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MHD Reiner-Rivlin Liquid Flow Through a Porous Cylindrical Annulus

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Abstract

The present work concerns the steady and unsteady flow of an incompressible Reiner-Rivlin liquid in the porous annular region of two concentric rotating cylinders, which is moving parallel to their axis, about the common axis of these cylinders under uniform magnetic field acted in perpendicular direction of the axis. The electrically conducting flow of Reiner-Rivlin liquid in the annular porous region is governed by the Brinkman equation with the consideration that the effective viscosity of liquid is same as viscosity of the liquid. Analytical expressions for velocity components, pressure gradient and volumetric flow rate are established. Effects of the magnetic field and other flow parameters on the axial and rotational velocity components and flow rate are discussed graphically.

Keywords: Reiner-Rivlin liquid; Porous medium; Brinkman equation; Hartmann Number; Flow rate; Unsteady flow; Reynolds number.

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1. Introduction

In the modern era, research work in non-linear fluid mechanics has a lot of significance due its industrial and medical applications. A lot of research work has been carried out to solve problems which arise in the non-linear fluid mechanics, related to the flow of non-Newtonian fluids like paint, coal tar, ligament, etc. With the study of non-Newtonian fluids, researchers have developed constitutive equations for such fluids which help in explaining the behavior of these fluids. The interest in the study of non-Newtonian fluids has increased, as more and more non-Newtonian fluids are being found in nature. One special kind of non-Newtonian fluid is Reiner-Rivlin liquid whose steady motion has importance due to its applications in petrochemical industries and bio-fluid mechanics. Reiner (1945) and Rivlin (1948) proposed constitutive relations for Reiner-Rivlin liquid.

Kapur (1962) studied flows of the non-Newtonian conducting Reiner-Rivlin liquids in the presence of magnetic fields and observed that cross viscosity does not effect the velocity and magnetic field though it effects the pressure field. Reiner-Rivlin liquid flow in the channel's inlet region studied by Kapur and Gupta (1964). Bagchi (1966) studied the flow through annular region of co-axial porous cylinder of Reiner-Rivlin liquid. Between a pair of porous concentric circular cylinders with magnetic field, unsteady flow of viscous conducting liquid was investigated by Mahapatra (1973). Polar flow past a Reiner-Rivlin fluid sphere was reported by Ramkissoon (1985). The study of Stokes flow past a Reiner-Rivlin liquid sphere was carried out by Ramkissoon (1989). Between two inclined porous planes, unsteady, immiscible flow of viscoelastic Reiner-Rivlin flow was investigated by Sengupta et al. (1992).

Between porous walls of two coaxial circular cylinders, the hydromagnetic flow of Reiner-Rivlin fluid was investigated by Panja et al. (1996). Sengupta and Kundu (2003) investigated MHD flow of Reiner-Rivlin liquid with porous walls of coaxial rotating cylinders of rotating boundary. Applying Happel and Kuwabara boundary conditions, Deo (2004) investigated creeping flow over an assemblage of porous cylinders. Oscillating Couette flow of a viscoelastic Rivlin Ericksen liquid was investigated by Chakraborty and Panja (2009). Gupta and Deo (2010) studied Stokes flow of micropolar fluid over a porous sphere under assumption of non-zero boundary condition for microrotation. Sahoo (2012) studied steady revolving flow of Reiner-Rivlin liquid. Axially-symmetric creeping flow of micropolar fluid past a sphere covered with thin liquid film was studied by Gupta and Deo (2013). Wall effects on Reiner-Rivlin liquid spheroid were investigated by Gupta and Jaiswal (2014). Jaiswal and Gupta (2015) investigated Brinkman flow of viscous fluid past a Reiner-Rivlin liquid sphere dipped in a saturated porous medium. Deo and Ansari (2016) investigated axisymmetric creeping flow through assemblage of porous cylindrical shells. Jaiswal and Gupta (2017) applied cell model technique to report analytical solution of incompressible fluid through assemblage of immiscible Reiner-Rivlin liquid droplets. Chakraborty (2017) was studied transient flow of Reiner-Rivlin liquid between two concentric porous cylinders with magnetic fluid. Das and Sahoo (2018) studied Reiner-Rivlin liquid flow between a pair of infinite rotating co-axial disks. Jaiswal (2019) worked on Stokes flow of Reiner-Rivlin liquid past a deformed sphere. One of the most recent works done in spherical geometry related Reiner-Rivlin liquid, is the study of flow of Reiner-Rivlin liquid spherical particle which is surrounded by a non-Newtonian liquid shell with permeable medium was investigated by Selvi et al. (2020). Due to lack of sufficient analytical solutions of MHD Reiner-Rivlin liquid flow problems for cylindrical geometry, motivate us to carry forward the present research work.

In this work, we have investigated incompressible, magnetohydrodynamics flow of Reiner-Rivlin liquid in the annular porous region of two revolving cylinders, under uniform magnetic field acted in perpendicular direction of the axis. Analytical expressions for velocity components, pressure gradient and volumetric flow rate of the Reiner-Rivlin liquid within the shell for both steady and unsteady cases are obtained. Effects of the magnetic field and other flow parameters on the axial and rotational velocity components and flow rate are discussed.

2. Mathematical formulation

The problem considered in this work related to the steady and unsteady flow of an electrically conducting Reiner-Rivlin liquid in the annular porous region of two coaxial, concentric rotating cylinders of radii a^* and b^* $(a^* < b^*)$. It is further assumed that the inner cylinder is revolved with angular velocity Ω_i^* and the outer cylinder is revolved with angular velocity Ω_o^*, and both cylinders are moving slowly along the common axis of cylinders. The velocities of inner and outer cylinders are $v_{a^*}^*$ and $v_{b^*}^*$, respectively, in the direction parallel to the axis of cylinders. The direction of applied uniform magnetic field $\tilde{\mathrm{B}}^*$ is perpendicular to the axis of the cylinders. By using cylindrical polar co-ordinates (r^*, θ, z^*), having z^* axis along the common axis of cylinders, and assuming that $v_{r^*}^*, v_\theta^*, v_{z^*}^*$ are the velocity components of Reiner-Rivlin liquid in the direction of r^*,θ,z^*, respectively. All the physical quantities are independent of θ due to symmetry of the flow and they are also independent of z^* as the cylinders are of infinite length assumed. The constitutive equations for Reiner-Rivlin liquid are given by:

$$T_{mn}^* = -P^*\delta_{mn} + \mu^* d_{mn}^* + \mu_c^* d_{mg}^* d_{gn}^* \qquad m, n = 1, 2, 3. \tag{1}$$

Here, P^* denotes pressure, T_{mn}^* represents stress tensor, δ_{mn} is Kronecker delta symbol, μ^* and μ_c^* represent viscosity and cross viscosity of the Reiner- Rivlin liquid, g is dummy index and d_{mn}^*= $v_{m,n}^* + v_{n,m}^*$ is the strain rate tensor.

