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Stability and Zero Velocity Curves in the Perturbed Restricted Problem of $2 + 2$ Bodies

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Abstract

The present study investigates the existence and linear stability of the equilibrium points in the restricted problem of $2+2$ bodies including the effect of small perturbations ϵ_1 and ϵ_2 in the Coriolis and centrifugal forces respectively. The less massive primary is considered as a straight segment and the more massive primary a point mass. The equations of motion of the infinitesimal bodies are derived. We obtain fourteen equilibrium points of the model, out of which six are collinear and rest non-collinear with the centers of the primaries. The position of the equilibrium points are affected by the small perturbation in centrifugal force, length and mass parameters, but there is no impact of small perturbation in Coriolis force on them. In addition, the obtained results are applied to Earth-22 Kalliope-dual satellite system. For this system, we calculate collinear and non-collinear equilibrium points and observed that the number of non-collinear equilibrium points depends on ϵ_2. Furthermore, for a set of values of the parameters ϵ_1 and ϵ_2, we have checked the stability of all the equilibrium points and concluded that all the equilibrium points are found to be unstable. The permissible regions of motion for the Earth-22 Kalliope-dual satellite system are also studied.

Keywords: Restricted $2 + 2$ body problem; Equilibrium points; Coriolis and centrifugal forces; Straight segment; Zero velocity curves

MSC 2010 No.: 37N05, 70F10, 70F15

1. Introduction

In celestial mechanics, the general three-body problem is to describe the motion of the three massive bodies moving under the mutual gravitational attraction. The restricted three-body problem is a kind of general three-body problem, in which the mass of one of the participating bodies is very small in comparison to the mass of other two bodies. It is an oldest and fascinating problem in the theory of astronomy that deals with the motion of an infinitesimal mass under the gravitational effect of the two massive bodies (called primaries). This problem holds five points of equilibrium, three are collinear and two non-collinear with the centers of the primary bodies. The non-collinear ones form an equilateral triangle with the centers of the primary bodies. The collinear equilibrium points are always unstable, whereas the non-collinear equilibrium points are stable for a critical value of the mass parameter $0 < \mu < \mu_c$, where $\mu_c = 0.0385209$ (Szebehely (1967b)).

The various generalizations of the restricted three-body problem have been solved so far by many researchers in this field. Szebehely (1967a) investigated the restricted three-body problem including the effect of small perturbations in the Coriolis and centrifugal forces. His linear stability investigation of the equilibrium points involved the effect of small perturbation in the Coriolis force by keeping centrifugal force unperturbed. He observed that under the effect of small perturbation in the Coriolis force ϵ, the collinear equilibrium points remain unstable; however, the non-collinear ones are stable for $0 < \mu < \mu_c$, where $\mu_c = \mu_0 + \left(16/3\sqrt{69}\right)\epsilon$ and $\mu_0 = 0.03852$.

Bhatnagar and Hallan (1978) examined the effect of small perturbations in the Coriolis and centrifugal forces on the existence and linear stability of the equilibrium points of the restricted three-body problem. They observed that the positions of the collinear and non-collinear equilibrium points are affected by the small perturbation in the centrifugal force, whereas the small perturbation in Coriolis force has no impact on them. They also investigated the stability of the equilibrium points and concluded that the collinear equilibrium points are remain unstable, while the non-collinear equilibrium points are stable for the condition $0 \le \mu < \mu_c$, where $\mu_c = \mu_0 + 4\left(36\epsilon - 19\epsilon'\right)/27\sqrt{69}$, $\mu_0 = 0.03852$, ϵ and ϵ' are the small perturbations in Coriolis and centrifugal forces respectively. Recently, the different perturbations in the restricted three-body problem have been studied by Kushvah (2008), Abouelmagd et al. (2013), Abouelmagd (2013), Abouelmagd and Guirao (2016).

In the solar system, the celestial bodies are not perfect spheres; they are either in the form of oblate, triaxial, or elongated in shape (like asteroids such as 216 Kleopatra and 22 Kalliope). In recent studies, many researchers have devoted their work to study the motion of the infinitesimal body near the small and irregular shaped celestial bodies. More recently, Kumar et al. (2019) studied the effect of the straight segment on the existence and stability of the equilibrium points in the Robe's restricted three-body problem (Robe (1977)). They obtained two collinear, infinite number of non-collinear and two out-of-plane equilibrium points and also discussed the parametric evolution on the equilibrium points. They also checked the linear stability of the equilibrium points and concluded that the collinear equilibrium points are conditionally stable, whereas the non-collinear and out-of-plane equilibrium points are unstable for all the values of the parameters involved in the model.

Researchers have worked on several variants of Robe's model, including the effect of small perturbations in the Coriolis and centrifugal forces, to examine the positions and stability of the equilibrium points. Some are Singh and Omale (2015), Kaur et al. (2020a), Kaur et al. (2020b), and Kaur et al. (2021). Recently, the existence and the stability of the equilibrium points in the restricted four-body problem (in which the motion of the test particle is studied under the gravitational effect of three massive bodies) was investigated under the effect of small perturbations in the Coriolis and centrifugal forces (Singh and Vincent (2015), Suraj et al. (2017), and Suraj et al. (2019)).

Similar to the restricted three-body problem, the restricted $2 + 2$ body problem describes the motion of two infinitesimal bodies under the gravitational attraction of the two massive bodies and their mutual gravitational attraction. The restricted three-body problem to the problem of $n + v$ bodies was initially carried out by Whipple and Szebehely (1984). In their study, they discussed the motion of the v infinitesimal bodies under the gravitational attraction of the n primary bodies. Further, Whipple (1984) studied the particular case of the problem of Whipple and Szebehely (1984) by taking $n = v = 2$. He obtained fourteen equilibrium points, six collinear and eight non-collinear and also discussed the linear stability of the obtained equilibrium points. Croustalloudi and Kalvouridis (2013) studied the effect of the involved parameters on the equilibrium points and their attracting regions in the restricted problem of $2 + 2$ bodies. The equilibrium points, regions of motion of the infinitesimal bodies and the stability of the equilibrium points in the restricted problem of $2 + 2$ bodies by considering the shape of the primary bodies have been studied by Kalvouridis and Mavraganis (1995), Kalvouridis (1997), and Prasad and Ishwar (1996).

A new variant of the Robe's restricted three-body problem to the Robe's restricted problem of $2+2$ bodies initially studied by Kaur and Aggarwal (2012). They considered two infinitesimal bodies inside the more massive primary and studied the existence and linear stability of the equilibrium points. They obtained four collinear and infinite number of non-collinear equilibrium points. They also concluded that all the equilibrium points are unstable for all the values of the involved parameters. Furthermore, the existence and stability of the equilibrium points in the perturbed Robe's restricted problem of $2 + 2$ bodies including the effect of small perturbations in the Coriolis and centrifugal forces have been studied by Kaur et al. (2016).

More recently, the effect of length parameter on the existence and linear stability of the equilibrium points in the restricted $2 + 2$ body problem has been investigated by Kumar et al. (2020). They obtained fourteen equilibrium points, six collinear and eight non-collinear. They also discussed how the length parameter affects the positions and stability of the equilibrium points. The Coriolis and centrifugal forces arise due the rotation of the coordinate system and these forces affect the nature of motion of the infinitesimal bodies. Thus, so far, many researchers have worked on restricted $2 + 2$ body problem, but nobody has studied the effect of length parameter and small perturbations in the Coriolis and centrifugal forces simultaneously. Therefore, motivated by the work of Szebehely (1967a), Whipple (1984), and Kumar et al. (2020), we have considered the restricted $2 + 2$ body problem under the combined effect of straight segment and small perturbations in the Coriolis and centrifugal forces. This work can be applied to the study of two infinitesimal bodies in the Earth-Asteroid system or Jupiter-Asteroid system. Thus, the considered model has practical applications in the field of astrophysics and astronomy.

This paper is divided into six sections. In Section 2, we describe the dynamical system and determine the equations of motion of the two infinitesimal bodies. Section 3 comprises the collinear and non-collinear equilibrium points of the dynamical system. The linear stability analysis of the equilibrium points is discussed in the Section 4. The application of the presented model is studied in Section 5. In Sections 6 and 7, the results of the problem are discussed with comparative study to other researchers.

