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Second-order Modified Nonstandard Explicit Runge-Kutta And Theta Methods for One-dimensional Autonomous Differential Equations

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Abstract

Nonstandard finite difference methods (NSFD) are used in physical sciences to approximate solutions of ordinary differential equations whose analytical solution cannot be computed. Traditional NSFD methods are elementary stable but usually only have first order accuracy. In this paper, we introduce two new classes of numerical methods that are of second order accuracy and elementary stable. The methods are modified versions of the nonstandard two-stage explicit Runge-Kutta methods and the nonstandard one-stage theta methods with a specific form of the nonstandard denominator function. Theoretical analysis of the stability and accuracy of both modified NSFD methods is presented. Numerical simulations that concur with the theoretical findings are also presented, which demonstrate the computational advantages of the proposed new modified nonstandard finite difference methods.

Keywords: Nonstandard; Finite difference; NSFD; Elementary stable; Dynamically consistent

MSC 2010 No.: 65L05, 65L06, 65L12, 65L20

1. Introduction

The behavior of dynamic systems in science, engineering, and economics is often modeled by ordinary differential equations (ODEs). Many of the ODEs are nonlinear and cannot be solved analytically. Therefore, discretization methods must be used to approximate the solutions to nonlinear ODEs. Conventional explicit finite difference methods require a time-step size restriction and may produce artificial fixed points and other undesirable behavior of the numerical solutions. Nonstandard finite difference methods (NSFD) were first proposed by Mickens approximately three decades ago (Mickens (1994)) to preserve certain important characteristics of the solutions to nonlinear ODEs. Throughout the last two decades, NSFD methods have been developed and applied to many specific problems in science and engineering (Chen and Kojouharov (1999); Dimitrov and Kojouharov (2005a); Lubuma and Patidar (2005); Dimitrov and Kojouharov (2008); Benz et al. (2009); Chen et al. (2009); Obaid et al. (2009); Suryanto et al. (2013)). Numerous research efforts have also been made to generalize these findings (Dimitrov and Kojouharov (2006); Dimitrov and Kojouharov (2007a); Dimitrov and Kojouharov (2007b); Dimitrov and Kojouharov (2011); Anguelov et al. (2014); Wood et al. (2015); Wood et al. (2017)). NSFD methods have also been developed, based on the standard theta methods (Anguelov and Lubuma (2001); Lubuma and Roux (2003); Dimitrov and Kojouharov (2005b); Dimitrov and Kojouharov (2007b); Anguelov et al. (2020)) and the standard two-stage explicit Runge-Kutta (ERK2) methods (Anguelov and Lubuma (2001); Dimitrov and Kojouharov (2005b)) that preserve the local dynamical properties of the solutions near equilibrium points, however, the above methods are only first order accurate. A second order accurate and elementary stable NSFD method was developed for ODEs with polynomial right-hand sides (Chen et al. (2006)); however, the method is implicit in its general form, which makes it computationally expensive.

In this work, we present a general modified nonstandard ERK2 method that is both of second order accuracy and elementary stable, as well as explicit. In addition, we present a second-order modified nonstandard one-stage theta method that is also of second order accuracy and elementary stable, thereby improving the order of accuracy of the underlying numerical method. Our approach is motivated by the second-order modified nonstandard two-stage theta method (Gupta et al. (2020); Kojouharov et al. (2021)). The resulting new numerical methods preserve the important features of their NSFD counterparts, while being of higher order accuracy, computationally simple and easy to implement.

The paper is organized as follows. The new NSFD methods are presented and analyzed in Section 2. Several simulation examples are then shown in Section 3 that validate the theoretical results. In Section 4, we provide our conclusions.

2. Main Results

Let us consider the autonomous differential equation:

$$\frac{dx}{dt} = f(x); \quad x(t_0) = x_0, \tag{1}$$

where $x : [t_0, T) \to \mathbb{R}$, $f \in C^2(\mathbb{R};\mathbb{R})$ is differentiable, $x_0 \in \mathbb{R}$, and assume that Equation (1) has a finite number of only hyperbolic equilibria. A general finite difference method for numerically solving Equation (1) on the interval $[t_0, T]$ can be written as

$$D_h(x_n) = F_h(f; x_n),\ k = 0, \cdots, N_t, \tag{2}$$

where $D_h(x_n) \approx \left.\dfrac{dx}{dt}\right|_{t=t_n}$, $x_n \approx x(t_n)$, and $F_h(f; x_n)$ approximates $f(x)$ in Equation (1), $t_n = t_0 + nh$, $n = 0, \cdots, N_t$, with mesh size $h > 0$. The numerical methods presented and analyzed in this paper satisfy the two main properties of NSFD methods, as formalized by Anguelov and Lubuma (2001), which are that the denominator function $\varphi(h)$ from the discretization of the derivative, i.e., $D_h(x_n) = \dfrac{x_{n+1} - x_n}{\varphi(h)}$ is a non-negative function of the form $\varphi(h) = h + \mathcal{O}(h^2)$, while the right-hand side function is discretized non-locally, i.e., $F_h(f; x_n) = g(x_n, x_{n+1}, h)$. Also, the new numerical methods are elementary stable, according to the definition in (Anguelov and Lubuma (2001)), which implies that the fixed points of the methods are the same as the equilibria of Equation (1) and vice-versa, and also that their local stability properties are the same for any value of the step-size h.

