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A New Type II Half Logistic-G family of Distributions with Properties, Regression Models, System Reliability and Applications

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Abstract

This study proposes a new family of distributions based on the half logistic distribution. With the new family, the baseline distributions gain flexibility through additional shape parameters. The important statistical properties of the proposed family are derived. A new generalization of the Weibull distribution is used to introduce a location-scale regression model for the censored response variable. The utility of the introduced models is demonstrated in survival analysis and estimation of the system reliability. Three data sets are analyzed. According to the empirical results, it is observed that the proposed family gives better results than other existing models.

Keywords: Half logistic-G family; Incomplete moment; Regression; Survival; Reliability; Simulation; Estimation;

MSC 2010 No.: 60E05, 62N01

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1. Introduction

The several generalizations of the well-known distributions have been proposed in the last decade. The reason of these generalizations is to add more flexibility to existing distributions by adding one or more shape parameters. These distributions have found importance application area in survival analysis because of their flexibility in failure rate modeling. For instance, the odd log-logistic family (OLL-G) proposed by Gleaton and Lynch (2006) has attracted attention by several authors. Researchers have introduced several generalizations of the OLL-G family. Cordeiro et al. (2016a) proposed the Zografos-Balakrishnan odd log-logistic family which is a generalization of the OLL-G family. Another generalization of the OLL-G family, generalized odd log-logistic family, was proposed by Cordeiro et al. (2017). Other generalizations of the OLL-G family can be cited as follows: the beta odd log-logistic generalized family by Cordeiro et al. (2016b), the Kumaraswamy odd log-logistic family by Alizadeh et al. (2015), a new generalized odd log-logistic family by Haghbin et al. (2017) and so on.

In this study, our focus is on the half-logistic (HL) distribution because of its tractable properties. It has simple probability density function (pdf) and cumulative distribution function (cdf). Also, it has only one parameter which controls the variation of the distribution. However, its flexibility is very limited to model the various characteristics of the data sets. For instance, it cannot be used to model left-skewed and bimodal data sets. To remove the drawback of this distribution, some researchers have proposed different generalization of the HL distribution such as exponentiated half-logistic-G by Cordeiro et al. (2014) and type I half-logistic-G by Cordeiro (2016c). Here, our aim is to add more flexibility to the HL distribution by adding shape parameter which help us to generate bimodal and left-skewed pdf shapes. Also, the HL distribution has drawbacks to model the lifetime data sets. By means of its new generalization, one can model the different shapes of the failure rate.

To generate a new family of distributions based on the HL distribution, we use the T-X idea of Alzaatreh et al. (2013). We call the new family as new type-II half-logistic-G family, shortly NTIIHL-G. The motivations to generate the NTIIHL-G family can be given as follows: (i) to build a new model for skewed and heavy-tailed data sets which are commonly seen in financial applications, (ii) to make the baseline distribution useful for the different failure shapes, (iii) to insert the skewness into the symmetric distributions for modeling the extremely right (left) skewed data sets.

The other parts of the presented paper can be summarized as follows: The proposed family is constructed in Section 2. The special cases of the NTIIHL-G family are given in Section 3. The mathematical properties of the NTIIHL-G family are studied in Section 4. The parameter estimation problem of the NTIIHL-G family is tackled in Section 5. The simulation study is given in Section 6 to see the asymptotic efficiency of the parameter estimation method. In Section 7, the location-scale regression model is defined by using the special member of the NTIIHL-G family. Estimation of the system reliability using a new generated distribution is discussed in Section 8. The univariate data fitting and application of the proposed regression model are given in Section 9. The important conclusions of the paper are given in Section 10.

2. Construction of the new family

Let the random variable X follow a HL distribution. The pdf and cdf of X are, respectively,

$$f\left(x\right)=\frac{2\,\lambda\,\exp\left(-\lambda x\right)}{\left(1+\exp\left(-\lambda x\right)\right)^{2}},\;\;x>0, \tag{1}$$

and

$$F\left(x\right)=\frac{1-\exp\left(-\lambda x\right)}{1+\exp\left(-\lambda x\right)},$$

where $\lambda > 0$ is a scale parameter. The pdf (1) has unimodal shape. Therefore, HL distribution is inadequate model to explain the characteristics of the data. The real data has a non-monotone hazard rate function (hrf) shapes such as bathtub or reversed-J shapes.

This study proposes a new generalization of the HL distribution, called as NTIIHL-G family of distributions. Let $G\left(x;\boldsymbol{\phi}\right)$ and $g\left(x;\boldsymbol{\phi}\right)$ are the cdf and pdf belonging to the baseline distribution, respectively. The cdf and pdf of the NTIIHL-G are defined by

$$F\left(x\right)=1-\int_{0}^{-\log\left[\frac{G(x;\boldsymbol{\phi})^{\alpha}}{G(x;\boldsymbol{\phi})^{\alpha}+\bar{G}(x;\boldsymbol{\phi})^{\alpha}}\right]}\frac{2\lambda\,\mathrm{e}^{-\lambda\,\mathrm{t}}}{\left[1+\mathrm{e}^{-\lambda\,\mathrm{t}}\right]^{2}}\,dt=\frac{2G(x;\phi)^{\alpha\lambda}}{G(x;\phi)^{\alpha\lambda}+\left[G(x;\phi)^{\alpha}+\overline{G}\left(x;\phi\right)^{\alpha}\right]^{\lambda}}, \tag{2}$$

and

$$f\left(x\right)=\frac{2\alpha\lambda g(x;\phi)G(x;\phi)^{\alpha\lambda-1}\overline{G}\left(x;\phi\right)^{\alpha-1}\left[G(x;\phi)^{\alpha}+\overline{G}\left(x;\phi\right)^{\alpha}\right]^{\lambda-1}}{\left\{G(x;\phi)^{\alpha\lambda}+\left[G(x;\phi)^{\alpha}+\overline{G}\left(x;\phi\right)^{\alpha}\right]^{\lambda}\right\}^{2}}, \tag{3}$$

where $\alpha > 0$ and $\lambda > 0$ are the shape parameters, $\overline{G}(x;\phi) = 1 - G(x;\phi)$ and $\boldsymbol{\phi}$ represents the vector of parameters for the baseline distribution. From now on, the density (3) is indicated by X ∼NTIIHL-$G(\alpha, \lambda, \boldsymbol{\phi})$. When $\alpha = 1$, we obtain the *new type half-logistic-*G (NTHL-G) family as a submodel of the NTIIHL-G family.

The reliability function (rf) and hrf of X are, respectively,

$$R\left(x;\alpha,\lambda,\phi\right)=\frac{\left[G(x;\phi)^{\alpha}+\bar{G}(x;\phi)^{\alpha}\right]^{\lambda}-G(x;\phi)^{\alpha\lambda}}{\left[G(x;\phi)^{\alpha}+\bar{G}(x;\phi)^{\alpha}\right]^{\lambda}+G(x;\phi)^{\alpha\lambda}},$$

and,

$$h(x;\alpha,\lambda,\boldsymbol{\phi})=\frac{2\alpha\lambda g(x;\phi)G(x;\phi)^{\alpha\lambda-1}\overline{G}\left(x;\phi\right)^{\alpha-1}\left[G(x;\phi)^{\alpha}+\overline{G}\left(x;\phi\right)^{\alpha}\right]^{\lambda-1}}{\left\{G(x;\phi)^{\alpha\lambda}+\left[G(x;\phi)^{\alpha}+\overline{G}\left(x;\phi\right)^{\alpha}\right]^{\lambda}\right\}\left\{\left[G(x;\phi)^{\alpha}+\bar{G}(x;\phi)^{\alpha}\right]^{\lambda}-G(x;\phi)^{\alpha\lambda}\right\}}. \tag{4}$$