2. Characterization of the dynamical system and equations of motion

Let S_1 and S_2 be two primary bodies of masses m_1 and m_2 respectively where $m_1 > m_2$. The more massive primary S_1 is considered as a point mass and the less massive body $S_{21}S_{22}$ with center at S_2 and length $2l$ is in a shape of a straight segment. In this setup, the line joining the centers of the primaries S_1 and S_2 is considered as $X-$axis and their common center of mass is set as the origin of the coordinate system. Both primaries are moving in circular orbits about their common center of mass with the same angular velocity ω. The line passing through origin and perpendicular to the plane of motion of S_1 and S_2 is considered as $Y-$axis. The $Z-$axis is the line that passes through origin and perpendicular to $XY-$plane. The $XY-$plane is moving in the anticlockwise direction about $Z-$axis. Further, we consider a rotating coordinate system Oxy that rotates with the angular velocity ω (which is same as that of the primaries). The coordinate system $Oxyz$ initially coincides with the inertial coordinate system $OXYZ$.

We consider two infinitesimal bodies S_3 and S_4 with masses m_3 and m_4 respectively (Figure 1). Here infinitesimal means the masses of S_3 and S_4 are very very small in comparison to the primary bodies S_1 and S_2, such that S_3 and S_4 do not influence the motion of S_1 and S_2, but are influenced by them. In this section, we determine the equations of motion of the infinitesimal bodies S_3 and S_4 under the gravitational effect of primary bodies and their mutual gravitational attraction.

Now, to make units dimensionless, we take sum of the masses of S_1 and S_2 and the distance between them as one unit. Also, the unit of time is chosen in such a way, that makes the gravitational constant G unity. Furthermore, the mass parameters are considered as

$$\frac{m_2}{m_1+m_2} = \mu,\ \frac{m_3}{m_1+m_2} = \mu_3,\ \frac{m_4}{m_1+m_2} = \mu_4.$$

Therefore, $m_1 = 1-\mu$, $m_2 = \mu$, $m_3 = \mu_3$ and $m_4 = \mu_4$.

Here, we introduce the small perturbations in the Coriolis and centrifugal forces in the terms of α_1 and α_2, respectively, where $\alpha_1 = 1+\epsilon_1$ and $\alpha_2 = 1+\epsilon_2$, $|\epsilon_1| << 1$, $|\epsilon_2| << 1$. The unperturbed values of α_1 and α_2 are unity. Thus, the equations of motion of the infinitesimal bodies S_3 and S_4 under the effect of small perturbations in the Coriolis and centrifugal forces in the dimensionless rotating coordinate system are

$$\left.\begin{aligned} \ddot{x}_j - 2n\alpha_1\dot{y}_j &= \Omega_{x_j} \\ \ddot{y}_j + 2n\alpha_1\dot{x}_j &= \Omega_{y_j} \\ \ddot{z}_j &= \Omega_{z_j} \end{aligned}\right\} j = 3, 4, \tag{1}$$

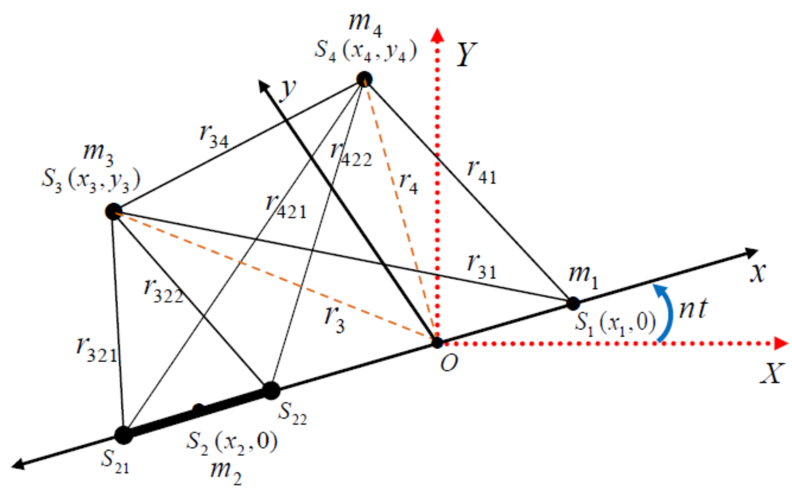

Figure 1. The configuration of the restricted problem of $2+2$ bodies in xy-plane when the less massive body is a straight segment

where

$$\Omega(x_j, y_j, z_j) = \sum_{j=3}^{4} \mu_j \left[\frac{\alpha_2}{2} n^2 \left(x_j^2 + y_j^2 \right) + \frac{(1-\mu)}{r_{j1}} + \frac{\mu}{2l} \log \left(\frac{r_{j21} + r_{j22} + 2l}{r_{j21} + r_{j22} - 2l} \right) + \frac{1}{2} \frac{\mu_{7-j}}{r_{34}} \right],$$

$$\begin{aligned} r_{j1}^2 &= (x_j - \mu)^2 + y_j^2 + z_j^2,\ r_{j21}^2 = \{x_j - (\mu - 1 - l)\}^2 + y_j^2 + z_j^2, \\ r_{j22}^2 &= \{x_j - (\mu - 1 + l)\}^2 + y_j^2 + z_j^2,\ r_{34}^2 = (x_3 - x_4)^2 + (y_3 - y_4)^2 + (z_3 - z_4)^2, \\ \alpha_1 &= 1 + \epsilon_1,\ |\epsilon_1| << 1,\ \alpha_2 = 1 + \epsilon_2,\ |\epsilon_2| << 1. \end{aligned}$$

Here, n is the mean motion of the primaries; and l is the dimensionless half length of the less massive body S_2; Ω_{x_j}, Ω_{y_j} and Ω_{z_j} are the partial derivatives of Ω with respect to x_j, y_j and z_j, respectively; the dot represents the differentiation with respect to time. ϵ_1 and ϵ_2 are the small perturbations in the Coriolis and centrifugal forces, respectively.

2.1. Mean motion of the primaries

The gravitational force between the primary bodies S_1 and S_2 is given by

$$F_{S_1 S_2} = G \frac{m_1 m_2}{(d_1 + d_2)^2 - l^2},$$

where $d_1 = OS_1$ and $d_2 = OS_2$.

Since the bodies S_1 and S_2 are moving in circular orbits around their common center of mass O, therefore

$$m_1 d_1 n^2 = G \frac{m_1 m_2}{(d_1 + d_2)^2 - l^2} = m_2 d_2 n^2, \tag{2}$$

On simplifying Equation (2), we get

$$(d_1 + d_2) n^2 = G \frac{(m_1 + m_2)}{(d_1 + d_2)^2 - l^2}.$$

Now, on using dimensionless variables defined earlier, we get $n^2 = 1 + l^2,\ 0 < l << 1$. Here, we have considered the terms containing l up to second order only. The impact of length parameter on the mean motion is shown in Figure 2. It is observed that on the increasing values of l, the mean motion also increases.

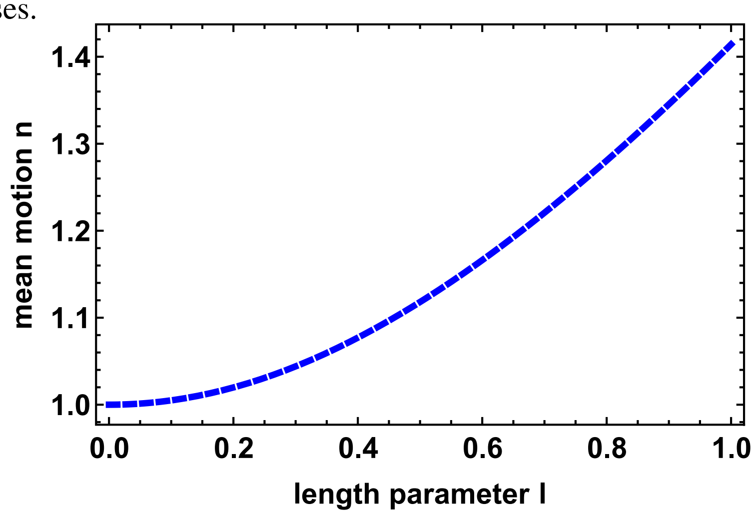

Figure 2. The effect of length parameter on the mean motion

3. Equilibrium points

In this section, we find the equilibrium points of the dynamical system defined in Section 2. The equilibrium point means where all the derivatives of all orders are zero, that is, we can obtain the equilibrium points by solving the differential equations $\dot{x}_j = 0$, $\dot{y}_j = 0$, $\dot{z}_j = 0$, $\ddot{x}_j = 0$, $\ddot{y}_j = 0$ and $\ddot{z}_j = 0$, where $j = 3, 4$. On substituting these conditions in the equations of motion (1), the positions of the equilibrium points under the effect of small perturbations in the Coriolis and centrifugal forces, are obtained by solving the following equations,

