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The lengthening pendulum: Adiabatic invariance and bursting solutions

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ABSTRACT

The adiabatic invariance of the action variable of a length varying pendulum is investigated in terms of the two different time scales that are associated with the problem. A length having a general polynomial variation in time is studied; an analytical solution for a pendulum with length which varies quadratically in time is obtained in the small angle approximation. We find that for length with quadratic time variation, the action neither converges (as it does for linear time variation), nor diverges (as it does for exponential time variation), but rather shows oscillatory behaviour with constant amplitude. It is then shown that for a pendulum length which has a combination of periodic and linear time variations, the action is no longer an adiabatic invariant and shows jumps with time. In the case in which the length varies sinusoidally in time, we demonstrate that the full nonlinear system exhibits bursting oscillations.

1. Introduction

The concept of adiabatic invariance [1,2] finds applications in many fields and the underlying ideas such as averaging methods were used even as long ago as by Lagrange, Laplace and Gauss in their studies of planetary motion [3,4]. It is a useful tool in studying the evolution of stellar systems [5], in plasma physics [6], in the design of accelerators [7], in numerical weather prediction [8], etc. Adiabatic invariants are quantities which are approximately conserved in a dynamical system having slow variation of parameter(s) [9–12] and these approximate constants of motion manifest when there are at least two widely separated time scales. In Hamiltonian mechanics, the action-angle variables are useful in determining the adiabatic invariants of a dynamical system.

A well-known mechanical system where adiabatic invariance has been discussed since long is the Lorentz pendulum in which the length of the pendulum is varied very slowly. In a recent paper by Sanchez-Soto and Zoido [13], this problem was studied in the small oscillation limit for two cases – when the length of the pendulum was varied uniformly, and when it was varied exponentially in time. The authors discussed adiabatic invariance using action-angle variables and the manner in which the long time behavior of the action variable I determines whether the system remains adiabatically invariant throughout the evolution. They showed that the condition of adiabatic change implied the validity of the asymptotic approximation of the Bessel functions and vice versa. They concluded that when the length varies uniformly in time, the action variable converges to a constant value (given by the initial value) for large enough times, while for length varying exponentially in time, the action variable diverges and only stays seemingly constant for a short duration, that too only if the initial pendulum length is sufficiently small, or in other words if the initial natural frequency of oscillation of the autonomous pendulum was sufficiently large.

In our present work we study the problem of adiabatic invariance of the action for a pendulum with length which has a polynomial time variation. A simple analytical solution is determined for a pendulum with length varying quadratically in time – the action is oscillatory. In the case of a pendulum having a combination of linear and periodic variations in time, we find that the action does not remain adiabatically invariant and shows jumps with time. We show that when the pendulum length is varied sinusoidally, the system exhibits bursting oscillations.

We begin by first revisiting the question of adiabatic invariance of the action variable of a pendulum with uniform length variation. We however approach the problem differently, using the method of multiple scales. This enables us to gain some further physical insights of the dynamical system. The idea of using the time scales as an alternative understanding of adiabatic invariance is motivated from the proof of adiabatic invariance of action of bounded time-varying Hamiltonian

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systems [14] (see also [15–17].

2. The uniformly varying pendulum revisited

The equation governing the oscillation of a pendulum with a bob of unit mass, and whose length l varies in time t is given by

$$\frac{d^2\theta}{dt^2}+\frac{2}{l}\frac{dl}{dt}\frac{d\theta}{dt}+\frac{g}{l}\sin\theta=0 \tag{1}$$

As in Ref. [13] we consider the uniform length variation described by

$$l(t)=l_0(1+\varepsilon t) \tag{2}$$

where l_0 is the initial length and ε is a small parameter. Replacing t with the dimensionless variable $\tau=\varepsilon t$, the resulting equation of motion for the pendulum for small oscillations, $\sin\theta\approx\theta$ reads

$$(1+\tau)\ddot{\theta}+2\dot{\theta}+\omega_0^2\theta=0 \tag{3}$$

where $\omega_0=\frac{1}{\varepsilon}\sqrt{\frac{g}{l_0}}$ and the dots denote differentiation with respect to τ.

As in ref.([13]), one might straightaway identify the two time scales of this pendulum with the time period of oscillation of the pendulum, proportional to $\frac{\theta_{max}}{\theta'_{max}}\approx\frac{(1+\tau)^{1/2}}{\omega_0}$, and the extension time of the pendulum given by $\frac{l}{l'}\approx(1+\tau)$, the primes denoting differentiation with respect to τ. Here θ'_{max} denotes the maximum of the angular velocity.

We observe that introducing a variable $z=\sqrt{1+\tau}$ and defining $\varphi=\varphi(z)$ such that $\varphi(z)=\theta(\tau)$, we can rewrite eqn. (3) as

$$z\varphi''+3\varphi'+4\omega_0^2z\varphi=0 \tag{4}$$

where the primes denote differentiation with respect to z. This enables us to introduce two different timescales which are present in the system by defining the (independent) variables $T=z$ and $u=\omega_0 z$. Since ω_0 varies as $1/\varepsilon$, it is generally large and so u represents the fast timescale while T represents the slow timescale. We demand that φ is a function of both T and u, i.e., $\varphi=\varphi(T,u)$.

