Applications and Applied Mathematics: An International Journal (AAM)

Volume 16 | Issue 2 | Article 6

12-2021

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Available at
http://pvamu.edu/aam
Appl. Appl. Math.
ISSN: 1932-9466

Applications and Applied Mathematics:
An International Journal
(AAM)

Vol. 16, Issue 2 (December 2021), pp. 881 – 893

Numerical Solution of the Time-space Fractional Diffusion Equation with Caputo Derivative in Time by a-polynomial Method

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Received: March 28, 2021; Accepted: May 17, 2021

Abstract

In this paper, a novel type of polynomial is defined which is equipped with an auxiliary parameter a. These polynomials are a combination of the Chebyshev polynomials of the second kind. The approximate solution of each equation is assumed as the sum of these polynomials, and then, with the help of the collocation points, the unknown coefficients of each polynomial, as well as auxiliary parameter, is obtained optimally. Now, by placing the optimal value of a in polynomials, the polynomials are obtained without auxiliary parameter, which is the restarted step of the present method. The time discretization is performed on fractional partial differential equations by $L1$ method. In the following, the convergence theorem of the method is proved.

Keywords: Caputo fractional derivative; Riemann-Liouville fractional derivative; Chebyshev polynomials; Collocation method; Diffusion; Chebyshev-Gauss points

MSC 2010 No.: 65M99, 35R11

1. Introduction

Fractional calculus is often considered a branch of mathematical analysis which deals with integral-differential equations. This branch of mathematics has a nearly 300-year history, which can be traced back to L'Hopital's letter to Leibnitz, in which he asked Leibnitz about a non-integer order derivative. Many applications of fractional calculus in various fields of engineering, physics, medicine, etc., are known and used: nuclear reactor dynamic (Ray (2015)), thermo-elasticity (Povstenko (2015)), biological tissues (Bueno-Orovio et al. (2014)), El Nino chaotic dynamical system (Samko et al. (1993)), Ebola epidemic model (Area et al. (2015)), cancer tumor modeling (Iyiola and Zaman (2014); Ghanbari et al. (2020)). Fractional calculus is often considered a branch of mathematical analysis which deals with integral-differential equations. This branch of mathematics has a nearly 300-year history, which can be traced back to L'Hopital's letter to Leibnitz, in which he asked Leibnitz about a non-integer order derivative. Many applications of fractional calculus in various fields of engineering, physics, medicine, etc., are known and used: nuclear reactor dynamic (Ray (2015)), thermo-elasticity (Povstenko (2015)), biological tissues (Bueno-Orovio et al. (2014)), El Nino chaotic dynamical system (Samko et al. (1993)), Ebola epidemic model (Area et al. (2015)), and cancer tumor modeling (Iyiola and Zaman (2014); Ghanbari et al. (2020)).

Many analytical and numerical methods have been used to solve fractional differential equations, including: variational iteration method (Drăgănescu (2006)), neural network method (Anastassiou (2018)), reproducing kernel method (Arqub and Momani (2019)), Hermite wavelets methods (Kumar et al. (2021a)), homotopy analysis method (Ray (2015)), pseudo-spectral method (Ejlali and Hosseini (2017)), Genocchi polynomials (Kumar et al. (2021b)), Bernstein polynomials (Asgari and Ezzati (2017); Entezari et al. (2019); Mirzaee and Samadyar (2017); Kumar et al. (2020)), q-transform analysis (Abelman et al. (2017)), and Inverse fractional Shehu transform method (Khalouta and Kadem (2019)).

In this paper, we present a new method for the numerical solution of the time-space fractional diffusion equation. A novel type of polynomial is defined which is equipped with an auxiliary parameter (Abbasbandy (2017)). These polynomials are a combination of the Chebyshev polynomials of the second kind. The approximate solution of each equation is assumed as the sum of these polynomials and then, with the help of the collocation points, the unknown coefficients of each polynomial, as well as auxiliary parameter, are obtained optimally. The time discretization is performed on fractional partial differential equations by the minimization method.

