Taylor's expansion. Finally, a fully or semi discretized equation is derived. This method makes it possible to have a complete view of the problem in order to monitor the calculations. Moreover, different boundary and initial conditions can be treated flexibly (Yam and Cheng (1993)). This method is somewhat old, as its first utilization is attributed to Euler in 1768 (Blazek (2005)), but it is widely applied today. Recently, Patel et al. (2019) have used FDM to solve 3rd order ODE numerically. Patil and Maniyeri (2019) have analyzed bio-heat transfer in human breast cyst by FDM. Dujardin et al (2018) have successfully applied FDM to determine electron energy levels in quantum dots arrays. Erickson et al. (2017) have developed a finite difference method for off-fault plasticity throughout the earthquake cycle. Sweilam et al. (2012) have successfully implemented finite difference method (FDM) and Pade-variational iteration method (Pade- VIM) solving the nonlinear fractional Riccati differential equation. Many other researchers have developed different kinds of this method including generalized FDM (Gu et al. (2019)), non-standard FDM (Sayevand et al. 2018), compact FDM (Biazar and Asayesh (2020), During and Pitkin (2019)), and some others.

The heat equation, a well-known parabolic partial differential equation, was developed by one of the most gifted scholars, Joseph Fourier at the beginning of the nineteenth century in order to model the conduction of heat through a certain media. It has been a powerful tool over the two past centuries for analyzing the dynamic motion of heat as well as for solving numerous diffusion-type problems in physics, biologics, geology, and so on (Narasimhan (1999)). Many researchers have studied this equation by different numerical methods. For example, Wang (2009) constructed the explicit difference schemes for the one-dimensional heat equation. Recently, Choi et al. (2020) have derived a general solution of the heat equation through the use of the similarity reduction method. The stability and stabilization of heat equation in non-cylindrical domain have been studied by Li and Gao (2020).

## 2. Description of the method

Here, the following well-known parabolic partial differential equation namely heat equation is considered.

$$\frac{\partial u}{\partial t} = \alpha \Delta u, \tag{1}$$

where $u$ is the temperature at a certain position and time $t$. $\Delta$ denotes the Laplace operator taken in the spatial variables, and $\alpha$ is a positive real coefficient called the diffusivity of the media. However, in dimensionless form of the equation, it is often, set $\alpha = 1$. In a (2+1)- dimensional Cartesian coordinates, (1) has the following form.

$$u_t = u_{xx} + u_{yy}. \tag{2}$$

This equation is called the two- dimensional sourceless heat equation (Polyanin 2002).

In order to discretize (2), by traditional FDM over a (2+1)-dimensional domain like $[a,b]\times[c,d]\times[0,l]$ $\Delta x = (b-a)/M$ , and $\Delta y = (c-d)/N$ ,

$\Delta t = l / T$ is constructed. The computational grid contains $(M+1)\times(N+1)\times(T+1)$ nodes $(x_i, y_j, t_n)$ where $x_i = a + i\,\Delta x$, $i = 0,\ldots,M$, $y_j = c + j\,\Delta y$, $j = 0,\ldots,N$, and $t_n = n\Delta t$, $n = 0,\ldots,T$.

Using finite difference approximations for each derivative term leads to the following explicit difference equation at interior points.

$$u_{i,j}^{n+1} = u_{i,j}^{n} + \Delta t\left(\frac{u_{i-1,j}^{n} - 2u_{i,j}^{n} + u_{i+1,j}^{n}}{\Delta x^2} + \frac{u_{i,j-1}^{n} - 2u_{i,j}^{n} + u_{i,j+1}^{n}}{\Delta y^2}\right), \tag{3}$$

where $u_{i,j}^{n}$ approximates $u(x_i, y_j, t_n)$.

By the assumption $h = \Delta x = \Delta y$, (3) can be expressed as follows.

$$u_{i,j}^{n+1} = (1-4r)u_{i,j}^{n} + \left(u_{i-1,j}^{n} + u_{i+1,j}^{n} + u_{i,j-1}^{n} + u_{i,j+1}^{n}\right), \tag{4}$$

where $r = \Delta t / h^2$ is the Courant number. This equation has a local truncation error of order 2, i.e., $O(h^2)$. The stability condition of (4) is $r \le \frac{1}{4}$. However, in one-dimensional case, the stability condition is $r \le \frac{1}{2}$ (Evans et al. (2012)). It can be said that by increasing dimensions, the interval for the Courant number becomes smaller.