We expand the solution of eqn. (4) as a series in ω_0^{-1}

$$\varphi(T,u)=\varphi_0(T,u)+\frac{\varphi_1(T,u)}{\omega_0}+\frac{\varphi_2(T,u)}{\omega_0^2}+\dots \tag{5}$$

We assume that φ_1/ω_0 and higher terms are small and use the chain rule for the derivatives in eqn. (4):

$$\frac{d\varphi}{dz}=\omega_0\frac{\partial\varphi}{\partial u}+\frac{\partial\varphi}{\partial T} \tag{6}$$

Then writing

$$\sin\varphi=\sin\varphi_0+\frac{\cos\varphi_0\cdot\varphi_1}{\omega_0}+\frac{\cos\varphi_0\cdot\varphi_2}{\omega_0^2}+\dots \tag{7}$$

We equate like powers of ω_0 in the resulting equation and obtain the leading behaviour of φ, which is contained in φ_0 by equating terms of order ω_0^2. This gives

$$\frac{\partial^2\varphi_0}{\partial u^2}+4\sin\varphi_0=0 \tag{8}$$

which in the small angle approximation becomes

$$\frac{\partial^2\varphi_0}{\partial u^2}+4\varphi_0=0 \tag{9}$$

Assuming that T remains constant over variations of u which are of order unity, this equation can be easily solved to give $\varphi_0=A(T)\cos 2u+B(T)\sin 2u$. The form of the T dependence of A and B may be found by equating terms with coefficient ω_0^1 in the differential equation. We obtain:

$$-T\left(\frac{\partial^2\varphi_1}{\partial u^2}+4\varphi_1\cos\varphi_0\right)=2\left(T\frac{\partial^2\varphi_0}{\partial T\partial u}+\frac{3}{2}\frac{\partial\varphi_0}{\partial u}\right) \quad . \tag{10}$$

This equation reduces for the small angle approximation to

$$\begin{aligned}-T\left(\frac{\partial^2\varphi_1}{\partial u^2}+4\varphi_1\right)&=2\left(T\frac{\partial^2\varphi_0}{\partial T\partial u}+\frac{3}{2}\frac{\partial\varphi_0}{\partial u}\right)\\&=\frac{2}{\sqrt{T}}\frac{\partial}{\partial T}\left(T^{3/2}\frac{\partial\varphi_0}{\partial u}\right)\end{aligned} \tag{11}$$

We set the right hand side to zero since otherwise, considering T constant as before and solving for u, the right hand side will produce terms proportional to $u\sin u$ and $u\cos u$ in the solution for φ_1. These secular terms will cause φ to blow up as $u\to\infty$ and render the solution absurd. To avoid this situation we set the right hand side to zero. This in turn implies that

$$\begin{aligned}0&=\frac{\partial}{\partial T}\left(T^{3/2}\frac{\partial\varphi_0}{\partial u}\right)\\&=\frac{\partial}{\partial T}\left(2T^{3/2}(-A(T)\sin 2u+B(T)\cos 2u)\right)\end{aligned} \tag{12}$$

For this to hold, we must have $A(T)\sim B(T)\sim T^{-3/2}$. This fixes the motion upto initial conditions. We now specify these by setting $\varphi(z=1)=\varphi_{max}$, $\varphi'(z=1)=0$. This gives

$$\varphi_0(T,u)=\frac{\varphi_{max}}{T^{3/2}}\cos 2(u-\omega_0) \tag{13}$$

or finally, after substituting for τ and θ,

$$\theta(\tau)=\frac{\theta_{max}}{(1+\tau)^{3/4}}\cos 2\omega_0(\sqrt{1+\tau}-1) \tag{14}$$

This is the same expression as has been obtained in Ref. [13] by approximating the exact solution with leading order terms in the asymptotic expansion of Bessel functions, in the adiabatic approximation. It is clear that $u=\omega_0\sqrt{1+\tau}$ is the timescale over which the pendulum oscillates, while $T=\sqrt{1+\tau}$ determines the scale at which the oscillation amplitude dies down.To discuss the temporal variation of the oscillation amplitude in the absence of the small angle approximation, we note that equation (10) is complicated by the presence of an extra $\cos\varphi_0$ term and also by the fact that we do not know φ_0 exactly. But we only want the T-dependence, and this can be achieved by requiring that the right hand side be 0 in order to prevent secular terms from emerging in the solution for φ_1. If we do so, φ_0 will have to vary as $T^{-3/2}$ and, apart from the fact that large oscillations must die out at large T, we will also have that the amplitude dependence of θ is the same whether the small-angle approximation is made or not.

It is interesting that τ itself, which is the timescale at which the length variation occurs, is not a natural timescale to describe the angular motion. This is why the change of variable from τ to $z=\sqrt{1+\tau}$ was necessary. By correctly fixing the τ-dependence, we are able to find the natural timescales of the angular variation directly. The advantage of this method is that in subsequent, more complicated calculations, it clearly separates the calculations pertaining to the oscillatory behaviour of θ (fast variation) from those related to the damping behaviour (slow variation). We can therefore make facilitative approximations with a clearer understanding of what features of the motion we can expect to accurately capture. The condition of adiabatic invariance can be defined as the case where the first time scale is much lower than the second, and hence during one time period of oscillation of the pendulum, the length will remain practically unchanged.