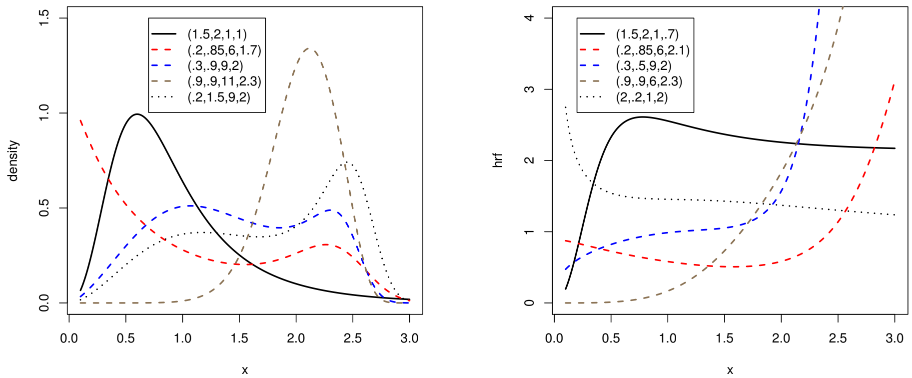

Figure 1. NTIIHL-W distribution: pdf (left), hrf (right)

3. Special models of NTIIHL-G

The special members of the NTIIHL-G family of distributions are examined for the famous baseline distributions such as normal and Weibull distributions.

3.1. The NTIIHL-W distribution

The pdf and cdf of the Weibull distribution are $g(x) = \frac{a}{c}\left(\frac{x}{c}\right)^{a-1}\exp\left(-\left(\frac{x}{c}\right)^{a}\right)$ and $G(x) = 1 - \exp\left(-\left(\frac{x}{c}\right)^{a}\right)$ where the parameter $c > 0$ is scale and $a > 0$ is the shape parameter. Inserting these quantities in (3), we have the following pdf for the new type II half logistic-Weibull (NTIIHL-W) distribution,

$$
\begin{aligned}
f(x;\alpha,\lambda,a,c) = {} & 2\alpha\lambda\frac{a}{c}\left(\frac{x}{c}\right)^{a-1}\left(1-\exp\left(-\left(\frac{x}{c}\right)^{a}\right)\right)^{\alpha\lambda-1}\exp\left(-\alpha\left(\frac{x}{c}\right)^{a}\right) \\
& \times\left\{\left(1-\exp\left(-\left(\frac{x}{c}\right)^{a}\right)\right)^{\alpha}+\exp\left(-\alpha\left(\frac{x}{c}\right)^{a}\right)\right\}^{\lambda-1} \\
& \times\left[\left(1-\exp\left(-\left(\frac{x}{c}\right)^{a}\right)\right)^{\alpha\lambda}+\left\{\left(1-\exp\left(-\left(\frac{x}{c}\right)^{a}\right)\right)^{\alpha}+\exp\left(-\alpha\left(\frac{x}{c}\right)^{a}\right)\right\}^{\lambda}\right]^{-2},
\end{aligned}
$$

where $x > 0$. The pdf and hrf shapes of the NTIIHL-W distribution are displayed in Figure 1. The pdf of the NTIIHL-W exhibits great flexibility with left and right skewness and bimodal shapes. Also, this distribution has the following hrf shapes: increasing, decreasing, unimodal and bathtub. Hemeda and Ahan ul Haq (2020) introduced a different generalization of the Weibull distribution, called as inverse Rayleigh Weibull (IRW) distribution. Also, Elgarhy et al. (2017) introduced the exponentiated Weibull exponential (EWE) distribution. However, the IRW and EWE distributions have only right-skewed pdf shapes. The NTIIHL-W distribution provides more flexibility than the IRW and EWE distributions (see Figure 1).

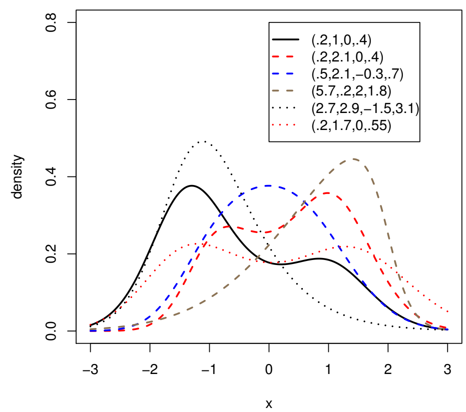

Figure 2. Pdf of NTIIHL-N distribution

3.2. The NTIIHL-N distribution

The several generalizations of the normal distribution have been proposed to add skewness into normal distribution. The detail information on generalized normal distributions can be found in Ma and Genton (2004), Arellano-Valle et al. (2010), and Rasekhi et al. (2016). Let $\phi(.)$ and $\Phi(.)$ represent the pdf and cdf of the standard normal distribution and $z = \frac{x-\mu}{\sigma}$. The pdf of new type II half logistic-Normal (NTIIHL-N) distribution is

$$f(x, \alpha, \lambda, \mu, \sigma) = \frac{2\alpha\lambda\phi(z)(\Phi(z))^{\alpha\lambda-1}(\Phi(-z))^{\alpha-1}\{(\Phi(z))^{\alpha} + (\Phi(-z))^{\alpha}\}^{\lambda-1}}{\left[(\Phi(z))^{\alpha\lambda} + \{(\Phi(z))^{\alpha} + (\Phi(-z))^{\alpha}\}^{\lambda}\right]^{2}}.$$

Figure 2 displays the pdf of NTIIHL-N distribution. It has unimodal and bimodal shapes.

4. Mathematical properties

This section gives the important statistical properties of the NTIIHL-G family of distributions.

4.1. Asymptotics

Proposition 4.1.

Let $a = \inf\{x|G(x) > 0\}$, when $x \to a$, the asymptotics of Equations (2), (3) and (4) are

$$\begin{aligned} F(x) &\sim 2\,G(x)^{\alpha\lambda} && \text{as} \quad \mathrm{x} \to \mathrm{a}, \\ f(x) &\sim 2\alpha\lambda\, g(x)\, G(x)^{\alpha\lambda-1}, && \text{as} \quad \mathrm{x} \to \mathrm{a}, \\ h(x) &\sim 2\alpha\lambda\, g(x)\, G(x)^{\alpha\lambda-1}, && \text{as} \quad \mathrm{x} \to \mathrm{a}. \end{aligned}$$

Proof:

Let $a = \inf\{x|G(x) > 0\}$, when $x \to a$, $[G(x)^\alpha + \bar{G}(x)^\alpha] \sim 1$, hence $G(x)^{\alpha\lambda} + \left[G(x)^\alpha + \bar{G}(x)^\alpha\right]^\lambda \sim 1$, then $F(x) \sim 2\,G(x)^{\alpha\lambda}$ as $x \to a$. By differentiation from last equation with respect to x, we obtain

$$f(x) \sim 2\alpha\lambda\, g(x)\, G(x)^{\alpha\lambda-1}, \qquad \text{as} \quad \mathrm{x} \to \mathrm{a},$$
$$h(x) \sim \frac{2\alpha\lambda\, g(x)\, G(x)^{\alpha\lambda-1}}{1-2\,G(x)^{\alpha\lambda}}, \qquad \text{as} \quad \mathrm{x} \to \mathrm{a}.$$

■

Proposition 4.2.