$$\Omega_{x_j} = 0,\ \Omega_{y_j} = 0,\ \Omega_{z_j} = 0,\ j = 3, 4,$$

that is,

$$\begin{aligned} n^2\left(1+\epsilon_2\right)x_3 &- \frac{(1-\mu)(x_3-\mu)}{r_{31}^3} \\ &- \frac{2\mu}{[(r_{321}+r_{322})^2-4l^2]} \times \left(\frac{x_3-\mu+1+l}{r_{321}} + \frac{x_3-\mu+1-l}{r_{322}}\right) - \frac{\mu_4(x_3-x_4)}{r_{34}^3} = 0, \end{aligned} \tag{3}$$

$$n^2\left(1+\epsilon_2\right)y_3 - \frac{(1-\mu)y_3}{r_{31}^3} - \frac{2\mu}{[(r_{321}+r_{322})^2-4l^2]}\left(\frac{y_3}{r_{321}} + \frac{y_3}{r_{322}}\right) - \frac{\mu_4(y_3-y_4)}{r_{34}^3} = 0, \tag{4}$$

$$\frac{(1-\mu)z_3}{r_{31}^3} + \frac{2\mu}{[(r_{321}+r_{322})^2-4l^2]}\left(\frac{z_3}{r_{321}} + \frac{z_3}{r_{322}}\right) + \frac{\mu_4(z_3-z_4)}{r_{34}^3} = 0, \tag{5}$$

and

$$n^2\,(1+\epsilon_2)\,x_4-\frac{(1-\mu)(x_4-\mu)}{r_{41}^3}$$
$$-\frac{2\mu}{[(r_{421}+r_{422})^2-4l^2]}\times\left(\frac{x_4-\mu+1+l}{r_{421}}+\frac{x_4-\mu+1-l}{r_{422}}\right)-\frac{\mu_3(x_4-x_3)}{r_{34}^3}=0, \tag{6}$$

$$n^2\,(1+\epsilon_2)\,y_4-\frac{(1-\mu)y_4}{r_{41}^3}-\frac{2\mu}{[(r_{421}+r_{422})^2-4l^2]}\times\left(\frac{y_4}{r_{421}}+\frac{y_4}{r_{422}}\right)-\frac{\mu_3(y_4-y_3)}{r_{34}^3}=0, \tag{7}$$

$$\frac{(1-\mu)z_4}{r_{41}^3}+\frac{2\mu}{[(r_{421}+r_{422})^2-4l^2]}\left(\frac{z_4}{r_{421}}+\frac{z_4}{r_{422}}\right)+\frac{\mu_3(z_4-z_3)}{r_{34}^3}=0. \tag{8}$$

On simplifying Equations (5) and (8) (as in Whipple (1984)), it is observed that z_3 and z_4 both are zero for all the values of the parameters involved. Therefore, all the equilibrium points will lie in $xy-$plane. Thus, we study the motion of the infinitesimal bodies S_3 and S_4 in $xy-$plane only.

It is also observed that Equations (3), (4), (6) and (7) are free from ϵ_1 which implies that, the small perturbation in the Coriolis force will not influence the position of the equilibrium points; however, the equilibrium points will be influenced by the small perturbation in the centrifugal force.

3.1. Collinear equilibrium points

The collinear equilibrium points are the solutions of the Equations (3), (4), (6), and (7) with $y_3 = y_4 = 0$, that is,

$$n^2\,(1+\epsilon_2)\,x_3-\frac{(1-\mu)(x_3-\mu)}{r_{31}^3}-\frac{2\mu}{[(r_{321}+r_{322})^2-4l^2]}\times\left(\frac{x_3-\mu+1+l}{r_{321}}+\frac{x_3-\mu+1-l}{r_{322}}\right)$$
$$-\frac{\mu_4(x_3-x_4)}{r_{34}^3}=0,$$

and

$$n^2\,(1+\epsilon_2)\,x_4-\frac{(1-\mu)(x_4-\mu)}{r_{41}^3}-\frac{2\mu}{[(r_{421}+r_{422})^2-4l^2]}\times\left(\frac{x_4-\mu+1+l}{r_{421}}+\frac{x_4-\mu+1-l}{r_{422}}\right)$$
$$-\frac{\mu_3(x_4-x_3)}{r_{34}^3}=0,$$

with

$$r_{j1}=|x_j-\mu|,\ r_{j21}=|x_j-(\mu-1-l)|,\ r_{j22}=|x_j-(\mu-1+l)|,\ j=3,4.$$

If either S_3 or S_4 is absent, the present model reduces to the restricted three-body problem under the combined effects of straight segment, small perturbations in the Coriolis and centrifugal forces. Without loss of generality, we take m_4 to be zero. Therefore, we have the following equation

$$n^2\,(1+\epsilon_2)\,x_3-\frac{(1-\mu)(x_3-\mu)}{r_{31}^3}$$
$$-\frac{2\mu}{[(r_{321}+r_{322})^2-4l^2]}\times\left(\frac{x_3-\mu+1+l}{r_{321}}+\frac{x_3-\mu+1-l}{r_{322}}\right)=0. \tag{9}$$

Now, we find the solutions of the Equation (9) (as in Szebehely (1967b)). We observe that Equation (9) has three real roots x_{L_1}, x_{L_2} and x_{L_3} that lie in the intervals $(\mu-2, \mu-1-l)$, $(\mu-1+l, \mu)$ and $(\mu, \mu+1)$, respectively, where

$$x_{L_1}=\mu-1-\xi_{L_1},\ x_{L_2}=\mu-1+\xi_{L_2},\ x_{L_3}=\mu+\xi_{L_3},$$

and ξ_{L_1}, ξ_{L_2} and ξ_{L_3} are the real roots of the Equations (10), (11), and (12), respectively.

$$\begin{aligned}&\left(1+\epsilon_2+l^2\right)\xi_{L_1}^5+(3-\mu)\left(1+\epsilon_2+l^2\right)\xi_{L_1}^4+\left[(3-2\mu)\left(1+\epsilon_2\right)+2\left(1-\mu\right)l^2\right]\xi_{L_1}^3\\&-\left[\mu+(\mu-1)\,\epsilon_2+2l^2\right]\xi_{L_1}^2-\left[2\mu+(3-2\mu)\,l^2\right]\xi_{L_1}-\mu=0,\end{aligned}\tag{10}$$

$$\begin{aligned}&\left(1+\epsilon_2+l^2\right)\xi_{L_2}^5-(3-\mu)\left(1+\epsilon_2+l^2\right)\xi_{L_2}^4+\left[(3-2\mu)\left(1+\epsilon_2\right)+2\left(1-\mu\right)l^2\right]\xi_{L_2}^3\\&-\left[\mu+(1-\mu)\,\epsilon_2-2l^2\right]\xi_{L_2}^2+\left[2\mu-(3-2\mu)\,l^2\right]\xi_{L_2}-\mu=0,\end{aligned}\tag{11}$$

and

$$\begin{aligned}&\left(1+\epsilon_2+l^2\right)\xi_{L_3}^5+(2+\mu)\left(1+\epsilon_2+l^2\right)\xi_{L_3}^4+\left[(1+2\mu)\left(1+\epsilon_2\right)+2\mu l^2\right]\xi_{L_3}^3\\&-\left(1-\mu-\mu\epsilon_2\right)\xi_{L_3}^2-2\left(1-\mu\right)\xi_{L_3}-(1-\mu)\left(1-l^2\right)=0.\end{aligned}\tag{12}$$

Thus, for $m_4 = 0$, we obtain three collinear equilibrium points $L_1\left(x_{L_1}, 0\right)$, $L_2\left(x_{L_2}, 0\right)$ and $L_3\left(x_{L_3}, 0\right)$. It is observed that the positions of these collinear equilibrium points are influenced by the small perturbation ϵ_2 in the centrifugal force. But there is no impact of small perturbation ϵ_1 in the Coriolis force on the collinear equilibrium points L_1, L_2 and L_3.