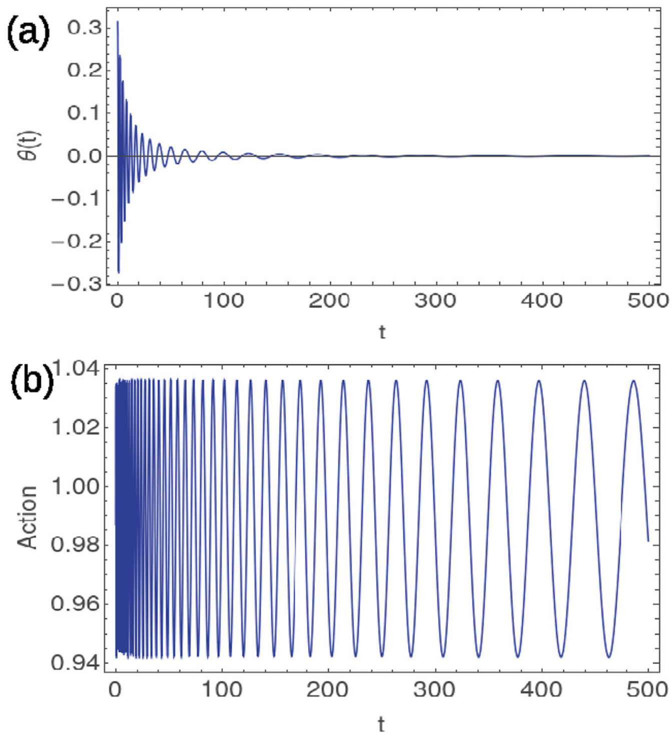

Fig. 1. (a) Time series for $\theta(t)$ and (b) Action vs time for a pendulum with quadratically varying length. Parameters: $\varepsilon = 0.1, l_0 = 1, \theta_0 = \frac{\pi}{10}, \theta^{'}(t=0) = 0$.

3. Pendulum with quadratically varying length

We now proceed with the analysis of a Lorentz pendulum which varies polynomially with time.Consider the general length variation:

$$l(t) = l_o(1+\varepsilon t)^n \ . \tag{15}$$

Defining $\omega_0 = \frac{\sqrt{\frac{g}{l_0}}}{\varepsilon}$ and $\tau = \varepsilon t$, we get the equation of motion (in terms of θ and τ) in the small angle approximation as:

$$\theta''(\tau) + \frac{2n}{1+\tau}\theta^{'}(\tau) + \frac{\omega_0^2}{(1+\tau)^n}\theta(\tau) = 0 \ . \tag{16}$$

As we show below, the $n = 2$ case is exactly analytically solvable. In this case, $\omega(t) = \frac{\omega_0}{1+\tau}$ and the equation to be solved is

$$\theta''(\tau) + \frac{4}{1+\tau}\theta^{'}(\tau) + \frac{\omega_0^2}{(1+\tau)^2}\theta(\tau) = 0 \tag{17}$$

The solution for the above equation (with the condition $\omega_0 > \frac{3}{2}$, $\theta(0) = \theta_o, \theta^{'}(0) = 0$) is

$$\theta(\tau) = \frac{\theta_o}{(1+\tau)^{3/2}}\left[\cos(\beta \ln(1+\tau)) + \frac{3}{2\beta}\sin(\beta \ln(1+\tau))\right] \tag{18}$$

where β is defined as $\beta = \sqrt{\omega_0^2 - \frac{9}{4}}$. Fig. (1(a)) depicts this solution. The angular velocity is given by

$$\theta^{'}(\tau) = \frac{-\omega_0^2\theta_0 \sin(\beta \ln(1+\tau))}{\beta(1+\tau)^{5/2}} \tag{19}$$

Hence, by the same notation as in Ref. [13], we get the following result for the maxima

$$\begin{aligned} \theta_{max}(\tau) &= \frac{\theta_0\omega_0}{\beta(1+\tau)^{3/2}} \\ \theta^{'}_{max}(\tau) &= \frac{\omega_0^2\theta_0}{\beta(1+\tau)^{5/2}} \end{aligned} \tag{20}$$

The time period of oscillation is

$$T_1 \approx \frac{1+\tau}{\omega_0} \ . \tag{21}$$

In this case $\frac{l}{l'} = \frac{1+\tau}{2}$. Thus for the quadratic variation, we get a pathological case where both the time periods vary with time in a similar way. We will see the implication of this situation in further analysis of adiabatic invariance.

3.1. Analysis of adiabatic invariance using action variable

We now analyse the adiabatic invariance of this case following the analysis in Ref. [13]. The action variable I for the pendulum with slowly varying length is given by:

$$I(t) = \frac{1}{2\pi}\oint p dq = \frac{E(t)}{\omega(t)} \tag{22}$$

where

$$E(t) = \frac{1}{2}l(l\theta'^2 + g\theta^2) \tag{23}$$

in the small angle approximation and the length $l(t)$ and $E(t)$ are assumed to be constant in one oscillation cycle. Using equations (18) and (19) in eqn. (22) and eqn. (23), we get

$$\begin{aligned} I(\tau) \;=\; & \frac{E(\tau)}{\omega(\tau)} = \frac{E_0}{\omega_0}\left[\frac{4\omega_0^2}{\varepsilon^2\beta^2}(\sin(\beta \ln(1+\tau)))^2\right. \\ & \left. + \left(\cos(\beta \ln(1+\tau)) + \frac{3}{2\beta}\sin(\beta \ln(1+\tau))\right)^2\right] \end{aligned} \tag{24}$$

