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The Existence and Uniqueness of Solution for Fractional Newell-Whitehead-Segel Equation Within Caputo-Fabrizio Fractional Operator

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Abstract

In this paper, we introduce and study the existence and uniqueness theorem of the solution for the fractional Newell-Whitehead-Segel equation within Caputo-Fabrizio fractional operator. Also, we propose a new numerical method known as natural reduced differential transform method (NRDTM) for solving this equation. We confirm our theoretical discussion with two numerical examples in order to achieve the validity and accuracy of the proposed method. The computations, associated with these examples, are performed by MATLAB software package.

Keywords: Newell-Whitehead-Segel equation; Caputo-Fabrizio fractional operator; Existence and uniqueness; Natural transform method; Reduced differential transform method

MSC 2010 No.: 34A08, 26A33

1. Introduction

Many researchers have paid attention to study the solutions of the Newell-Whitehead-Segel equation because it is one of the most important nonlinear partial differential equations that arise naturally in a number of physical applications, including fluid mechanics, solid-state physics, semiconductor physics, nonlinear optics, plasma physics, convection system, relativistic field theory, and classical mechanics (Hilal et al. (2020), Latif et al. (2020), Mahgoub (2016), Manaa (2011), Patade et al. (2015), Prakash et al. (2016)).

Recently, the subject of fractional calculus, that is, the theory of integrals and derivatives of any real or complex arbitrary order, has gained considerable popularity and importance, mainly due to its applications in diverse fields of science and engineering. The nonlinear partial differential equation can be modeled with fractional derivatives. Therefore, many definitions of fractional derivatives have been proposed and used to develop mathematical models for a wide variety of real world systems (Abu Arqub (2018), Abu Arqub (2019), Abu Arqub et al. (2019), Abu Arqub et al. (2021a), Abu Arqub et al. (2021b), Ardjouni (2019), Chandola et al. (2021), Djennadi et al. (2021), Hosseini et al. (2021a), Hosseini et al. (2021b), Owolabi (2018a), Owolabi (2018b), Younus et al. (2020)).

The aim of this study is to prove the existence and uniqueness of the solution for the fractional Newell-Whitehead-Siegel equation and to use a new numerical method called the natural reduced differential transform method (NRDTM) to get an approximate analytical solution to this equation.

The NRDTM is one of the important numerical methods for obtaining approximate analytical solutions of fractional partial differential equations. The properties of this method are the ability to combine two different methods: the natural transform method and the reduced differential transform, as well as to provide an approximate solution in the form of a rapidly converging series with easily computable components and without the need for linearization, discretization, perturbation or any other restriction.

The rest of the paper is arranged as follows. The main results of the existence and uniqueness of the solution for the fractional Newell-Whitehead-Segel equation are presented in Section 2. The fundamental idea of the NRDTM to solve the considered equation is defined in Section 3. The convergence analysis of the NRDTM is illustrated in Section 4. The performance and efficiency of the proposed method are demonstrate by solving two important examples in Section 5. Finally, the conclusion and some important results of the paper are presented in Section 6.

2. Existence and uniqueness results

In this section, we aim to demonstrate the existence and uniqueness of the solution for fractional Newell-Whitehead-Segel equation within Caputo-Fabrizio fractional operator in the following form,

$$\mathcal{D}_t^{\alpha} u = a u_{xx} + bu - cu^p, \tag{1}$$

with the initial condition

$$u(x,0) = u_0(x), \tag{2}$$

where $\mathcal{D}_t^{\alpha}$ is the Caputo-Fabrizio fractional operator of order $0 < \alpha \leq 1$, a, b and c are real constants with $a, c > 0$, p is a positive integer and $u = \{u(x,t), x \in \mathbb{R}, t > 0\}$.

We transform Equation (1) by applying the Caputo-Fabrizio fractional integral (Losada et al. (2015)), to obtain

$$u(x,t)-u(x,0)=\mathcal{I}^{\alpha}\left(au_{xx}+bu-cu^{p}\right). \tag{3}$$

Equivalently,

$$\begin{aligned} u(x,t)-u(x,0)=&\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left(au_{xx}+bu-cu^{p}\right)\\ &+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_{0}^{t}\left(au_{xx}+bu-cu^{p}\right)d\tau. \end{aligned} \tag{4}$$

Theorem 2.1.