In (3+1)-dimensional Cartesian coordinates, (1) has the following form.

$$u_t = u_{xx} + u_{yy} + u_{zz}. \tag{5}$$

This equation governs on heat diffusion in a three-dimensional media (Polyanin 2002). Applying the traditional finite difference method for (5) leads to

$$u_{i,j,k}^{n+1} = u_{i,j,k}^{n} + \Delta t\left(\begin{array}{l}\dfrac{u_{i-1,j,k}^{n} - 2u_{i,j,k}^{n} + u_{i+1,j,k}^{n}}{\Delta x^2}\\ + \dfrac{u_{i,j-1,k}^{n} - 2u_{i,j,k}^{n} + u_{i,j+1,k}^{n}}{\Delta y^2} + \dfrac{u_{i,j,k-1}^{n} - 2u_{i,j,k}^{n} + u_{i,j,k+1}^{n}}{\Delta z^2}\end{array}\right).$$

By considering $h := \Delta x = \Delta y = \Delta z$, the last formula can be restated as

$$u_{i,j,k}^{n+1} = (1-6r)u_{i,j,k}^{n} + \left(u_{i-1,j,k}^{n} + u_{i+1,j,k}^{n} + u_{i,j-1,k}^{n} + u_{i,j+1,k}^{n} + + u_{i,j,k-1}^{n} + u_{i,j,k+1}^{n}\right). \tag{6}$$

Notice that $u_{i,j,k}^{n}$ approximates $u(x_i, y_j, z_k, t_n)$. The local truncation error of this equation is also $O(h^2)$. It can be shown that (6) is stable while $r \le \frac{1}{6}$. This interval is smaller than those in one- and two-dimensional cases. However, by using the Crank-Nicolson method, an implicit unconditionally stable equation can be obtained. In this research a new approach is addressed that leads an explicit approach with larger stability domain.

In the following, by using Taylor expansion, new formulas are developed for (2+1)- and (3+1)-dimensional equations.

### 2.1. (2+1)-dimensional equation

Consider the (2+1)- dimensional parabolic equation

$$u_t = u_{xx} + u_{yy}, \qquad (x, y, t) \in \Omega \times (0, T], \tag{7}$$

with the following initial and Dirichlet boundary conditions.

$$\begin{cases} u(x, y, 0) = G(x, y), & (x, y) \in \Omega, \\ u(x, y, t) = H(x, y, t), & (x, y, t) \in \partial\Omega \times (0, T], \end{cases} \tag{8}$$

where $\Omega \equiv [a, b] \times [c, d]$ and $\partial\Omega$ is the boundary of the domain.

Let $h = \Delta x = \Delta y$. Using Taylor expansions for functions of two variables gives

$$\begin{aligned} u_{i+1,j+1} = u_{i,j} &+ h\left(u_x + u_y\right)_{i,j} + \frac{h^2}{2!}\left(u_{xx} + 2u_{xy} + u_{yy}\right)_{i,j} \\ &+ \frac{h^3}{3!}\left(u_{xxx} + 3u_{xxy} + 3u_{xyy} + u_{yyy}\right)_{i,j} \\ &+ \frac{h^4}{4!}\left(u_{xxxx} + 4u_{xxxy} + 6u_{xxyy} + 4u_{xyyy} + u_{yyyy}\right)_{i,j} + \cdots, \end{aligned} \tag{9}$$

$$\begin{aligned} u_{i+1,j-1} = u_{i,j} &+ h\left(u_x - u_y\right)_{i,j} + \frac{h^2}{2!}\left(u_{xx} - 2u_{xy} + u_{yy}\right)_{i,j} \\ &+ \frac{h^3}{3!}\left(u_{xxx} - 3u_{xxy} + 3u_{xyy} - u_{yyy}\right)_{i,j} \\ &+ \frac{h^4}{4!}\left(u_{xxxx} - 4u_{xxxy} + 6u_{xxyy} - 4u_{xyyy} + u_{yyyy}\right)_{i,j} + \cdots, \end{aligned} \tag{10}$$

$$u_{i-1,j+1} = u_{i,j} + h\left(-u_x + u_y\right)_{i,j} + \frac{h^2}{2!}\left(u_{xx} - 2u_{xy} + u_{yy}\right)_{i,j} + \frac{h^3}{3!}\left(-u_{xxx} + 3u_{xxy} - 3u_{xyy} + u_{yyy}\right)_{i,j} + \frac{h^4}{4!}\left(u_{xxxx} - 4u_{xxxy} + 6u_{xxyy} - 4u_{xyyy} + u_{yyyy}\right)_{i,j} + \cdots, \tag{11}$$

$$u_{i-1,j-1} = u_{i,j} + h\left(-u_x - u_y\right)_{i,j} + \frac{h^2}{2!}\left(u_{xx} + 2u_{xy} + u_{yy}\right)_{i,j} + \frac{h^3}{3!}\left(-u_{xxx} - 3u_{xxy} - 3u_{xyy} - u_{yyy}\right)_{i,j} + \frac{h^4}{4!}\left(u_{xxxx} + 4u_{xxxy} + 6u_{xxyy} + 4u_{xyyy} + u_{yyyy}\right)_{i,j} + \cdots, \tag{12}$$

where indices $x$ and $y$ with different numbers of iterations denote different orders of derivatives respect to $x$ and $y$. Adding both sides of equations (9)-(12) to each other leads to the following difference equation.