Here, $E_0 = \frac{1}{2}l_0^2\omega_0^2\varepsilon^2\theta_0^2$ denotes the initial oscillation energy. From eqn. (24), it is evident that the action variable $I(\tau)$ is an oscillatory function, with oscillation amplitude

$$\frac{E_{osc}(0)}{\omega_o}\sqrt{\frac{9}{4\beta^2} + \frac{1}{4}\left(\frac{\omega_o^2}{\varepsilon^2\beta^2} + \frac{9}{4\beta^2} - 1\right)^2}$$

which is independent of τ. Hence the action does not die down and keeps oscillating for infinite time, as opposed to the case where the length increases linearly. We plot a suitably normalised action

$$\tilde{I}(t) = \sqrt{l(t)}\left(l(t)^2\theta'^2(t) + gl(t)\theta^2(t)\right)$$

. All further plots of action are for this suitably normalised action variable. In Fig. (1(b)), where the action is plotted for the pendulum with length varying quadratically in time, the action neither converges (as it did for linear length variation)(ref. [13]) nor diverges (as it did for exponential length variation (ref. [13]), and as was expected, follows an oscillatory behaviour with constant amplitude. For higher degree polynomial variation, which we explore in the next section, we expect that the action will diverge.

4. Pendulum with general polynomially varying length

For a general polynomial variation of the pendulum length, the governing equation in the small angle limit is eqn. (16). For $n < 1$ and $n > 2$, we found it hard to get analytical results and we therefore proceeded to numerically determine the behaviour of the action for different polynomial variations.It is clear from Fig. 2 that the action diverges for all polynomial cases for $n > 2$ (i.e. all variations steeper than quadratic).

5. General algorithm for determining adiabatic invariance of action

The discussion above suggests that one can proceed in a systematic way to determine if the action for a lengthening pendulum is indeed adiabatically invariant, given any general variation of length $l(t)$. The following general algorithm may be followed:

1. Numerically plot θ vs t and θ' vs t for the different cases of variation.
2. Find the temporal variation of the envelope of the oscillating functions θ and θ'. Asymptotically, that will provide us with the functional dependence of θ_{max} and θ'_{max} on time, and their ratio will give us an estimate of the functional dependence of the time period of oscillation T_1 on time.
3. Compare T_1 with the timescale associated with the length variation $T_2 = \frac{l(t)}{l'(t)}$. If asymptotically, $T_1 > T_2$, then the action will not remain adiabatically invariant with time.

6. Sinusoidally varying length, bursting behaviour

We next consider the case of the Lorentz pendulum whose length varies sinusoidally in time. This situation has analogues with the motion of a swing. Here, the distance between the point of suspension of the swing and the center of gravity of the person swinging plays the role of the length of the pendulum. The exact differential equation, under certain approximations, takes the form of the Mathieu equation.

The dynamics of a pendulum whose length varies periodically in time was studied in Ref. [18] where the parameter space of the system was explored and the transitions to chaos were investigated numerically.

Here we show that this system exhibits bursting oscillations. Then we discuss the variation of the action with time for the system.

Consider a Lorentz pendulum whose length varies sinusoidally about a mean length l_m as

$$l(t) = l_m(1+\varepsilon\cos(\Omega t)) \quad , \tag{25}$$

where ε is small.

Setting $\Omega t = \tau$, our original differential equation (1) can be rewritten as

