Astrometry in the 21st Century. From Hipparchus to Einstein

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Summary. — Astrometry is that fundamental part of astronomy which allows to determine the geometric, kinematical, and dynamical properties of celestial objects, including our own Galaxy, which is assembled and shaped by gravity. The knowledge of star positions was already important at the times of Hipparchus (190 -120 BC) and his predecessors, Timocharis and Aristillos. Their cataloging (approximately 150 years earlier) of star positions enabled Hipparchus to update the observations with a precision of nearly half a degree and thus to discover the phenomenon of equinox precession.
Nowadays a big jump is mandatory: positions, motions, and distances exist in the realm of the Einstein Theory and null geodesics represent our unique physical links to the stars through a curved space-time, namely a varying background geometry. Astrometry must be equipped with all of the proper tools of General Relativity to define the observables and the measurements needed for compiling astronomical catalogs at the microarcosecond accuracy and beyonds.
The Astrometry of 21st century, endowed with a fully relativistic framework, is fully fledged for new potential applications in astrophysics, can lead the way to forefront discoveries in fundamental physics, and is becoming the pillar of Local Cosmology. In this respect, is more appropriate, in the 21st century, to refer to it as "Gravitational Astrometry".

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1. – Notation and conventions

Through the text c is the speed of light in vacuum and G is the Newtonian gravitational constant. When it is not made explicit, geometrized units are used ($c = G = 1$). Four-tangent vectors are denoted in bold as $\mathbf{u}$ or u^α with $\alpha = \{0, 1, 2, 3\}$. Given a coordinate system (t, x^i) with $i = 1, 2, 3$, the metric of space-time is denoted by $g_{\alpha\beta}(t, x^i)$ and the signature adopted for is $(- + ++)$. Then, anywhere $g_{00} < 0$.

Einstein convention on repeated indices is used and the quantity $||$ stands for the modulo with respect to the chosen metric; whereas, the quantity $(|)$ stand for scalar

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product, namely $g_{\alpha\beta}u^\alpha u^\beta$. For any function $f(x^\alpha)$, $f_{,\alpha}$ and $\partial_\alpha f$ denote the partial derivative of f with respect to x^α and ∇_α the covariant derivative: $\nabla_\alpha u^\beta = \partial_\alpha u^\beta + \Gamma^\beta_{\mu\alpha}u^\mu$, where $\Gamma^\beta_{\mu\alpha}$ are the affine connection coefficient compatible with the chosen metric. Milliarsecond or microarcsecond is shortened as mas and μas respectively.

2. – Introduction

Earth, planets, stars and galaxies are all in relative motion. Once collected all of the emitted light signals from the same source, one can recover the objects relative sky coordinates by comparing their recorded positions. Classical astrometric observations are prerequisites to studying such motions, which, however, can be registered only with respect to some *absolute* reference, in principle a Newtonian Space and Time framework. For these reasons, fundamental catalogues have been established with the intention that the catalogue objects represent as closely as possible fiducial points that do not show linear and rotational accelerations [1].

As a matter of fact the application of astrometry spreads over several fields of astrophysics: (i) stellar astrophysics, where the measurement of parallax and proper motions, of orbital motions in binary and multiple systems, of apparent stellar diameter, allows to derive physical constraints on stellar models, including their internal structure and evolution; (ii) kinematics and dynamics of stellar groups, where proper motions and radial velocities are indispensable for the study of the motion of stellar clusters, to isolate stellar associations, and to analyze stellar motions in the Galactic gravitational potential; (iii) establishment and maintenance of reference frames by determining the positions and motions of reference stars; (iv) geodesy, where precise measurements of the orbital positions of artificial satellites from the ground allow a very detailed determination of the gravitational field of the Earth (for example tectonic plate motions), and of the parameters describing its rotation, which are monitored simultaneously by observing artificial satellite, the Moon and quasars.