$\mathcal{K}(x,t,u,a,b,c,p)$ satisfy the Lipschitz condition and is contraction if the following inequality

$$0<aL_1^2+b+cp\lambda^{p-1}\leq 1, \tag{5}$$

where

$$\mathcal{K}(x,t,u,a,b,c,p)=au_{xx}+bu-cu^{p}. \tag{6}$$

Proof:

Let u and v be two bounded functions. From Equation (6) and triangular inequality, we have

$$\begin{aligned} \left\|\mathcal{K}(x,t,u,a,b,c,p)-\mathcal{K}(x,t,v,a,b,c,p)\right\| &\leq a\left\|u_{xx}-v_{xx}\right\|+b\left\|u-v\right\|+c\left\|v^{p}-u^{p}\right\|\\ &\leq a\left\|\partial_{xx}\left(u-v\right)\right\|+b\left\|u-v\right\|+c\left\|v^{p}-u^{p}\right\|. \end{aligned} \tag{7}$$

Because of the assumption that u and v are bounded, there is a positive constant $\lambda_1,\lambda_2>0$ such that for all (x,t), $\|u\|\leq\lambda_1$ and $\|v\|\leq\lambda_2$.

Let $\lambda=\max\{\lambda_1,\lambda_2\}$. Then, their first order derivative function ∂_x satisfies the Lipschitz condition and there is a number $L_1\geq 0$ such that

$$\begin{aligned} \left\|\mathcal{K}(x,t,u,a,b,c,p)-\mathcal{K}(x,t,v,a,b,c,p)\right\| &\leq aL_1^2\left\|u-v\right\|+b\left\|u-v\right\|+cp\lambda^{p-1}\left\|v-u\right\|\\ &\leq\left(aL_1^2+b+cp\lambda^{p-1}\right)\left\|u-v\right\|. \end{aligned} \tag{8}$$

Taking $L=aL_1^2+b+cp\lambda^{p-1}$, we get

$$\left\|\mathcal{K}(x,t,u,a,b,c,p)-\mathcal{K}(x,t,v,a,b,c,p)\right\|\leq L\left\|u-v\right\|. \tag{9}$$

Therefore, $\mathcal{K}(x,t,u,a,b,c,p)$ satisfy the Lipschitz condition and if $0<L\leq 1$, then it is a contraction, and the theorem is proved. ■

Now we can state the main result.

Theorem 2.2.

If the following condition is provided,

$$\frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)}+\frac{2\alpha Lt}{(2-\alpha)M(\alpha)}<1, \tag{10}$$

then, the fractional Newell-Whitehead-Segel equation (1) with the initial condition (2) admits a unique solution that is continuous.

Proof:

To prove it, using the expression (6), we consider Equation (4),

$$\begin{aligned}u(x,t)-u(x,0)=&\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\mathcal{K}(x,t,u,a,b,c,p)\\&+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t\mathcal{K}(x,t,u,a,b,c,p)d\tau,\end{aligned} \tag{11}$$

which suggest the following recurrence formula,

$$u_0(x,0)=u(x,0)=u_0(x), \tag{12}$$
$$u_n(x,t)=\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\mathcal{K}(x,t,u_{n-1},a,b,c,p)+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t\mathcal{K}(x,t,u_{n-1},a,b,c,p)d\tau.$$

Let

$$\widetilde{u}(x,t)=\lim_{n\longrightarrow\infty}u_n(x,t). \tag{13}$$

Our aim now is to show that $\widetilde{u}(x,t)=u(x,t)$ is a solution that is continuous. To this end, let us set

$$\mathcal{U}_n(x,t)=u_n(x,t)-u_{n-1}(x,t). \tag{14}$$

It is clear that

$$u_n(x,t)=\sum_{m=0}^{n}\mathcal{U}_n(x,t). \tag{15}$$

In addition, in more detail way, we have

$$\begin{aligned}\mathcal{U}_n(x,t)=&\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left(\mathcal{K}(x,t,u_{n-1},a,b,c,p)-\mathcal{K}(x,t,u_{n-2},a,b,c,p)\right)\\&+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t\left(\mathcal{K}(x,t,u_{n-1},a,b,c,p)-\mathcal{K}(x,t,u_{n-2},a,b,c,p)\right)d\tau.\end{aligned} \tag{16}$$