We find the collinear equilibrium points for the case when S_3 and S_4 both are present. We use the perturbation technique in the terms of small parameters $\epsilon_j, j = 3, 4$ to the solutions x_{L_1}, x_{L_2} and x_{L_3}. Therefore,

$$\begin{aligned}x_3=&x_{L_k}+a_1\epsilon_4+a_2\epsilon_4^2+a_3\epsilon_4^3+\cdots,\\x_4=&x_{L_k}+b_1\epsilon_3+b_2\epsilon_3^2+b_3\epsilon_3^3+\cdots,\ k=1,2,3,\end{aligned}$$

where

$$\epsilon_j=\frac{\mu_j}{(\mu_3+\mu_4)^{2/3}},\quad j=3,4.$$

Following the same procedure as given in Whipple (1984) and considering only linear terms of ϵ_3 and ϵ_4, six collinear equilibrium points

$$\left.\begin{aligned}&L_1^1((x_3^{11},0),(x_4^{11},0)),\\&L_1^2((x_3^{12},0),(x_4^{12},0)),\end{aligned}\right\}\text{around } L_1,$$

$$\left.\begin{aligned}&L_2^1((x_3^{21},0),(x_4^{21},0)),\\&L_2^2((x_3^{22},0),(x_4^{22},0)),\end{aligned}\right\}\text{around } L_2,$$

$$\left.\begin{aligned}&L_3^1((x_3^{31},0),(x_4^{31},0)),\\&L_3^2((x_3^{32},0),(x_4^{32},0)),\end{aligned}\right\}\text{around } L_3,$$

are obtained. The abscissae of these collinear equilibrium points are

$$\left.\begin{aligned} x_3^{k1} &= x_{L_k} + \frac{\mu_4}{[W_{xx}^{L_k}(\mu_3+\mu_4)^2]^{1/3}},\ x_4^{k1} = x_{L_k} - \frac{\mu_3}{[W_{xx}^{L_k}(\mu_3+\mu_4)^2]^{1/3}},\\ x_3^{k2} &= x_{L_k} - \frac{\mu_4}{[W_{xx}^{L_k}(\mu_3+\mu_4)^2]^{1/3}},\ x_4^{k2} = x_{L_k} + \frac{\mu_3}{[W_{xx}^{L_k}(\mu_3+\mu_4)^2]^{1/3}},\end{aligned}\right\} k=1,2,3,$$

where

$$\begin{aligned} W(x,y) &= \frac{1}{2}n^2\alpha_2\left(x^2+y^2\right) + \frac{(1-\mu)}{r_{31}} + \frac{\mu}{2l}\log\left(\frac{r_{321}+r_{322}+2l}{r_{321}+r_{322}-2l}\right),\\ n^2 &= 1+l^2,\ \alpha_2 = 1+\epsilon_2,\ |\epsilon_2| << 1,\ r_{31}^2 = (x-\mu)^2+y^2,\\ r_{321}^2 &= \{x-(\mu-1-l)\}^2+y^2,\ r_{322}^2 = \{x-(\mu-1+l)\}^2+y^2. \end{aligned}$$

Here, $W(x,y)$ is the potential function of the perturbed restricted three-body problem when the less massive body is a straight segment. And $W_{xx}^{L_k}$ is the second order partial derivative of W with respect to x calculated at the equilibrium point L_k.

3.2. Non-collinear equilibrium points

The non-collinear equilibrium points are the solutions of the Equations (3), (4), (6), and (7) when y_3 and y_4 both are non zero, that is,

$$\begin{aligned} &n^2(1+\epsilon_2)x_3 - \frac{(1-\mu)(x_3-\mu)}{r_{31}^3}\\ &\quad - \frac{2\mu}{[(r_{321}+r_{322})^2-4l^2]} \times \left(\frac{x_3-\mu+1+l}{r_{321}} + \frac{x_3-\mu+1-l}{r_{322}}\right) - \frac{\mu_4(x_3-x_4)}{r_{34}^3} = 0, \end{aligned} \tag{13}$$

$$n^2(1+\epsilon_2)y_3 - \frac{(1-\mu)y_3}{r_{31}^3} - \frac{2\mu}{[(r_{321}+r_{322})^2-4l^2]}\left(\frac{y_3}{r_{321}} + \frac{y_3}{r_{322}}\right) - \frac{\mu_4(y_3-y_4)}{r_{34}^3} = 0, \tag{14}$$

and

$$\begin{aligned} &n^2(1+\epsilon_2)x_4 - \frac{(1-\mu)(x_4-\mu)}{r_{41}^3}\\ &\quad - \frac{2\mu}{[(r_{421}+r_{422})^2-4l^2]} \times \left(\frac{x_4-\mu+1+l}{r_{421}} + \frac{x_4-\mu+1-l}{r_{422}}\right) - \frac{\mu_3(x_4-x_3)}{r_{34}^3} = 0, \end{aligned} \tag{15}$$

$$n^2(1+\epsilon_2)y_4 - \frac{(1-\mu)y_4}{r_{41}^3} - \frac{2\mu}{[(r_{421}+r_{422})^2-4l^2]} \times \left(\frac{y_4}{r_{421}} + \frac{y_4}{r_{422}}\right) - \frac{\mu_3(y_4-y_3)}{r_{34}^3} = 0. \tag{16}$$

In the absence of the infinitesimal body S_4, the Equations (13), (14), (15), and (16) are reduced to the following equations

$$\begin{aligned} &n^2(1+\epsilon_2)x_3 - \frac{(1-\mu)(x_3-\mu)}{r_{31}^3}\\ &\quad - \frac{2\mu}{[(r_{321}+r_{322})^2-4l^2]} \times \left(\frac{x_3-\mu+1+l}{r_{321}} + \frac{x_3-\mu+1-l}{r_{322}}\right) = 0, \end{aligned} \tag{17}$$

$$n^2\left(1+\epsilon_2\right)y_3-\frac{(1-\mu)y_3}{r_{31}^3}-\frac{2\mu}{\left[(r_{321}+r_{322})^2-4l^2\right]}\left(\frac{y_3}{r_{321}}+\frac{y_3}{r_{322}}\right)=0. \tag{18}$$

On solving Equations (17) and (18), considering only linear terms of ϵ_2 and up to second order terms of length parameter l, two non-collinear equilibrium points $L_4(x_{L_4}, y_{L_4})$ and $L_5(x_{L_5}, y_{L_5})$ are obtained, where

$$x_{L_4,L_5}=\mu-\frac{1}{2}+\frac{(\mu+3)}{24(\mu-1)}l^2,\ y_{L_4,L_5}=\pm\left[\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{3}}\left(\frac{2}{3}\epsilon_2+\frac{(23\mu-19)}{24(\mu-1)}l^2\right)\right].$$

It is clear that the non-collinear equilibrium points L_4 and L_5 are influenced by the parameters μ, ϵ_2 and l, whereas the they are independent of the parameter ϵ_1.

Following the procedure of Whipple (1984) when S_3 and S_4 both are present, the solutions of the Equations (3), (4), (6), and (7) are obtained as

$$x_j^{k1}=x_{L_k}+\frac{\alpha_k\mu_{7-j}}{\left[(\mu_3+\mu_4)^2(W_{xy}^{L_k}\alpha_k+W_{yy}^{L_k})\right]^{\frac{1}{3}}\left(1+\alpha_k^2\right)^{\frac{1}{2}}},$$
$$y_j^{k1}=y_{L_k}-\frac{\mu_{7-j}}{\left[(\mu_3+\mu_4)^2(W_{xy}^{L_k}\alpha_k+W_{yy}^{L_k})\right]^{\frac{1}{3}}\left(1+\alpha_k^2\right)^{\frac{1}{2}}},$$

$$x_j^{k2}=x_{L_k}-\frac{\alpha_k\mu_{7-j}}{\left[(\mu_3+\mu_4)^2(W_{xy}^{L_k}\alpha_k+W_{yy}^{L_k})\right]^{\frac{1}{3}}\left(1+\alpha_k^2\right)^{\frac{1}{2}}},$$
$$y_j^{k2}=y_{L_k}+\frac{\mu_{7-j}}{\left[(\mu_3+\mu_4)^2(W_{xy}^{L_k}\alpha_k+W_{yy}^{L_k})\right]^{\frac{1}{3}}\left(1+\alpha_k^2\right)^{\frac{1}{2}}},$$