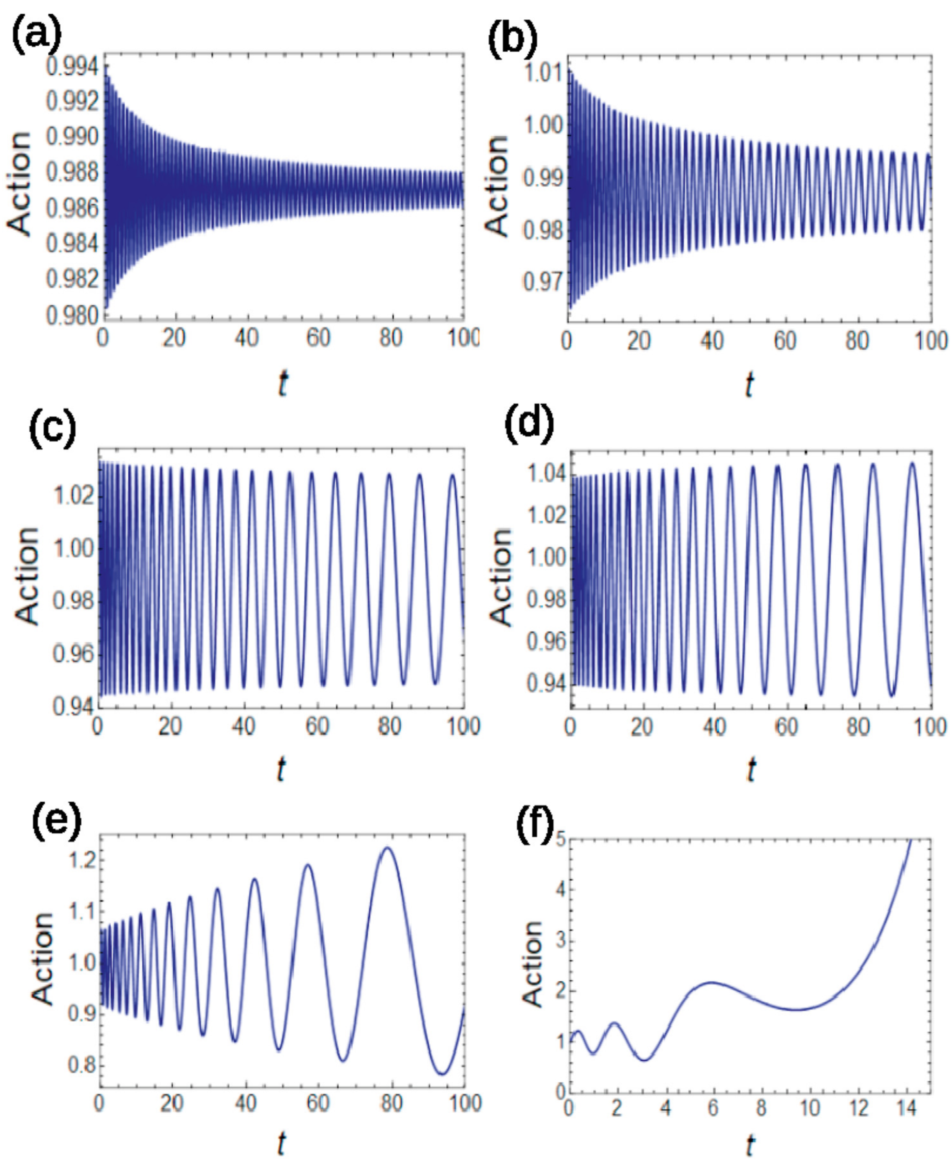

Fig. 2. Action vs time for a pendulum with a general polynomial variation of length: $l(t) = l_o(1+\varepsilon t)^n$. Parameters: $\varepsilon = 0.1, l_o = 1, \theta_o = \frac{\pi}{10}, \theta'(t=0) = 0$. (a) n=0.3, (b) n=1, (c) n=1.9, (d) n=2.1, (e) n=3, (f) n=8.

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{l}\frac{dl}{d\tau}\frac{d\theta}{d\tau} + \frac{g}{\Omega^2 l}\sin\theta = 0 \tag{26}$$

Using eqn. (25), this simplifies to:

$$\theta'' - \frac{2\varepsilon\sin\tau}{1+\varepsilon\cos\tau}\theta' + \omega^2\frac{\sin\theta}{1+\varepsilon\cos\tau} = 0\ . \tag{27}$$

which reduces in the small angle approximation $\sin\theta \approx \theta$, to

$$\theta'' - \frac{2\varepsilon\sin\tau}{1+\varepsilon\cos\tau}\theta' + \omega^2\frac{\theta}{1+\varepsilon\cos\tau} = 0\ . \tag{28}$$

where the primes denote differentiation with respect to τ, and we have defined $\omega^2 = \frac{\Omega_0^2}{\Omega^2}$, where $\Omega_0 = \sqrt{\frac{g}{l_m}}$.

We introduce a new variable

$$\chi = \theta(1+\varepsilon\cos\tau). \tag{29}$$

and rewrite eqn. (28) in terms of χ. This leads to the expression:

$$\chi'' + \frac{\omega^2+\varepsilon\cos\tau}{1+\varepsilon\cos\tau}\chi = 0\quad . \tag{30}$$

For small ε one can make the approximation:

$$\frac{1}{1+\varepsilon\cos\tau} \approx 1 - \varepsilon\cos\tau \tag{31}$$

in equation (30) and neglect $O(\varepsilon^2)$ terms to obtain

$$\chi'' + [\omega^2 - \varepsilon\cos\tau(\omega^2-1)]\chi = 0\quad . \tag{32}$$

we get

$$\ddot{\chi} + [4\omega^2 - 4\varepsilon\cos(2\alpha)(\omega^2-1)]\chi = 0\ , \tag{33}$$

where the dots now denote differentiation with respect to α. The solution to this equation is given by the Mathieu function, and θ is given by $\theta = \frac{\chi}{(1+\varepsilon\cos\tau)}$. Note that the parameter ω couples the frequency of length-oscillation of the pendulum with the frequency of oscillation of the pendulum at its mean length, i.e., $\omega^2 = \frac{\Omega_0^2}{\Omega^2}$. An interesting case arises for $\omega = 1$, when the frequency at which the length of the pendulum changes matches the frequency of oscillation of a simple pendulum of mean length l_m. The phase plot for this situation is shown in Fig. (3(b)) and 3(a) shows the corresponding time series plots.The solution to the approximate equation (28) is then found to be

$$\theta(\tau) = \frac{B\cos\tau}{1+\varepsilon\cos\tau}, \tag{34}$$

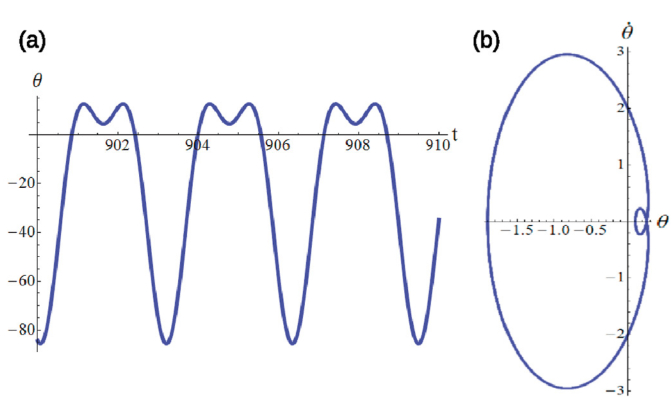

Fig. 3. (a) Time series for $\theta(t)$ and (b) phase plot ($\dot{\theta}$ vs θ) for a pendulum with sinusoidally varying length. Here, $\omega = 1$, $\varepsilon = 0.9$.