Taking the norm on both sides of Equation (16) and triangular inequality gives

$$\begin{aligned}\|\mathcal{U}_n(x,t)\| &= \|u_n(x,t)-u_{n-1}(x,t)\| \\ &\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left\|\mathcal{K}(x,t,u_{n-1},a,b,c,p)-\mathcal{K}(x,t,u_{n-2},a,b,c,p)\right\| \\ &\quad+\frac{2\alpha}{(2-\alpha)M(\alpha)}\left\|\int_0^t \mathcal{K}(x,t,u_{n-1},a,b,c,p)-\mathcal{K}(x,t,u_{n-2},a,b,c,p)d\tau\right\| \\ &\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left\|\mathcal{K}(x,t,u_{n-1},a,b,c,p)-\mathcal{K}(x,t,u_{n-2},a,b,c,p)\right\| \\ &\quad+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left\|\mathcal{K}(x,t,u_{n-1},a,b,c,p)-\mathcal{K}(x,t,u_{n-2},a,b,c,p)\right\| d\tau. \quad (17)\end{aligned}$$

Using Theorem 2.1 yields

$$\|\mathcal{U}_n(x,t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L\left\|u_{n-1}-u_{n-2}\right\| + \frac{2\alpha}{(2-\alpha)M(\alpha)}L\int_0^t \left\|u_{n-1}-u_{n-2}\right\| d\tau, \quad (18)$$

which is equivalent to

$$\|\mathcal{U}_n(x,t)\| \leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}L\left\|\mathcal{U}_{n-1}(x,t)\right\| + \frac{2\alpha}{(2-\alpha)M(\alpha)}L\int_0^t \left\|\mathcal{U}_{n-1}(x,t)\right\| d\tau. \quad (19)$$

The recursive principle applied to Equation (19) gives

$$\|\mathcal{U}_n(x,t)\| \leq \left[\left(\frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)}\right)^n + \left(\frac{2\alpha Lt}{(2-\alpha)M(\alpha)}\right)^n\right]u(x,0), \quad (20)$$

which proves that the solution exists and is continuous.

To prove that

$$u(x,t) = \lim_{n\longrightarrow\infty} u_n(x,t), \quad (21)$$

is the solution of Equations (1) and (2), we let

$$\mathcal{V}_n(x,t) = \widetilde{u}(x,t) - u_n(x,t), n\in\mathbb{N}. \quad (22)$$

Consequently, from (13), the difference $\mathcal{V}_n(x,t)$ between $\widetilde{u}(x,t)$ and $u_n(x,t)$ should tend to zero as $n \longrightarrow \infty$. Indeed,

$$\begin{aligned}\widetilde{u}(x,t)-u_n(x,t) &= \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left(\mathcal{K}(x,t,u,a,b,c,p)-\mathcal{K}(x,t,u_n,a,b,c,p)\right) \\ &\quad+\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left(\mathcal{K}(x,t,u,a,b,c,p)-\mathcal{K}(x,t,u_n,a,b,c,p)\right)d\tau. \quad (23)\end{aligned}$$

Using Theorem 2.1, we obtain

$$\begin{aligned}\|\widetilde{u}(x,t)-u_n(x,t)\| &\leq \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\|\mathcal{K}(x,t,u,a,b,c,p)-\mathcal{K}(x,t,u_n,a,b,c,p)\| \\ &\quad +\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \|\mathcal{K}(x,t,u,a,b,c,p)-\mathcal{K}(x,t,u_n,a,b,c,p)\|\,d\tau \\ &\leq \frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)}\|u-u_n\|+\frac{2\alpha L}{(2-\alpha)M(\alpha)}\|u-u_n\|\int_0^t d\tau \\ &\leq \frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)}\|\mathcal{V}_n\|+\frac{2\alpha Lt}{(2-\alpha)M(\alpha)}\|\mathcal{V}_n\|. \end{aligned} \tag{24}$$

So indeed, when $n \longrightarrow \infty$, then $\mathcal{V}_n \longrightarrow 0$, and the right hand side gives

$$\lim_{n\longrightarrow\infty} u_n(x,t)=\widetilde{u}(x,t). \tag{25}$$

With the above, we can take $u(x,t)=\widetilde{u}(x,t)$ as a solution to the Equations (1) and (2) that is continuous. In reality,

$$\begin{aligned} & u(x,t)-\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\mathcal{K}(x,t,u,a,b,c,p)-\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \mathcal{K}(x,t,u,a,b,c,p)d\tau \\ &= \mathcal{V}_n(x,t)+\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\left(\mathcal{K}(x,t,u_{n-1},a,b,c,p)-\mathcal{K}(x,t,u,a,b,c,p)\right) \\ &\quad +\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \left(\mathcal{K}(x,t,u_{n-1},a,b,c,p)-\mathcal{K}(x,t,u,a,b,c,p)\right)d\tau, \end{aligned} \tag{26}$$