$$u_{i+1,j+1} + u_{i+1,j-1} + u_{i-1,j+1} + u_{i-1,j-1} = 4u_{i,j} + 2h^2(u_{xx} + u_{yy})_{i,j} + O\left(h^4\right).$$

Hence, a second order approximation of $\Delta u$ for the grid point $(x_i, y_j)$ is obtained as follows.

$$(u_{xx} + u_{yy})_{i,j} = \frac{u_{i-1,j+1} + u_{i-1,j-1} - 4u_{i,j} + u_{i+1,j+1} + u_{i+1,j-1}}{2h^2} + O\left(h^2\right). \tag{13}$$

Applying (13) and forward difference approximation for the first derivative of time in (7) results in the following difference equation.

$$u_{i,j}^{n+1} = \left(1-2r\right)u_{i,j}^{n} + \frac{r}{2}\left(u_{i-1,j-1}^{n} + u_{i-1,j+1}^{n} + u_{i+1,j+1}^{n} + u_{i+1,j-1}^{n}\right), \tag{14}$$

where $r = \dfrac{\Delta t}{h^2}$.

Figure 1 shows the mesh points used in the new finite difference equation (14). Blue points are the known values of the function $u$ at the $n^{th}$ time step. The red point is the unknown value of the function $u$ at the $\left(n+1\right)^{th}$ time step which is going to be found.

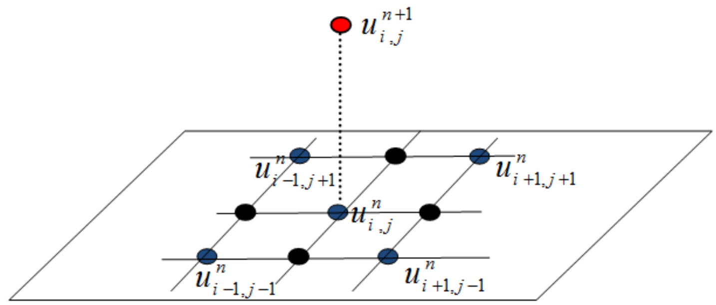


Figure 1. The mesh points used in the new finite difference equation (14)

## 2.2. (3+1)-dimensional equation

Consider the (3+1)-dimensional parabolic equation

$$u_t = u_{xx} + u_{yy} + u_{zz}, \qquad (x,y,z,t) \in \Omega \times (0,T], \tag{15}$$

with initial and Dirichlet boundary conditions.

$$\begin{cases} u(x,y,z,0) = G(x,y,z), & (x,y,z) \in \Omega \\ u(x,y,z,t) = H(x,y,z,t), & (x,y,t) \in \partial\Omega \times (0,T], \end{cases} \tag{16}$$

where $\Omega \equiv [a,b] \times [c,d] \times [e,f]$ and $\partial\Omega$ is the boundary of the domain.

Similar to the previous subsection, it is assumed that $h = \Delta x = \Delta y = \Delta z$. Using Taylor expansions for functions of three variables, following equations are obtained.

$$\begin{aligned} u_{i+1,j+1,k+1} &= u_{i,j,k} + h\left(u_x + u_y + u_z\right)_{i,j,k} + \frac{h^2}{2!}\left(u_{xx} + u_{yy} + u_{zz} + 2(u_{xy} + u_{xz} + u_{yz})\right)_{i,j,k} \\ &+ \frac{h^3}{3!}\left(u_{xxx} + u_{yyy} + u_{zzz} + 3(u_{xxy} + u_{xxz} + u_{xyy} + u_{xzz} + u_{yyz} + u_{yzz}) + 6u_{xyz}\right)_{i,j,k} + \cdots, \end{aligned} \tag{17}$$

$$\begin{aligned} u_{i+1,j+1,k-1} &= u_{i,j,k} + h\left(u_x + u_y - u_z\right)_{i,j,k} + \frac{h^2}{2!}\left(u_{xx} + u_{yy} + u_{zz} + 2(u_{xy} - u_{xz} - u_{yz})\right)_{i,j,k} \\ &+ \frac{h^3}{3!}\left(u_{xxx} + u_{yyy} - u_{zzz} + 3(u_{xxy} - u_{xxz} + u_{xyy} + u_{xzz} - u_{yyz} + u_{yzz}) - 6u_{xyz}\right)_{i,j,k} + \cdots, \end{aligned} \tag{18}$$