In Section 2, the basic concepts are defined. In Section 3, the Caputo derivative discretization and the method implementation are explained. The convergence theorems of the method are expressed along with proof in Section 4. The numerical results of the present method are displayed in two practical examples in Section 5.

2. Basic Concepts

Various types of fractional derivatives and integrals have been introduced in mathematics to date, including: Riemann, Liouville, Riesz, Letnikov, Grünwald, Weyl, Marchaud, and Caputo. In this paper, the Caputo and Riemann-Liouville fractional derivatives have been used.

In this paper, we present a new method for the numerical solution of the following time-space fractional diffusion equation (Li and Zeng (2015)):

$$\begin{cases} {}_CD_{0,t}^{\gamma}U = \left(L^{(\alpha)}U\right)(x,t) + g(x,t), & (x,t) \in (a,b)\times(0,T], \\ U(x,0) = \varphi_0(x), & x \in (a,b), \\ U(a,t) = 0, U(b,t) = 0, & t \in (0,T], \end{cases} \tag{1}$$

where $L^{(\alpha)} = c(x,t)_{RL}D_{a,x}^{\alpha} + d(x,t)_{RL}D_{x,b}^{\alpha}$, $0 < \gamma \leqslant 1$, $0 < \alpha < 2$ and $c, d > 0$. Also, ${}_CD_{0,t}^{\gamma}$ is the Caputo fractional derivative of order α, ${}_{RL}D_{a,x}^{\alpha}$ and ${}_{RL}D_{x,b}^{\alpha}$ are the left and right Riemann-Liouville derivatives of order α (Li and Zeng (2015)).

For solving Equation (1), a novel class of functions, a-polynomials, are used. These functions are defined as

$$A_0(t) = 1,\ A_n(t) = atU_{n-1}(t) + U_n(t),\ n \geqslant 1,$$

where U_n is the second kind of Chebyshev polynomial and a is an auxiliary real parameter (see Abbasbandy (2017), Hajishafieiha and Abbasbandy (2020b), and Hajishafieiha and Abbasbandy (2020a) for more properties).

3. Method of solution

In this section, we first implement the time discretization scheme by L1 method on Equation (1). Then, at each point t_k, the approximate solution is approximated by the sum of the a-polynomials. This approximation is obtained by meshing the spatial points at the point t_k. In other words, at any point, t_k, a nonlinear equations system is obtained by the collocation method, which by solving the system, the unknown coefficients in the series of the approximate solution are obtained. Finally, by interpolating points $(x_i, t_k, u(x_i, t_k))$ with cubic B-spline polynomials, the approximate solution of the problem is obtained on the domain $(a, b) \times [0, T]$.

3.1. Discretization of time

The approximation of the finite difference for the Caputo derivative by the $L1$ method is studied by Li and Zeng (2015). This method is conditionally stable. In this method,

$${}_CD_{0,t}^{\gamma}f(t)\big|_{t=t_k} = \sum_{j=0}^{k-1} b_{k-j-1}(f_{j+1} - f_j) + O(\Delta t^{2-\gamma}),\ 0 < \gamma < 1, \tag{2}$$

where $b_j = \frac{\Delta t^{-\gamma}}{\Gamma(2-\gamma)}\left[(j+1)^{1-\gamma} - j^{1-\gamma}\right]$. To discrete the interval $[0, T]$, suppose

$$t_k = k\tau,\ k = 0, 1, 2, \cdots, M,$$

where $\tau = \frac{T}{M}$, T is final time. Using the $L1$ finite difference method, for time discretization, we use (2) instead of ${}_C D_{0,t}^{\gamma} f(t)$:

$$ {}_C D_{0,t}^{\gamma} U(t)\Big|_{t=t_k} = \sum_{j=0}^{k-1} b_{k-j-1}(u^{j+1} - u^j) = L^{(\alpha)} u^{k+1} + g^{k+1}, \tag{3}$$

where $u^{k+1} = u(x, t_{k+1})$ and $g^{k+1} = g(x, t_{k+1})$. Now, for each point t_k, a nonlinear equations system is obtained.