hence, the application of the Lipschitz condition to $\mathcal{K}$, we have

$$\begin{aligned} &\left\| u(x,t)-\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\mathcal{K}(x,t,u,a,b,c,p)-\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \mathcal{K}(x,t,u,a,b,c,p)d\tau\right\| \\ &\leq \|\mathcal{V}_n(x,t)\|+\left[\frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)}+\frac{2\alpha Lt}{(2-\alpha)M(\alpha)}\right]\|\mathcal{V}_{n-1}(x,t)\|. \end{aligned} \tag{27}$$

Taking the limit when $n \longrightarrow \infty$ and considering the initial condition, we have

$$\begin{aligned} u(x,t) &= u(x,0)+\frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\mathcal{K}(x,t,u,a,b,c,p) \\ &\quad +\frac{2\alpha}{(2-\alpha)M(\alpha)}\int_0^t \mathcal{K}(x,t,u,a,b,c,p)d\tau. \end{aligned} \tag{28}$$

Finally, for the uniqueness, we consider u and v be two different solutions to the Equations (1) and (2). Then, the Lipschitz condition for $\mathcal{K}$ gives

$$\begin{aligned}\|u(x,t)-v(x,t)\| \leq \frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)}\|u(x,t)-v(x,t)\| \\ +\frac{2\alpha Lt}{(2-\alpha)M(\alpha)}\|u(x,t)-v(x,t)\|.\end{aligned} \tag{29}$$

This leads to

$$\|u(x,t)-v(x,t)\|\left(1-\frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)}-\frac{2\alpha Lt}{(2-\alpha)M(\alpha)}\right)\leq 0. \tag{30}$$

Therefore, $\|u(x,t)-v(x,t)\|=0$, if

$$\frac{2(1-\alpha)L}{(2-\alpha)M(\alpha)}-\frac{2\alpha Lt}{(2-\alpha)M(\alpha)}<1, \tag{31}$$

and the theorem is proved. ∎

3. Fundamental idea of the NRDTM

In this section, we consider the fractional Newell-Whitehead-Segel equation within Caputo-Fabrizio fractional operator in order to demonstrate the fundamental idea of the NRDTM.

Theorem 3.1.

Consider the fractional Newell-Whitehead-Segel equation (1) with the initial condition (2). Then, the NRDTM-solution of equations (1) and (2) is given in the form of infinite series as follows,

$$u(x,t)=\sum_{k=0}^{\infty}U_k(x), \tag{32}$$

where $U_k(x)$ is the reduced differential transformed function of $u(x,t)$.

Proof:

To prove it, we consider the fractional Newell-Whitehead-Segel equation (1) with the initial condition (2).

Take the natural transform on two sides of (1) and using the natural transform of the Caputo-Fabrizio fractional derivative (Zhou et al. (2021)) to get

$$\mathcal{N}^{+}[u]=\frac{1}{s}u_0(x)+\frac{s-\alpha(s-v)}{s}\mathcal{N}^{+}\left[au_{xx}+bu-cu^{p}\right]. \tag{33}$$

Then, we apply the inverse natural transform on two sides of (33), and we have

$$u=u_0(x)+\mathcal{N}^{-1}\left(\frac{s-\alpha(s-v)}{s}\mathcal{N}^{+}\left[au_{xx}+bu-cu^{p}\right]\right). \tag{34}$$

Now, we apply the reduced differential transform method (Keskin et al. (2009)) to Equation (34). We get the following recurrence relation,

$$U_0(x) = u_0(x), \tag{35}$$

$$U_{k+1}(x) = \mathcal{N}^{-1}\left(\frac{s-\alpha(s-v)}{s}\mathcal{N}^{+}\left[a\frac{\partial^2}{\partial x^2}U_k(x) + bU_k(x) - cA_k(x)\right]\right), \tag{36}$$

where $A_k(x)$ is transformed form of the nonlinear terms u^p.

The first nonlinear terms are as follows,

$$A_0 = U_0^p, \tag{37}$$

$$A_1 = pU_0^{p-1}U_1, \tag{38}$$

$$A_2 = pU_0^{p-1}U_2 + \frac{p(p-1)}{2!}U_0^{p-2}U_1^2, \tag{39}$$

$$A_3 = pU_0^{p-1}U_3 + p(p-1)U_0^{p-2}U_1U_2 + \frac{p(p-1)(p-2)}{3!}U_0^{p-3}U_1^3, \tag{40}$$

$$\vdots$$

and so on.