When $x \to \infty$, the asymptotics of Equations (2), (3) and (4) are

$$1-F(x) \sim \frac{\lambda}{2}\,\bar{G}(x)^\alpha, \qquad \text{as} \quad \mathrm{x} \to \infty,$$
$$f(x) \sim \frac{\alpha\lambda}{2}\; g(x)\,\bar{G}(x)^{\alpha-1}, \qquad \text{as} \quad \mathrm{x} \to \infty,$$
$$h(x) \sim \frac{\alpha\, g(x)}{\bar{G}(x)}, \qquad \text{as} \quad \mathrm{x} \to \infty.$$

The above quantities are useful to examine the effects of the parameters on the tails of the distribution.

Proof:

When $x \to \infty$, $[G(x)^\alpha + \bar{G}(x)^\alpha] \sim 1$ and $G(x)^{\alpha\lambda} + \left[G(x)^\alpha + \bar{G}(x)^\alpha\right]^\lambda \sim 2$, then $1 - F(x) \sim 1 -\ G(x)^{\alpha\lambda}$. Also $1 - t^{\alpha\lambda} \sim \alpha\lambda(1-t)$ as $t \to 1$. Then, $1 - F(x) \sim \alpha\lambda\,\bar{G}(x)$ as $x \to \infty$. By differentiation from last equation with respect to x, we obtain

$$f(x) \sim \alpha\lambda\, g(x), \qquad \text{as} \quad \mathrm{x} \to \infty,$$
$$h(x) \quad \sim \tfrac{g(x)}{\bar{G}(x)}, \qquad \text{as} \quad \mathrm{x} \to \infty.$$

■

4.2. Quantile function

Generating random variables from a continuous probability distribution are done by the quantile function (qf). Therefore, it is important to obtain the qf in explicit form. The qf is the solution of equation $F(x) = U$ where $U \sim U(0, 1)$. The qf of the NTIIHL-G is

$$X_U = Q_G\left(\frac{U^{\frac{1}{\alpha\lambda}}}{U^{\frac{1}{\alpha\lambda}} + \left\{[1+(1-U)]^{\frac{1}{\lambda}} - U^{\frac{1}{\lambda}}\right\}^{\frac{1}{\alpha}}}\right).$$

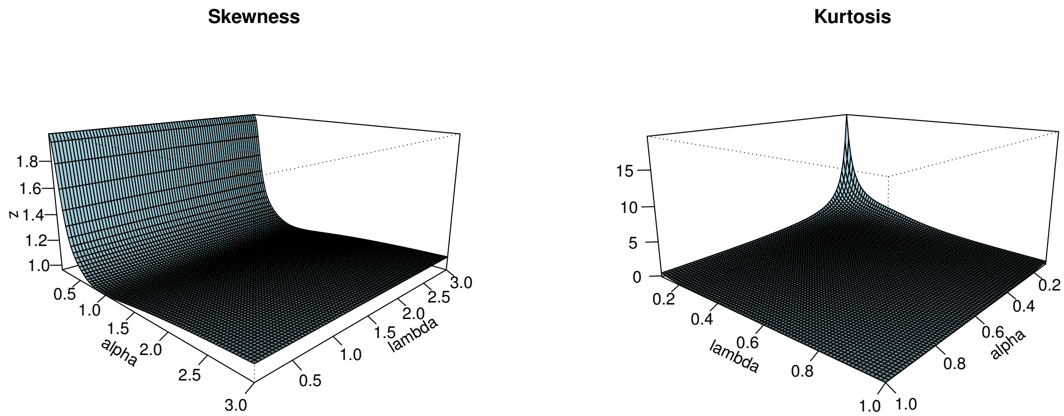

Figure 3. The skewness (left) and kurtosis (right) values of the NTIIHL-W distribution

The Bowley's skewness and Moors's kurtosis measures are defined based on the quantiles of the distribution. These measures are given, respectively, by

$$S = \frac{Q(1/4)+Q(3/4)-2Q(1/2)}{Q(3/4)-Q(1/4)}, K = \frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}.$$

Based on the above quantities, the skewness and kurtosis values of the NTIIHL-W distribution are plotted in Figure 3. The shape and scale parameters of the baseline distribution, Weibull, are taken as $a = 9$ and $c = 2$, respectively. From Figure 3, we conclude that when the parameter α increases, the skewness decreases and kurtosis increases; when the parameter λ increases, skewness and kurtosis decreases. Note that the parameter λ has no critical effect on the skewness measure of the NTIIHL-W distribution.

4.3. Useful expansions

For the simplification, we use $g(x;\phi) = g(x)$ and $G(x;\phi) = G(x)$. Firstly, we define the pdf and cdf of the Exponentiated-G ("Exp-G") distribution which are given by

$$H_\gamma(x) = G(x)^\gamma \qquad \text{and} \qquad h_\gamma(x) = \gamma\, g(x)\, G(x)^{\gamma-1},$$

respectively. This model is known as the Lehmann type I distribution and denoted as $\text{Exp}^\gamma(G)$. Now, an expansion for the cdf of the NTIIHL-G family is obtained. The power series for the quantity u^λ ($\lambda > 0$ real, $0 < u < 1$) is

$$u^\lambda = \sum_{k=0}^{\infty} s_k(\lambda)\, u^k,$$

where

$$s_k(\lambda) = \sum_{j=k}^{\infty} (-1)^{k+j} \binom{\lambda}{j} \binom{j}{k}.$$

Using power series, we have ($\alpha > 0$)

$$2G(x)^{\alpha\lambda} = \sum_{k=0}^{\infty} a_k \, G(x)^k,$$

where $a_k = 2\, s_k(\alpha\lambda)$ and

$$G(x)^{\alpha\lambda} + \left[G(x)^{\alpha} + \bar{G}(x)^{\alpha}\right]^{\lambda} = \sum_{k=0}^{\infty} b_k \, G(x)^k,$$

where $b_k = s_k(\alpha\lambda) + h_k(\alpha, \lambda)$. We have the following equation from the ratio of the two power series given above

$$F(x) = \frac{\sum_{k=0}^{\infty} a_k \, G(x)^k}{\sum_{k=0}^{\infty} b_k \, G(x)^k} = \sum_{k=0}^{\infty} c_k \, \Pi_k(x), \tag{5}$$

where $\Pi_k(x) = G(x)^k$ is the Exp-G cdf, $c_0 = \frac{a_0}{b_0}$ and the coefficients c_k's (for $k \geq 1$) can be determined by

$$c_k = \frac{1}{b_0} \left[a_k - \frac{1}{b_0} \sum_{r=1}^{k} b_r \, c_{k-r} \right].$$

Differentiating (5), we have the pdf of X which is given by

$$f(x) = \sum_{k=0}^{\infty} c_{k+1} \, \pi_{k+1}(x), \tag{6}$$

where $\pi_{k+1}(x) = (k+1)\, g(x) G(x)^k$ is the Exp-G density function with power parameter $(k+1)$. Equation (6) shows that the NTIIHL-G can be expressed as a linear combination of the Exp-G densities. Using this property, several properties of the NTIIHL-G can be obtained easily. One can refer to the following studies for the properties of the Exp-G densities: Mudholkar and Srivastava (1993) and Mudholkar et al. (1995), Gupta et al. (1998), Gupta and Kundu (1999), Nadarajah (2005), Shirke and Kakade (2006), Nadarajah and Gupta (2007) and Nadarajah and Kotz (2006).