for $j=3,4$; $k=4$ at L_4 and $k=5$ at L_5, and

$$x_j^{k3}=x_{L_k}+\frac{\beta_k\mu_{7-j}}{\left[(\mu_3+\mu_4)^2\left(W_{xx}^{L_k}+\frac{W_{xy}^{L_k}}{\beta_k}\right)\right]^{\frac{1}{3}}\left(1+\frac{1}{\beta_k^2}\right)^{\frac{1}{2}}},$$
$$y_j^{k3}=y_{L_k}-\frac{\mu_{7-j}}{\left[(\mu_3+\mu_4)^2\left(W_{xx}^{L_k}+\frac{W_{xy}^{L_k}}{\beta_k}\right)\right]^{\frac{1}{3}}\left(1+\frac{1}{\beta_k^2}\right)^{\frac{1}{2}}},$$

$$x_j^{k4}=x_{L_k}-\frac{\beta_k\mu_{7-j}}{\left[(\mu_3+\mu_4)^2\left(W_{xx}^{L_k}+\frac{W_{xy}^{L_k}}{\beta_k}\right)\right]^{\frac{1}{3}}\left(1+\frac{1}{\beta_k^2}\right)^{\frac{1}{2}}},$$
$$y_j^{k4}=y_{L_k}+\frac{\mu_{7-j}}{\left[(\mu_3+\mu_4)^2\left(W_{xx}^{L_k}+\frac{W_{xy}^{L_k}}{\beta_k}\right)\right]^{\frac{1}{3}}\left(1+\frac{1}{\beta_k^2}\right)^{\frac{1}{2}}},$$

for $j=3,4$; $k=4$ at L_4 and $k=5$ at L_5, where

$$\alpha_k=\frac{(-1)^{k+1}+(-1)^k\sqrt{1+12(\mu-1/2)^2}}{2\sqrt{3}(\mu-1/2)},\ \beta_k=\frac{(-1)^{k+1}-(-1)^k\sqrt{1+12(\mu-1/2)^2}}{2\sqrt{3}(\mu-1/2)}.$$

Here, $W_{xy}^{L_k}$, $W_{yy}^{L_k}$ and $W_{xx}^{L_k}$ are the second order partial derivatives of W calculated at the non-collinear equilibrium points $L_k, k=4,5$. Thus, for the restricted problem of $2+2$ bodies under the combined effect of straight segment, small perturbations in Coriolis and centrifugal forces, eight non-collinear equilibrium points are obtained and denoted as follows:

$$\left.\begin{array}{l} L_4^p\left((x_3^{4p},y_3^{4p}),(x_4^{4p},y_4^{4p})\right) \dashrightarrow \text{about } L_4 \\ L_5^p\left((x_3^{5p},y_3^{5p}),(x_4^{5p},y_4^{5p})\right) \dashrightarrow \text{about } L_5 \end{array}\right\} p=1,2,3,4.$$

Out of eight non-collinear equilibrium points, four lie around L_4 and remaining lie around L_5.

Hence, for the present dynamical system fourteen equilibrium points are obtained, which is in contrast to the restricted three-body problem (Szebehely (1967b)) where five equilibrium points exist.

4. Stability of the equilibrium points

In this section, we investigate the linear stability of the equilibrium points obtained in Section 3. To check the stability, we displace the infinitesimal body a little from an equilibrium point. If the infinitesimal body oscillates about the equilibrium point, we say such a point a stable equilibrium point. However, if the motion of an infinitesimal body is a rapid departure from the equilibrium point, it is called as unstable equilibrium point.

Without loss of generality, we determine the stability of the infinitesimal body S_3. Let $E\left(x_{30},y_{30}\right)$ be one of the equilibrium points corresponding to S_3. The small perturbations from the equilibrium point are considered as ξ_3 and η_3 in the x and y axes respectively. That is,

$$x_3=x_{30}+\xi_3,\ y_3=y_{30}+\eta_3.$$

Substituting theses values in the first two equations of Equations (1) and applying Taylor series expansion and considering only linear terms of ξ_3 and η_3, we get the system of variational equations as:

$$\left.\begin{aligned} \ddot{\xi}_3-2n\alpha_1\dot{\eta}_3&=\frac{1}{\mu_3}\left(\xi_3\Omega^0_{x_3x_3}+\eta_3\Omega^0_{x_3y_3}\right), \\ \ddot{\eta}_3+2n\alpha_1\dot{\xi}_3&=\frac{1}{\mu_3}\left(\xi_3\Omega^0_{x_3y_3}+\eta_3\Omega^0_{y_3y_3}\right), \end{aligned}\right\} \tag{19}$$

where the superscript 0 denotes that the partial derivatives of Ω are evaluated at the equilibrium point $E\left(x_{30},y_{30}\right)$.

The characteristic equation corresponding to the system of Equations (19) is given by

$$\begin{vmatrix} \lambda_3^2-\frac{1}{\mu_3}\Omega^0_{x_3x_3} & -2n\alpha_1\lambda_3-\frac{1}{\mu_3}\Omega^0_{x_3y_3} \\ 2n\alpha_1\lambda_3-\frac{1}{\mu_3}\Omega^0_{x_3y_3} & \lambda_3^2-\frac{1}{\mu_3}\Omega^0_{y_3y_3} \end{vmatrix}=0,$$

or

$$\lambda_3^4+\left(4n^2\alpha_1^2-\frac{1}{\mu_3}\Omega^0_{x_3x_3}-\frac{1}{\mu_3}\Omega^0_{y_3y_3}\right)\lambda_3^2+\frac{1}{\mu_3^2}\left(\Omega^0_{x_3x_3}\Omega^0_{y_3y_3}-\left(\Omega^0_{x_3y_3}\right)^2\right)=0. \tag{20}$$

The roots of the characteristic Equation (20) have an important role to check the stability of the equilibrium points. The equilibrium point is said to be stable equilibrium point, if all the four roots of the characteristic Equation (20) are either imaginary or complex with negative real parts. Therefore, for the stable equilibrium point, the following three conditions must be satisfied simultaneously:

$$\left.\begin{array}{l}\left(4n^2\alpha_1^2 - \frac{1}{\mu_3}\Omega^0_{x_3x_3} - \frac{1}{\mu_3}\Omega^0_{y_3y_3}\right) > 0,\ \frac{1}{\mu_3^2}\left(\Omega^0_{x_3x_3}\Omega^0_{y_3y_3} - \left(\Omega^0_{x_3y_3}\right)^2\right) > 0,\\ \left(4n^2\alpha_1^2 - \frac{1}{\mu_3}\Omega^0_{x_3x_3} - \frac{1}{\mu_3}\Omega^0_{y_3y_3}\right)^2 - \frac{4}{\mu_3^2}\left(\Omega^0_{x_3x_3}\Omega^0_{y_3y_3} - \left(\Omega^0_{x_3y_3}\right)^2\right) > 0.\end{array}\right\} \tag{21}$$

5. Application

In this section, we study the application of the present model to the Earth-22 Kalliope-dual satellite system. In this physical model, we consider the more massive primary as the Earth and the infinitesimal bodies are as satellites. The less massive body is considered as 22 Kalliope. Furthermore, for the the distance between the primary bodies, the minimum orbit intersection distance (MOID) is considered. The required physical data has been taken from NASA database (https://ssd.jpl.nasa.gov/sbdb.cgi), Croustalloudi and Kalvouridis (2013) and Kaur et al. (2020a).

(1) **Earth-22 Kalliope-dual satellite system:**

Mass of the Earth: $m_1 = 5.97237 \times 10^{24}$ kg, mass of 22 Kalliope: $m_2 = 8.42 \times 10^{18}$ kg, distance between Earth and 22 Kalliope $= 1.63844$ A.U. $= 245107135$ km, length of 22 Kalliope: $2l = 215$ km, mass of S_3: $m_3 = 475$ kg, mass of S_4: $m_4 = 245$ kg.

In dimensionless system: $\mu = 1.40982 \times 10^{-6}$, $l = 4.38584 \times 10^{-7}$, $\mu_3 = 7.95328 \times 10^{-23}$, $\mu_4 = 4.10222 \times 10^{-23}$.