From Equations (35) and (36), we have

$$U_0(x) = u_0(x), \tag{41}$$

$$U_1(x) = \mathcal{N}^{-1}\left(\frac{s-\alpha(s-v)}{s}\mathcal{N}^{+}\left[a\frac{\partial^2}{\partial x^2}U_0(x) + bU_0(x) - cA_0(x)\right]\right), \tag{42}$$

$$U_2(x) = \mathcal{N}^{-1}\left(\frac{s-\alpha(s-v)}{s}\mathcal{N}^{+}\left[a\frac{\partial^2}{\partial x^2}U_1(x) + bU_1(x) - cA_1(x)\right]\right), \tag{43}$$

$$U_3(x) = \mathcal{N}^{-1}\left(\frac{s-\alpha(s-v)}{s}\mathcal{N}^{+}\left[a\frac{\partial^2}{\partial x^2}U_2(x) + bU_2(x) - cA_2(x)\right]\right), \tag{44}$$

$$U_4(x) = \mathcal{N}^{-1}\left(\frac{s-\alpha(s-v)}{s}\mathcal{N}^{+}\left[a\frac{\partial^3}{\partial x^3}U_3(x) + bU_3(x) - cA_3(x)\right]\right), \tag{45}$$

$$\vdots$$

and so on.

Hence, the NRDTM-solution of Equations (1) and (2) is given as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x). \tag{46}$$

The proof is complete. ■

4. Convergence analysis of the NRDTM

In this section, we study the convergence of the NRDTM when it is used in Equations (1) and (2).

Theorem 4.1.

The series solution $\sum\limits_{k=0}^{\infty} U_k(x)$, given in Equation (32), converges if $\exists 0 < \delta < 1$ such that

$$\|U_{k+1}\| \leq \delta \|U_k\|, \forall k \in \mathbb{N} \cup \{0\}. \tag{47}$$

Proof:

Let $(C[l], \| \, . \, \|)$ be the Banach space of all continuous functions on l with the norm $\|U_k(x)\|$. Also assume that $\|U_0(x)\| < \eta_0$, where η_0 is a positive number. Define the sequence of partial sums $\{S_n\}_{n=0}^{\infty}$ as

$$S_n = U_0 + U_1 + U_2 + ... + U_n. \tag{48}$$

We want to prove that $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in this Banach space. To achieve this goal, we take

$$\|S_{n+1} - S_n\| = \|U_{n+1}\| \leq \delta \|U_n\| \leq \delta^2 \|U_{n-1}\| \leq ... \leq \delta^{n+1} \|U_0\| \leq \delta^{n+1}\eta_0. \tag{49}$$

For every $n, m \in \mathbb{N}, n \geq m$, we get

$$\begin{aligned} \|S_n - S_m\| &= \|(S_n - S_{n-1}) + (S_{n-1} - S_{n-2}) + ... + (S_{m+1} - S_m)\| \\ &\leq \|(S_n - S_{n-1})\| + \|(S_{n-1} - S_{n-2})\| + ... + \|(S_{m+1} - S_m)\| \\ &\leq \delta^n \|U_0\| + \delta^{n-1} \|U_0\| + ... + \delta^{m+1} \|U_0\| \\ &\leq \left(\delta^{n-m-1} + \delta^{n-m-2} + ... + 1\right) \delta^{m+1} \|U_0\| \\ &\leq \left(\frac{1 - \delta^{n-m}}{1 - \delta}\right) \delta^{m+1} \|U_0\|, \end{aligned} \tag{50}$$

and because $0 < \delta < 1$, we get

$$\lim_{n,m \to \infty} \|S_n - S_m\| = 0. \tag{51}$$

Therefore, $\{S_n\}_{n=0}^{\infty}$ is a Cauchy sequence in the Banach space $(C[l], \| \, . \, \|)$. Then, the series solution $\sum\limits_{k=0}^{\infty} U_k(x)$, defined in Equation (32), converges and completes the proof. ■

If the series $\sum\limits_{k=0}^{\infty} U_k(x)$ converges then it is an exact solution of the fractional Newell-Whitehead-Segel equation (1).

Theorem 4.2.