4.4. General properties

The moments play important role to characterize the distributions such as their dispersion, skewness and kurtosis measures. The r^{th} moment of X is given by

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r \, f(x) \, dx.$$

Using (6), we have

$$\mu_r' = \sum_{k=0}^{\infty} c_{k+1} E(Y_{k+1}^r),$$

where

$$E\left(Y_\gamma^r\right) = \gamma \int_{-\infty}^{\infty} x^r\, g\left(x\right) G\left(x\right)^{\gamma-1} dx.$$

The above quantity can be numerically calculated based on the baseline qf such as $Q_G(u;\phi) = G^{-1}(x;\phi)$ as $E(Y_\alpha^n) = \alpha \int_0^1 Q_G(u;\phi)^n\, u^{\alpha-1} du$. For $r=1$ we have $E(X)=\mu$. For the NTIIHL-W model, we have

$$\mu_r' = \sum_{k,h=0}^{\infty} c_{k+1} \frac{(k+1)\left(-1\right)^h}{\left(1/c\right)^r \left(h+1\right)^{(r+a)/a}} \binom{k}{h} \Gamma\left(1+\frac{r}{a}\right),\ \forall\, r > -a.$$

The moment generating function (mgf) of the NTIIHL-W, $M_X\left(t\right)$ is obtained from (6) as $M_X\left(t\right) = \sum_{j=0}^{\infty} c_{k+1}\, \tau\left(t,k\right)$,where $\tau\left(t,k\right) = \int_0^1 \exp\left[t\, Q_G\left(u\right)\right] u^k du$ and $Q_G(u)$ is the qf corresponding to $G\left(x;\phi\right)$, i.e., $Q_G(u) = G^{-1}(u;\phi)$. The mgf of the NTIIHL-W is

$$M_X\left(t\right) = \sum_{k,r,h=0}^{\infty} c_{k+1} \frac{(k+1)\left(-1\right)^h t^r}{\left(1/c\right)^r r!\left(h+1\right)^{(r+a)/a}} \binom{k}{h} \Gamma\left(1+\frac{r}{a}\right),\ \forall\, r > -a.$$

The r^{th} incomplete moment of X is defined by $m_r(y) = \int_{-\infty}^{y} x^r f(x) dx$. Using (6), it is possible to define that $m_r(y) = \sum_{k=0}^{\infty} c_{k+1}\, m_{r,k}(y)$, where $m_{r,\gamma}(y) = E(Y_\gamma^r) = \int_0^{G(y;\varphi)} Q_G^r(u;\phi)\, u^{\gamma-1}\, du$. For the NTIIHL-W model we have

$$m_r(y) = \sum_{k,h=0}^{\infty} c_{k+1} \frac{(k+1)\left(-1\right)^h}{\left(1/c\right)^r \left(h+1\right)^{(r+a)/a}} \binom{k}{h} \gamma\left(1+\frac{r}{a}, \left(\frac{1}{ct}\right)^a\right),\ \forall\, r > -a.$$

Additionally, we compute the mean and variance of the NTIIHL-W distribution computationally for the parameters $a=9$ and $c=2$. The effects of the parameters α and λ on the mean and variance of the NTIIHL-W distributions are investigated. Figure 4 displays the mean and variance values of the NTIIHL-W distribution. As seen from these results, it is clear that the when α increases, the mean and variance decrease; when the parameter λ increases, the mean increases and the variance decreases.

5. Maximum Likelihood Estimation

Assume that the random sample, $x_1, \ldots, x_n$, follows the NTIIHL-G family with the unknown parameters α, λ and ϕ, where ϕ is a $q \times 1$ baseline parameter vector. Let $\mathbf{\Psi} = (\alpha, \lambda, \phi^{\intercal})^{\intercal}$ be a $(q+2)\times 1$ parameter vector. Based on these definitions, the log-likelihood function of the NTIIHL-G family is

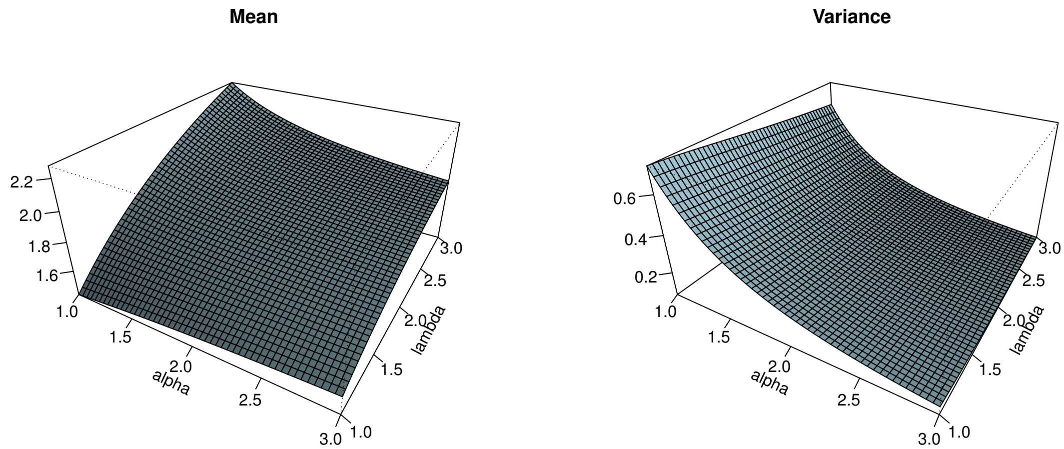

Figure 4. The mean (left) and variance (right) of the NTIIHL-W distribution

$$\ell = n\log(2) + n\log(\alpha) + n\log\lambda + \sum_{i=1}^{n}\log g(x_i;\phi) + (\alpha\lambda - 1)\sum_{i=1}^{n}\log G(x_i;\phi)$$
$$+ (\alpha - 1)\sum_{i=1}^{n}\log \overline{G}(x_i;\phi) + (\lambda - 1)\sum_{i=1}^{n}\log s_i - 2\sum_{i=1}^{n}\log z_i, \tag{7}$$

where $s_i = \left[G(x_i;\phi)^{\alpha} + \overline{G}(x_i;\phi)^{\alpha}\right]$ and $z_i = G(x_i;\phi)^{\alpha\lambda} + s_i^{\lambda}$. The log-likelihood function in (7) is maximized using the optim function of the R software. The other choice is to take partial derivatives of (7) with respect to the parameters and obtain the maximum likelihood estimators (MLEs) of the parameters of the NTIIHL-G family based on the joint solution of these score vectors. The score vector components, say $\mathbf{U}(\mathbf{\Psi}) = \frac{\partial\ell}{\partial\mathbf{\Psi}} = (\frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial\,\phi_r})^{\intercal} = (U_{\alpha}, U_{\lambda}, U_{\phi_r})^{\intercal}$, are given by

$$U_{\alpha} = \frac{n}{\alpha} + \lambda\sum_{i=1}^{n}\log G(x_i;\phi) + \sum_{i=1}^{n}\log\overline{G}(x_i;\phi) + (\lambda-1)\sum_{i=1}^{n}\frac{t_i}{s_i} - 2\sum_{i=1}^{n}\frac{d_i}{z_i},$$

$$U_{\lambda} = \frac{n}{\lambda} + \alpha\sum_{i=1}^{n}\log G(x_i;\phi) + \sum_{i=1}^{n}\log s_i - 2\sum_{i=1}^{n}\frac{p_i}{z_i},$$

and (for $r = 1, ..., q$)

$$U_{\phi_r} = \sum_{i=1}^{n}\frac{g_r'(x_i;\phi)}{g(x_i;\phi)} + (\alpha\lambda-1)\sum_{i=1}^{n}\frac{G_r'(x_i;\phi)}{G(x_i;\phi)} + (\alpha-1)\sum_{i=1}^{n}\frac{G_r'(x_i;\phi)}{\overline{G}(x_i;\phi)}$$
$$+ (\lambda-1)\sum_{i=1}^{n}\frac{m_{i,r}}{s_i} - 2\sum_{i=1}^{n}\frac{w_{i,r}}{z_i},$$