For the Earth-22 Kalliope-dual satellite system, we calculate the position and stability of the equilibrium points, by taking different values of small perturbations in Coriolis and centrifugal forces ϵ_1 and ϵ_2, respectively.

Table 1. The abscissae of the collinear equilibrium points $L_1^1\left((x_3^{11},0),(x_4^{11},0)\right)$ and $L_1^2\left((x_3^{12},0),(x_4^{12},0)\right)$ about L_1 for the Earth-22 Kalliope-dual satellite system

ϵ_2	L_1^1		L_1^2	
	x_3^{11}	x_4^{11}	x_3^{12}	x_4^{12}
-0.04	-1.015689989	-1.015690021	-1.015690011	-1.015689979
-0.02	-1.010849990	-1.010850019	-1.010850010	-1.010849981
0.02	-1.006069993	-1.006070013	-1.006070007	-1.006069987
0.04	-1.005049994	-1.005050011	-1.005050006	-1.005049989

The effect of small perturbation in the centrifugal force on the position of the collinear equilibrium points L_1^1, L_1^2, L_2^1, L_2^2, L_3^1 and L_3^2 are calculated for the different values of $\epsilon_2 = -0.04$, -0.02,

Table 2. The abscissae of the collinear equilibrium points $L_2^1\left((x_3^{21},0),(x_4^{21},0)\right)$ and $L_2^2\left((x_3^{22},0),(x_4^{22},0)\right)$ about L_2 for the Earth-22 Kalliope-dual satellite system

	L_2^1		L_2^2	
ϵ_2	x_3^{21}	x_4^{21}	x_3^{22}	x_4^{22}
-0.04	-0.9949379942	-0.9949380112	-0.9949380058	-0.9949379888
-0.02	-0.9939249933	-0.9939250130	-0.9939250067	-0.9939249870
0.02	-0.9893629904	-0.9893630185	-0.9893630096	-0.9893629815
0.04	-0.9850039894	-0.9850040206	-0.9850040106	-0.9850039794

Table 3. The abscissae of the collinear equilibrium points $L_3^1\left((x_3^{31},0),(x_4^{31},0)\right)$ and $L_3^2\left((x_3^{32},0),(x_4^{32},0)\right)$ about L_3 for the Earth-22 Kalliope-dual satellite system

	L_3^1		L_3^2	
ϵ_2	x_3^{31}	x_4^{31}	x_3^{32}	x_4^{32}
-0.04	1.0137000120	1.0136999770	1.0136999880	1.0137000230
-0.02	1.0067600120	1.0067599770	1.0067599880	1.0067600230
0.02	0.9934210116	0.9934209776	0.9934209884	0.9934210224
0.04	0.9870120115	0.9870119777	0.9870119885	0.9870120223

Table 4. The positions of the non-collinear equilibrium points $L_4^p\left((x_3^{4p},y_3^{4p}),(x_4^{4p},y_4^{4p})\right)$, $p=1,2,3,4$ about L_4 for the Earth-22 Kalliope-dual satellite system

ϵ_2		(x_3^{4p},y_3^{4p})		(x_4^{4p},y_4^{4p})
-0.04	(x_3^{41},y_3^{41})	$(-0.4999990059, 0.8814209898)$	(x_4^{41},y_4^{41})	$(-0.4999990114, 0.8814209802)$
	(x_3^{42},y_3^{42})	$(-0.4999989941, 0.8814210102)$	(x_4^{42},y_4^{42})	$(-0.4999998986, 0.8814210198)$
	(x_3^{43},y_3^{43})	$(*,*)$	(x_4^{43},y_4^{43})	$(*,*)$
	(x_3^{44},y_3^{44})	$(*,*)$	(x_4^{44},y_4^{44})	$(*,*)$
-0.02	(x_3^{41},y_3^{41})	$(-0.4999990059, 0.8737229898)$	(x_4^{41},y_4^{41})	$(-0.4999990114, 0.8737229803)$
	(x_3^{42},y_3^{42})	$(-0.4999989941, 0.8737230102)$	(x_4^{42},y_4^{42})	$(-0.4999989886, 0.8737230197)$
	(x_3^{43},y_3^{43})	$(*,*)$	(x_4^{43},y_4^{43})	$(*,*)$
	(x_3^{44},y_3^{44})	$(*,*)$	(x_4^{44},y_4^{44})	$(*,*)$
0.02	(x_3^{41},y_3^{41})	$(-0.4999990058, 0.8583269900)$	(x_4^{41},y_4^{41})	$(-0.4999990112, 0.8583269805)$
	(x_3^{42},y_3^{42})	$(-0.4999989942, 0.8583270100)$	(x_4^{42},y_4^{42})	$(-0.4999989888, 0.8583270195)$
	(x_3^{43},y_3^{43})	$(-0.4999988659, 0.8583269226)$	(x_4^{43},y_4^{43})	$(-0.4999987400, 0.8583268499)$
	(x_3^{44},y_3^{44})	$(-0.4999991341, 0.8583270774)$	(x_4^{44},y_4^{44})	$(-0.4999992600, 0.8583271501)$
0.04	(x_3^{41},y_3^{41})	$(-0.4999990058, 0.8506289900)$	(x_4^{41},y_4^{41})	$(-0.4999990112, 0.8506289807)$
	(x_3^{42},y_3^{42})	$(-0.4999989942, 0.8506290100)$	(x_4^{42},y_4^{42})	$(-0.4999989888, 0.8506290193)$
	(x_3^{43},y_3^{43})	$(-0.4999988935, 0.8506289385)$	(x_4^{43},y_4^{43})	$(-0.4999987935, 0.8506288808)$
	(x_3^{44},y_3^{44})	$(-0.4999991065, 0.8506290615)$	(x_4^{44},y_4^{44})	$(-0.4999992065, 0.8506291192)$

0.02 and 0.04, that are shown in Tables 1, 2 and 3. It is observed that, the small perturbation in centrifugal force has a substantial effect on the position of the collinear equilibrium points. On the increasing values of ϵ_2 from -0.04 to 0.04, the abscissae of four collinear equilibrium points L_1^1, L_1^2, L_2^1 and L_2^2 increase, whereas the abscissae of remaining collinear equilibrium points L_3^1 and L_3^2 decrease.

Furthermore, we study the effect of small perturbation in centrifugal force on the position of the

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Table 5. The positions of the non-collinear equilibrium points $L_5^p\left((x_3^{5p}, y_3^{5p}), (x_4^{5p}, y_4^{5p})\right)$, $p = 1, 2, 3, 4$ about L_5 for the Earth-22 Kalliope-dual satellite system

ϵ_2		(x_3^{5p}, y_3^{5p})		(x_4^{5p}, y_4^{5p})
-0.04	(x_3^{51}, y_3^{51})	$(-0.4999989941, -0.8814210102)$	(x_4^{51}, y_4^{51})	$(-0.4999998986, -0.8814210198)$
	(x_3^{52}, y_3^{52})	$(-0.4999990059, -0.8814209898)$	(x_4^{52}, y_4^{52})	$(-0.4999990114, -0.8814209802)$
	(x_3^{53}, y_3^{53})	$(*, *)$	(x_4^{53}, y_4^{53})	$(*, *)$
	(x_3^{54}, y_3^{54})	$(*, *)$	(x_4^{54}, y_4^{54})	$(*, *)$
-0.02	(x_3^{51}, y_3^{51})	$(-0.4999989941, -0.8737230102)$	(x_4^{51}, y_4^{51})	$(-0.4999989886, -0.8737230197)$
	(x_3^{52}, y_3^{52})	$(-0.4999990059, -0.8737229898)$	(x_4^{52}, y_4^{52})	$(-0.4999990114, -0.8737229803)$
	(x_3^{53}, y_3^{53})	$(*, *)$	(x_4^{53}, y_4^{53})	$(*, *)$
	(x_3^{54}, y_3^{54})	$(*, *)$	(x_4^{54}, y_4^{54})	$(*, *)$
0.02	(x_3^{51}, y_3^{51})	$(-0.4999989942, -0.8583270100)$	(x_4^{51}, y_4^{51})	$(-0.4999989888, -0.8583270195)$
	(x_3^{52}, y_3^{52})	$(-0.4999990058, -0.8583269900)$	(x_4^{52}, y_4^{52})	$(-0.4999990112, -0.8583269805)$
	(x_3^{53}, y_3^{53})	$(-0.4999991341, -0.8583270774)$	(x_4^{53}, y_4^{53})	$(-0.4999992600, -0.8583271501)$
	(x_3^{54}, y_3^{54})	$(-0.4999988659, -0.8583269226)$	(x_4^{54}, y_4^{54})	$(-0.4999987400, -0.8583268499)$
0.04	(x_3^{51}, y_3^{51})	$(-0.4999989942, -0.8506290100)$	(x_4^{51}, y_4^{51})	$(-0.4999989888, -0.8506290193)$
	(x_3^{52}, y_3^{52})	$(-0.4999990058, -0.8506289900)$	(x_4^{52}, y_4^{52})	$(-0.4999990112, -0.8506289807)$
	(x_3^{53}, y_3^{53})	$(-0.4999991065, -0.8506290615)$	(x_4^{53}, y_4^{53})	$(-0.4999992065, -0.8506291192)$
	(x_3^{54}, y_3^{54})	$(-0.4999988935, -0.8506289385)$	(x_4^{54}, y_4^{54})	$(-0.4999987935, -0.8506288808)$