2.1. General second-order modified nonstandard ERK2 methods

The following result holds for the general modified nonstandard ERK2 method:

Theorem 2.1.

Let $f \in C^2(\mathbb{R};\mathbb{R})$ and $\varphi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ satisfy the following conditions:

(i) $\varphi(h, x) = h + \mathcal{O}(h^3)$ for all $h > 0$,

(ii) $0 < \varphi(h, x) < \dfrac{2}{|f_x(x^*)|}$, for all hyperbolic equilibria x^* of Equation (1) with $h > 0$ and $x \in \mathbb{R}$.

Then, the following general modified nonstandard ERK2 method for approximating the solution of Equation (1):

$$x_{n+1} = x_n + \varphi(h, x_n)\left\{(1-\omega)f(x_n) + \omega f\left(x_n + \frac{1}{2\omega}f(x_n)\varphi(h, x_n)\right)\right\},\ 0 < \omega \leq 1, \tag{3}$$

is elementary stable and of second-order accuracy, provided the method does not introduce additional fixed points other than those of Equation (1).

Proof:

Using a Taylor series expansion about t_n we obtain

$$
\begin{aligned}
&x(t_{n+1}) - \left[x(t_n) + \varphi(h, x(t_n))\left\{(1-\omega)f(x(t_n)) + \omega f\left(x(t_n) + \frac{1}{2\omega}f(x(t_n))\varphi(h, x(t_n))\right)\right\}\right] \\
&= \left[x(t_n) + hx'(t_n) + \frac{h^2}{2}x''(t_n) + \frac{h^3}{6}x'''(t_n) + \mathcal{O}(h^4)\right] \\
&\quad - \left[x(t_n) + \varphi(h, x(t_n))\left\{(1-\omega)f(x(t_n)) + \omega\left(f(x(t_n)) + \frac{1}{2\omega}\varphi(h, x(t_n))f(x(t_n))f_x(x(t_n))\right.\right.\right. \\
&\quad \left.\left.\left. + \frac{1}{2}\left(\frac{1}{2\omega}f(x(t_n))\varphi(h, x(t_n))\right)^2 f_{xx}(x(t_n)) + \mathcal{O}(h^3)\right)\right\}\right] \\
&= (h - \varphi(h, x(t_n)))f(x(t_n)) + \left(\frac{h^2}{2} - \frac{\varphi^2(h, x(t_n))}{2}\right) f(x(t_n))f_x(x(t_n)) \\
&\quad + \frac{h^3}{6}\left(f_{xx}(x(t_n))f^2(x(t_n))) + f_x^2(x(t_n))f(x(t_n))\right) \\
&\quad - \frac{1}{8\omega}\varphi^3(h, x(t_n))f^2(x(t_n))f_{xx}(x(t_n)) + \mathcal{O}(h^4).
\end{aligned}
$$

Substituting the expression for $\varphi(h, x(t_n))$ from condition (i) in Equation (4) yields

$$
\begin{aligned}
&x(t_{n+1}) - \left[x(t_n) + \varphi(h, x(t_n))\left\{(1-\omega)f(x(t_n)) + \omega f\left(x(t_n) + \frac{1}{2\omega}f(x(t_n))\varphi(h, x(t_n))\right)\right\}\right] \\
&= \mathcal{O}(h^3),
\end{aligned}
$$

which proves the second order accuracy of Method (3).

Next, applying Method (3) to the linearized version of Equation (1) in a neighborhood of an equilibrium point x^*, i.e., for $f(x) = f_x(x^*)x$, yields

$$x_{n+1} = x_n + \varphi(h, x_n)\left\{(1-\omega)f_x(x^*)x_n + \omega f_x(x^*)\left(x_n + \frac{1}{2\omega}f_x(x^*)x_n\varphi(h, x_n)\right)\right\}.$$

For clarity of presentation, we present the analysis of the method only for one of the most popular choices of parameter $\omega = \frac{1}{2}$, which represents a modified nonstandard version of the classical Heun's method (Quarteroni et al. (2007)). For other values of ω the proof follows similarly. After simplifying the above equation, we get

$$x_{n+1} = \left[1 + \varphi(h, x_n)f_x(x^*) + \frac{\varphi^2(h, x_n)f_x^2(x^*)}{2}\right]x_n.$$

Therefore, to show that the modified nonstandard ERK2 method (3), for $\omega = \frac{1}{2}$, is elementary stable, we need to prove that

$$\left|1 + \varphi(h, x_n)f_x(x^*) + \frac{\varphi^2(h, x_n)f_x^2(x^*)}{2}\right| < 1,$$

if and only if x^* is a stable equilibrium point.