Since cylinders are rotating about their common axis and moving slowly along axial direction, so radial velocity $v_{r^*}^* = 0$. For steady case, the equations of motion (Sattar and Waheedullah (2013)) for Reiner-Rivlin liquid are:

$$\frac{\partial T_{r^*r^*}^*}{\partial r^*} + \frac{\partial T_{z^*r^*}^*}{\partial z^*} + \frac{1}{r^*}\frac{\partial T_{\theta r^*}}{\partial \theta} + \frac{T_{r^*r^*}^* - T_{\theta\theta}^*}{r^*} + \rho^*\frac{{v^*}_\theta^2}{r^*} = 0, \tag{2}$$

$$\frac{1}{r^*}\frac{\partial (r^* T_{r^*\theta}^*)}{\partial r^*} + \frac{\partial T_{\theta z^*}^*}{\partial z^*} + \frac{1}{r^*}\frac{\partial T_{\theta\theta}^*}{\partial \theta} + \frac{T_{\theta r^*}^*}{r^*} - \frac{\mu^*}{K^*}v_\theta^* - \sigma^* B^{*2} v_\theta^* = 0, \tag{3}$$

$$\frac{1}{r^*}\frac{\partial (r^* T_{r^*z^*}^*)}{\partial r^*} + \frac{1}{r^*}\frac{\partial T_{\theta z^*}^*}{\partial \theta} + \frac{\partial T_{z^*z^*}^*}{\partial z^*} - \frac{\mu^*}{K^*}v_{z^*}^* - \sigma^* B^{*2} v_{z^*}^* = 0. \tag{4}$$

For the unsteady case, the equations of motion for Reiner-Rivlin liquid are:

$$\frac{\partial T_{r^*r^*}^*}{\partial r^*} + \frac{\partial T_{z^*r^*}^*}{\partial z^*} + \frac{1}{r^*}\frac{\partial T_{\theta r^*}}{\partial \theta} + \frac{T_{r^*r^*}^* - T_{\theta\theta}^*}{r^*} + \rho^*\frac{{v^*}_\theta^2}{r^*} = 0, \tag{5}$$

$$\frac{1}{r^*}\frac{\partial (r^* T_{r^*\theta}^*)}{\partial r^*} + \frac{\partial T_{\theta z^*}^*}{\partial z^*} + \frac{1}{r^*}\frac{\partial T_{\theta\theta}^*}{\partial \theta} + \frac{T_{\theta r^*}^*}{r^*} - \frac{\mu^*}{K^*}v_\theta^* - \sigma^* B^{*2} v_\theta^* = \rho^*\frac{\partial v_\theta^*}{\partial t^*}, \tag{6}$$

$$\frac{1}{r^*}\frac{\partial (r^* T_{r^*z^*}^*)}{\partial r^*} + \frac{1}{r^*}\frac{\partial T_{\theta z^*}^*}{\partial \theta} + \frac{\partial T_{z^*z^*}^*}{\partial z^*} - \frac{\mu^*}{K^*}v_{z^*}^* - \sigma^* B^{*2} v_{z^*}^* = \rho^*\frac{\partial v_z^*}{\partial t^*}. \tag{7}$$

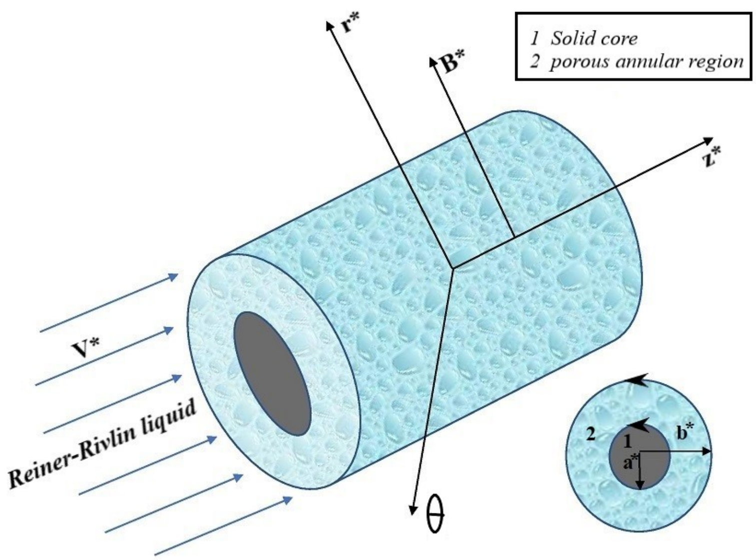

Figure 1. Schematic diagram of the problem

Here, ρ^* and μ^* are density and viscosity of the liquid, respectively. Also, σ^* is the electrical conductivity of the liquid, $B^* = |\, \tilde{\mathbf{B}}^* \,|$ and K^* is the permeability of the porous medium.

Due to the taken velocity $(0, v_\theta^*(r^*), v_{z^*}^*(r^*))$ of the liquid, the equation of continuity is satisfied. Now, we will evaluate components of rate of strain tensor by using the relation

$$d_{mn}^* = v_{m,n}^* + v_{n,m}^*, \qquad m,n = 1,2,3, \tag{8}$$

which come out as

$$\begin{gathered} d_{r^*r^*}^* = d_{\theta\theta}^* = d_{z^*z^*}^* = d_{\theta z^*}^* = d_{z^*\theta}^* = 0, \\ d_{r^*\theta}^* = d_{\theta r^*}^* = r^* \frac{d}{dr^*}\left(\frac{v_\theta^*}{r^*}\right), d_{r^*z^*}^* = d_{z^*r^*}^* = \frac{dv_{z^*}^*}{dr^*}. \end{gathered} \tag{9}$$

Using above values (9), we will obtain expressions for stress components with the help of Equation (1) as follows:

$$T_{r^*r^*}^* = -P^* + \mu_c^* \left\{ r^{*2} \left(\frac{d}{dr^*}\left(\frac{v_\theta^*}{r^*}\right)\right)^2 + \left(\frac{dv_{z^*}^*}{dr^*}\right)^2 \right\}, \tag{10}$$

$$T_{\theta\theta}^* = -P^* + \mu_c^* r^{*2} \left(\frac{d}{dr^*}\left(\frac{v_\theta^*}{r^*}\right)\right)^2, \tag{11}$$

$$T_{z^*z^*}^* = -P^* + \mu_c^* \left(\frac{dv_{z^*}^*}{dr^*}\right)^2, \tag{12}$$

$$T_{\theta z^*}^* = \mu_c^* r^* \frac{d}{dr^*}\left(\frac{v_\theta^*}{r^*}\right) \frac{dv_{z^*}^*}{dr^*} = T_{z^*\theta}^*, \tag{13}$$

$$T^*_{r^*z^*} = \mu^* \left(\frac{dv^*_{z^*}}{dr^*} \right) = T^*_{z^*r^*}, \tag{14}$$

$$T^*_{r^*\theta} = \mu^* r^* \frac{d}{dr^*} \left(\frac{v^*_\theta}{r^*} \right) = T^*_{\theta r^*}. \qquad . \tag{15}$$

Here, it is noted that pressure P^* of the liquid depends on the time t^* for unsteady case.