$$\begin{aligned} u_{i+1,j-1,k+1} &= u_{i,j,k} + h\left(u_x - u_y + u_z\right)_{i,j,k} + \frac{h^2}{2!}\left(u_{xx} + u_{yy} + u_{zz} + 2(-u_{xy} + u_{xz} - u_{yz})\right)_{i,j,k} \\ &+ \frac{h^3}{3!}\left(u_{xxx} - u_{yyy} + u_{zzz} + 3(-u_{xxy} + u_{xxz} + u_{xyy} + u_{xzz} + u_{yyz} - u_{yzz}) - 6u_{xyz}\right)_{i,j,k} + \cdots, \end{aligned} \tag{19}$$

$$u_{i+1,j-1,k-1}=u_{i,j,k}+h\left(u_x-u_y-u_z\right)_{i,j,k}+\frac{h^2}{2!}\left(u_{xx}+u_{yy}+u_{zz}+2(-u_{xy}-u_{xz}+u_{yz})\right)_{i,j,k}$$
$$+\frac{h^3}{3!}\left(u_{xxx}-u_{yyy}-u_{zzz}+3(-u_{xxy}-u_{xxz}+u_{xyy}+u_{xzz}-u_{yyz}-u_{yzz})+6u_{xyz}\right)_{i,j,k}+\cdots, \tag{20}$$

$$u_{i-1,j+1,k+1}=u_{i,j,k}+h\left(-u_x+u_y+u_z\right)_{i,j,k}+\frac{h^2}{2!}\left(u_{xx}+u_{yy}+u_{zz}+2(-u_{xy}-u_{xz}+u_{yz})\right)_{i,j,k}$$
$$+\frac{h^3}{3!}\left(-u_{xxx}+u_{yyy}+u_{zzz}+3(u_{xxy}+u_{xxz}-u_{xyy}-u_{xzz}+u_{yyz}+u_{yzz})-6u_{xyz}\right)_{i,j,k}+\cdots, \tag{21}$$

$$u_{i-1,j+1,k-1}=u_{i,j,k}+h\left(-u_x+u_y-u_z\right)_{i,j,k}+\frac{h^2}{2!}\left(u_{xx}+u_{yy}+u_{zz}+2(-u_{xy}+u_{xz}-u_{yz})\right)_{i,j,k}$$
$$+\frac{h^3}{3!}\left(-u_{xxx}+u_{yyy}-u_{zzz}+3(u_{xxy}-u_{xxz}-u_{xyy}-u_{xzz}-u_{yyz}+u_{yzz})+6u_{xyz}\right)_{i,j,k}+\cdots, \tag{22}$$

$$u_{i-1,j-1,k+1}=u_{i,j,k}+h\left(-u_x-u_y+u_z\right)_{i,j,k}+\frac{h^2}{2!}\left(u_{xx}+u_{yy}+u_{zz}+2(u_{xy}-u_{xz}-u_{yz})\right)_{i,j,k}$$
$$+\frac{h^3}{3!}\left(-u_{xxx}-u_{yyy}+u_{zzz}+3(-u_{xxy}+u_{xxz}-u_{xyy}-u_{xzz}+u_{yyz}-u_{yzz})+6u_{xyz}\right)_{i,j,k}+\cdots, \tag{23}$$

$$u_{i-1,j-1,k-1}=u_{i,j,k}+h\left(-u_x-u_y-u_z\right)_{i,j,k}+\frac{h^2}{2!}\left(u_{xx}+u_{yy}+u_{zz}+2(u_{xy}+u_{xz}+u_{yz})\right)_{i,j,k}$$
$$+\frac{h^3}{3!}\left(-u_{xxx}-u_{yyy}-u_{zzz}+3(-u_{xxy}-u_{xxz}-u_{xyy}-u_{xzz}-u_{yyz}-u_{yzz})-6u_{xyz}\right)_{i,j,k}+\cdots. \tag{24}$$

Adding two sides of equations (17)-(24) to each other results in

$$u_{i+1,j+1,k+1}+u_{i+1,j+1,k-1}+u_{i+1,j-1,k+1}+u_{i+1,j-1,k-1}+u_{i-1,j+1,k+1}$$
$$+u_{i-1,j+1,k-1}+u_{i-1,j-1,k+1}+u_{i-1,j-1,k-1}=8u_{i,j}+4h^2(u_{xx}+u_{yy}+u_{zz})_{i,j,k}+O\left(h^4\right). \tag{26}$$

This formula gives a second order approximation for $\Delta u$ in the grid point $(x_i, y_j, z_k)$ as follows:

$$(u_{xx}+u_{yy}+u_{zz})_{i,j}=$$
$$\frac{u_{i+1,j+1,k+1}+u_{i+1,j+1,k-1}+u_{i+1,j-1,k+1}+u_{i+1,j-1,k-1}-8u_{i,j}+u_{i-1,j+1,k+1}+u_{i-1,j+1,k-1}+u_{i-1,j-1,k+1}+u_{i-1,j-1,k-1}}{4h^2}$$
$$+O\left(h^2\right).$$