On the other hand, the relationship between kinematic and astrophysical properties of stars lead to understand the formation and the evolution of the Milky Way. Moreover, stellar proper motions and radial velocities are the basic observational data for determining the Galactic gravitational field and its evolution [2, 1]

Basically, two approaches of observing stars are used: a) surveying a large number of stars by continuously scanning the sky with a wide field of view or b) pointing at selected star fields. In the first case a large number of stars can be detected and a global fundamental catalog obtained, after processing a very large amount of data. The second case allows longer integration times with the goal of achieving high accuracy for fainter objects.

Besides these basic observation principles, in order to achieve high astrometric accuracy in space a new approach to modeling is required, which must be compliant with General Theory of Relativity, because of influence on the space-time curvature due to the Sun and all of the Solar System masses, which in turn affect the observations and the observer.

In fact, the gravitational field due to the relative motion of the solar system bodies and their tidal interactions, generate several light deflection effects which modify light propagation, thus compromising the correct determination of parallaxes and proper motions from our local gravitational fields, namely from our local point-of-view. Even more in case one has to trace back the direction of light to the position of the star from within such ever-present and ever-changing gravitational fields. Consequently, also the retarded time contributions should be taken into account, namely the moment when the gravitational field of the source actually began to propagate along the incoming light cone. In the static case, the major effects are the deflections of light due to the monopole mass of the planets: already at the first post-Newtonian approximation they produce overlapping contributions up to the order of several μas; the contribution amounts to just 1 μas, for example, at 180 degrees from the limb of the Sun and at 90 degrees from that of Jupiter (table I).

TABLE I. – *Post-Newtonian (pN) effects on the light deflection due to the mass monopole and quadrupole at the order of 1 μas. The values in parentheses are the maximum angular distances between the perturbing body and photon at which the effect still attains 1 μarcsec. Values computed in [3].*

Perturbing body	monopole (μas)	quadrupole (μas)
Sun	1.75" (180°)	~ 1
Mercury	83 (9′)	-
Venus	493 (4.5°)	-
Earth	574 (124°)	0.6
Moon	26 (5°)	-
Mars	116 (25′)	0.2
Jupiter	16300 (90°)	240 (3′)
Saturn	5800 (18°)	95 (51″)
Uranus	2100 (72′)	25 (6″)
Neptune	2600 (51′)	10 (3″)
Io	30 (11″/32″)	-
Europa	19 (11″/32″)	-
Ganymede	35 (11″/32″)	-
Callisto	28 (11″/32″)	-
Titan	32 (11″/32″)	-
Triton	20 (11″/32″)	-
Pluto	7 (11″/32″)	-
Charon	4 (11″/32″)	-
Iapetus	1-3 (11″/32″)	-
Ceres	1-3 (11″/32″)	-

Therefore, if aiming at the μas level, one is obliged to consider gravity properly when compiling stellar catalogues, as it will be shown in the sections below. This necessarily implies the dismission of Newtonian straight lines and the adoption of the measurement toolkit provided by General Relativity (GR), in particular to solve the null geodesics linking stars to observers. Moreover, as the electromagnetic signals deliver physical information about a lot of phenomena interacting with the propagating light, relativistic astrometry promotes new gravitational tests on GR/alternative theories of gravity and current cosmological models. This opens a novel perspective for astrometry, which is presented in the last sections: from having been the old branch of astronomy to being "Gravitational Astrometry", i.e. part of fundamental physics as well as an all-sky relativistic tool to peer the surrounding Universe.

Part of what is reported here has already been discussed in the literature. The present review, on the other hand, aims (i) to collect the fundamentals of Relativistic Astrometry in few basic steps (ii) to pinpoint the potential of the new astrometry mostly dominated by null geodesics solutions (iii) to broad the application of the actual weak gravitational astronomy domain to other different regimes (iii) to highlight the role of null geodesics and GR measurement protocol for astronomy in the prospect of advanced future space missions.

3. – Astrometric measurements through ages

We can say that astrometry was invented when humankind started to glance repeatedly at the changing sky, and based the fundamental measurement of time on the comparison of the periodic appearance of the Sun, the Moon and the stars.