For steady case, with the help of above expressions of stresses (10)-(15), one can write Equations (2)-(4) in the following form:

$$\mu^*_c \frac{d}{dr^*} \left(\left(r^* \frac{d}{dr^*} \left(\frac{v^*_\theta}{r^*} \right) \right)^2 + \left(\frac{dv^*_{z^*}}{dr^*} \right)^2 \right) + \frac{\mu^*_c}{r^*} \left(\frac{dv^*_{z^*}}{dr^*} \right)^2 + \rho^* \frac{v^{*2}_\theta}{r^*} = \frac{dP^*}{dr^*}, \tag{16}$$

$$\frac{\mu^*}{r^*} \frac{d}{dr^*} \left(r^{*2} \frac{d}{dr^*} \left(\frac{v^*_\theta}{r^*} \right) \right) - \frac{\mu^* v^*_\theta}{K^*} - \sigma^* B^{*2} v^*_\theta + \mu^* \frac{d}{dr^*} \left(\frac{v^*_\theta}{r^*} \right) = 0, \tag{17}$$

$$-\frac{\mu^* v^*_{z^*}}{K^*} - \sigma^* B^{*2} v^*_{z^*} + \frac{\mu^*}{r^*} \frac{d}{dr^*} \left(r^* \frac{dv^*_{z^*}}{dr^*} \right) = 0. \tag{18}$$

Similarly, for unsteady case from Equations $(5)-(7)$, we obtain:

$$\mu^*_c \frac{\partial}{\partial r^*} \left(\left(r^* \frac{\partial}{\partial r^*} \left(\frac{v^*_\theta}{r^*} \right) \right)^2 + \left(\frac{\partial v^*_{z^*}}{\partial r^*} \right)^2 \right) + \frac{\mu^*_c}{r^*} \left(\frac{\partial v^*_{z^*}}{\partial r^*} \right)^2 + \rho^* \frac{v^{*2}_\theta}{r^*} = \frac{\partial P^*}{\partial r^*}, \tag{19}$$

$$\frac{\mu^*}{r^*} \frac{\partial}{\partial r^*} \left(r^{*2} \frac{\partial}{\partial r^*} \left(\frac{v^*_\theta}{r^*} \right) \right) - \frac{\mu^* v^*_\theta}{K^*} - \sigma^* B^{*2} v^*_\theta + \mu^* \frac{\partial}{\partial r^*} \left(\frac{v^*_\theta}{r^*} \right) = \rho^* \frac{\partial v^*_\theta}{\partial t^*}, \tag{20}$$

$$-\frac{\mu^* v^*_{z^*}}{K^*} - \sigma^* B^{*2} v^*_{z^*} + \frac{\mu^*}{r^*} \frac{\partial}{\partial r^*} \left(r^* \frac{\partial v^*_{z^*}}{\partial r^*} \right) = \rho^* \frac{\partial v^*_z}{\partial t^*}. \tag{21}$$

Introduced are the following non-dimensional quantities,

$$\begin{gathered} r = \frac{r^*}{a^*}, v_\theta = \frac{v^*_\theta}{V^*}, v_z = \frac{v^*_{z^*}}{V^*}, P = \frac{P^*}{\mu^* V^*/a^*}, t = \frac{\rho^* V^{*2} t^*}{\mu^*}, \\ K = \frac{K^*}{a^{*2}}, \, \Omega_i = \frac{\Omega^*_i}{V^*/a^*}, \, \Omega_o = \frac{\Omega^*_o}{V^*/a^*}, \, \ell = \frac{b^*}{a^*}, \end{gathered} \tag{22}$$

where V^* represents characteristic velocity and a^* represents characteristic length.

Then, the non-dimensional form of equations of motion $(16)-(18)$ for steady flow are:

$$\frac{dP}{dr} = S \frac{d}{dr} \left(\left(r \frac{d}{dr} \left(\frac{v_\theta}{r} \right) \right)^2 + \left(\frac{dv_z}{dr} \right)^2 \right) + S \frac{1}{r} \left(\frac{dv_z}{dr} \right)^2 + R_e \frac{v^2_\theta}{r}, \tag{23}$$

$$\frac{1}{r} \frac{d}{dr} \left[(r)^2 \frac{d}{dr} \left(\frac{v_\theta}{r} \right) \right] - \xi^2 v_\theta + \frac{d}{dr} \left(\frac{v_\theta}{r} \right) = 0, \tag{24}$$

$$\frac{1}{r} \frac{d}{dr} \left(r \frac{dv_z}{dr} \right) - \xi^2 v_z = 0. \tag{25}$$

For unsteady flow, the non-dimensional form of equations of motion $(19)-(21)$ are:

$$\frac{\partial P}{\partial r}=S\frac{\partial}{\partial r}\left(\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right)^2+\left(\frac{\partial v_z}{\partial r}\right)^2\right)+S\frac{1}{r}\left(\frac{\partial v_z}{\partial r}\right)^2+R_e\frac{v_\theta^2}{r}, \tag{26}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[(r)^2\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]-\xi^2 v_\theta+\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)=R_e^2\frac{\partial v_\theta}{\partial t}, \tag{27}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)-\xi^2 v_z=R_e^2\frac{\partial v_z}{\partial t}. \tag{28}$$

Here, $S=\frac{\mu_c^* V^*}{\mu^* a^*}$, is a non-dimensional number, $H=\sqrt{\frac{\sigma^*}{\mu^*}}B^* a^*$, is the Hartmann number, $R_e=\frac{\rho^* V^* a}{\mu^*}$, is the Reynolds number of the liquid and $\xi^2=\frac{1}{K}+H^2$.

3. Analytical solution

For steady flow, expressions of velocity components are obtained by solving ordinary differential equations (24) and (25). These equations can be expressed in the form of modified Bessel's differential equations of order one and zero, respectively, as:

$$\frac{d^2 v_\theta}{dr^2}+\frac{1}{r}\frac{dv_\theta}{dr}-\left(\frac{1}{r^2}+\xi^2\right)v_\theta=0, \tag{29}$$

$$\frac{d^2 v_z}{dr^2}+\frac{1}{r}\frac{dv_z}{dr}-\xi^2 v_z=0. \tag{30}$$

The general solutions of Equations (29) and (30) are, respectively,

$$v_\theta=A_1 I_1(\xi r)+A_2 K_1(\xi r), \tag{31}$$

$$v_z=C_1 I_0(\xi r)+C_2 K_0(\xi r), \tag{32}$$

where I_0, I_1 and K_0, K_1 are modified Bessel's functions of first and second kinds of order zero and one, respectively.