Applying the last approximation for the second order spatial partial derivatives and also forward difference formula for the first derivative of time for (15) leads to

$$\begin{aligned} u_{i,j,k}^{n+1} &= (1-2r)u_{i,j,k}^{n} \\ &\quad + \frac{r}{4}\Big(u_{i-1,j+1,k+1}^{n} + u_{i-1,j+1,k-1}^{n} + u_{i-1,j-1,k+1}^{n} + u_{i-1,j-1,k-1}^{n} \\ &\qquad + u_{i+1,j+1,k+1}^{n} + u_{i+1,j+1,k-1}^{n} + u_{i+1,j-1,k+1}^{n} + u_{i+1,j-1,k-1}^{n}\Big), \end{aligned} \tag{27}$$

where $r = \dfrac{\Delta t}{h^2}$.

In Figure 2, nine interior mesh points used in the new finite difference equation are illustrated in three spatial dimensions. The blue points are the known values of the function $u$ at the $n^{th}$ time step. However, it is impossible to show the point of the unknown value of the function $u$ at the $(n+1)^{th}$ time step.

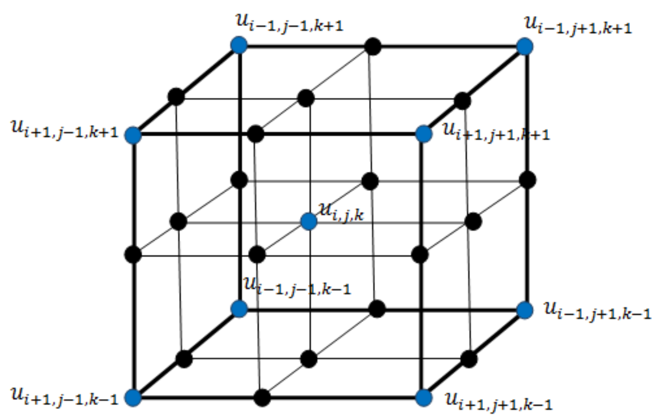


**Figure** 2. Mesh points used in the new finite difference equation (27)

## 3. Consistency, stability, and convergence

In this section, proofs of consistency, stability, and convergence of the proposed numerical method are presented.

### 3.1. Consistency

To prove the consistency of the proposed scheme for the heat equation, some assumptions on the regularity of the exact solution are needed. This regularity is expressed as the existence of Taylor approximations of the exact solution up to some appropriate order (Boldo et al. (2014)).

The consistency of the new finite difference equations are investigated by generalizing the definition of consistency for (1+1)- dimensional equations presented by Evans et al. (2012) to (2+1)- and (3+1) dimensional equations.

Let $L(\Phi)=0$ be a partial differential equation in independent variables $x$ , $y$ , and $t$ with exact solution $\Phi$. Let $F(\varphi)=0$ be the finite difference equation with exact solution $\varphi$, and let $\nu$ be any continuous function of $x$ , $y$ , and $t$ , with a sufficient number of continuous derivatives such that $L(\nu)$ can be evaluated at $\left(x_i, y_j, t_n\right)$. The truncation error is $T_{i,j}^{n}(\upsilon)=F\left(\upsilon_{i,j}^{n}\right)-L\left(\upsilon_{i,j}^{n}\right)$ at point $\upsilon_{i,j}^{n}=\upsilon\left(x_i, y_j, t_n\right)$. The difference equation is said to be consistent with partial differential equation when $T_{i,j}^{n}(\upsilon)\to 0$ as $\Delta x\to 0,\ \Delta y\to 0,\ \Delta t\to 0$.
If it is put $\upsilon=\Phi$ $(L(\Phi)=0)$ then $T_{i,j}^{n}(\Phi)=F\left(\Phi_{i,j}^{n}\right)$ and the truncation error coincides with $\tau_{i,j}^{n}$ , the local truncation error at the mesh point $\left(x_i, y_j, t_n\right)$. Therefore, the difference equation is consistence if the limiting value of the local truncation error is zero as $\Delta x\to 0,\ \Delta y\to 0,\ \Delta t\to 0$ . This definition of consistency can be expressed for a partial differential equation in 4 independent variables $x$ , $y$ , $z$ , and $t$ in a similar manner. Then, $\tau_{i,j,k}^{n}$ would be the local truncation error at the mesh point $\left(x_i, y_j, z_k, t_n\right)$.