Let x^* be a stable fixed point. Then, $|f_x(x^*)| = -f_x(x^*)$. Therefore, from condition (ii), we have

$$0 < \varphi(h, x_n) < -\frac{2}{f_x(x^*)}.$$

This implies

$$-2 < f_x(x^*)\varphi(h, x_n) < 0. \tag{4}$$

Therefore, we get

$$\frac{\varphi(h, x_n)(-f_x(x^*))}{2} < 1,$$

which yields

$$1 + \frac{\varphi(h, x_n)(f_x(x^*))}{2} > 0. \tag{5}$$

Then, by multiplying Equation (4) with $1 + \frac{\varphi(h, x_n)(f_x(x^*))}{2} > 0$, we get

$$-2\left(1 + \frac{\varphi(h, x_n)(f_x(x^*))}{2}\right) < \varphi(h, x_n)f_x(x^*)\left(1 + \frac{\varphi(h, x_n)(f_x(x^*))}{2}\right) < 0.$$

Since $-\varphi(h, x_n)(f_x(x^*)) > 0$, then $-2 < -2 - \varphi(h, x_n)(f_x(x^*))$. Using this fact in the above equation, we now have

$$-2 < \varphi(h, x_n)f_x(x^*)\left(1 + \frac{\varphi(h, x_n)(f_x(x^*))}{2}\right) < 0.$$

By adding 1 throughout, we then get

$$-1 < 1 + \varphi(h, x_n)f_x(x^*) + \frac{\varphi^2(h, x_n)(f_x^2(x^*))}{2} < 1.$$

Finally, we obtain

$$\left|1 + \varphi(h, x_n)f_x(x^*) + \frac{\varphi^2(h, x_n)f_x^2(x^*)}{2}\right| < 1.$$

Now, if x^* is an unstable fixed point, then $|f_x(x^*)| = f_x(x^*)$. Therefore, we get

$$\left|1 + \varphi(h, x_n)f_x(x^*) + \frac{\varphi^2(h, x_n)f_x^2(x^*)}{2}\right| > 1,$$

since $\varphi(h, x_n)f_x(x^*) > 0$. ■

The next lemma provides a more concrete way for constructing the nonstandard denominator function φ under particular circumstances.

Lemma 2.1.

Let $\phi : \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

(1) $\phi(h) = h + \mathcal{O}(h^3)$,
(2) $0 < \phi(h) < 1$ for all $h > 0$.

Construct the denominator function as $\varphi(h, x_n) = \dfrac{\phi(hq)}{q}$, where the parameter q is chosen as follows:

(1) Case 1: If the hyperbolic equilibrium points are easily computable, then choose $q > \dfrac{|f_x(x^*)|}{2}$, for all hyperbolic equilibria x^* of Equation (3).

(2) Case 2: If the right hand side function is such that the hyperbolic equilibria are tedious to calculate, then choose a variable $q = |f_x(x_n)|$ at each time step, which yields a variable denominator function $\varphi(h, x_n)$.

With this choice of the denominator function φ, Method (3) is elementary stable and of second-order accuracy.

Proof:

To prove the above lemma, it is sufficient to show that $\varphi(h, x_n)$ satisfies properties (i) and (ii) of Theorem 2.1. Since $\varphi(h, x_n) = \dfrac{\phi(hq)}{q} = h + \mathcal{O}(h^3)$, this shows that condition (i) of Theorem 2.1 holds. Next, we prove that condition (ii) of Theorem 2.1 is also satisfied.

(1) Case 1: Since $0 < \phi(h) < 1$, we have $0 < \dfrac{\phi(qh)}{q} < \dfrac{1}{q} < \dfrac{2}{|f_x(x^*)|}$.
(2) Case 2: Since f is continuously differentiable, f_x is continuous, and, hence, $|f_x|$ is also continuous. Let $\epsilon = \dfrac{|f_x(x^*)|}{2}$, where x^* is a hyperbolic equilibrium. By continuity of $|f_x|$, there exists a $\delta(x^*)$ such that, if $|x_n - x^*| < \delta(x^*)$, then

$$||f_x(x_n)| - |f_x(x^*)|| < \epsilon = \frac{|f_x(x^*)|}{2},$$

This gives

$$-\frac{|f_x(x^*)|}{2} < |f_x(x_n)| - |f_x(x^*)| < \frac{|f_x(x^*)|}{2}. \tag{6}$$

Then, using the left inequality in (6), we get

$$\frac{|f_x(x^*)|}{2} < |f_x(x_n)| = q. \tag{7}$$

Thus, we have

$$\frac{1}{q} < \frac{2}{|f_x(x^*)|}.$$

Note that Equation (7) implies, if $|x_n - x^*| < \delta(x^*)$, then $f_x(x_n) \neq 0$ whenever $f_x(x^*) \neq 0$. Since $0 < \phi(h) < 1$, we have

$$0 < \frac{\phi(hq)}{q} < \frac{1}{q},$$

and, therefore,

$$0 < \varphi(h, x_n) < \frac{1}{q} < \frac{2}{|f_x(x^*)|}.$$

■

Remark 2.1.