Using expressions of velocities $(31)-(32)$, we find the expression for the pressure gradient by the Equation (23) as given below:

$$\begin{aligned}\frac{dP}{dr}=(\left(\frac{S}{4r^3}\right)&((\xi r I_0(\xi r)-2I_1(\xi r)+\xi r I_2(\xi r))A_1-(\xi r K_0(\xi r)+2K_1(\xi r)+\xi r K_2(\xi r)A_2)\\ &\times((-2\xi r I_0(\xi r)+(4+3\xi^2 r^2)I_1(\xi r)+\xi r(-2I_2(\xi r)+\xi r I_3(\xi r)))A_1+(2\xi r K_0(\xi r)\\ &+(4+3\xi^2 r^2)K_1(\xi r)+\xi r(2K_2(\xi r)+\xi r K_3(\xi r)))A_2)+\xi^2 r^2 S(I_1(\xi r)C_1\\ &-K_1(\xi r)C_2)((\xi r I_0(\xi r)+I_1(\xi r)+\xi r I_2(\xi r))C_1+(\xi r K_0(\xi r)-K_1(\xi r)\\ &+\xi r K_2(\xi r))C_2)+r^2(I_1(\xi r)A_1+K_1(\xi r)A_2)^2 R_e)).\end{aligned} \tag{33}$$

For unsteady flow, to obtain analytical expressions for velocity components, we will apply the method of separation of variables. Expressions of velocity components are obtained by solving partial differential equations (27) and (28). These equations can be expressed in the form as:

$$\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}-\left(\frac{1}{r^2}+\xi^2\right)v_\theta={R_e}^2\frac{\partial v_\theta}{\partial t}, \tag{34}$$

$$\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}-\xi^2 v_z={R_e}^2\frac{\partial v_z}{\partial t}. \tag{35}$$

The solutions of Equations (34) and (35) are, respectively,

$$v_\theta=e^{-\frac{\delta_1^2 t}{R_e^2}}[M_1 I_1(\eta_1 r)+M_2 K_1(\eta_1 r)], \tag{36}$$

$$v_z=e^{-\frac{\delta_2^2 t}{R_e^2}}[N_1 I_0(\eta_2 r)+N_2 K_0(\eta_2 r)], \tag{37}$$

where $\eta_1=\sqrt{(\xi^2-\delta_1^2)}$ and $\eta_2=\sqrt{(\xi^2-\delta_2^2)}$. Here, M_1, M_2, N_1 and N_2 are arbitrary constants, and δ_1 and δ_2 are separation constants.

Using expressions of velocities $(36)-(37)$, we find the expression for the pressure gradient by the equation (33) as given below:

$$\begin{aligned}\frac{\partial P}{\partial r}&=(-2Se^{\frac{-2\delta_1^2 t}{R_e^2}}((M_1(2I_1(\eta_1 r)-\eta_1 r I_0(\eta_1 r)))+M_2(2K_1(\eta_1 r)+\eta_1 r K_0(\eta_1 r))))\\&\times(((M_1(\eta_1^2 r^2+6)I_1(\eta_1 r))-3\eta_1 r I_0(\eta_1 r))+M_2((\eta_1^2 r^2+6)K_1(\eta_1 r)+3\eta_1 r K_0(\eta_1 r)))/r\\&+(4Se^{\frac{-2\delta_1^2 t}{R_e^2}}(M_1(2I_1(\eta_1 r)-\eta_1 r I_0(\eta_1 r))+M_2(2K_1(\eta_1 r)+\eta_1 r K_0(\eta_1 r)))^2)/r\\&+R_e e^{\frac{-2\delta_1^2 t}{R_e^2}}(M_1 I_1(\eta_1 r)+M_2 K_1(\eta_1 r))^2/r+\eta_2^3 Se^{\frac{-2\delta_2^2 t}{R_e^2}}(N_1(I_0(\eta_2 r)+I_2(\eta_2 r))\\&+N_2(K_0(\eta_2 r)+K_2(\eta_2 r)))(N_1 I_1(\eta_2 r)-N_2 K_1(\eta_2 r))\\&+\eta_2^2 Se^{\frac{-2\delta_2^2 t}{R_e^2}}(N_1 I_1(\eta_2 r)-N_2 K_1(\eta_2 r))^2/r.\end{aligned} \tag{38}$$

It is observed that pressure gradient depends on the ratio of cross viscosity to viscosity of the Reiner-Rivlin liquid and Reynolds number.

3.1. Boundary conditions

The values of arbitrary constants $A_1, A_2, C_1, C_2, M_1, M_2, N_1, N_2$ are appearing in the above expressions of velocity components (31), (32) and (36), (37) for both steady and unsteady cases, respectively can be determined by applying following boundary conditions:

Conditions at the boundary of inner cylinder (r= 1):
For steady case

$$v_\theta=\Omega_i, \qquad v_z=v_a. \tag{39}$$

For unsteady case at $t=0$

$$v_\theta=\Omega_i, \qquad v_z=v_a, \tag{40}$$

where Ω_i is non-dimensional angular velocity of inner cylinder and v_a=$\frac{v^*_{a*}}{V^*}$, is the non-dimensional velocity of inner cylinder in the direction parallel to the axis of cylinders.

Conditions at the boundary of outer cylinder (r= ℓ):
For steady case

$$v_\theta = \Omega_o \ell, \qquad v_z = v_b. \tag{41}$$

For unsteady case at $t = 0$

$$v_\theta = \Omega_o \ell, \qquad v_z = v_b, \tag{42}$$

where Ω_o is non-dimensional angular velocity of outer cylinder and v_b=$\frac{v^*_{b*}}{V^*}$ is the non-dimensional velocity of outer cylinder in the direction parallel to the axis of cylinders.

3.2. Determination of arbitrary constants

For steady case, applying conditions (39) and (41) in the Equations (31) and (32), we get:

$$A_1 = \frac{-\Omega_i K_1(\ell\xi) + \ell\Omega_o K_1(\xi)}{\triangle_1}, \qquad A_2 = \frac{\Omega_i I_1(\ell\xi) - \ell\Omega_o I_1(\xi)}{\triangle_1}, \tag{43}$$

$$C_1 = \frac{-v_b K_0(\xi) + v_a K_0(\ell\xi)}{\triangle_2}, \qquad C_2 = \frac{v_b I_0(\xi) - v_a I_0(\ell\xi)}{\triangle_2}, \tag{44}$$

where

$$\triangle_1 = I_1(\ell\xi)K_1(\xi) - K_1(\ell\xi)I_1(\xi), \qquad \triangle_2 = I_0(\xi)K_0(\ell\xi) - K_0(\xi)I_0(\ell\xi).$$

Thus, substituting these values of constants in equations (31) and (32), we get analytical expressions for velocity components of the liquid.

Also, for unsteady case using conditions (40) and (42) in Equations (36) and (37), we obtain

$$M_1 = \frac{-\Omega_i K_1(\ell\eta_1) + \ell\Omega_o K_1(\eta_1)}{\triangle_3}, \qquad M_2 = \frac{\Omega_i I_1(\ell\eta_1) - \ell\Omega_o I_1(\eta_1)}{\triangle_3}, \tag{45}$$

$$N_1 = \frac{-v_b K_0(\eta_2) + v_a K_0(\ell\eta_2)}{\triangle_4}, \qquad N_2 = \frac{v_b I_0(\eta_0) - v_a I_0(\ell\eta_2)}{\triangle_4}, \tag{46}$$

where

$$\triangle_3 = I_1(\ell\eta_1)K_1(\eta_1) - K_1(\ell\eta_1)I_1(\eta_1), \qquad \triangle_4 = I_0(\eta_2)K_0(\ell\eta_2) - K_0(\eta_2)I_0(\ell\eta_2).$$

Therefore, inserting these values of constants in Equations (36) and (37), we get velocity components of the liquid.