where

$$\begin{aligned}
t_i &= \tfrac{\log G(x_i;\phi)}{G(x_i;\phi)^{-\alpha}} + \tfrac{\log \overline{G}(x_i;\phi)}{\overline{G}(x_i;\phi)^{-\alpha}}, \quad g_r'(x_i;\phi) = \frac{\partial g(x_i;\phi)}{\partial \phi_r}, \\
d_i &= \lambda \left[\tfrac{\log G(x_i;\phi)}{G(x_i;\phi)^{-\alpha\lambda}} + \tfrac{t_i}{s_i^{1-\lambda}} \right], \quad G'(x_i;\phi) = \frac{\partial G(x_i;\phi)}{\partial \phi_r}, \\
p_i &= \tfrac{\alpha \log G(x_i;\phi)}{G(x_i;\phi)^{-\alpha\lambda}} + \tfrac{\log s_i}{s_i^{-\lambda}}, \quad m_{i,r} = \frac{G(x_i;\phi)^{\alpha-1} - \overline{G}(x_i;\phi)^{\alpha-1}}{\alpha^{-1}\left[G'(x_i;\phi)\right]^{-1}}, \text{ and} \\
w_{i,r} &= \lambda \left[\tfrac{G(x_i;\phi)^{\alpha\lambda-1}}{\alpha^{-1}[G'(x_i;\phi)]^{-1}} + \tfrac{m_{i,r}}{s_i^{1-\lambda}} \right].
\end{aligned}$$

One can obtain the MLE of the parameter vector, $\mathbf{\Psi} = (\widehat{\alpha}, \widehat{\lambda}, \widehat{\phi}^{\intercal})^{\intercal}$, by means of the simultaneous solution of the score vectors for zero. However, as seen from the score vectors, they contain non-linear functions which make the solution of the MLE impossible. Therefore, the log-likelihood function has to be maximized directly.

6. Simulation study

The asymptotic efficiencies of the MLEs of the NTIIHL-N parameters are investigated by means of a simulation study. The below measures are considered to evaluate the simulation results.

(1) Bias $\rightarrow \widehat{Bias}_\eta(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\eta}_i - \eta)$
(2) Means square error (MSE) $\rightarrow \widehat{MSE}_\eta(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\eta}_i - \eta)^2$
(3) Average length (AL) $\rightarrow AL_\eta(n) = \frac{3.919928}{N}\sum_{i=1}^{N} s_{\hat{\eta}_i}$
(4) Coverage probability (CP) $\rightarrow CP_\eta(n) = \frac{1}{N}\sum_{i=1}^{N} I(\hat{\eta}_i - 1.95996 s_{\hat{\eta}_i}, \hat{\eta}_i + 1.95996 s_{\hat{\eta}_i})$ where $\eta = \alpha, \lambda, \mu, \sigma$ and $s_{\hat{\eta}_i}$ is the standard error of the estimated parameter.

We generate random variables from the NTIIHL-N distribution with sample sizes $n = 50, 55, \ldots, 900$. The simulation is repeated $N = 1000$ times. The simulation is implemented with the parameters $\alpha = 0.5, \lambda = 2, \mu = 0.5, \sigma = 2$. The simulation results are displayed in Figure 5. We expect to see that the estimated biases and MSEs approach the zero for large sample sizes. Additionally, the estimated CP should be near 0.95 and AL should be a decreasing function of the sample size. As seen from the results displayed in Figure 5, the estimated biases and MSEs are near the zero and also CP approach the desired value, 0.95. The estimated ALs always decrease when the sample size increases. These results verify the consistency property of the MLE method.

7. The log-new type II half-logistic-Weibull (LNTIIHL-W) regression model

In the last decade, the location-scale regression models have gained attention and have found an application area in the survival modeling. Now, we introduce a new log location-scale regression

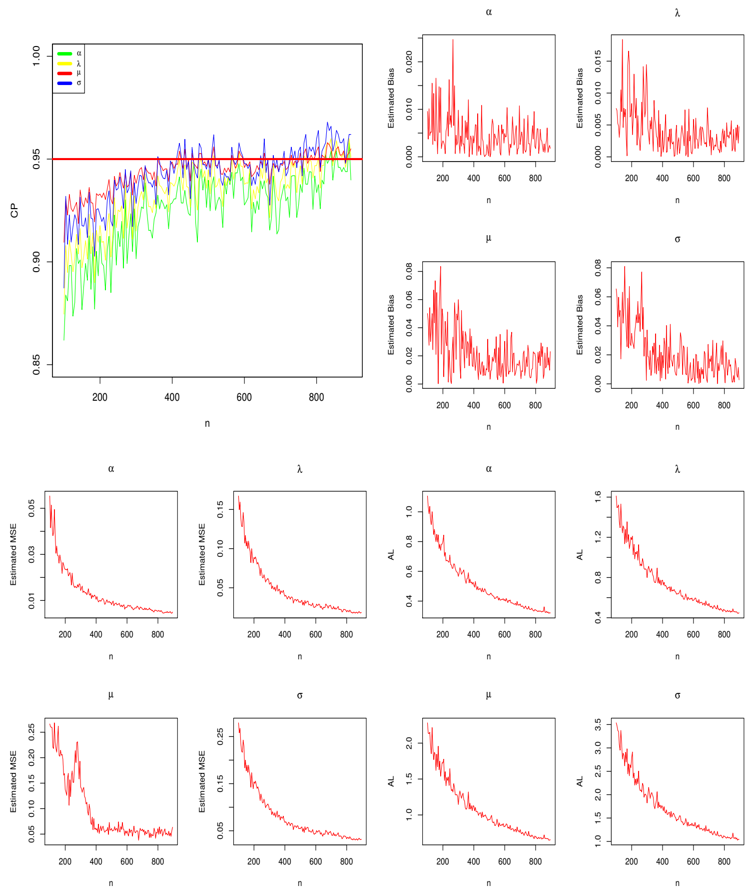

Figure 5. The estimated biases, MSEs, CPs and ALs for the NTIIHL-W distribution

model based on the NTIIHL-W density. Let X be a random variable following the NTIIHL-W distribution with four parameters $\alpha > 0$, $\lambda > 0$, $a > 0$ and $c > 0$, introduced in Section 3.1. Define a random variable $Y = \log(X)$ and re-parametrizations on the parameters of the NTIIHL-W such as $a = 1/\sigma$ and $c = \exp(\mu)$. Then, we have

$$\begin{aligned} f(y) =& \frac{2\alpha\lambda}{\sigma}\exp\left[\left(\frac{y-\mu}{\sigma}\right)-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\left\{1-\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right\}^{\alpha\lambda-1}\left(\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right)^{\alpha-1} \\ &\times\left[\left\{1-\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right\}^{\alpha}+\left(\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right)^{\alpha}\right]^{\lambda-1} \\ &\times\left[\left\{\left\{1-\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right\}^{\alpha\lambda}+\left[\left\{1-\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right\}^{\alpha}+\left(\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right)^{\alpha}\right]^{\lambda}\right\}^{2}\right]^{-1}, \end{aligned} \tag{8}$$

where $y \in \Re$, $\mu \in \Re$, $\sigma > 0$, $\alpha > 0$ and $\lambda > 0$. The density (8) is called as LNTIIHL-W and denoted as $Y \sim$ LNTIIHL-W$(\alpha, \lambda, \sigma, \mu)$ where μ is the location and σ is the scale parameters. The pdf shapes of LNTIIHL-W distribution are displayed in Figure 6 which reveals that the distribution can be used to model bimodal, left skewed and nearly symmetric data sets.