The local truncation error of (14) is as follows:

$$\tau_{i,j}^{n}=\left(\frac{\partial\Phi}{\partial t}-\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right)\right)_{i,j}^{n}+\frac{k^2}{2}\left(\frac{\partial^2\Phi}{\partial t^2}\right)_{i,j}^{n}-\frac{h^2}{6}\left(\frac{\partial^4\Phi}{\partial x^4}+6\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}\right)_{i,j}^{n}$$
$$+\frac{k^3}{6}\left(\frac{\partial^3\Phi}{\partial t^3}\right)_{i,j}^{n}-\frac{h^4}{180}\left(\frac{\partial^6\Phi}{\partial x^6}+15\left(\frac{\partial^6\Phi}{\partial x^2\partial y^4}+\frac{\partial^6\Phi}{\partial x^4\partial y^2}\right)+\frac{\partial^6\Phi}{\partial y^6}\right)_{i,j}^{n}+\cdots.$$

As

$$\left(\frac{\partial\Phi}{\partial t}-\left(\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right)\right)_{i,j}^{n}=0,$$

the principal part of $\tau_{i,j}^{n}$ would be as

$$\frac{k^2}{2}\left(\frac{\partial^2\Phi}{\partial t^2}\right)_{i,j}^{n}-\frac{h^2}{6}\left(\frac{\partial^4\Phi}{\partial x^4}+6\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}\right)_{i,j}^{n},$$

which implies $\tau_{i,j}^{n}=O\left(k\right)+O\left(h^2\right)$. According to the definition of consistency, the explicit approximation (14) is consistent with the differential equation (7).

The local truncation error of (27) is as the following.

$$\tau_{i,j,k}^{n}=\left(\frac{\partial \Phi}{\partial t}-\left(\frac{\partial^{2}\Phi}{\partial x^{2}}+\frac{\partial^{2}\Phi}{\partial y^{2}}+\frac{\partial^{2}\Phi}{\partial z^{2}}\right)\right)_{i,j,k}^{n}+\frac{k^{2}}{2}\left(\frac{\partial^{2}\Phi}{\partial t^{2}}\right)_{i,j,k}^{n}$$
$$-\frac{h^{2}}{3}\left(6\left(\frac{\partial^{4}\Phi}{\partial x^{2}\partial y^{2}}+\frac{\partial^{4}\Phi}{\partial x^{2}\partial z^{2}}+\frac{\partial^{4}\Phi}{\partial y^{2}\partial z^{2}}\right)+\frac{\partial^{4}\Phi}{\partial x^{4}}+\frac{\partial^{4}\Phi}{\partial y^{4}}+\frac{\partial^{4}\Phi}{\partial z^{4}}\right)_{i,j,k}^{n}+\cdots.$$

Since

$$\left(\frac{\partial \Phi}{\partial t}-\left(\frac{\partial^{2}\Phi}{\partial x^{2}}+\frac{\partial^{2}\Phi}{\partial y^{2}}+\frac{\partial^{2}\Phi}{\partial z^{2}}\right)\right)_{i,j,k}^{n}=0,$$

the principal part of $\tau_{i,j,k}^{n}$ would be as

$$\frac{k^{2}}{2}\left(\frac{\partial^{2}\Phi}{\partial t^{2}}\right)_{i,j,k}^{n}-\frac{h^{2}}{3}\left(6\left(\frac{\partial^{4}\Phi}{\partial x^{2}\partial y^{2}}+\frac{\partial^{4}\Phi}{\partial x^{2}\partial z^{2}}+\frac{\partial^{4}\Phi}{\partial y^{2}\partial z^{2}}\right)+\frac{\partial^{4}\Phi}{\partial x^{4}}+\frac{\partial^{4}\Phi}{\partial y^{4}}+\frac{\partial^{4}\Phi}{\partial z^{4}}\right)_{i,j,k}^{n}.$$

Hence $\tau_{i,j,k}^{n}=O(k)+O(h^{2})$. As a result, the explicit approximation (27) is consistent with the partial differential equation (15).

### 3.2. Stability analysis

Here, the stability of the suggested method is analyzed by the standard Fourier stability analysis. It is assumed that the numerical solution of (14), by using the Fourier series, can be expressed as follows.

$$u_{i,j}^{n}=\chi^{n}e^{ik_x\Delta xI}e^{jk_y\Delta yI}, \tag{28}$$

where $I=\sqrt{-1}$, $\chi^{n}$ is the amplitude at the time level $n$, and $k_x$, $k_y$ are the wave numbers in the $x$ - and $y$ - directions, respectively (Tian and Yu (2010)). Substituting (28) into (14) gives

$$\chi=(1-2r)+\frac{r}{2}\left(e^{-(\xi+\eta)I}+e^{-(\xi-\eta)I}+e^{(\xi+\eta)I}+e^{(\xi-\eta)I}\right), \tag{29}$$

where $\xi:=k_x\Delta x$ and $\eta=k_y\Delta y$. According to the formula

$$e^{-\hat{i}\alpha}-2+e^{\hat{i}\alpha}=-4\sin^{2}\frac{\alpha}{2}, \tag{30}$$

(29) can be expressed trigonometrically as follows:

$$\chi=1-2r\left(\sin^{2}\frac{\lambda}{2}+\sin^{2}\frac{\gamma}{2}\right),$$

where $\lambda := \xi + \eta$ and $\gamma := \xi - \eta$.