For example, the function $\phi(h) = \tanh(h)$ satisfies the conditions of Lemma 2.1 and can be used to construct a denominator function $\varphi(h, x_n)$ that ensures the second-order accuracy and the elementary stability of Method (3).

2.2. Second-order modified nonstandard theta methods

The following result holds for the second-order modified nonstandard one-stage theta method.

Theorem 2.2.

Let $f \in C^2(\mathbb{R}; \mathbb{R})$ and $\varphi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ satisfy the following conditions:

(i) $\varphi(h, x) = h + (1 - 2\theta) f_x(x) \dfrac{h^2}{2} + \mathcal{O}(h^3)$.

(ii) $0 < \varphi(h, x) < \dfrac{2}{|2\theta - 1||f_x(x^*)|}$, $0 \leq \theta \leq 1$, $\theta \neq \frac{1}{2}$, for all hyperbolic equilibria x^* of (1) with $h > 0$ and $x \in \mathbb{R}$.

Then, the modified nonstandard one-stage theta method:

$$\frac{x_{n+1} - x_n}{\varphi(h, x_n)} = f\left(\theta x_{n+1} + (1 - \theta) x_n\right), \tag{8}$$

for approximating the solution of Equation (1) is both elementary stable and of second-order accuracy.

Proof:

First we consider the modified nonstandard one-stage theta method (8). We use a Taylor series expansion about t_n to obtain

$$\begin{aligned}
&x(t_{n+1}) - [x(t_n) + \varphi(h, x(t_n)) f\left(\theta x(t_{n+1}) + (1-\theta)x(t_n)\right)] \\
&= \left[x(t_n) + hx'(t_n) + \frac{h^2}{2}x''(t_n) + \mathcal{O}(h^3)\right] \\
&\quad - \left[x(t_n) + \varphi(h, x(t_n)) f\left(\theta x(t_n) + \theta h x'(t_n) + \theta\frac{h^2}{2}x''(t_n) + \mathcal{O}(h^3) + x(t_n) - \theta x(t_n)\right)\right] \\
&= hx'(t_n) + \frac{h^2}{2}x''(t_n) - \varphi(h, x(t_n)) f\left(x(t_n) + \theta h x'(t_n) + \theta\frac{h^2}{2}x''(t_n) + \mathcal{O}(h^3)\right) + \mathcal{O}(h^3).
\end{aligned}$$

Introducing the notation $\bar{h} = \theta h x'(t_n) + \theta\dfrac{h^2}{2}x''(t_n) + \mathcal{O}(h^3)$ yields

$$\begin{aligned}
&x(t_{n+1}) - [x(t_n) + \varphi(h, x(t_n)) f\left(\theta x(t_{n+1}) + (1-\theta)x(t_n)\right)] \\
&= hx'(t_n) + \frac{h^2}{2}x''(t_n) - \varphi(h, x(t_n)) f\left(x(t_n) + \bar{h}\right) + \mathcal{O}(h^3) \\
&= hx'(t_n) + \frac{h^2}{2}x''(t_n) - \varphi(h, x(t_n)) \left(f(x(t_n)) + \bar{h} f_x(x(t_n) + \mathcal{O}(h^2)\right) + \mathcal{O}(h^3) \\
&= hx'(t_n) + \frac{h^2}{2}x''(t_n) - \left[h + (1-2\theta) f_x(x(t_n))\frac{h^2}{2}\right] \left(f(x(t_n)) + \bar{h} f_x(x(t_n) + \mathcal{O}(h^2)\right) + \mathcal{O}(h^3) \\
&= \mathcal{O}(h^3).
\end{aligned}$$

Therefore, the numerical method (8) is of second-order accuracy.

Next, we show that the numerical method (8) is elementary stable. This is equivalent to showing

$$\left|\frac{1 + (1-\theta) f_x(x^*)\varphi(h, x_n)}{1 - \theta f_x(x^*)\varphi(h, x_n)}\right| < 1,$$

if and only if x^* is a stable hyperbolic equilibrium.