where B is determined by initial conditions for θ. For small ε,

$$\theta(\tau) \approx B(\cos\tau - \varepsilon\cos 2\tau + \varepsilon) \tag{35}$$

As noted earlier, there are two frequencies contributing to θ under the small angle approximation. When ε is small, it is largely the frequency of oscillation of the pendulum at its mean length. However, when ε is larger, we see that the oscillatory nature of the amplitude is evident from the double peaks in Fig. (3(a)). Also, the dip in θ occurs where the amplitude of oscillation of the pendulum of length l_m (the mean length of this pendulum) is maximum. Depending upon the initial conditions, note that as ε is increased, the oscillation loses symmetry about $\theta = 0$. The two peaks in the amplitudes correspond to the loop seen in the limit cycle (Fig. (3(b))). This is a feature which is seen only in the small angle approximation.

Interestingly, the full nonlinear equation (27) admits bursting oscillations for some parameters. To our knowledge, bursting solutions have not been reported earlier for the Lorentz pendulum.Bursting oscillatory activity is well known in varied systems – in mechanical systems, in Josephson junctions, in ecological population dynamics, in the Belousov-Zhabotinsky reaction, and in living systems in single and coupled neurons of different kinds and in endocrine cells. Various kinds of regular bursts exist which have been classified on the basis of the bifurcation mechanisms producing them [19–21]. Known burst mechanisms include slow oscillations regulating fast oscillatory activity in a system — distinct fast and slow time scales are simultaneously present in such systems. Some recent literature discussing bursting oscillations governed by an interplay between slow-fast dynamics include [22–24]. The method developed in Ref. [23] to study mixed mode oscillations in excited systems with rationally related low excitation frequencies was used in Ref. [24] to obtain amplitude modulated bursting.

In the system in eqn. (27), the oscillations of the damped pendulum of mean length l_m provide one (rapid) time scale while the slow variations of the lengthening pendulum provide the other (slow) time scale. The system exhibits bursting oscillations shown in Fig. (4). The dynamical mechanisms underlying these bursts would be explored in a separate future work and are beyond the scope of this present study.As we found it hard to get analytical results for the action for the periodically varying-length pendulum, this was obtained numerically and plotted in Fig. (5). It is clear that the variations of θ and the action I are modulated by the same periodic behavior as variation of length. More interestingly, the action is no longer invariant, in the sense that the variations continue for all time.

7. Pendulum with a combination of linear and periodic length variations

We consider now a variation in length of the form

$$l(t) = l_m(1 + \varepsilon_1 t + \varepsilon_2\cos(\Omega t))\quad . \tag{36}$$

The case where $\varepsilon_1 = 0$ where the variation is purely sinusoidal, was discussed in the previous Section.Now consider a variation where $\varepsilon_1 \neq 0$. This could be a model of a yoyo type pendulum with a rubber band, i.e. the pendulum lengthens and bobs during oscillation. If the length

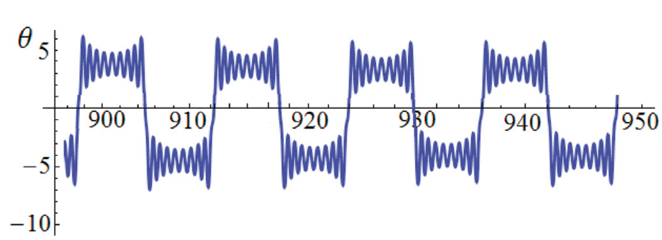

Fig. 4. Bursting oscillations in a pendulum with sinusoidal variation of length. Here, $\omega = 0.1$, $\varepsilon = 0.99$, $\theta_0 = 0.69$, $\dot{\theta}_0 = 0.02$.