4. Flow Rate

The volumetric flow rate of the Reiner-Rivlin liquid flowing through porous annular region of cylinders for both cases (steady flow and unsteady flow) can be evaluated using the formula

$$Q^* = 2\pi \int_a^b v^*_{z^*} r^* dr^*. \tag{47}$$

Non-dimensional volumetric flow rate (Q) is given by

$$Q = \frac{Q^*}{a^{*2}V^*} = 2\pi \int_1^\ell v_z r dr. \tag{48}$$

For steady case, inserting expression of the axial velocity in Equation (48) and on integration, one can find that

$$Q = 2\pi \frac{C_1(-I_1(\xi) + \ell I_1(\xi\ell)) + C_2(K_1(\xi) - \ell K_1(\xi\ell))}{\xi}. \tag{49}$$

By putting values of constants C_1 and C_2 in Equation (49), we get:

$$\begin{aligned} Q = 2\pi &\frac{(-1 + \xi I_1(\xi)K_0(\xi\ell) + \xi I_0(\xi\ell)K_1(\xi))v_a}{\xi^2(I_0(\xi\ell)K_0(\xi) - I_0(\xi)K_0(\xi\ell))} \\ &+ \frac{(-1 + \xi\ell I_1(\xi\ell)K_0(\xi) + \xi\ell I_0(\xi)K_1(\xi\ell))v_b}{\xi^2(I_0(\xi\ell)K_0(\xi) - I_0(\xi)K_0(\xi\ell))}. \end{aligned} \tag{50}$$

For unsteady case, substituting expression of axial velocity in Equation (48) and then on integration, we obtain

$$Q = \frac{2\pi e^{-\frac{\delta_2^2 t}{R_e^2}}}{\eta_2} \left(N_1(-I_1(\eta_2) + \ell I_1(\eta_2\ell)) + N_2(K_1(\eta_2) - \ell K_1(\eta_2\ell))\right). \tag{51}$$

By substituting values of constants N_1andN_2 in Equation (51), we get:

$$\begin{aligned} Q = 2\pi e^{-\frac{\delta_2^2 t}{R_e^2}} &\left(\frac{(-1 + \eta_2 I_1(\eta_2)K_0(\eta_2\ell) + \eta_2 I_0(\eta_2\ell)K_1(\eta_2))v_a}{\eta_2^2(I_0(\eta_2\ell)K_0(\eta_2) - I_0(\eta_2)K_0(\eta_2\ell))} \right. \\ &\left. + \frac{(-1 + \eta_2\ell I_1(\eta_2\ell)K_0(\eta_2) + \eta_2\ell I_0(\eta_2)K_1(\eta_2\ell))v_b}{\eta_2^2(I_0(\eta_2\ell)K_0(\eta_2) - I_0(\eta_2)K_0(\eta_2\ell))} \right). \end{aligned} \tag{52}$$

5. Discussion of results

In this section, we will discuss the effect of magnetic field on the velocity of the Reiner-Rivlin liquid for both steady and unsteady cases. The influence of velocities of cylinders on the flow rate of liquid in the porous annular region of cylinders. We will also discuss the effect of Reynolds number on the velocity of the liquid for unsteady flow. Also, see the influence of Reynolds number on flow rate of liquid in the porous annular region of cylinders with respect to time during unsteady flow. For convenience, take the ratio of the radii of cylinders $\ell = 2$ and non-dimensional permeability $K = 100$.

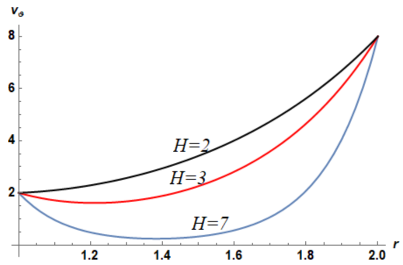

Figure 2(a). Variation of velocity v_θ with r for angular velocities Ω_i=2, Ω_o=4

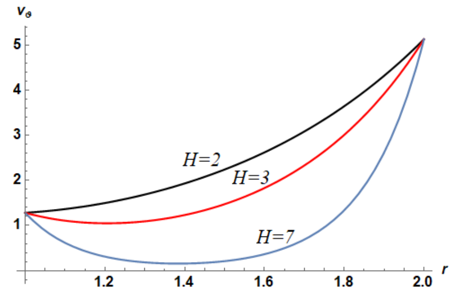

Figure 2(b). Variation of velocity v_θ with r for angular velocities $\Omega_i = 2, \Omega_o = 4$ at $t = 4$

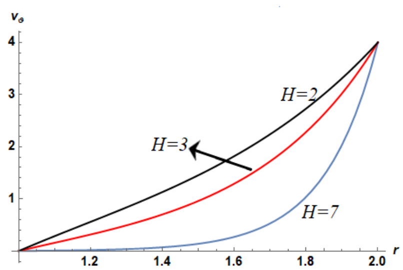

Figure 3(a). Variation of velocity v_θ with r for angular velocities Ω_i=0, Ω_o=2

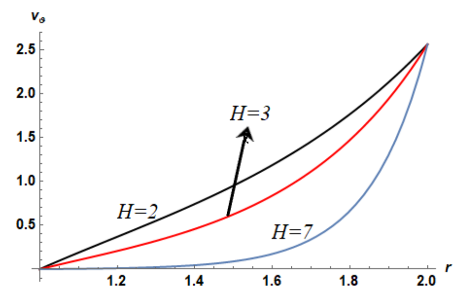

Figure 3(b). Variation of velocity v_θ with r for angular velocities Ω_i=0, Ω_o=2 at $t = 4$

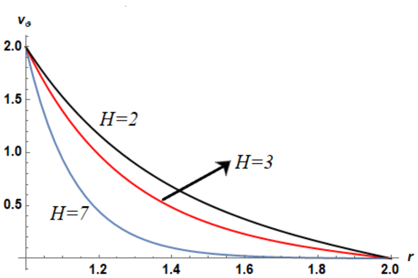

Figure 4(a). Variation of velocity v_θ with r for angular velocities Ω_i=2, Ω_o=0

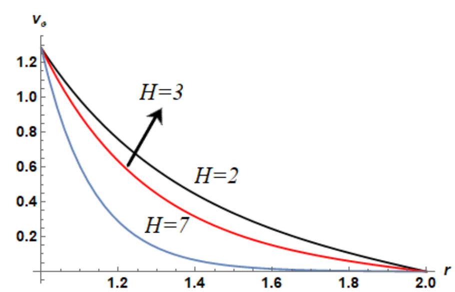

Figure 4(b). Variation of velocity v_θ with r for angular velocities Ω_i=2, Ω_o=0 at $t = 4$

5.1. Effect of magnetic field on rotational velocity of the liquid

Now, we will discuss the impact of magnetic field on the rotational velocity components of the liquid for both steady and unsteady cases. To study the behaviour of velocity component in steady flow field, we take Ω_i=2, Ω_o=4, K=100 and draw plot between v_θ and r for different values of $H = 2, 3, 7$ (Figure 2(a)). For fixed permeability of porous region, and fixed values of all parameters with Ω_i<Ω_o, the rotational velocity of liquid decreases first and then increases in the annular region of flow field. Also, it is observed that increasing the value of magnetic field, rotational velocity decreases.