First, if x^* is a stable equilibrium point, then $f_x(x^*) < 0$. We now consider two cases:

(1) If $0 \leq \theta < 1/2$, we have $|2\theta - 1| = 1 - 2\theta$. From condition (ii), we then obtain

$$0 < \varphi(h, x_n) < -\frac{2}{(1-2\theta) f_x(x^*)}.$$

This implies

$$-2 < (1-2\theta) f_x(x^*)\varphi(h, x_n) < 0,$$

which then gives us

$$-2 + 2\theta f_x(x^*)\varphi(h, x_n) < f_x(x^*)\varphi(h, x_n) < 2\theta f_x(x^*)\varphi(h, x_n). \tag{9}$$

Since $f_x(x^*) < 0$, we have that $1 - \theta f_x(x^*)\varphi(h, x_n) > 1$. Thus, by dividing (9) by $1 - \theta f_x(x^*)\varphi(h, x_n)$ and adding 1 throughout, we get

$$-1 < 1 + \frac{f_x(x^*)\varphi(h, x_n)}{1 - \theta f_x(x^*)\varphi(h, x_n)} < \frac{1 + \theta f_x(x^*)\varphi(h, x_n)}{1 - \theta f_x(x^*)\varphi(h, x_n)} < 1, \quad \text{since } f_x(x^*) < 0.$$

This now implies that

$$-1 < \frac{1 + (1-\theta) f_x(x^*)\varphi(h, x_n)}{1 - \theta f_x(x^*)\varphi(h, x_n)} < 1. \tag{10}$$

(2) If $1/2 < \theta \leq 1$, we have $|2\theta - 1| = 2\theta - 1$. From condition (ii), we then obtain,

$$0 < \varphi(h, x_n) < -\frac{2}{(2\theta - 1) f_x(x^*)}.$$

This implies

$$-\frac{2}{(2\theta - 1)} < f_x(x^*)\varphi(h, x_n) < 0, \text{ since } f_x(x^*) < 0.$$

After multiplying by $(1-\theta)$ and adding 1 throughout, we then obtain

$$1 - \frac{2(1-\theta)}{|2\theta - 1|} < 1 + (1-\theta) f_x(x^*)\varphi(h, x_n) < 1. \tag{11}$$

Now, we have

$$1 - \frac{2(1-\theta)}{2\theta - 1} = \frac{4\theta - 3}{2\theta - 1}.$$

Since $\theta > 1/2$, we have $4\theta - 3 > -1$. Additionally, we have $2\theta - 1 < 1$. Thus,

$$-1 < \frac{4\theta - 3}{2\theta - 1}.$$

From (11), we have

$$-1 < 1 + (1-\theta) f_x(x^*)\varphi(h, x_n) < 1.$$

Again, since $f_x(x^*) < 0$, we have $1 - \theta f_x(x^*)\varphi(h, x_n) > 1$, which implies

$$-1 < \frac{1 + (1-\theta) f_x(x^*)\varphi(h, x_n)}{1 - \theta f_x(x^*)\varphi(h, x_n)} < 1. \tag{12}$$

Thus, Equations (10) and (12) give us

$$\left| \frac{1 + (1-\theta) f_x(x^*)\varphi(h, x_n)}{1 - \theta f_x(x^*)\varphi(h, x_n)} \right| < 1, \ 0 \leq \theta \leq 1,$$

whenever x^* is a stable equilibrium point.

Next, if x^* is an unstable equilibrium point, then $f_x(x^*) > 0$. This implies

$$1 + (1-\theta) f_x(x^*)\varphi(h, x_n) > 1 - \theta\varphi(h, x_n) f_x(x^*). \tag{13}$$

We now again consider two cases:

(1) If $0 \leq \theta < 1/2$, then $1 - 2\theta > 0$. This implies

$$2 + (1 - 2\theta) f_x(x^*)\varphi(h, x_n) > 0.$$

Thus,

$$1 + (1-\theta) f_x(x^*)\varphi(h, x_n) > \theta\varphi(h, x_n) f_x(x^*) - 1. \tag{14}$$

Equations (13) and (14) imply
$$\left|\frac{1+(1-\theta)f_x(x^*)\varphi(h,x_n)}{1-\theta f_x(x^*)\varphi(h,x_n)}\right| > 1. \tag{15}$$

(2) If $1/2 < \theta \le 1$, then $2\theta - 1 > 0$. From condition (ii), we then obtain
$$0 < \varphi(h,x_n) < \frac{2}{(2\theta-1)f_x(x^*)}.$$
This implies
$$2 + (1-2\theta)f_x(x^*)\varphi(h,x_n) > 0.$$
Therefore, using similar arguments as in the previous case, we obtain
$$\left|\frac{1+(1-\theta)f_x(x^*)\varphi(h,x_n)}{1-\theta f_x(x^*)\varphi(h,x_n)}\right| > 1. \tag{16}$$

Thus, from Equations (15) and (16), we now have
$$\left|\frac{1+(1-\theta)f_x(x^*)\varphi(h,x_n)}{1-\theta f_x(x^*)\varphi(h,x_n)}\right| > 1,\ 0 \le \theta \le 1,$$
whenever x^* is an unstable equilibrium point. ■