The sf to density (8) is

$$s(y) = \frac{\left[\left\{1-\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right\}^{\alpha}+\left(\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right)^{\alpha}\right]^{\lambda}-\left\{1-\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right\}^{\alpha\lambda}}{\left[\left\{1-\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right\}^{\alpha}+\left(\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right)^{\alpha}\right]^{\lambda}+\left\{1-\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right\}^{\alpha\lambda}}. \tag{9}$$

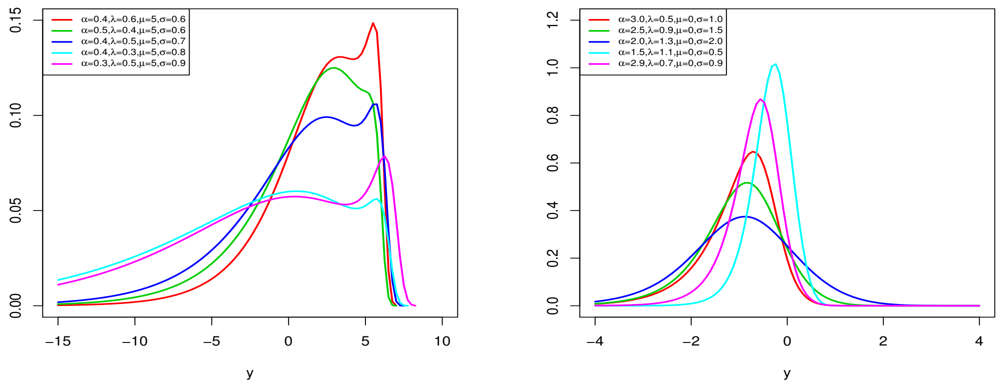

Figure 6. The pdf plots of the LNTIIHL-W distribution

Under the following transformation, $Z = (Y - \mu)/\sigma$, the corresponding pdf is

$$\begin{aligned} f(z) &= 2\alpha\lambda \exp\left[z - \exp(z)\right]\left\{1 - \exp\left[-\exp(z)\right]\right\}^{\alpha\lambda-1}\left(\exp\left[-\exp(z)\right]\right)^{\alpha-1} \\ &\times \left[\left\{1 - \exp\left[-\exp(z)\right]\right\}^{\alpha} + \left(\exp\left[-\exp(z)\right]\right)^{\alpha}\right]^{\lambda-1} \\ &\times \left[\left\{\left\{1 - \exp\left[-\exp(z)\right]\right\}^{\alpha\lambda} + \left[\left\{1 - \exp\left[-\exp(z)\right]\right\}^{\alpha} + \left(\exp\left[-\exp(z)\right]\right)^{\alpha}\right]^{\lambda}\right\}^{2}\right]^{-1}. \end{aligned} \tag{10}$$

Using the density given in (8), LNTIIHL-W regression model is introduced. Let y_i be a dependent variable and $\mathbf{x}_i^T = (x_{i1}, ..., x_{ip})$ be a vector of explanatory variable. Then, the regression model is given by

$$y_i = \boldsymbol{x}_i^T \boldsymbol{\beta} + \sigma z_i, \mathrm{i} = 1,..., \mathrm{n}, \tag{11}$$

where the error term z_i has the density function in (10). The vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_p)^T$ represents the regression parameters and $\sigma > 0$, $\alpha > 0$ and $\lambda > 0$ are unknown parameters. The location of the response variable y_i is modeled with identity link such as $\mu_i = \boldsymbol{x}_i^T \boldsymbol{\beta}$. The LNTIIHL-W regression model contain the following model as its submodel.

- **Log-new type half-logistic-Weibull (LNTHL-W) regression model**
  For $\alpha = 1$, the survival function is

$$s(y) = \frac{1 - \left\{1 - \exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right\}^{\lambda}}{1 + \left\{1 - \exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right\}^{\lambda}}. \tag{12}$$

Now, we discuss the parameter estimation of the LNTIIHL-W regression model. Let $y_1, y_2, ..., y_n$ be a random sample from the LNTIIHL-W distribution and it is defined as $y_i = \min\{\log(x_i), \log(c_i)\}$ where c_i is censoring time and x_i is the lifetime. We use two sets, F and C to identify the log-lifetime and log-censoring, respectively. The log-likelihood function of the model given in (11) is $l(\boldsymbol{\tau}) = \sum\limits_{i \in F} l_i(\boldsymbol{\tau}) + \sum\limits_{i \in C} l_i^{(c)}(\boldsymbol{\tau})$. So, the log-likelihood function of the LNTIIHL-W regression model is

$$\begin{aligned}
\ell(\boldsymbol{\tau}) &= r\log\left(\tfrac{\alpha\lambda}{\sigma}\right)+\sum_{i\in F}(z_i-u_i)+(\alpha\lambda-1)\sum_{i\in F}\log\left[\{1-\exp[-u_i]\}\right]+(\alpha-1)\sum_{i\in F}\log\left[(\exp[-u_i])\right]\\
&+(\lambda-1)\sum_{i\in F}\log\left[\left[\{1-\exp[-u_i]\}^{\alpha}+(\exp[-u_i])^{\alpha}\right]\right]\\
&-\sum_{i\in F}\log\left[\left[\left\{\{1-\exp[-u_i]\}^{\alpha\lambda}+\left[\{1-\exp[-u_i]\}^{\alpha}+(\exp[-u_i])^{\alpha}\right]^{\lambda}\right\}^{2}\right]\right]\\
&+\sum_{i\in C}\log\left\{\left[\{1-\exp[-u_i]\}^{\alpha}+(\exp[-u_i])^{\alpha}\right]^{\lambda}-\{1-\exp[-u_i]\}^{\alpha\lambda}\right\}\\
&-\sum_{i\in C}\log\left\{\left[\{1-\exp[-u_i]\}^{\alpha}+(\exp[-u_i])^{\alpha}\right]^{\lambda}+\{1-\exp[-u_i]\}^{\alpha\lambda}\right\},
\end{aligned} \tag{13}$$

where $\boldsymbol{\tau}=\left(\alpha,\lambda,\sigma,\boldsymbol{\beta}^{T}\right)^{T}$, $u_i=\exp(z_i)$, $z_i=(y_i-x_i^{T}\beta)/\sigma$ and r is the number of uncensored observations. The unknown parameters of the LNTIIHL-W regression model are estimated by direct maximization of the given log-likelihood.

As mentioned before, LNTHL-W and LNTIIHL-W regression models are nested. So, likelihood ratio (LR) test can be used to decide which model performs better than other one. The LR test statistic is defined as $w=2\{\ell(\widehat{\tau})-\ell(\widetilde{\tau})\}$ where $\ell(\widehat{\tau})$ and $\ell(\widetilde{\tau})$ are the estimated log-likelihood values for the null and alternative models, respectively. Here, we have the following hypothesis: $H_0:\ \alpha=1$ versus $H_1:\ \alpha\neq 1$. For these hypothesis testing, the LR test statistic is distributed as χ^2_k distribution with one degree-of-freedom.