For stability, it is sufficient to show that $|\chi| \le 1$. Therefore,

$$-1 \le 1 - 2r\left(\sin^2\frac{\lambda}{2} + \sin^2\frac{\gamma}{2}\right) \le 1.$$

The right-hand side of this inequality is trivial. However, the left-hand side yields to the condition $r \le \frac{1}{2}$. This interval is obviously larger than the one related to the traditional method (4).

A similar procedure is implemented in three spatial dimensions. First, stability condition of (6) is obtained. Suppose that the numerical solution of (6) can be expressed as follows.

$$u^n = \chi^n e^{ik_x \Delta x I} e^{jk_y \Delta y I} e^{kk_z \Delta z I}, \tag{31}$$

where $I = \sqrt{-1}$, $\chi^n$ is the amplitude at the time level $n$, and $k_x$, $k_y$, $k_z$ are the wave numbers in the $x-$, $y-$, and $z-$directions, respectively. Substituting (31) into (6) leads to

$$\chi = (1 - 6r) + r\left(e^{-\xi I} + e^{\xi I} + e^{-\eta I} + e^{\eta I} + e^{-\zeta I} + e^{\zeta I}\right),$$

where $\xi = k_x \Delta x$, $\eta := k_y \Delta y$, and $\zeta = k_z \Delta z$. According to the formula (30), the trigonometric expression of the above equation is as follows:

$$\chi = 1 - 4r\left(\sin^2\frac{\xi}{2} + \sin^2\frac{\eta}{2} + \sin^2\frac{\zeta}{2}\right).$$

This equation implies $r \le \frac{1}{6}$.

Putting (31) into (27) gives

$$\begin{aligned}\chi = (1 - 2r) + \frac{r}{4}\Big(&e^{-(\xi-\eta-\zeta)I} + e^{-(\xi-\eta+\zeta)I} + e^{-(\xi+\eta-\zeta)I} + e^{-(\xi+\eta+\zeta)I} \\ &+ e^{(\xi+\eta+\zeta)I} + e^{(\xi+\eta-\zeta)I} + e^{(\xi-\eta+\zeta)I} + e^{(\xi-\eta-\zeta)I}\Big).\end{aligned} \tag{32}$$

Using formula (30) and the condition $|\chi| \le 1$ for the stability, results in $r \le \frac{1}{4}$.

### 3.3. Convergence

A numerical scheme is called convergent, if in the limit of infinitesimal discretization, the bound on the discretization error is also infinitesimally small. Under these conditions, the numerical solution converges or approaches the analytic solution. This idea is formally articulated by the Lax equivalence theorem (Lax and Richtmyer (1956)), which states that if a numerical method is consistent and stable, then it is convergent (Tekriwal et al. (2021)).

The Lax equivalence theorem (Evans et al. (2012)) states that "Let $\Phi$ be the exact solution of a partial differential equation and $\varphi$ be the exact solution of the related finite difference equation. When the finite difference equation is consistent with a well posed initial-value problem then stability is necessary and sufficient for convergence of $\varphi$ to $\Phi$ as the mesh lengths tend to zero." The consistency and stability of the method has been shown in sections 3.1 and 3.2. Thus, according to the Lax equivalence theorem it can be said the method is convergent.

## 4. Numerical experiments

In this section, the proposed formulas are applied for two and three spatial dimensional diffusion equations.

For simplicity, the space domain $\Omega=[0,1]\times[0,1]$ for the (2+1)- dimensionl equations and $\Omega=[0,1]\times[0,1]\times[0,1]$ for (3+1)- dimensionl equations are fixed. All computations are performed by Maple 2018.

**Example 1.**

Consider two-dimensional diffusion equation (7) while Dirichlet boundary condition (8) and initial condition (9), $G(x,y)$ and $H(x,y,t)$ functions, are properly given with the following exact solution.

$$u(x,y,t)=e^{-2t}\sin x\,\sin y. \tag{33}$$

This problem is solved by the new difference formula (14) for various values of *r*. The absolute maximum error is computed as follows:

$$Error=\max_{1\le i\le M-1,\ 1\le j\le N-1}\left|u_{ij}^{n}-u_{exact}(x_i,y_j,t_n)\right|.$$

Numerical results are presented in Tables 1 and 2 at $t=1$. In Table 1, the spatial mesh size is fixed $h:=\Delta x=\Delta y=1/4$ and different values for $\Delta t$ are considered. In Table 2, the time mesh size is fixed at $\Delta t=1/1000$ and different values for $h$ are considered. Tables 1, and 2 confirm this fact that while $r$ is less than $1/2$, good approximations are obtained by the new approaches.