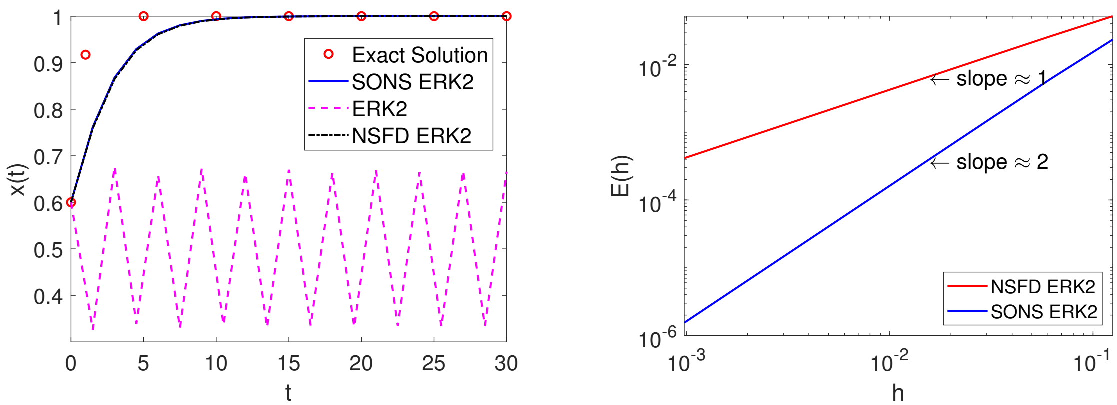
Remark 2.2.

Under the conditions of Theorem 2.2, the modified nonstandard two-stage theta method, considered in Kojouharov et al. (2021) and given as
$$\frac{x_{n+1}-x_n}{\varphi(h,x_n)} = \theta f(x_{n+1}) + (1-\theta)f(x_n), \tag{17}$$
is also both elementary stable and second-order accurate.

3. Numerical Simulations

To illustrate the advantages of the proposed new modified NSFD methods, we first consider the following logistic growth equation:
$$\frac{dx}{dt} = ax\left(1-\frac{x}{K}\right), \tag{18}$$
where $a > 0$ is the intrinsic growth rate constant and $K > 0$ is the carrying capacity of the environment. Here, $x^* = 0$ is an unstable equilibrium and $x^* = 1$ is a stable equilibrium of Equation (18).

We present a set of numerical simulations for $a = 2$ and $K = 1$. The new second-order modified nonstandard explicit Runge-Kutta (SONS ERK2) method (3) with $\omega = \frac{1}{2}$, is numerically compared to the standard ERK2 method and the NSFD ERK2 method (Anguelov and Lubuma (2001); Dimitrov and Kojouharov (2005b)). We use the nonstandard denominator function
$$\varphi(h,x) = \frac{\tanh(qh)}{q},\ \text{with } q = 2.5 > \frac{\max\{|f_x(0)|, |f_x(1)|\}}{2} = 1.$$



(a) Comparison of SONS ERK2, NSFD ERK2, and ERK2

(b) Error plot: SONS ERK2 vs NSFD ERK2

Figure 1. Numerical solutions of Equation (18) with $q = 2.5,\ x_0 = 0.6$, and using $h = 1.5$ in (a)

The SONS ERK2 method is of second-order accuracy and elementary stable, while the ERK2 method is also second-order accurate but unstable for $h > \dfrac{2}{a} = 1$, and the NSFD ERK2 method is elementary stable but only first-order accurate. Accordingly, for $h = 1.5$, we see in Figure 1(a) that the ERK2 method does not converge to the exact solution whereas both the SONS ERK2 and NSFD ERK2 methods correctly mimic the behavior of the exact solution.

To better visualize the second-order accuracy of the new SONS ERK2 method (3), we denote the numerical solution for a given mesh size h as x^h. Let us define the l^∞ error as