8. Estimation of System Reliability using NTIIHL-W model

The hrf is used to measure the item's tendency to fail. Several probability distributions are used for system reliability estimations. Due to its simplicity, the exponential and Weibull distributions are commonly used for modeling lifetime data. In this section, we investigate the usefulness of the NTIIHL-W distribution in terms of the estimation of the system reliability. Let T represents the lifetime of units, the rf and hrf of the NTIIHL-W distribution are given, respectively,

$$R(t;\alpha,\lambda,a,c)=\frac{\left[\{1-d\}^{\alpha}+(d)^{\alpha}\right]^{\lambda}-\{1-d\}^{\alpha\lambda}}{\left[\{1-d\}^{\alpha}+(1-d)^{\alpha}\right]^{\lambda}+\{1-d\}^{\alpha\lambda}}, \tag{14}$$

and

$$\begin{aligned}
h(t;\alpha,\lambda,a,b) &= 2\alpha\lambda\left\{\tfrac{a}{c}\left(\tfrac{t}{c}\right)^{a-1}d\right\}\{1-d\}^{\alpha\lambda-1}(d)^{\alpha-1}\\
&\times\left[\{1-d\}^{\alpha}+(d)^{\alpha}\right]^{\lambda-1}\left\{\{1-d\}^{\alpha\lambda}+\left[\{1-d\}^{\alpha}+(d)^{\alpha}\right]^{\lambda}\right\}^{-1}\\
&\times\left\{\left[\{1-d\}^{\alpha}+(d)^{\alpha}\right]^{\lambda}-\{1-d\}^{\alpha\lambda}\right\}^{-1},
\end{aligned} \tag{15}$$

where $d=\exp\left[-\left(\frac{t}{c}\right)^{a}\right]$, $\alpha>0$, $\lambda>0$, $a>0$ are the shape parameters and $c>0$ is the scale parameter. Under the normal system conditions, the mean-time-between-failure (MTBF) represents the estimated elapsed time between failures. The MTBF of the NTIIHL-W distribution can

Table 1. Parameter estimates and fitting summary of the models

Models	Parameter Estimates (Standard Errors)				p-value	AIC	BIC
Exponential	0.002				0.656	1196.20	1198.66
(c)	(0.0002)						
Weibull	1.170	400.930			0.720	1195.26	1195.40
(a, c)	(0.103)	(38.729)					
NTIIHL-W	0.168	0.278	21.540	887.200	0.980	1181.96	1191.78
(α, λ, a, c)	(0.049)	(0.073)	(1.658)	(1.916)			

be given by,

$$\begin{aligned} MTBF &= \int_0^\infty tF(t)dt = \int_0^\infty R(t)dt \\ &= \int_0^\infty \frac{\left[\left\{1-\exp\left[-\left(\frac{t}{c}\right)^a\right]\right\}^\alpha + \left(\exp\left[-\left(\frac{t}{c}\right)^a\right]\right)^\alpha\right]^\lambda - \left\{1-\exp\left[-\left(\frac{t}{c}\right)^a\right]\right\}^{\alpha\lambda}}{\left[\left\{1-\exp\left[-\left(\frac{t}{c}\right)^a\right]\right\}^\alpha + \left(1-\exp\left[-\left(\frac{t}{c}\right)^a\right]\right)^\alpha\right]^\lambda + \left\{1-\exp\left[-\left(\frac{t}{c}\right)^a\right]\right\}^{\alpha\lambda}}. \end{aligned} \tag{16}$$

Hereby, we demonstrate the usefulness of the NTIIHL-W distribution in terms of the estimation of the system reliability by means of real data application. The MLE method is used to estimate the parameters of the NTIIHL-W distribution for modeling the failure data set of software. The data set can be found in Lyu (1996). The data set contains 86 observations and represents the time-between-failures (time unit in miliseconds) of a software which can be also found in **reliaR** R package.

We compare the performance of NTIIHL-W distribution with Weibull and exponential distributions. The parameter estimations and fitting summary of the used models are given in Table 1. We use Akaike Information Criteria (AIC) and Bayesian Information Criteria (BIC) as well as Kolmogorov-Smirnov (K-S) test with corresponding p-value to decide the best model. The reported results in Table 1 indicate that NTIIHL-W distribution provides better fits than the Weibull and exponential distributions based on the and AIC and BIC values. Additionally, the p-value of the K-S test is the highest for the NTIIHL-W distribution.

MTBF value is calculated as 375.052 by using the Equation (16) for the NTIIHL-W distribution. We present the plots of the fitted density, hrf and rf with the probability-probability (P-P) plot for the NTIIHL-W model in Figure 7. It is clear that NTIIHL-W provides superior fits to used data set.

9. Applications

9.1. Turbocharger failure time

We compare the NTIIHL-W model with Kw-Weibull (Cordiero et al., 2010), Beta-Weibull (Lee et al., 2007), Generalized Modified Weibull (Carrasco et al., 2008) and (P-A-L) Extended Weibull

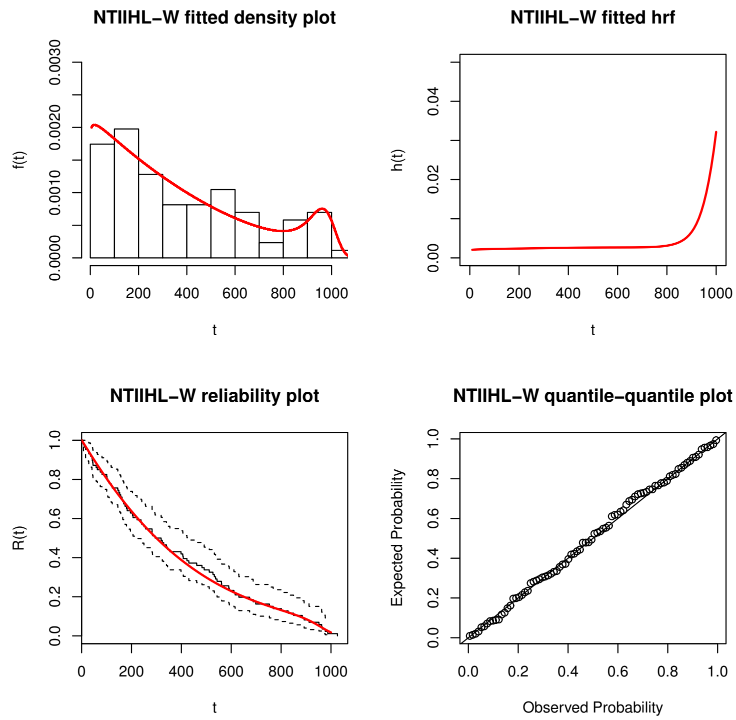

Figure 7. Fitted plots of NTIIHL-W distribution

Table 2. Summary statistics of the data set

Data set	Mean	Median	SD	Skewness	Kurtosis
Turbocharger failure time	6.2	6.5	1.9	-0.66	2.64

(Al-Zahrani et al., 2015), Type I half-logistic-Weibull (Cordeiro et al., 2016c) and Exponentiated half-Logistic weibull (Cordeiro et al., 2014) Distributions. The following measures are considered to decide the best model: AIC, Consistent Akaike Information Criterion (CAIC), BIC, Hannan-Quinn information criterion (HQIC). The model with the smallest values of these measures is chosen as the best model. The descriptive statistics of the used data set is given in Table 2.

The data contains the time-to-failure of turbocharger, given in the work of Xu et al. (2003). Table 3 contains the estimated parameters of the fitted models. The standard errors are given in parentheses. The results of the model selection criteria are given in Table 4. Since the NTIIHL-W model has the lowest values of the model selection criteria, we conclude that the proposed distribution is the best choice for the used data among others. Figure 8(a) displays the histogram of the data with fitted pdfs of the competitive models. Figure 8(b) displays the fitted functions of the NTIIHL-W distribution. Especially, Figure 8(b) proves the suitability of the NTIIHL-W distribution for the used data set. The proposed distribution provides nearly perfect fit for the data.