Suppose that the series solution $\sum\limits_{k=0}^{\infty} U_k(x)$, converges to the solution $u(x, t)$. If the truncated series $\sum\limits_{k=0}^{m} U_k(x)$ is used as an approximation to the solution $u(x, t)$, then the maximum absolute truncated error is computed as

$$\left\| u(x,t) - \sum_{k=0}^{m} U_k(x) \right\| \leq \frac{1}{1-\delta}\delta^{m+1} \left\| U_0 \right\|. \tag{52}$$

Proof:

According to Theorem (4.1), by following the inequality equation (50), we have

$$\left\| S_n - S_m \right\| \leq \left(\frac{1-\delta^{n-m}}{1-\delta} \right) \delta^{m+1} \left\| U_0 \right\|, \tag{53}$$

for $n \geq m$. Also, since $0 < \delta < 1$, we have $1 - \delta^{n-m} < 1$, therefore, the inequality equation (53) can be changed to

$$\left\| S_n - S_m \right\| \leq \frac{1}{1-\delta}\delta^{m+1} \left\| U_0 \right\|. \tag{54}$$

It is evident when $n \to \infty$, $S_n \to u(x,t)$. Thus, the inequality equation (52) is obtained.

This completes the proof. ∎

5. Examples

In this section, we discuss two important examples to demonstrate the performance and efficiency of the NRDTM.

Example 5.1.

Consider the following linear fractional Newell-Whitehead-Segel equation

$$\mathcal{D}_t^{\alpha} u = u_{xx} - 2u, \tag{55}$$

with the initial condition

$$u(x,0) = e^{x}, \tag{56}$$

where $\mathcal{D}_t^{\alpha}$ is the Caputo-Fabrizio fractional operator of order $0 < \alpha \leq 1$.

For $\alpha = 1$, the exact solution of Equations (55) and (56) is (Latif et al. (2020))

$$u(x,t) = e^{x-t}.$$

Following the description of the NRDTM presented in Section 3, the following recurrence relation is obtained

$$U_0(x) = e^{x}, \tag{57}$$

$$U_{k+1}(x) = \mathcal{N}^{-1}\left(\frac{s-\alpha(s-v)}{s} \mathcal{N}^{+}\left[\frac{\partial^2}{\partial x^2} U_k(x) - 2U_k(x) \right] \right). \tag{58}$$

From Equations (57) and (58), we obtain

$$U_0(x) = e^x, \tag{59}$$

$$U_1(x) = -e^x\left(1-\alpha+\alpha t\right), \tag{60}$$

$$U_2(x) = e^x\left((1-\alpha)^2 + 2\alpha(1-\alpha)t + \alpha^2\frac{t^2}{2!}\right), \tag{61}$$

$$U_3(x) = -e^x\left((1-\alpha)^3 + 3\alpha(1-\alpha)^2 t + 3\alpha^2(1-\alpha)\frac{t^2}{2!} + \alpha^3\frac{t^3}{3!}\right), \tag{62}$$

$$\vdots$$

and so on.

So, the NRDTM-solution of Equations (55) and (56) is given by

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) = U_0(x) + U_1(x) + U_2(x) + U_3(x) + \ldots \tag{63}$$

In particular when $\alpha \to 1$, we get the solution in the form

$$u(x,t) = e^x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots\right), \tag{64}$$

which converge rapidly to the exact solution

$$u(x,t) = e^{x-t}. \tag{65}$$

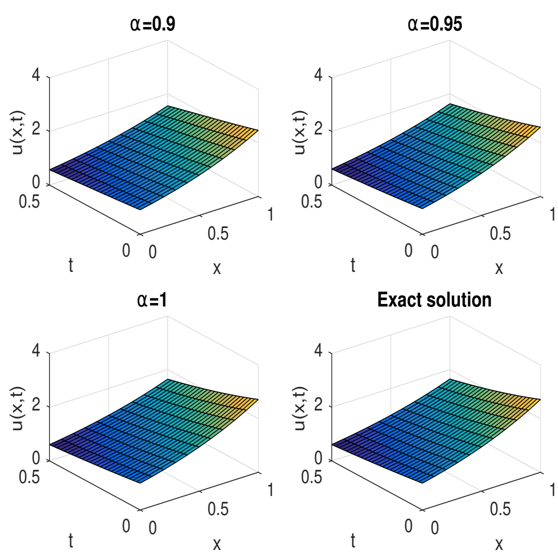

Figure 1. 3D Plots graphs of the $4-$term NRDTM-approximate solutions and exact solution for Equations (55) and (56)