$$E(h) = \|x^h - x\|_\infty,$$

where

$$\|y\|_\infty = \max_{k=0,\cdots,N_t} |y_k|,$$

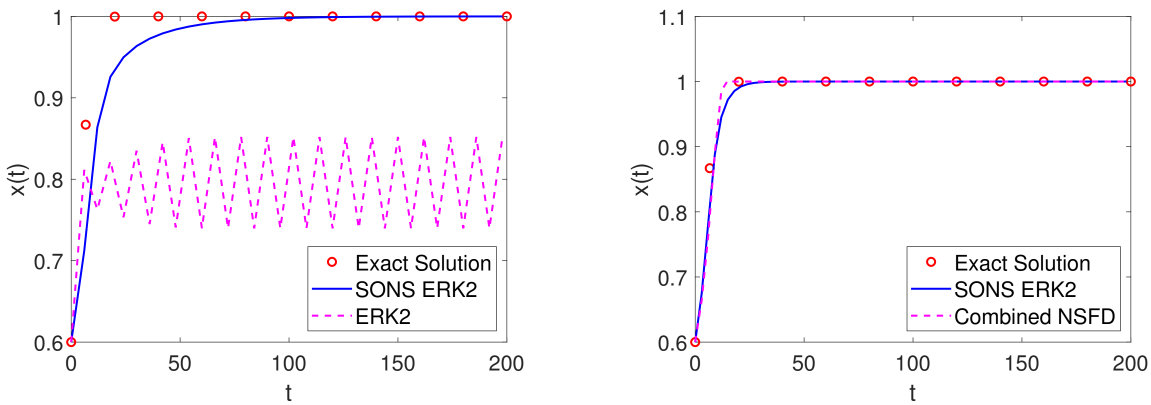
represents the discrete l_∞ norm of the vector y, and x represents the exact solution of Equation (1). Figure 1(b) shows the error plot for NSFD ERK2 and SONS ERK2 methods, where the slopes of the error lines are 1 and 2, respectively. This numerically verifies that the SONS ERK2 method is second-order accurate while the NSFD ERK2 method is only first-order accurate.

As a second example, we consider the following differential equation which is a modification of the predator pit model in population ecology (Yeargers et al. (1996), p. 115):

$$\frac{dx}{dt} = -\left(x - b + \frac{1}{2}\right)(x - b)\left(x - b - \frac{1}{2}\right). \tag{19}$$

Equation (19) has $x^* = b$ as an unstable equilibrium while $x^* = b \pm \frac{1}{2}$ are both stable equilibria, with $\max\{|f_x(x^*)|\} = \frac{1}{2}$. To support the results of Theorem 2.1, we perform numerical simulations using the nonstandard denominator function

$$\varphi(h, x) = \frac{\tanh(qh)}{q}, \text{ with } q = 0.26 > \frac{\max\{|f_x(x^*)|\}}{2} = 0.25.$$



(a) Comparison of SONS ERK2 and ERK2

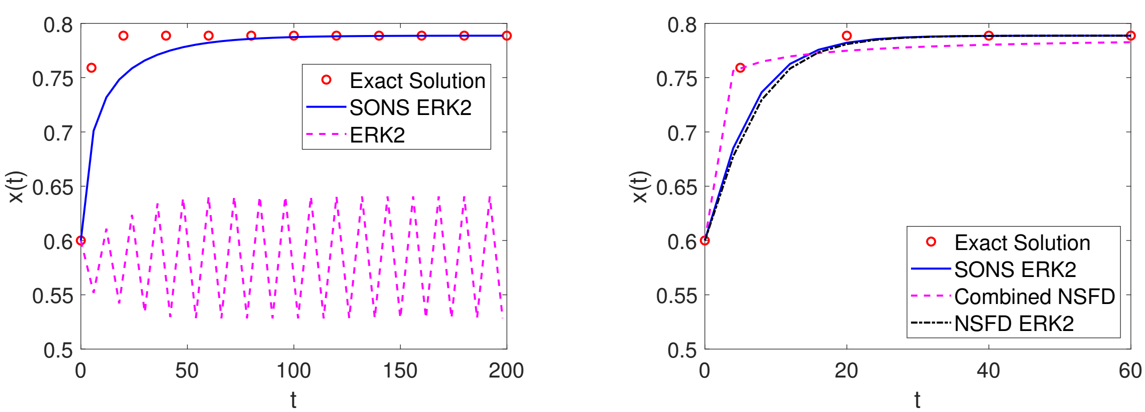
(b) Comparison of SONS ERK2 and Combined NSFD

Figure 2. Numerical solutions of Equation (19) for $b = \frac{1}{2}$, with $q = 0.26, x_0 = 0.6$, and using $h = 6$ in (a) and $h = 3$ in (b)

First, we consider $b = \frac{1}{2}$, which results in the right-hand side function

$$f(x) = -x^3 + \frac{3}{2}x^2 - \frac{1}{2}x.$$

Figure 2(a) shows a comparison of the SONS ERK2 method with the standard ERK2 method, for $h = 6$ and initial condition $x_0 = 0.6$. Simulations show that the ERK2 method does not converge to the exact solution for large values of h, while the SONS ERK2 method preserves the local stability properties of the equilibrium $x^* = 1$ for any value of the step-size h. Figure 2(b) shows a comparison of the SONS ERK2 method with the combined NSFD method (Chen et al. (2006)) for $h = 3$. The two numerical methods reproduce the correct behavior of the exact solution as they are both of second order accuracy and elementary stable; however, the combined NSFD method is implicit in nature and, therefore, not as computationally easy to implement.