**Table 1.** Maximum errors for example 1 at t=1 with fixed mesh size *h*=1/4

| $\Delta t$ | $r$ | Maximum error of the proposed method | Maximum error of the traditional FDM |
|---|---|---|---|
| 0.01 | 0.16 | 6.2124e-5 | 2.2694e-5 |
| 0.02 | 0.32 | 5.3463e-6 | 0.0103 |
| 0.03125 | 0.50 | 5.7434e-5 | 6.6160e+6 |

| | | | |
|---|---|---|---|
| 0.04 | 0.64 | 1.0953e-4 | 7.6984e+7 |
| 0.05 | 0.80 | 0.1016 | 6.2211e+6 |

**Table 2.** Maximum errors for example 1 at t=1 with fixed time step $\Delta t = 1/1000$

| $h$ | $r$ | Maximum error of the proposed method | Maximum error of the traditional FDM |
|---|---|---|---|
| 1 / 10 | 0.1 | 1.3043e-5 | 9.0370e-7 |
| 1 / 15 | 0.225 | 2.659e-6 | 3.4302e-6 |
| 1 / 20 | 0.40 | 9.1446e-7 | 2.4576e+327 |
| 1 / 22 | 0.484 | 1.7031e-6 | 1.5734e+444 |
| 1 / 25 | 0.625 | 2.1636e+161 | 2.8692e+588 |

**Example 2.**

Consider the (3+1)- dimensional equation (15), while Dirichlet boundary condition (16) and initial condition (17), $G(x,y,z)$ and $H(x,y,z,t)$ functions, are properly given by the following exact solution:

$$u(x,y,t) = e^{-2t}\sin x\,\sin y\,\sin z\,. \tag{34}$$

This equation is solved by new formula (27) for various values of $r$. The maximum of absolute error defined as the following is computed for both the traditional and proposed finite difference methods.

$$Error = \max_{1\le i\le M-1,\ 1\le j\le N-1, 1\le k\le P-1} \left| u_{ijk}^{n} - u_{exact}(x_i, y_j, z_k, t_n) \right|,$$

The numerical results are presented in Tables 3 and 4 for different values of $r$ at $t = 1$.

**Table 3.** Maximum errors for example 2 at t=1 with fixed mesh size $h$=1/4

| $\Delta t$ | $r$ | Maximum error of the proposed method | Maximum error of the traditional FDM |
|---|---|---|---|
| 0.01 | 0.16 | 3.7983e-5 | 9.5403e-6 |
| 0.02 | 0.32 | 1.1571e-5 | 2.3619e+11 |
| 0.03125 | 0.50 | 1.7412e-5 | 2.5158e+13 |
| 0.04 | 0.64 | 3.9456e-5 | 2.8294e+12 |
| 0.05 | 0.80 | 4.9453e-3 | 1.17164e+11 |

**Table 4.** Maximum errors for example 2 at t=1 with fixed time step $\Delta t = 1/1000$

| $h$ | $r$ | Maximum error of the proposed method | Maximum error of the traditional FDM |
|---|---|---|---|
| 1 / 10 | 0.1 | 8.0826e-5 | 8.1007e-7 |
| 1 / 15 | 0.225 | 2.0135e-5 | 4.8923e10+211 |
| 1 / 20 | 0.40 | 7.5320e-7 | 4.0798e+564 |
| 1 / 22 | 0.484 | 5.3472e-6 | 1.0825e+667 |
| 1 / 25 | 0.625 | 9.3995e+7 | 5.7277e+798 |

As it is established, in the traditional finite difference method, for stability, the Courant number must be less than or equal to 1/6. While this value should be less than or equal to 1/4 in the new method. Tables 3 and 4 demonstrate this fact. However, in Tables 3 and 4, when $r$ is even a bit more than 1/4, good approximations are obtained by the new formula. One may say it is because the stability condition is the sufficient but not necessary condition.

## 5. Conclusions

In this paper, following the main strategy of finite differences method, that is to say, implementation of Taylor expansion, a new explicit scheme of discretization of two and three-dimensional parabolic equations is obtained. Unlike the traditional FDM in which approximations of spatial derivatives are determined separately, this new method proposes only one formula to approximate the summation of all second-order spatial derivatives. On the other hand, similar to the traditional FDM, this new method is of the second-order accuracy. Stability of the proposed scheme is analyzed by Fourier method. It is shown that this scheme, like other explicit schemes, is stable provided that the Courant number, $r$, is bounded. However, the interval found for $r$ is enlarged. It is larger than those obtained in the literature for the traditional methods. This can be the superiority of the suggested method to the traditional ones. This method is implemented for two examples. One for the (2+1)- dimensional parabolic equation and the other for (3+1)-dimensional equation. The results are shown in Tables 1 - 4. In these tables, the maximum of absolute errors of both methods are presented in order to compare them with each other. Comparison of the results reveals the efficiency of the suggested method. They also confirm the validity of the obtained intervals for the Courant number $r$.

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