Table 3. Estimated parameters of the fitted models

Model	Estimates (Standard Error)
NTIIHL-W	0.225, 1.339, 12.503, 7.170
(α, λ, a, c)	(0.077), (0.823), (8.126), (0.996)
Kw-W	122.106, 451.971, 0.181, 64.685
(a, b, λ, c)	(58.395), (376.933), (0.019), (39.853)
B-W	0.557, 0.066, 0.297, 3.643
(a, b, λ, c)	(0.196), (0.011), (0.001), (0.002)
GM-W	0.003, 0.982, 0.510, 0.941
$(\alpha, \gamma, \lambda, \beta)$	(0.001), (1.037), (0.258), (0.247)
PALEW	5.419, 3.756, 0.250, 737.875
(α, β, ν, p)	(1.117), (1.180), (0.391), (1194.629)
TIHLW	0.167, 0.271, 3.396
(λ, α, c)	(1.828), (0.873), (0.471)
EHLW	0.093, 0.740, 0.262, 4.135
(λ, α, b, c)	(0.017), (0.139), (0.004), (0.005)

Table 4. Results of the model selection criteria

Model	Goodness of fit criteria			
	AIC	BIC	$HQIC$	$CAIC$
NTIIHL-W	165.064	171.819	167.506	166.206
Kw-W	177.899	184.654	180.342	179.042
B-W	171.908	178.663	174.350	173.050
GM-W	168.163	174.918	170.605	169.306
PALEW	166.196	172.952	168.639	167.339
TIHLW	169.317	174.384	171.149	169.984
EHLW	169.284	176.039	171.727	170.427

9.2. Leukaemia data

This application is on the data set about the length of remission in weeks for two groups of leukaemia patients, treated and control. The data set were analyzed previously by Cox (1992). The response variable y_i is the log weeks of remission and explanatory variable x_i is the treatment which takes value 1 for drug and 0 for placebo. The aim of is to model the weeks of remission with the treatment. The following regression model is fitted to data set

$$y_i = \beta_0 + \beta_1 x_i + \sigma z_i,$$

where y_i has the LNTIIHL-W density (8). Table 5 contains the estimated parameters of the fitted regression models. The standard errors are given in $(\cdot)$ and corresponding p-values are in $[\cdot]$. The regression parameter β_1 is found statistically significant at any significance level for all fitted

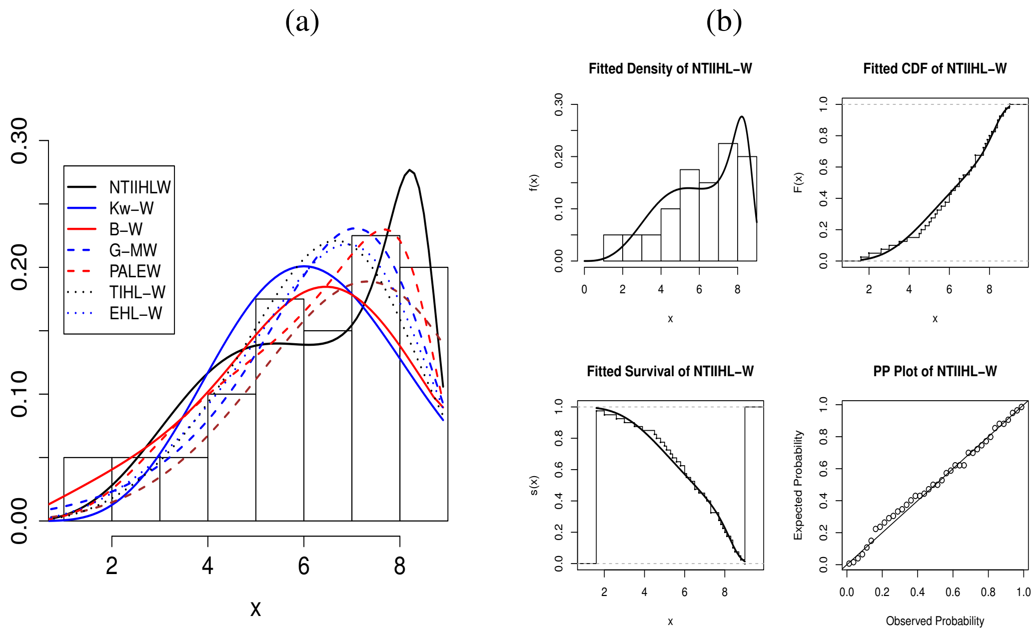

Figure 8. (a) Comparison of fitted models on the histogram of the data and (b) fitted functions of NTIIHL-W model

Table 5. The estimated parameters of the fitted regression models

Model	α	λ	σ	β_0	β_1	AIC
LNTIIHL-W	0.251	0.378	0.081	1.645	1.327	98.756
	(0.085)	(0.001	(0.134)	(0.025)	(0.025)	
				[< 0.001]	[< 0.001]	
LNTHL-W	1.000	3.555	1.429	0.126	1.319	102.573
	-	(1.146)	(0.629)	(0.872)	(0.321)	
				[0.885]	[< 0.001]	
LW	-	-	0.732	0.981	1.267	
			(0.108)	(0.430)	(0.311)	100.128
				[0.022]	[< 0.001]	

regression models. However, the LNTIIHL-W regression model has the lowest value of the AIC which proves that the LNTIIHL-W regression model is the best model among others. Since the parameter β_1 is statistically significant, we conclude that there is a difference between treatment groups for the weeks of remission.

Additionally, we perform the LR test to compare the LNTIIHL-W regression model with its sub-model, LNTHL-W regression model. We test the hypothesis $H_0 : \alpha = 1$ against $H_1 : \alpha \neq 1$. The LR test statistic is obtained as 5.816 with the corresponding p-value is 0.016 which indicates that the LNTIIHL-W regression model is better than the LNTHL-W regression model. The suitability of the fitted LNTIIHL-W regression model is proved graphically in Figure 9. As seen from Figure 9, the proposed regression model provides excellent fit to the estimated Kaplan-Meier curve and it

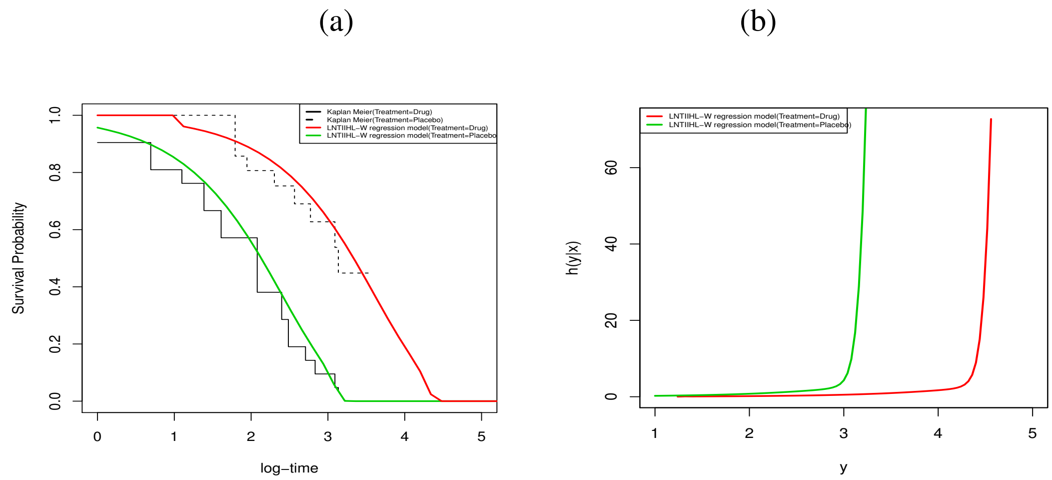

Figure 9. (a) Estimated sf and (b) hrf for the LNTIIHL-W regression model

is an evidence for the difference between treatment groups.

10. Conclusion and future work

In this study, a new type II half-logistic-G (NTIIHL-G) family is defined. The proposed family is applied to different fields such as system reliability and survival analysis. An application on real data set is given for the estimation of system reliability based on a new extension of the Weibull distribution. Besides, two applications are also given for survival data analysis with covariates and without covariates. In the light of the results of these applications, we conclude that the proposed family work well for survival and system reliability analysis. As a future work of the presented study, a financial risk model based on the NTIIHL-G family by using the generalized Pareto distribution as a baseline distribution is planned.

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