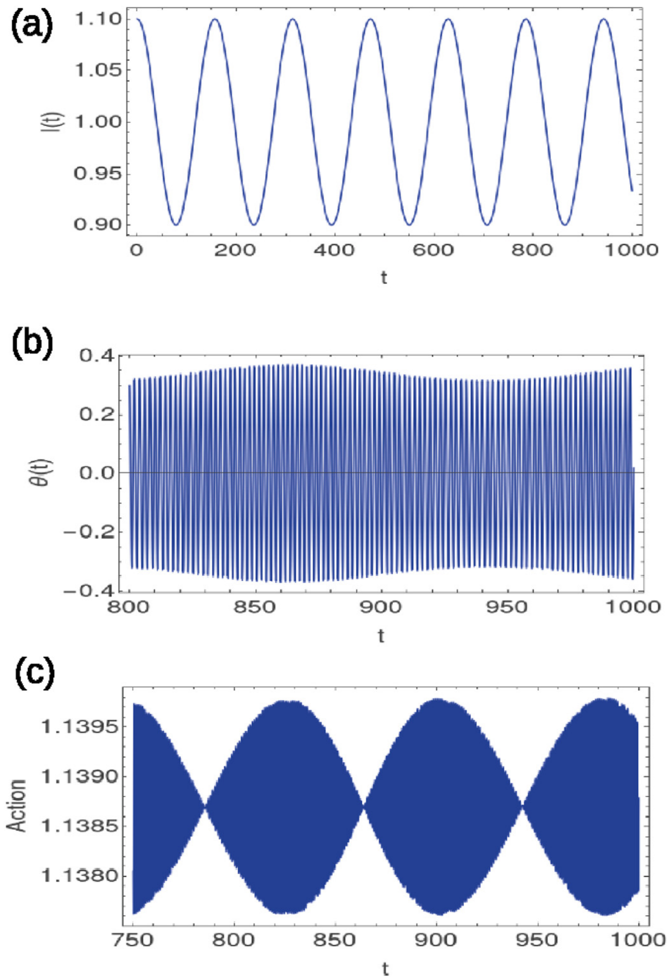

Fig. 5. Pendulum with sinusoidal length variation. (a)pendulum length $l(t)$, (b)$\theta(t)$, (c)Action $I(t)$. Parameters: $\varepsilon = 0.1, \Omega = 0.04, l_m = 1, \theta_o = \frac{\pi}{10}, \theta^{'}(t=0) = 0$.

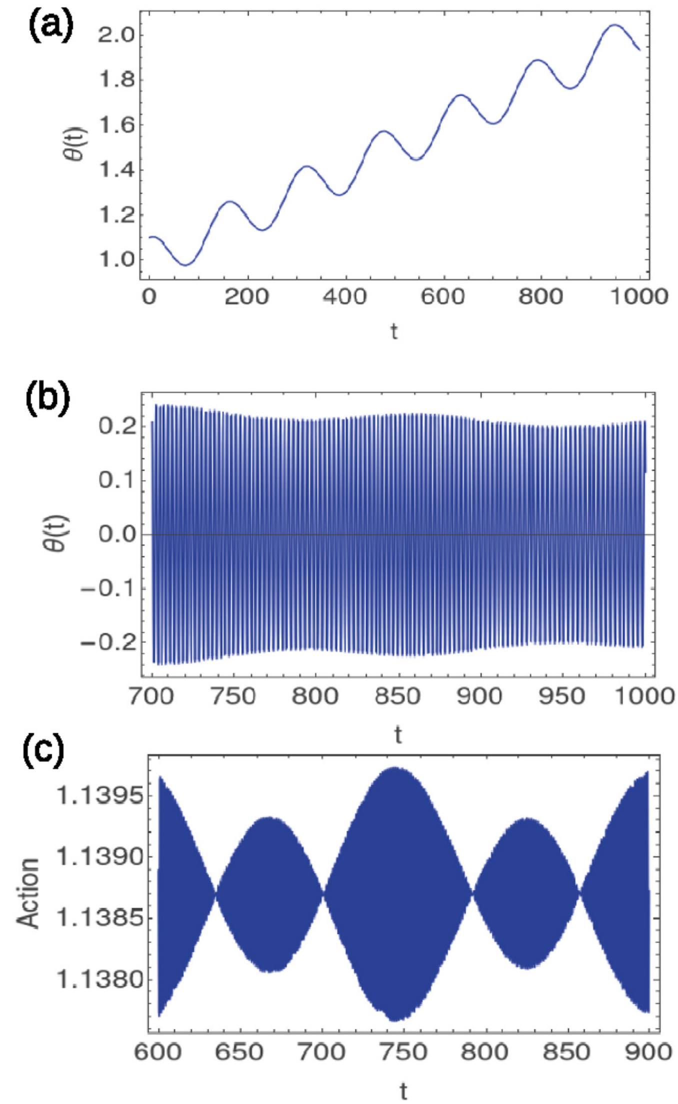

Fig. 6. Combination of linear and periodic time variation of pendulum length, with $\varepsilon_1 < \varepsilon_2$. (a)pendulum length $l(t)$, (b)$\theta(t)$, (c)Action $I(t)$. Parameters: $\varepsilon_1 = 0.001$, $\varepsilon_2 = 0.1$, $\Omega = 0.04$, $l_m = 1, \theta_o = \frac{\pi}{10}, \theta^{'}(t=0) = 0$.

variation (ε_1 term) is much slower than oscillation of the length (coming from the ε_2 term), then we expect a similar behaviour for the action as the previous case. This can be seen in Fig. (6(a, b, c)). However, if we plot the action for this case for a longer time, we get a radically very different behaviour of the action (shown in Fig. (7)).

Note that this case is essentially similar to uniform variation of length of pendulum, with a sinusoidal modulation on top of the uniform variation. For a uniformly varying pendulum, it has been shown in Ref. [13] that the action is adiabatically invariant. But with a sinusoidal modulation added to the uniform lengthening, the action doesn't remain adiabatically invariant any more; in fact it undergoes a jump. This was observed for many parameter values. One of the clearest instances of the jump is shown in Fig. (8).It is clear from Fig. (8) that the length variation is largely uniform, only with small sinusoidal modulation. But as can be seen from the action, there is a sudden jump at $t \approx 200$.

It was shown in numerical studies in Ref. [18] that in certain parameter regimes, the periodically length varying pendulum shows chaotic behavior. It has also been observed in other studies [25] that jumps or non conservation of adiabatic invariance are associated with chaos. In the case of the pendulum having uniform length variation with sinusoidal modulation however, for the cases shown in Figs. (7) and (8) and for the several other parameter values we studied, we checked that the system did not display chaotic behaviour and did not show any sensitivity to the choice of the initial conditions.