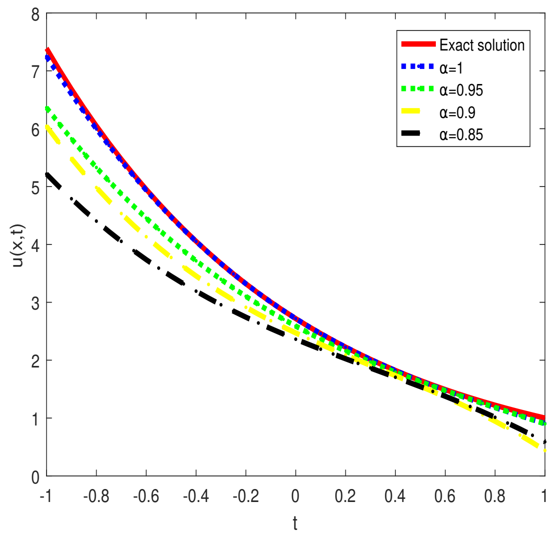

Figure 2. 2D Plots graphs of the $4-$term NRDTM-approximate solutions and exact solution for Equations (55) and (56) when $x = 1$

Table 1. The numerical values of the exact solution and $4-$term NRDTM-approximate solutions for different values of α when $x = 1$

t	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.95$	$\alpha = 1$	Exact solution	$\lvert u_{exact} - u_{NRDTM} \rvert$
0.01	2.1521	2.3448	2.5655	2.6912	2.6912	1.1304×10^{-9}
0.02	2.1382	2.3273	2.5424	2.6645	2.6645	1.8050×10^{-8}
0.03	2.1242	2.3098	2.5194	2.6379	2.6379	9.1194×10^{-8}
0.04	2.1104	2.2924	2.4967	2.6117	2.6117	2.8765×10^{-7}
0.05	2.0965	2.2752	2.4742	2.5857	2.5857	7.0087×10^{-7}

Example 5.2.

Consider the following nonlinear fractional Newell-Whitehead-Segel equation

$$\mathcal{D}_t^{\alpha} u = u_{xx} + 2u - 3u^2, \tag{66}$$

with the initial condition

$$u(x, 0) = \lambda, \tag{67}$$

where $\mathcal{D}_t^{\alpha}$ is the Caputo-Fabrizio fractional operator of order $0 < \alpha \leq 1$.

For $\alpha = 1$, the exact solution of Equations (66) and (67) is (Latif et al. (2020))

$$u(x, t) = \frac{-2\lambda e^{2t}}{-2 + 3\lambda(1 - e^{2t})}. \tag{68}$$

Following the description of the NRDTM presented in Section 3, the following recurrence relation is obtained,

$$U_0(x) = \lambda, \tag{69}$$

$$U_{k+1}(x) = \mathcal{N}^{-1}\left(\frac{s-\alpha(s-v)}{s}\mathcal{N}^{+}\left[\frac{\partial^2}{\partial x^2}U_k(x) + 2U_k(x) - 3A_k(x)\right]\right), \tag{70}$$

where $A_k(x)$ is transformed form of the nonlinear terms, u^2.

From Equations (37)-(40), the first nonlinear terms are given as

$$A_0 = U_0^2, \tag{71}$$

$$A_1 = 2U_0U_1, \tag{72}$$

$$A_2 = 2U_0U_2 + U_1^2 \tag{73}$$

$$A_3 = 2U_0U_3 + 2U_1U_2 \tag{74}$$

$$\vdots$$

and so on.

From Equations (69) and (70), we obtain

$$U_0(x) = \lambda, \tag{75}$$

$$U_1(x) = (2\lambda - 3\lambda^2)(1-\alpha+\alpha t), \tag{76}$$

$$U_2(x) = 2(2\lambda - 3\lambda^2)(1-3\lambda)\left((1-\alpha)^2 + 2\alpha(1-\alpha)t + \alpha^2\frac{t^2}{2!}\right), \tag{77}$$

$$U_3(x) = (2\lambda - 3\lambda^2)\Big[\left(45\lambda^2 - 30\lambda + 4\right)(1-\alpha)^3 + \left(135\lambda^2 - 90\lambda + 12\right)\alpha(1-\alpha)^2 t$$
$$(72\lambda^2 - 48\lambda + 6)\alpha^2(1-\alpha)t^2 + 2(27\lambda^2 - 18\lambda + 2)\alpha^3\frac{t^3}{3!}\Big], \tag{78}$$

$$\vdots$$

and so on.