3.2. Method implementation

To implement the proposed method, first u^{k+1} is approximated as follows:

$$u^{k+1} \cong \sum_{n=0}^{N} c_n^{k+1} A_n(x).$$

By replacing the above approximation per t_k in (3), discretization on the spatial domain $[a, b]$, and the initial and boundary conditions of this problem, a nonlinear equations system with $N+2$ equations and $N + 2$ unknowns c_n^{j+1}, $n = 0, 1, 2, \cdots, N$ and the unknown parameter a are obtained. To obtain a, we can minimize the L_2 norm of the residual in the augmented nonlinear system by the least squares method. Now we put the parameter a in a-polynomials and then with these polynomials we repeat the method with one unknown (a) less. Finally, by interpolating points $(x_i, t_k, u(x_i, t_k))$ with cubic B-spline polynomials, the approximate solution of the problem is obtained on the domain $(a, b) \times [0, T]$.

4. The convergence theorem

Suppose that $\Lambda = [-1, 1]$ and $L^2_\omega(\Lambda)$ be a function Hilbert space with the standard inner product and $\omega(t) = \sqrt{1 - t^2}$ is positive weight function. Let N be a positive integer. We will consider the subspace of $L^2_\omega(\Lambda)$ by using the second kind of Chebyshev polynomials as

$$S_N = span\{U_0, U_1, \ldots, U_N\}.$$

We define $L^2_\omega(\Lambda)$-orthogonal projection as follows:

$$P_N : L^2_\omega(\Lambda) \to S_N,$$
$$(P_N v)(t) = \sum_{i=0}^{N} c_i U_i(t),$$

such that $(P_N v - v, \varphi)_\omega = 0$, $\forall \varphi \in S_N$. To estimate $\|P_N v - v\|_\omega$, we have the space interpolation:

$$H^r_{\omega,R}(\Lambda) = \left\{ v |\ v \text{ is measurable and } \|v\|_{r,\omega,R} < \infty \right\},$$

where $r > 0$ is any real number, and

$$\|v\|_{r,\omega,R} = \left(\sum_{i=0}^{r} \left\| (t+2)^{\frac{r}{2}+i} \frac{d^i v}{dt^i} \right\|_\omega^2 \right)^{1/2}. \tag{4}$$

We define the Sturm-Liouville operator of the second-kind Chebyshev polynomials, R, as

$$R(U_n(t)) = -\omega^{-1}(t)\frac{d}{dt}(\omega^3(t)\frac{d}{dt}U_n(t)), \tag{5}$$

and we know that, U_n is the eigenfunction of the singular Sturm-Liouville problem

$$[(1-t^2)^{-1/2}\frac{d}{dt}((1-t^2)^{3/2}\frac{d}{dt}) + n(n+2)]U_n(t) = 0,$$

for $n = 0, 1, 2, \ldots$ (see Hanson and Yakovlev (2002), Chapter 5).

Proposition 4.1.

R^m is a continuous mapping from $H^{2m}_{\omega,R}(\Lambda)$ to $L^2_\omega(\Lambda)$.

Proof:

For showing this, we will prove that

$$R^m v(t) = \sum_{k=1}^{2m}(t+2)^{m+k}q_k(t)\frac{d^k v(t)}{dt^k}, \tag{6}$$

where q_k is a rational bounded uniformly function on the whole interval Λ. It is proved by induction. For $m = 1$, we have

$$\begin{aligned} Rv(t) &= 3t\frac{dv}{dt} - (1-t^2)\frac{d^2v}{dt^2} \\ &= (t+2)^2\left(\frac{3t}{(t+2)^2}\right)\frac{dv}{dt} + (t+2)^3\left(\frac{t-1}{(t+2)^2}\right)\frac{d^2v}{dt^2}. \end{aligned}$$

Suppose that for $m \leqslant n$, the relation (6) is satisfied. One can easily prove that this relation is established for $m = n+1$. ■

Proposition 4.2.

For any real $r \geq 0$, $v \in H^r_{\omega,R}(\Lambda)$, $v = \sum\limits_{n=0}^{\infty}\hat{v}_n U_n(t)$ then

$$\|P_N v - v\|_\omega \leq cN^{-r}\|v\|_{r,\omega,R}, \tag{7}$$

for some real constant c.