We will study the effect of magnetic field on rotational velocity for unsteady flow as well, when $t = 4, \Omega_i = 2, \Omega_o = 4, R_e = 1.5, \delta_1 = 0.5$ for different values of Hartmann number $H = 2, 3, 7$. It is observed that for $\Omega_i < \Omega_o$, the behavior of rotational velocity (Figure 2(b)) for unsteady case at a constant time is similar to the behavior of rotational velocity (Figure 2(a)) for steady case. But, the rotational velocity for unsteady case for time $t > 0$ is lesser in magnitude comparison to the rotational velocity for the case of steady flow.

Now, when we take Ω_i=0, Ω_o=2, i.e. inner cylinder is not rotating but outer cylinder is rotating for different values of Hartmann numbers H, then rotational velocity increases continuously for fixed magnetic field and it decreases with increasing magnetic field for steady case (Figure 3(a)). Figure-3(b) narrates the effect of magnetic field on rotational velocity liquid for unsteady case when inner cylinder is not rotating, i.e., Ω_i=0 at fixed time $t = 4$. It is observed that rotational velocity increases for fixed magnetic field and decreases with increasing magnetic field at a constant time.

Now, when we take Ω_i=2, Ω_o=0, i.e., the outer cylinder is not rotating but the inner cylinder is rotating with different values of H, then rotational velocity decreases continuously for fixed magnetic field and it decreases with increasing magnetic field in steady flow field (Figure 4(a)). Figure 4(b) shows the effect of magnetic field on rotational velocity of liquid for unsteady case when outer cylinder is not rotated, i.e., $\Omega_o = 0$ at fixed time $t = 4$. It is observed that rotational velocity decreases for fixed magnetic field and decreases with increasing magnetic field at any constant time.

5.2. Influence of magnetic field on the axial velocity of the liquid

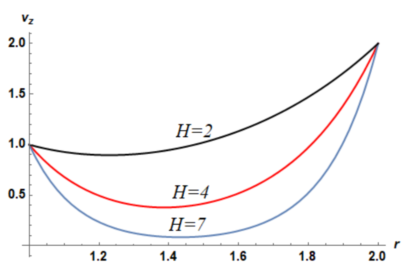

Figure 5(a). Variation of velocity v_z with r for axial velocities v_a=1, v_b=2

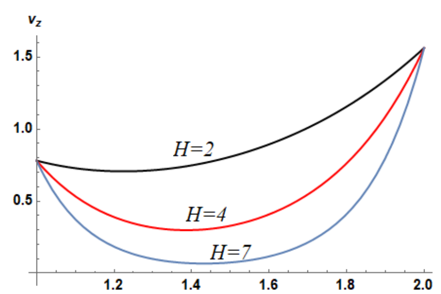

Figure 5(b). Variation of velocity v_z with r for axial velocities v_a=1, v_b=2 at t=5

As we have found, the angular velocities of the inner cylinder and the outer cylinder do not affect the z^* component of velocity by expression of $v^*_{z^*}$. So, rotation of inner and outer cylinders affects only θ component of velocity of Reiner-Rivlin liquid in both cases. If we take $v_a = 1$, $v_b = 2$ with the values of $H = 2, 4, 7$, then axial velocity of liquid decreases first and then increases continuously for fixed magnetic field for steady flow field. Also, axial velocity decreases with increasing magnetic field (Figure 5(a)).

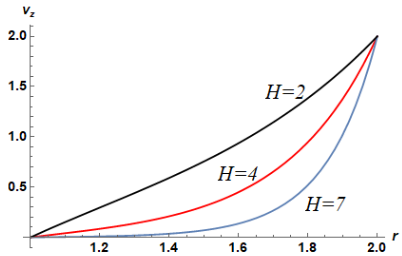

Figure 6(a). Variation of velocity v_z with r for axial velocities v_a=0, v_b=2

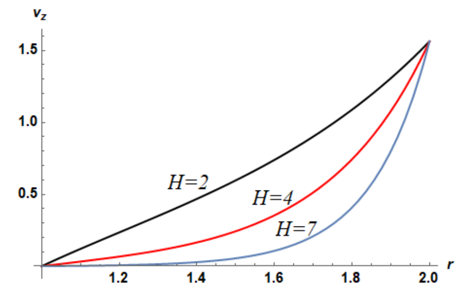

Figure 6(b). Variation of velocity v_z with r for axial velocities v_a=0, v_b=2 at $t = 5$

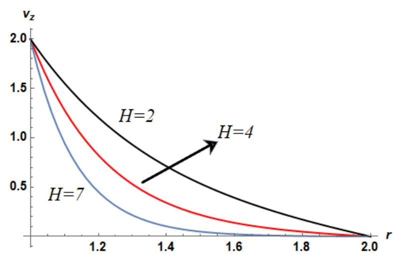

Figure 7(a). Variation of velocity v_z with r for axial velocities v_a=2, v_b=0

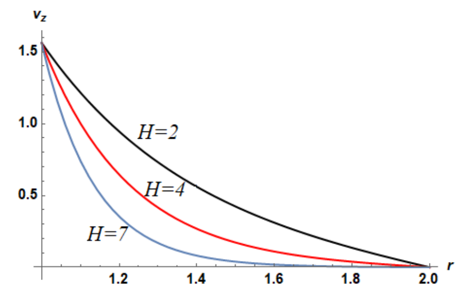

Figure 7(b). Variation of velocity v_z with r for axial velocities v_a=2, v_b=0 at $t = 5$

Figure 5(b) sketch out the effect of magnetic field on axial velocity component of liquid for unsteady case when $v_a = 1, v_b = 2, R_e = 1.5, \delta_2 = 1/3, t = 5$ for various values of Hartmann number $H = 2, 3, 7$. It is observed that for $v_a < v_b$, the behavior of axial velocity (Figure 5(b)) for unsteady case at a constant time is similar to the behavior of axial velocity (Figure 5(a)) for steady flow. But the axial velocity for unsteady case for time $t > 0$ is lesser in magnitude comparison to the axial velocity for case of steady flow.

Now, when we take $v_a = 0$, $v_b = 2$, i.e., the inner cylinder is fixed and the outer cylinder is moving with several values of H, then axial velocity increases continuously with fixed magnetic field and axial velocity decreases with increasing the magnetic field for steady flow (Figure 6(a)). Figure 6(b) depicts the influence of magnetic field on axial velocity of liquid for unsteady case when $v_a = 0$, $v_b = 2$, $t = 5$ with different values of Hartmann number $H = 2, 3, 7$. It is found that axial velocity increases continuously with fixed magnetic field and axial velocity decreases with increasing the Hartmann number at any constant time (Figure 6(b)).