So, the NRDTM-solution of Equations (66) and (67) is given by

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x) = U_0(x) + U_1(x) + U_2(x) + U_3(x) + ... \tag{79}$$

In particular when $\alpha \longrightarrow 1$, we get the solution in the form

$$u(x,t) = \lambda + (2\lambda - 3\lambda^2)t + 2(2\lambda - 3\lambda^2)(1-3\lambda)\frac{t^2}{2!} + 2(2\lambda - 3\lambda^2)(27\lambda^2 - 18\lambda + 2)\frac{t^3}{3!} + ... \tag{80}$$

which converge rapidly to the exact solution

$$u(x,t) = \frac{-2\lambda e^{2t}}{-2 + 3\lambda(1-e^{2t})}. \tag{81}$$

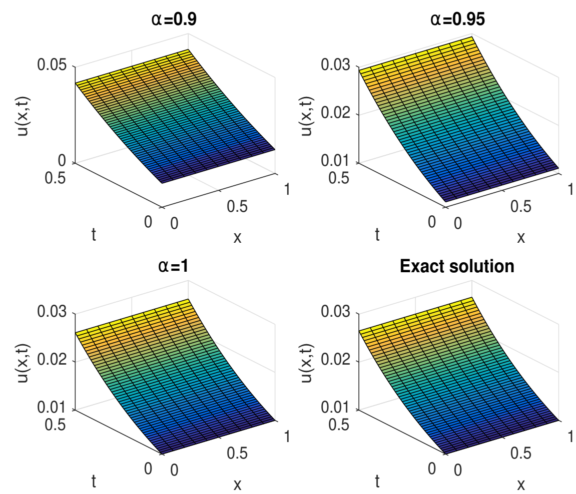

Figure 3. 3D Plots graphs of the $4-$term NRDTM-approximate solutions and exact solution for Equations (66) and (67) when $\lambda = 0.01$

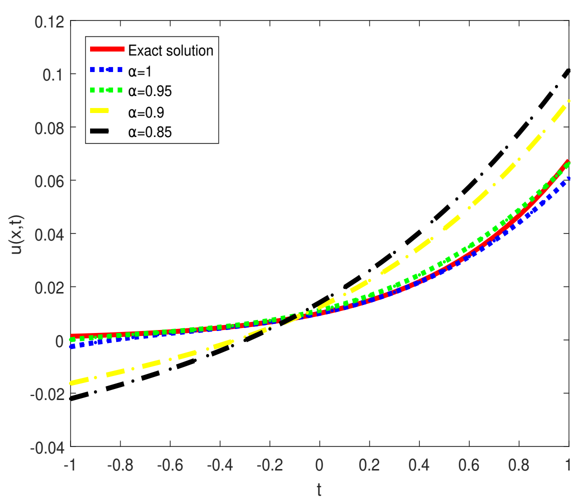

Figure 4. 2D Plots graphs of the $4-$term NRDTM-approximate solutions and exact solution for Equations (66) and (67) when $\lambda = 0.01$

Table 2. The numerical values of the exact solution and $4-$term NRDTM-approximate solutions for different values of α when $\lambda = 0.01$

t	$\alpha = 0.75$	$\alpha = 0.85$	$\alpha = 0.95$	$\alpha = 1$	Exact solution	$\lvert u_{exact} - u_{NRDTM} \rvert$
0.01	0.018850	0.014371	0.011321	0.010199	0.010199	5.2620×10^{-11}
0.02	0.019252	0.014685	0.011556	0.010402	0.010402	8.4387×10^{-10}
0.03	0.019659	0.015005	0.011796	0.010609	0.010609	4.2837×10^{-9}
0.04	0.020071	0.015330	0.012040	0.010819	0.010819	1.3577×10^{-8}
0.05	0.020488	0.015660	0.012290	0.011034	0.011034	3.3241×10^{-8}

6. Conclusion

This study presents the existence and uniqueness of the solution for the fractional Newell-Whitehead-Segel equation within Caputo-Fabrizio fractional operator. Moreover, we have developed NRDTM to obtain approximate solutions of this equation successfully. The approximate solutions are compared with exact solutions and also with other existing solutions in the literature. It is observed that the obtained approximate solutions for the first four terms are very precise and converge very rapidly to the exact solutions. This assures us that the proposed method is reliable, simple and effective to find approximate solutions of many fractional partial differential equations.

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