Proof:

First, we suppose that $r = 2m$. It is easy to see that $(U_n, U_m)_\omega = \frac{\pi}{2}\delta_{n,m}$. Hence, by (5) and integration by parts,

$$
\begin{aligned}
\hat{v}_n &= \frac{2}{\pi}\int_\Lambda v(t)U_n(t)\omega(t)dt \\
&= \frac{2}{\pi n(n+2)}\int_\Lambda v(t)RU_n(t)\omega(t)d\eta \\
&= -\frac{2}{\pi n(n+2)}\int_\Lambda v(t)\frac{d}{dt}(\omega^3(t)\frac{d}{dt}U_n(t))dt \\
&= \frac{2}{\pi n(n+2)}\int_\Lambda \omega^3(t)\frac{d}{dt}v(t)(\frac{d}{dt}U_n(t))dt \\
&= -\frac{2}{\pi n(n+2)}\int_\Lambda \frac{d}{dt}(\omega^3(t)\frac{d}{dt}v(t))U_n(t)dt \\
&= \frac{2}{\pi n(n+2)}\int_\Lambda Rv(t)U_n(t)\omega(t)dt \\
&= \dots \\
&= \frac{2}{\pi n^m(n+2)^m}\int_\Lambda R^m v(t)U_n(t)\omega(t)dt.
\end{aligned} \tag{8}
$$

Now according to (6) and (8) and the definition of $H^r_{\omega,R}(\Lambda)$, we have:

$$
\begin{aligned}
\|P_N v - v\|^2_\omega &= \sum_{n=N+1}^{\infty} \hat{v}_n \|U_n\|^2_\omega \\
&\leqslant cN^{-4m}\sum_{n=N+1}^{\infty}\left(\frac{\int_\Lambda R^m v(t)U_n(t)\omega(t)dt}{\|U_n\|^2_\omega}\right)^2 \|U_n\|^2_\omega \\
&\leqslant cN^{-4m}\|R^m v\|^2_\omega \leqslant cN^{-4m}\|v\|^2_{r,\omega,R}.
\end{aligned}
$$

Next, we put $r = 2m+1$. From $(\omega U_n', \omega)_\omega = 0$ and integration by part, we have:

$$
\begin{aligned}
\hat{v}_n &= \frac{2}{\pi n^m(n+2)^m}\int_\Lambda R^m v(t)U_n(t)\omega(t)dt \\
&= -\frac{2}{\pi n^{m+1}(n+2)^{m+1}}\int_\Lambda R^m v(t)\frac{d}{dt}(\omega^3(t)\frac{d}{dt}U_n(t))dt \\
&= -\frac{2}{\pi n^{m+1}(n+2)^{m+1}}\int_\Lambda \frac{d}{dt}(R^m v(t))\frac{d}{dt}U_n(t)\omega^3(t)dt.
\end{aligned} \tag{9}
$$

Now using $(\omega U_n', \omega U_m')_\omega = \frac{1}{2}n(n+2)\delta_{n,m}$, and (6), the following inequality is obtained:

$$\begin{aligned}
\|P_N v - v\|_\omega^2 &= \sum_{n=N+1}^{\infty} \hat{v}_n^2 \|U_n\|_\omega^2 \\
&= \sum_{n=N+1}^{\infty} \frac{4}{\pi^2 (n(n+2))^{2m+2}} \left(\int_\Lambda \frac{d}{dt}(R^m v(t)) \frac{d}{dt} U_n(t) \omega^3(t) dt \right)^2 \\
&= \sum_{n=N+1}^{\infty} \frac{4}{\pi^2 (n(n+2))^{2m+2}} \left(\frac{\int_\Lambda \frac{d}{dt}(R^m v(t)) \frac{d}{dt} U_n(t) \omega^3(t) dt}{\left\| \frac{d}{dt} U_n \right\|_{\omega^3}^2} \right)^2 \left\| \frac{d}{dt} U_n \right\|_{\omega^3}^2 \\
&\leqslant c N^{-2(2m+1)} \sum_{n=N+1}^{\infty} \left(\frac{\int_\Lambda \frac{d}{dt}(R^m v(t)) \frac{d}{dt} U_n(t) \omega^3(t) dt}{\left\| \frac{d}{dt} U_n \right\|_{\omega^3}^2} \right)^2 \left\| \frac{d}{dt} U_n \right\|_{\omega^3}^2 \\
&\leqslant c N^{-2(2m+1)} \left\| \frac{d}{dt}(R^m v) \right\|_{\omega^3}^2 \leqslant c N^{-2(2m+1)} \left\| \frac{d}{dt}(R^m v)(t+2)^{7/2} \right\|_{\omega}^2 \\
&\leqslant c N^{-2(2m+1)} \|v\|_{r,\omega,R}^2 .
\end{aligned}$$