(a) Comparison of SONS ERK2 and ERK2

(b) SONS ERK2, Combined NSFD, and NSFD ERK2

Figure 3. Numerical solutions of Equation (19) for $b = \frac{1}{2\sqrt{3}}$, with $x_0 = 0.6$, and using $h = 6, q = 0.26$ in (a) and $h = 4, q = 0.45$ in (b)

Next, we consider $b = \frac{1}{2\sqrt{3}}$, that results in the right-hand side function

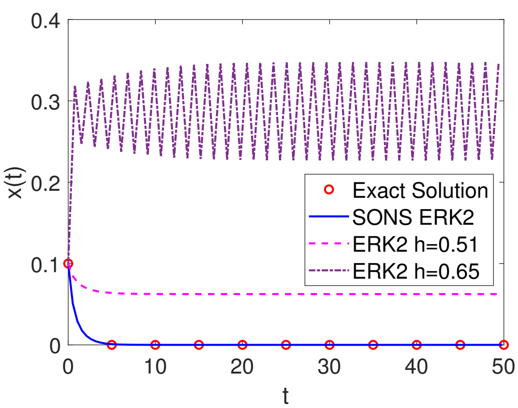
$$f(x) = -x^3 + \frac{\sqrt{3}}{2}x^2 - \frac{1}{12\sqrt{3}}.$$

A similar set of numerical comparisons was performed as in the case with $b = \frac{1}{2}$ and the same results were obtained, as shown in Figure 3. In this case, the combined NSFD method, which is of second-order accuracy and elementary stable, does not require a nonstandard denominator function, since the right-hand side function $f(x)$ does not contain a first term (Chen et al. (2006)). However, it is again an implicit method, and therefore still not as computationally easy to implement as the explicit SONS ERK2 method.

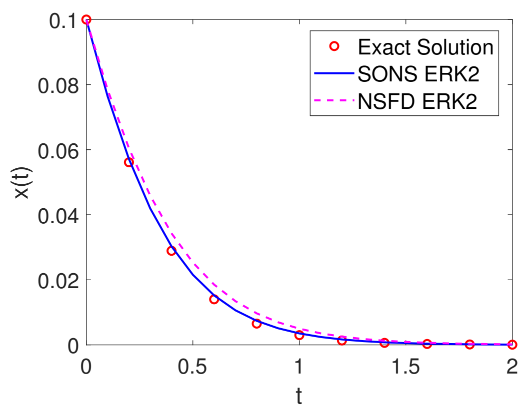
In the third example, we consider the Michaelis-Menten model (Allen (2007)), where the rate of change of the nutrient concentration $x(t)$ used by a cell for growth and development is modeled by the following differential equation:

$$\frac{dx}{dt} = -\frac{k_{max}x}{k_n + x}. \tag{20}$$

Here, the parameter $k_{max} > 0$ is the maximum rate of uptake by the cell of the nutrient and $k_n > 0$ is the half-saturation constant. Given that $f(x) = -\frac{k_{max}x}{k_n + x}$, yields $f'(0) = -k_{\max}/k_n < 0$ and, therefore, $x^* = 0$ is a stable equilibrium of Equation (20). In the numerical simulations, we take $k_n = 0.2, k_{\max} = 0.8$, with an initial condition $x(0) = 0.1$, and $q = 0.25$ for the comparison of our method with the NSFD ERK2 method.



(a) Comparison of SONS ERK2 and ERK2



(b) SONS ERK2 and NSFD ERK2 for $q = 0.25, h = 0.1$

Figure 4. Numerical solutions of Equation (20) with $x_0 = 0.1$

Figure 4(a) shows a comparison of the SONS ERK2 method with the ERK2 method, where we see that the ERK2 method introduces artificial equilibria for $h = 0.51$ and becomes unstable when $h = 0.65$, while the SONS ERK2 method behaves very well for arbitrary large values of h. Figure 4(b) shows a comparison of our method with the nonstandard ERK2 method which is elementary stable but only of first-order accuracy, and therefore, the numerical solution of the SONS ERK2 method converges faster to the stable equilibrium $x^* = 0$.

Remark 3.1.

The modified nonstandard theta methods (8) and (17) with $\theta = 0$ reduce to the same second-order nonstandard explicit Euler (SONSEE) method (Gupta et al. (2020)), which is presented, analyzed, and numerically investigated in Gupta et al. (2020) and Kojouharov et al. (2021).

4. Conclusion

In this paper, second-order modified nonstandard Runge-Kutta and theta methods for one-dimensional autonomous differential equations were constructed and analyzed. The new numerical methods were developed based on modifications of the nonstandard denominator functions used in NSFD methods. The methods were shown to be of second order accuracy, which is an improvement in the accuracy of their NSFD counterparts, while preserving their stability properties. Using a set of numerical simulations, the two-stage modified nonstandard explicit Runge-Kutta methods were compared to the NSFD ERK2 method, the standard ERK2 method, and the combined NSFD method, which verified the theoretical results and demonstrated the strengths of the proposed new numerical methods.

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