Again, when we take $v_a = 2$, $v_b = 0$, then axial velocity decreases continuously with fixed magnetic field and axial velocity decreases with increasing the magnetic field for steady case (Figure 7(a)). Figure 7(b) shows the effect of magnetic field on axial velocity of liquid for unsteady case when $v_a = 2$, $v_b = 0$, $t = 5$ with various values of Hartmann numbers $H = 2, 3, 7$, then axial velocity decreases continuously with constant magnetic field and axial velocity decreases with increasing the magnetic field at any stationary time (Figure 7(b)).

5.3. Influence of magnetic field on flow rate

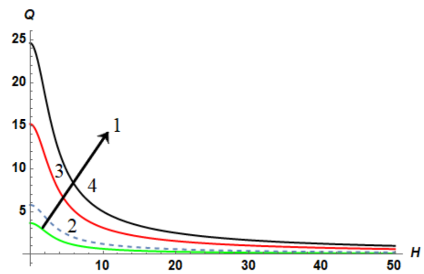

Figure 8(a). Variation of flow rate with magnetic field for ℓ=2, K=100 when: (1) v_a=1, v_b=0, (2) v_a=0, v_b=1, (3) v_a=1, v_b=2, (4) v_a=2, v_b=3

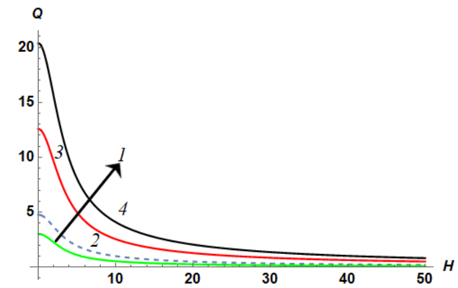

Figure 8(b). Variation of flow rate with magnetic field for ℓ=2, K=100 when: (1) v_a=1, v_b=0, (2) v_a=0, v_b=1, (3) v_a=1, v_b=2, (4) v_a=2, v_b=3 at $t = 4$

The effect of magnetic field on the flow rate for different values of non-dimensional axial velocities is discussed under steady flow (Figure 8(a)). It is observed that for fixed axial velocities of cylinders, flow rate decreases with increasing the magnetic field. It is also seen from the figure that greater flow rate for outer cylinder axial movement compared to the same value of inner cylinder axial movement. Also, it is observed that when the difference of axial velocities of cylinders is same for two different combinations of axial velocities of cylinders, then the flow rate is more for axial velocity combination for whose outer cylinder axial velocity is greater. So, flow rate increases more rapidly with increasing axial velocity of the outer cylinder in comparison to increasing axial velocity of the inner cylinder.

Figure 8(b) elaborates the effect of magnetic field on the flow rate for different values of unsteady axial velocities. It is observed that the behavior of flow rate for unsteady case at constant time is similar to the behavior of flow rate for steady case (Figure 8(a)). But, the flow rate for unsteady case in magnitude for time $t > 0$ is lesser than the flow rate for steady case.

5.4. Influence of Reynolds number on rotational velocity of the liquid

Figure 9 discusses the influence of Reynolds number on the rotational velocity of liquid during unsteady case when $\Omega_i = 2, \Omega_o = 4, \delta_1 = 0.5, t = 1$ for different values of Reynolds number $R_e = 1, 1.5, 5$. It is observed that rotational velocity of liquid increases with increasing R_e at any instant.

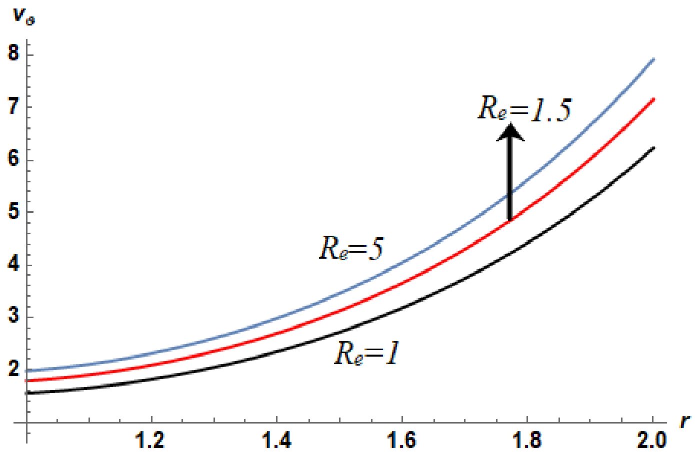

Figure 9. Variation of velocity v_θ with r for angular velocities $\Omega_i = 2, \Omega_o = 4$ at $t = 1$

5.5. Influence of Reynolds number on axial velocity of the liquid

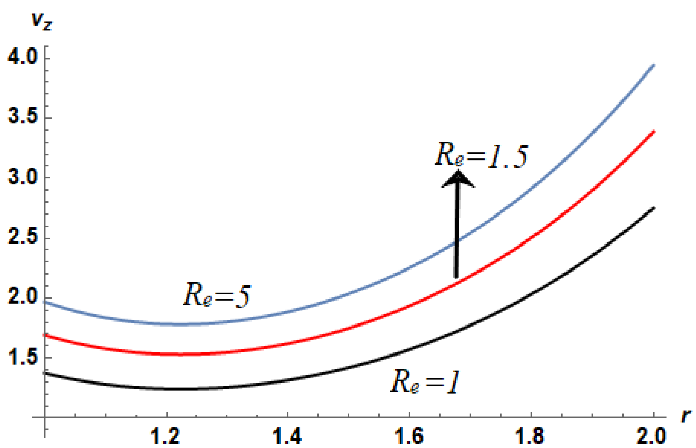

Figure 10. Variation of velocity v_z with r for angular velocities $v_a = 2, v_b = 4$ at $t = 1.5$

Figure 10 shows effect of Reynolds number on the axial velocity of liquid in unsteady flow field when $v_a = 2, v_b = 4, \delta_2 = 1/3, t = 1.5$ for different values of Reynolds number $R_e = 1, 1.5, 5$. It is found that axial velocity of liquid increases with increasing Reynolds number at any instant.

5.6. Influence of time on the rotational velocity of the liquid

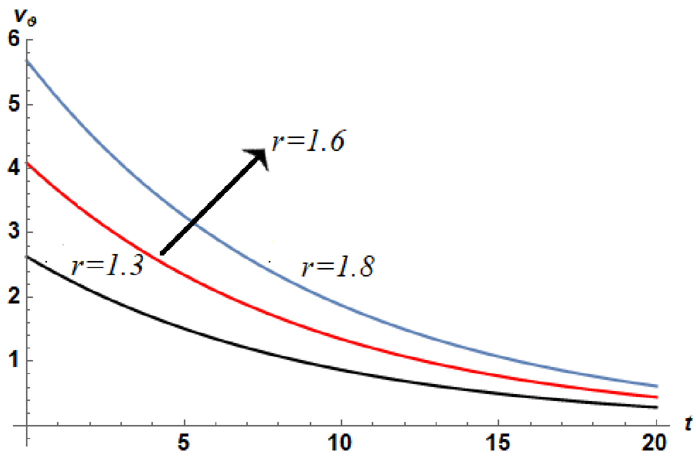

Figure 11. Variation of velocity v_θ with time for angular velocities $\Omega_i = 2, \Omega_o = 4$

Here, we will discuss variation of rotational velocity of liquid with respect to time when $\Omega_i = 2, \Omega_o = 4, R_e = 1.5, \delta_1 = 0.5, H = 2$ for various values of $r = 1.3, 1.6, 1.8$. It is observed that rotational velocity decreases as time t increases for a fixed position.