The general result follows from the previous results and space interpolation. ■

Theorem 4.1.

For any real $r > 0$, $y \in H^r_{\omega,R}(\Lambda)$, we have:

$$\|y_N - y\|_\omega \leqslant \tilde{a} c (N-2)^{-r} \|y\|_{r,\omega,R}. \tag{10}$$

Proof:

Using $A_n(t) = (1 + \frac{a}{2})U_n(t) + \frac{a}{2}U_{n-2}(t)$, and Proposition (4.2), we get the following inequality:

$$\begin{aligned}
\|y_N - y\|_\omega &= \left\| \sum_{i=N+1}^{\infty} c_i U_i(t) \right\|_\omega = \left\| \sum_{i=N+1}^{\infty} c_i ((1 + \frac{a}{2}) U_i(t) + \frac{a}{2} U_{i-2}(t)) \right\|_\omega \\
&\leqslant \left| 1 + \frac{a}{2} \right| \left\| \sum_{i=N+1}^{\infty} c_i U_i(t) \right\|_\omega + \left| \frac{a}{2} \right| \left\| \sum_{i=N+1}^{\infty} c_i U_{i-2}(t) \right\|_\omega \\
&\leqslant \left| 1 + \frac{a}{2} \right| \underbrace{c' N^{-r} \|y\|_{r,\omega,R}}_{Equation\ (7)} + \left| \frac{a}{2} \right| \underbrace{c'' (N-2)^{-r} \|y\|_{r,\omega,R}}_{Equation\ (7)} \leqslant \tilde{a} c (N-2)^{-r} \|y\|_{r,\omega,R}.
\end{aligned}$$

where $\tilde{a} = \max\{|1 + \frac{a}{2}|, |\frac{a}{2}|\},\ c = \max\{c', c''\}$. ■

This theorem shows that the a-polynomial approximation has exponential convergence. The similar theorems which have been proved in this section can be seen in Guo et al. (2002) for the Chebyshev polynomials of the first kind.

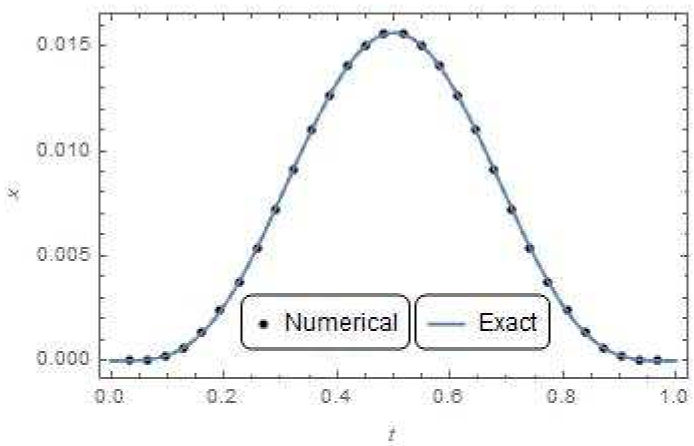

Figure 1. Graph of the exact and numerical solutions in Example 5.1 at $\tau = 0.01$, $N = 15$, $(\gamma, \alpha) = (0.8, 1.8)$ and $T = 1$

5. Numerical results

In this section, two examples of this problem are analyzed numerically. These examples are a special type of the fractional telegraph equation (Zhao and Li (2012)). We use the Chebyshev-Gauss-Lobato (CGL) collocation points to provide the numerical solution. We also use the L_2 error norm to compare the error of the proposed method with other methods.