eight non-collinear equilibrium points L_4^p, L_5^p, $p = 1, 2, 3, 4$. From Tables 4 and 5, we observe that the small perturbation in centrifugal force has a substantial effect on the number and position of non-collinear equilibrium points. For $\epsilon_2 = -0.04$ and -0.02, only four non-collinear equilibrium points exist, two around L_4 and two around L_5. However, for $\epsilon_2 = 0.02$ and 0.04 eight non-collinear equilibrium points exist, four around L_4 and four around L_5. In Tables 4 and 5, $(*, *)$ represents the non-collinear equilibrium point does not exist for the corresponding values of ϵ_2.

Table 6. The stability of the equilibrium points $L_1^{1,2}$, $L_2^{1,2}$, $L_3^{1,2}$, $L_{4,5}^p$, $p = 1, 2, 3, 4$ for the Earth-22 Kalliope-dual satellite system

$\epsilon_1 \rightarrow$ $\epsilon_2 \downarrow$	-0.2	-0.1	$+0.1$	$+0.2$
-0.04	unstable	unstable	unstable	unstable
-0.02	unstable	unstable	unstable	unstable
$+0.02$	unstable	unstable	unstable	unstable
$+0.04$	unstable	unstable	unstable	unstable

We have also checked the linear stability of the equilibrium points for the Earth-22 Kalliope-dual satellite system. The stability of the equilibrium points is influenced by both the parameters ϵ_1 and ϵ_2. We have determined the roots of the characteristic equation (20) at all the equilibrium points for a considered set of values of ϵ_1 and ϵ_2. It is observed that for all the equilibrium points, there always exist a positive real root for the combinations of $\epsilon_1 = -0.2, -0.1, 0.1, 0.2$ and $\epsilon_2 = -0.04$, -0.02, 0.02, 0.04. It means all the equilibrium points are unstable, that is shown in Table 6.

5.1. Zero velocity curves

This section is devoted to the study of the permissible regions of motion of the infinitesimal body. To do so, first we find the Jacobian integral corresponding to Equations (1), that is, obtained as

$$C = 2\Omega\left(x_3, y_3, z_3, x_4, y_4, z_4\right) - \sum_{j=3}^{4} \mu_j \left(\dot{x}_j^2 + \dot{y}_j^2 + \dot{z}_j^2\right), \tag{22}$$

where C is the Jacobian constant.

The zero velocity curves are the projection of the zero velocity surfaces onto the xy-plane. These curves are obtained by $2\Omega\left(x_3, y_3, x_4, y_4\right) = C$. The effect of small perturbation ϵ_2 in the centrifugal force on the Jacobian constant C is determined for the Earth-22 Kalliope-dual satellite system, that is given in Table 7. In this model, the function Ω is dependent on four variables x_3, y_3, x_4 and y_4. First, we fix the position of the infinitesimal body $S_4(-0.9850040206, 0)$ to study the regions of motion of S_3 in two dimensional. In Figure 3, the zero velocity curves are plotted for the Earth-22 Kalliope-dual satellite system and fixed value of $\epsilon_2 = 0.04$. We investigate the effect of Jacobian constant on the regions of motion of the infinitesimal body S_3. In Figure 3, the region with light blue color is the permissible region and the white region is corresponding to the forbidden region.

Table 7. The Jacobian constant evaluated at the equilibrium points for the Earth-22 Kalliope-dual satellite system

Equilibrium points	$\epsilon_2 = -0.04$	$\epsilon_2 = -0.02$	$\epsilon_2 = +0.02$	$\epsilon_2 = +0.04$
L_1^1, L_1^2	3.56799×10^{-22}	3.59274×10^{-22}	3.64174×10^{-22}	3.66612×10^{-22}
L_2^1, L_2^2	3.56967×10^{-22}	3.59352×10^{-22}	3.64098×10^{-22}	3.66448×10^{-22}
L_3^1, L_3^2	3.56777×10^{-22}	3.59238×10^{-22}	3.6406×10^{-22}	3.66424×10^{-22}
L_4^1, L_4^2, L_5^1, L_5^2	3.56777×10^{-22}	3.59237×10^{-22}	3.6406×10^{-22}	3.66424×10^{-22}
L_4^3, L_4^4, L_5^3, L_5^4	$\cdots\cdots$	$\cdots\cdots$	3.6406×10^{-22}	3.66424×10^{-22}

In Figure 3 panel (a), the zero velocity curves are plotted for $C = 4.979 \times 10^{-22}$ and it is observed that a circular region (with light blue color) is formed around the primary m_1 in which the infinitesimal body can move. All the equilibrium points and less massive primary m_2 lie in the white region in which the infinitesimal body can not move. The zoomed portions of the equilibrium points are shown in panels (g), (h), (i), (j) and (k). The positions of the infinitesimal bodies S_3 and S_4 are shown by solid diamond with black and magenta color respectively. Further, on decreasing the value of $C = 4.243 \times 10^{-22}$ (in panel (b)), a circular strip (with white color) is formed that consists of m_2 and fourteen equilibrium points. The circular region around m_1 increases, but the infinitesimal body can not move from one primary to other primary. In panel (c), we take $C = 4.125 \times 10^{-22}$ and observe that the permissible regions increase, but the infinitesimal body can not move from m_1 to other places.

Furthermore, panel (d) is drawn for $C = 3.7524 \times 10^{-22}$ and noticed that the size of circular strip (with white color) decrease. However the connectivity between the primary bodies is not possible. On further decreasing the value of the Jacobian constant $C = 3.67810 \times 10^{-22}$ in panel (e), we observe that the permissible regions of motion increase. In the last panel (f), the zero velocity