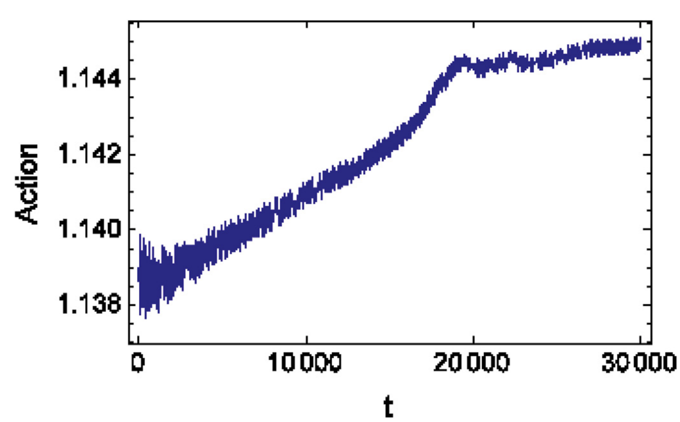

Fig. 7. The action in Fig. (6c)) plotted for a long time. Parameters: $\varepsilon_1 = 0.001$, $\varepsilon_2 = 0.1$, $\Omega = 0.04$, $l_m = 1$, $\theta_o = \frac{\pi}{10}, \theta^{'}(t=0) = 0$.

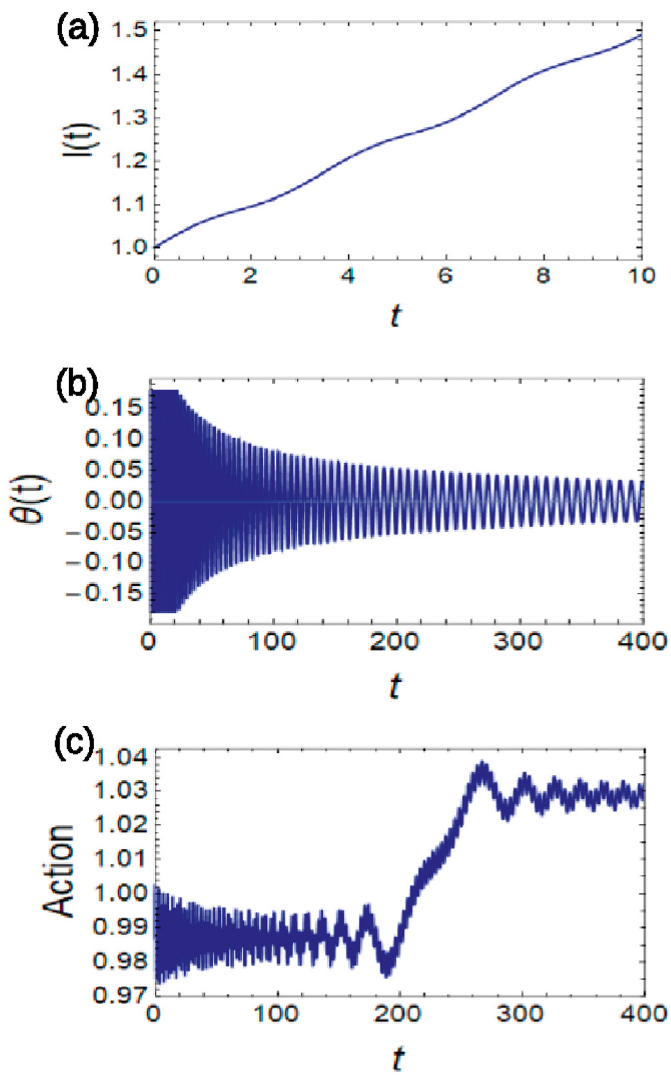

Fig. 8. Jump in the action for $\varepsilon_1 > \varepsilon_2$(a) Variation of Length; (b) Variation of θ; (b) Variation of Action. Parameters: $\varepsilon_1 = 0.05$, $\varepsilon_2 = 0.01$, $\Omega = 1.8$, $l_m = 1$, $\theta_0 = \frac{\pi}{10}$, $\theta'(t = 0) = 0$.

8. Conclusions

We have studied adiabatic invariance of the action for the length varying pendulum for different cases of length variation. Some original ideas about ascertaining adiabatic invariance of action for lengthening pendulum have been explored and an interesting result about polynomial variation has been obtained. For pendulum length having quadratic time variation, the action neither converges nor diverges, but shows oscillatory behaviour with constant amplitude. In the small angle approximation, an analytical result is obtained for this. For the pendulum in which the length is slowly varied sinusoidally, we show that the system exhibits bursting oscillations. For the case of the pendulum having a combination of periodic and linear time variations, the action shows jumps. The system does not show any signature of chaos. The action for all these cases is not an adiabatic invariant.

CRediT authorship contribution statement

Subhayan Sahu: Investigation, Formal analysis, Methodology, Visualization, Validation, Writing – review & editing. **Shriya Pai:** Investigation, Formal analysis, Methodology, Visualization, Validation, Writing – review & editing. **Naren Manjunath:** Investigation, Formal analysis, Methodology, Visualization, Validation, Writing – review & editing. **Janaki Balakrishnan:** Conceptualization, Investigation, Formal analysis, Methodology, Project administration, Supervision, Visualization, Validation, Writing – review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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