Example 5.1.

Take the time-space fractional partial differential equation (1), where c and d are unity constant functions and $(a, b) = (0, 1)$, $T = 1$, and $\varphi_0(x) = 2x^4(1-x^4)$. The function $g(x, t)$ will be chosen so that the exact solution is $U(x, t) = (t^{2+\gamma} + t + 2)x^4(1 - x^4)$.

In Tables 1 and 2, the L_2 error norm is calculated at $\tau = 0.01$, $T = 1$ and different values of N for $\gamma = 0.5$ and $\gamma = 0.8$, respectively. According to the obtained results, increasing the number of collocation points reduces the L_2 error norm at $\tau = 0.01$ and $T = 1$ for $\Gamma = 0.5$ and $\gamma = 0.8$. Therefore, the convergence theorem proved in the previous section is confirmed. In Table 3, the L_2 error norm of the present method is compared with the fractional finite difference method (FFDM) (Li and Zeng (2015)) at $T = 1$ and different values N, τ, α and γ. It can be seen that due to the smaller number of N and the higher value of τ in the proposed method, the error of the proposed method is still better than the fractional finite difference method.

Figure 1 shows the graph of the exact and numerical solutions at $\tau = 0.01$, $N = 15$, $(\gamma, \alpha) = (0.8, 1.8)$ and $T = 1$. Figure 2 shows graph of the exact and numerical solutions $u(x, t)$ at $\tau = 0.01$, $N = 15$ and $(\gamma, \alpha) = (0.8, 1.8)$.

Example 5.2.

Take the time-space fractional partial differential equation (1), where c and d are unity constant functions and $(a, b) = (0, 1)$, $T = 1$, and $\varphi_0(x) = 0$. The function $g(x, t)$ will be chosen so that

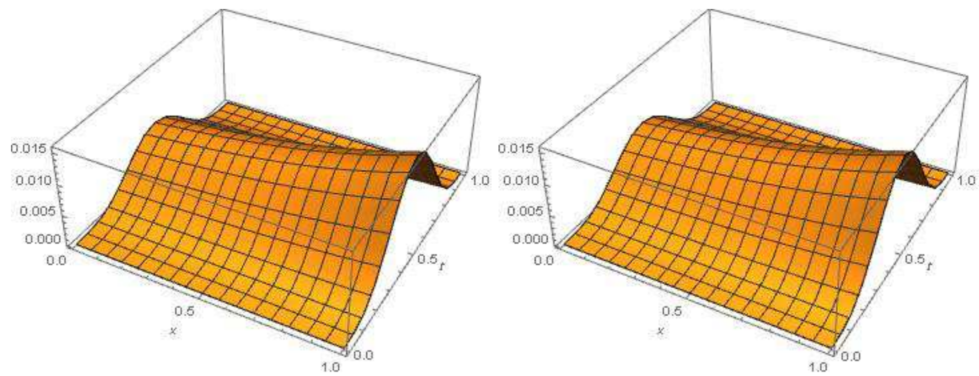

Figure 2. Graph of the exact and numerical solutions $u(x,t)$ in Example 5.1 at $\tau = 0.01$, $N = 15$ and $(\gamma, \alpha) = (0.8, 1.8)$ (Left: Exact solution, Right: Numerical solution)