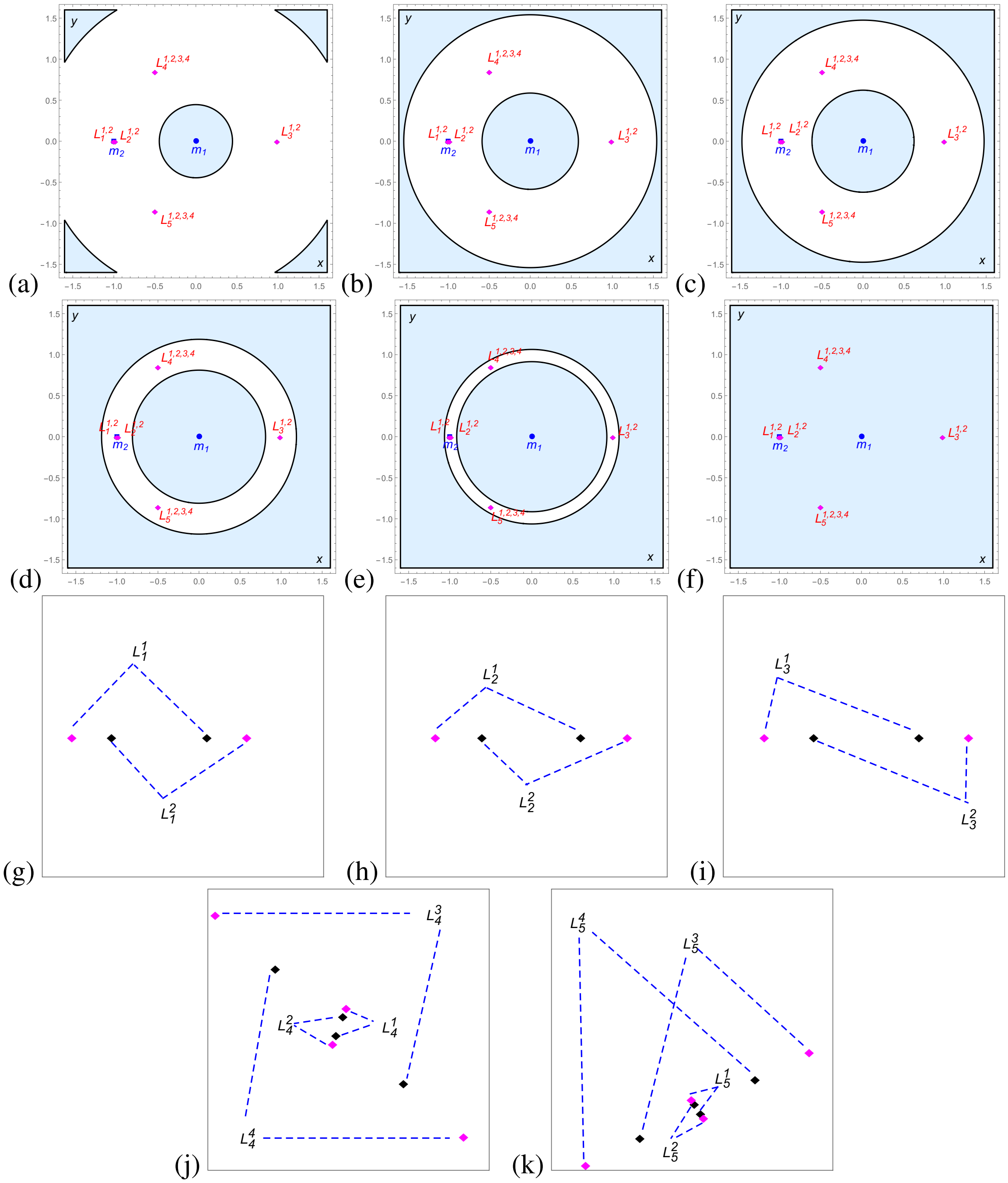

Figure 3. The zero velocity curves for the Earth-22 Kalliope-dual satellite system when $\epsilon_2 = 0.04$ and different values of Jacobian constant (a) $C = 4.979 \times 10^{-22}$ (b) $C = 4.243 \times 10^{-22}$ (c) $C = 4.125 \times 10^{-22}$ (d) $C = 3.7524 \times 10^{-22}$ (e) $C = 3.67810 \times 10^{-22}$ (f) $C = 3.66424 \times 10^{-22}$. (g) The zoomed region of $L_1^{1,2}$. (h) The zoomed region of $L_2^{1,2}$. (i) The zoomed region of $L_3^{1,2}$. (j) The zoomed region of $L_4^{1,2,3,4}$. (k) The zoomed region of $L_5^{1,2,3,4}$. The positions of S_3 and S_4 are shown by solid diamond with black and magenta color respectively. The light blue region represents the permissible region, while the white region represents the forbidden region.

curves are drawn for $C = 3.66424 \times 10^{-22}$ and we observe that the forbidden region disappears and infinitesimal body can move anywhere in whole $xy-$plane.

6. Discussion

In this paper, we have studied the combined effects of straight segment and small perturbations in the Coriolis and centrifugal forces on the existence and linear stability of the equilibrium points in the restricted problem of $2+2$ bodies. We have obtained fourteen equilibrium points $L_1^{1,2}$, $L_2^{1,2}$, $L_3^{1,2}$, $L_4^{1,2,3,4}$ and $L_5^{1,2,3,4}$, six collinear and eight non-collinear for the present model. In the absence of infinitesimal body S_4, the present model has five equilibrium points $L_k, k = 1, 2, 3, 4, 5$. The equilibrium points L_1, L_2 and L_3 are collinear, while L_4 and L_5 are non-collinear with the centers of the primaries S_1 and S_2. The collinear equilibrium points L_1^1 and L_1^2 lie in the vicinity of the collinear equilibrium point L_1. For the equilibrium point L_1^1, the infinitesimal body S_3 lies in the right side of L_1, and S_4 lies in the left side of L_1. However, in the case of equilibrium point L_1^2, the infinitesimal body S_3 lies in the left side of L_1, and S_4 lies in the right side of L_1. Similarly, the equilibrium points L_2^1, L_2^2 and L_3^1, L_3^2 lies around the equilibrium points L_2 and L_3 respectively. The non-collinear equilibrium points $L_4^{1,2,3,4}$ and $L_5^{1,2,3,4}$ lies in the neighborhood of the non-collinear equilibrium points L_4 and L_5. All the equilibrium points are influenced by the length, mass and small perturbation in centrifugal force parameters. The linear stability analysis of the equilibrium points is also performed, and it is observed that the stability of the equilibrium points depends on the small perturbations in Coriolis and centrifugal forces, length and mass parameters.

The present model is applied to the Earth-22 Kalliope-dual satellite system including the effect of small perturbations in Coriolis and centrifugal force. We have observed that the small perturbation in the centrifugal force has a substantial effect on the existence and position of the equilibrium points, whereas the small perturbation in the Coriolis force has no impact on them. We have calculated the effect of small perturbation in the centrifugal force on the position of the collinear equilibrium points for the different values of $\epsilon_2 = -0.04$, -0.02, 0.02 and 0.04 and are shown in Tables 1, 2 and 3. It is noticed that, on the increasing values of ϵ_2 from -0.04 to 0.04, the abscissae of four collinear equilibrium points L_1^1, L_1^2, L_2^1 and L_2^2 increase, whereas the abscissae of remaining collinear equilibrium points L_3^1 and L_3^2 decrease. Further, we have calculated the effect of small perturbation in the centrifugal force on the existence and positions of the non-collinear equilibrium points. From Tables 4 and 5, it is observed that for $\epsilon_2 = -0.04$ and -0.02, only four non-collinear equilibrium points exist, but for $\epsilon_2 = 0.02$ and 0.04, eight non-collinear equilibrium points exist.

For the Earth-22 Kalliope-dual satellite system, the stability of the equilibrium points is also performed. The stability of the equilibrium points depends on both the parameters ϵ_1 and ϵ_2. Therefore, we have considered a set of values of ϵ_1 and ϵ_2 in their ranges to check the stability, that is shown in Table 6. We observe that for all the equilibrium points and the combinations of the values of $\epsilon_1 = -0.2, -0.1, 0.1, 0.2$ and $\epsilon_2 = -0.04, -0.02, 0.02, 0.04$, there always exist one positive real root of the characteristic equation, which implies the instability of the equilibrium points. The zero velocity curves of the infinitesimal body S_3 for a known value of S_4 for the Earth-22 Kalliope-dual satellite system are are studied. It is observed that the Jacobian constant increases, as we increase the small perturbation ϵ_2 in centrifugal force. The zero velocity curves are drawn for a fixed value of $\epsilon_2 = 0.04$ and it is concluded that on decreasing the value of C, the permissible regions of motion increase, whereas the forbidden regions of motion decrease.

Our results are different from classical case of the restricted three-body problem (Szebehely (1967b)) in which five points of equilibrium exist. If we neglect the effect of perturbations in the Coriolis and centrifugal forces, the results of Kumar et al. (2020) can be obtained. The results of Whipple (1984) can be obtained by taking $\epsilon_1 = 0$, $\epsilon_2 = 0$ and $l = 0$ in the present analysis.

7. Conclusion

The effect of small perturbations in the Coriolis and centrifugal forces on the existence and linear stability of the equilibrium points in the restricted $2 + 2$ body problem has been investigated. Fourteen equilibrium points are obtained for the present model. It is observed that, the position of the equilibrium points are influenced by the length parameter, mass parameters and perturbation in the centrifugal force parameter. Whereas, the stability of the equilibrium points is affected by the small perturbations in Coriolis and centrifugal forces, length and mass parameters. The position of the equilibrium points and their stability are evaluated numerically for the Earth-22 Kalliope-dual satellite system. The zero velocity curves are drawn and it is observed that on decreasing the value of Jacobian constant, the permissible regions of motion increase, while the forbidden regions of motion decrease. It is also noticed that the Jacobian constant increases, as we increase the small perturbation in centrifugal force.

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