5.7. Influence of time on the axial velocity of the liquid

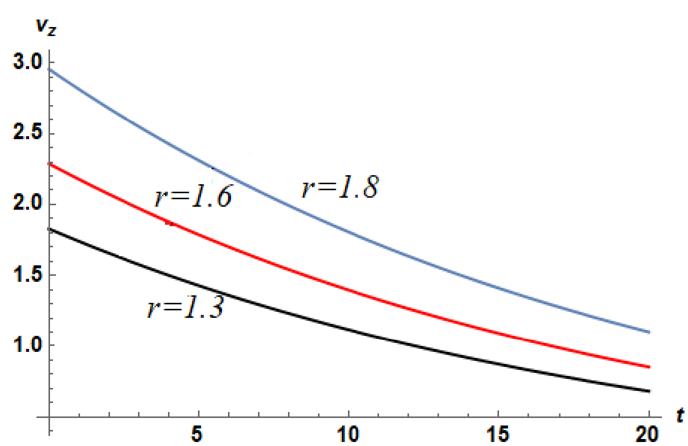

Figure 12. Variation of velocity v_z with time for axial velocities $v_a = 2, v_b = 4$

Here, we will elaborate variation of axial velocity of liquid with respect to time when $v_a = 2, v_b = 4, R_e = 1.5, \delta_2 = 1/3, H = 2$ for various values of $r = 1.3, 1.6, 1.8$. It is observed that axial velocity decreases with time for a fixed position.

5.8. Influence of time on flow rate of the liquid

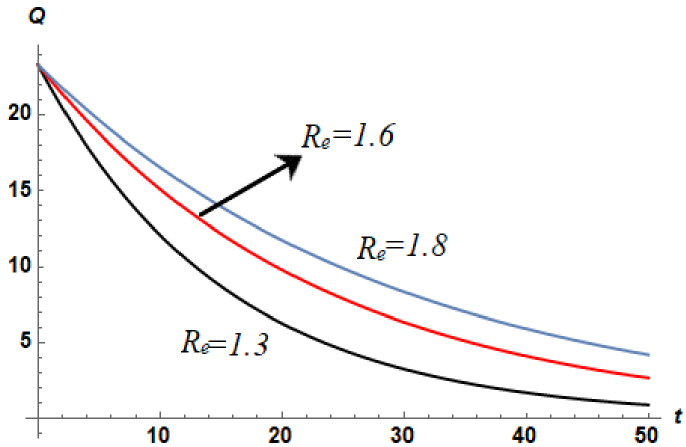

Figure 13. Variation of flow rate Q with time for axial velocities $v_a = 2, v_b = 4$

Figure 13 shows effect of time t on volumetric flow rate Q when $v_a = 2, v_b = 4, \delta_2 = 1/3, H = 2$ for different values of Reynolds number $R_e = 1.3, 1.6, 1.8$. It is noticed from the figure that the flow rate decreases with increasing time for all values of R_e and it increases with increasing the values of R_e.

6. Conclusion

The problem of steady and unsteady flow of a Reiner-Rivlin liquid in the porous annular region of two concentric rotating cylinders, which is moving parallel to their axes, about the common axis of these cylinders under uniform magnetic field acted in perpendicular direction of the axis, is studied. Analytical expressions for velocity components, pressure gradient and flow rate are reported.

- For a constant value of permeability and non-zero angular velocities of inner and outer cylinders, the rotational velocity of liquid decreases and then increases from a certain point for all values of magnetic field. Also, it is observed that rotational velocity decreases with increasing magnetic field for steady flow, whereas, for unsteady flow it shows the similar variation with lesser magnitude at a certain time.
- The axial velocity of the liquid is affected by the magnetic field in both steady and unsteady case. It is observed that with increasing magnetic field, axial velocity decreases in both cases. Also, for a non-zero constant axial velocities of inner and outer cylinders, the axial velocity of the liquid decreases and then increases in both cases.
- It is observed that pressure gradient depends on the ratio of cross viscosity to the viscosity of the Reiner-Rivlin liquid and Reynolds number whereas, velocities are independent from cross viscosity of the liquid.
- The rotational velocity and axial velocity of the liquid increases with increasing Reynolds number at a stationary time for unsteady flow. It is also observed that flow rate of the liquid decreases with increasing magnetic field for both cases.

- The rotational velocity, axial velocity and flow rate of the liquid are decreases with respect to time for a constant Reynolds number and at fixed position for unsteady flow.

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Caption of figures

Figure 1: Schematic diagram of the problem

Figure 2(a): Variation of velocity v_θ with r for angular velocities Ω_i=2, Ω_o=4

Figure 2(b): Variation of velocity v_θ with r for angular velocities $\Omega_i = 2, \Omega_o = 4$ at $t = 4$

Figure 3(a): Variation of velocity v_θ with r for angular velocities Ω_i=0, Ω_o=2

Figure 3(b): Variation of velocity v_θ with r for angular velocities Ω_i=0, Ω_o=2 at $t = 4$

Figure 4(a): Variation of velocity v_θ with r for angular velocities Ω_i=2, Ω_o=0

Figure 4(b): Variation of velocity v_θ with r for angular velocities Ω_i=2, Ω_o=0 at $t = 4$

Figure 5(a): Variation of velocity v_z with r for axial velocities v_a=1, v_b=2

Figure 5(b): Variation of velocity v_z with r for axial velocities v_a=1, v_b=2 at t=5

Figure 6(a): Variation of velocity v_z with r for axial velocities v_a=0, v_b=2

Figure 6(b): Variation of velocity v_z with r for axial velocities v_a=0, v_b=2 at $t = 5$

Figure 7(a): Variation of velocity v_z with r for axial velocities v_a=2, v_b=0

Figure 7(b): Variation of velocity v_z with r for axial velocities v_a=2, v_b=0 at $t = 5$

Figure 8(a): Variation of flow rate with magnetic field for ℓ=2, K=100 when: (1) v_a=1, v_b=0, (2) v_a=0, v_b=1, (3) v_a=1, v_b=2, (4) v_a=2, v_b=3

Figure 8(b): Variation of flow rate with magnetic field for ℓ=2, K=100 when: (1) v_a=1, v_b=0, (2) v_a=0, v_b=1, (3) v_a=1, v_b=2, (4) v_a=2, v_b=3 at $t = 4$

Figure 9: Variation of velocity v_θ with r for angular velocities $\Omega_i = 2, \Omega_o = 4$ at $t = 1$

Figure 10: Variation of velocity v_z with r for angular velocities $v_a = 2, v_b = 4$ at $t = 1.5$

Figure 11: Variation of velocity v_θ with time for angular velocities $\Omega_i = 2, \Omega_o = 4$

Figure 12: Variation of velocity v_z with time for axial velocities $v_a = 2, v_b = 4$

Figure 13: Variation of flow rate Q with time for axial velocities $v_a = 2, v_b = 4$