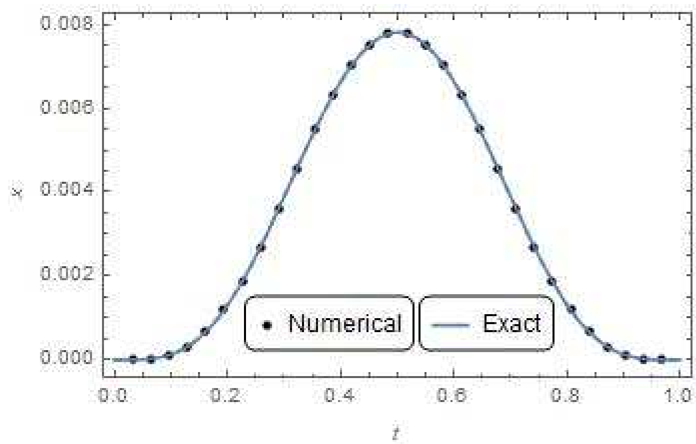

Figure 3. Graph of the exact and numerical solutions in Example 5.2 at $\tau = 0.01$, $N = 20$, $(\gamma, \alpha) = (0.5, 1.5)$ and $T = 1$

the exact solution is $U(x,t) = (t^{2.5} + t)x^4(1 - x^4)$.

In Table 4, the L_2 error norm is calculated at $\tau = 0.01$, $T = 1$ and different values of N. According to the obtained results, increasing the number of collocation points reduces the L_2 error norm at $\tau = 0.01$ and $T = 1$. Therefore, the convergence theorem proved in the previous section is confirmed. In Table 5, the L_2 error norm of the present method is compared with the fractional finite difference method (Li and Zeng (2015)) at $T = 1$ and different values N, τ, α and γ. It can be seen that due to the smaller number of N and the higher value of τ in the proposed method, the error of the proposed method is still better than the fractional finite difference method.

Figure 3 shows the graph of the exact and numerical solutions at $\tau = 0.01$, $N = 20$, $(\gamma, \alpha) = (0.5, 1.5)$ and $T = 1$. Figure 4 shows graph of the exact and numerical solutions $u(x,t)$ at $\tau = 0.01$, $N = 20$ and $(\gamma, \alpha) = (0.5, 1.5)$.

Table 1. Results of L_2 error norm in Example 5.1 at $\tau = 0.01$, $T = 1$ and $\gamma = 0.5$

N	a	$\alpha = 1.2$	a	$\alpha = 1.5$	a	$\alpha = 1.8$
5	0.99991	4.2673e-4	1.00002	5.8764e-4	1.00024	7.3510e-4
10	0.99997	4.6334e-7	0.99996	1.8147e-7	1.00205	9.2696e-8
15	1	4.7099e-7	1	1.8452e-7	0.99997	9.4210e-8
20	1	4.7593e-7	1	1.8606e-7	1	9.5839e-8
25	1	4.7812e-7	1	1.8778e-7	1	9.6071e-8

Table 2. Results of L_2 error norm in Example 5.1 at $\tau = 0.01$, $T = 1$ and $\gamma = 0.8$

N	a	$\alpha = 1.2$	a	$\alpha = 1.5$	a	$\alpha = 1.8$
5	0.99989	4.2674e-4	0.99997	5.8772e-4	1.00012	7.3520e-4
10	0.99997	3.8232e-6	0.99997	1.5454e-6	1.00208	8.2271e-7
15	1	3.8856e-6	0.99997	1.6222e-6	0.99997	8.3611e-7
20	1	3.9240e-6	1	1.6360e-6	1	8.5072e-7
25	1	3.9428e-6	1	1.5606e-6	1	8.5274e-7

Table 3. Comparison of L_2 error norm of the present method in Example 5.1 with fractional finite difference method (FFDM) at $T = 1$ and different values N, τ, α and γ

Method	N	τ	(γ, α) $(0.5, 1.5)$	a	(γ, α) $(0.8, 1.8)$	a
FFDM	8	10^{-3}	6.3601e-4	-	5.9009e-4	-
Present method	5	10^{-2}	5.8764e-4	1.00002	7.3520e-4	1.00012
FFDM	16	10^{-3}	2.2826e-4	-	1.6719e-4	-
Present method	10	10^{-2}	1.8147e-7	0.99996	8.2271e-7	1.002081
FFDM	32	10^{-3}	6.5588e-5	-	4.3810e-5	-
Present method	15	10^{-2}	1.8452e-7	1	8.3611e-7	0.99997
FFDM	64	10^{-3}	1.7418e-5	-	1.1180e-5	-
Present method	20	10^{-2}	1.8778e-7	1	8.5072e-7	11

6. Conclusion

In this paper, a-polynomials are used to numerically solve a specific type of fractional partial differential equations. The results obtained from the proposed method were compared with other methods used to solve them in different tables. In solving the time-space fractional diffusion equa-

Table 4. Results of L_2 error norm in Example 5.2 at $\tau = 0.01$ and $T = 1$

N	(γ, α) $(0.2, 1.2)$	a	(γ, α) $(0.5, 1.5)$	a	(γ, α) $(0.8, 1.8)$	a
5	2.0981e-4	-1.94639	2.8872e-4	-1.98183	3.6056e-4	-1.98034
10	5.0059e-8	-2.47892	1.8147e-7	-2.48306	6.2429e-7	-1.75972
15	5.0886e-8	-1.84856	1.8452e-7	-1.87973	6.3448e-7	-1.68443
20	5.1440e-8	-1.69363	1.8714e-7	-1.51669	6.4589e-7	-1.68385
25	5.1731e-8	-1.50459	1.8719e-7	-1.38683	6.4743e-7	-1.36792

Table 5. Comparison of L_2 error norm of the present method in Example 5.2 with fractional finite difference method (FFDM) at $T = 1$ and different values N, τ, α and γ

Method	N	τ	(γ, α) $(0.5, 1.5)$	a	(γ, α) $(0.8, 1.8)$	a
FFDM	8	10^{-3}	3.0921e-4	-	2.9140e-4	-
Present method	5	10^{-2}	2.8872e-4	-1.98183	3.6056e-4	-1.98034
FFDM	16	10^{-3}	1.1228e-4	-	8.2738e-5	-
Present method	10	10^{-2}	1.8147e-7	-2.48306	6.2429e-7	-1.75972
FFDM	32	10^{-3}	3.2374e-5	-	2.1694e-5	-
Present method	15	10^{-2}	1.8452e-7	-1.87973	6.3448e-7	-1.68443
FFDM	64	10^{-3}	8.6250e-6	-	5.5311e-6	-
Present method	20	10^{-2}	1.8714e-7	-1.51669	6.4589e-7	-1.68385

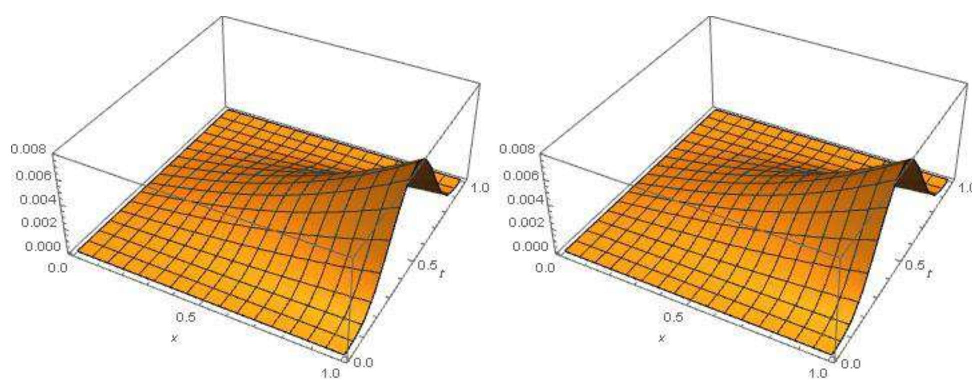

Figure 4. Graph of the exact and numerical solutions $u(x, t)$ in Example 5.2 at $\tau = 0.01$, $N = 20$ and $(\gamma, \alpha) = (0.5, 1.5)$ (Left: Exact solution, Right: Numerical solution)

tions, due to the choice of the small number of collocation points and the higher value of τ than the other method, the proposed method shows less error and more advantage. The simplicity of using a-polynomials in fractional derivatives could be one of the advantages of the proposed method, which creates less complexity to solve.

Acknowledgment:

We are grateful to the anonymous reviewers for their helpful comments, which undoubtedly led to the definite improvement in the manuscript.

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