Astrometric observations through ages have provided the material for fundamental catalogues to record the motions of celestial bodies. The first star catalog was produced by Hipparchus, comprising about 1000 stars together with the first classification based on the definition of magnitude.

As well known, Hipparchus could measure the precession of the equinox by comparing the different star longitudes at the autumnal equinox observed long before by the Babylonians and the Chaldeans. His measurements, nearly 36 arcsecond per year, were not so far from the value $\sim$ 50 arcsecond per year provided by current observations. Angles in the sky were the key to understand such tiny movements related to relative motion of the Earth with respect to the Moon, the planets and the Sun, where the word "relative" is referred to the gravitational pull that those bodies exert on our planet.

As a matter of fact, by continuously observing star positions, Hipparcos somehow recognized the role of gravity moving the objects toward a center as quoted by Plutarco: "Indeed, motion according to nature guides every body, if it is not deviated by something else " in *De facie quae in orbe lunae apparet* [4]. And somehow this has been inherited by the new astrometry, which requires GR, the standard theory of gravity, to fully accomplish its application to astronomy. According to this theory, in fact, geometry is a manifestation of gravity and the geometry dictates how stars move and galaxies are assembled.

Not differently from the past, any positions and their variations on the sky are still measured via angles as the celestial objects are projected on an ideal sphere with the observer at its center. Angle measurements are essentially based on rules of spherical trigonometry, and the triangulation method (trigonometric parallax) allows to measure directly the (inverse) distance of sources outside the Solar System.

3·1. *From Earth to Space, measurements of smaller and smaller angles*. – The application of astrometry can be divided into two main groups: (i) small field astrometry and (ii) global astrometry. In small field astrometry the position of the celestial body is referred to neighboring stars within the field of view θ of the telescope. It comprises very narrow ($\theta < 10''$), narrow ($\theta < 0.5°$), and wide field astrometry ($\theta < 5°$). The first one deals with single or multiple stars and analyzes their relative motions. Narrow field astrometry is used for the determination of relative parallaxes and for the detection of invisible companions. Among the goals of wide field astrometry there are the study of internal motion of stellar clusters and the computation of quasi-inertial coordinates of a given target with respect to the fundamental stars visible in the field of view. The main techniques are direct detection by charge coupled device (CCD), speckle interferometry, Michelson interferometry, and radio interferometry [7]. Semi-global astrometry, instead, makes possible to construct fundamental catalogs by measuring positions separated by

Table II. – *List of the main achievements in Astrometry through ages [5, 6].*

Assyria, Babylonia (1000 BC): systematic observations, discovery of periodic phenomena (visibility of Venus, lunar eclipses);
Pythagoras (580 - 500 BC): Earth, Sun, planets as a sphere, Venus as a planet (Phosphorus and Hesperus);
Plato (427 - 347 BC): motion of the Sun, Moon and planets visible to the naked eye, movements in astronomy associated to "geometry";
Eudoxus, **Euclid** (400 - 300 BC): basis of spherical astronomy and celestial globes;
Aristarchus of Samos (310 - 230 BC): first measurement of the distance and diameter of the Sun and Moon via trigonometric method, heliocentric universe;
Callippus (330 BC): measurement of the length (in days) of the season and equinoxes, indicating an apparent variable velocity of the Sun;
Eratosthenes (200 BC): system of longitudes and latitudes, earth's circumference (40,500 Km), catalog of 675 stars;
Hipparchus (150 BC): compilation of a catalogue containing the positions of 1025 stars, reported later by Ptolemy in the *Almagest* (100-170 AD), distance of the Moon from Earth of about 62-72 Earth radii (correct value from 55 to 63);
Ulugh Beg's astronomers (1420 -1437):*Zij-i Sultani* catalogue of about 1000 stars without using Ptolemy's Almagest;
Regiomontanus (1462): *Epitome of the Almagest*, new planetary observations and correction of *Alfonsine Tables*;
Copernicus (1500 AD): affirmation of the heliocentric theory;
Tycho Brahe (1600): improvement of the instrumentation, accurate observations about 20" (Kepler's laws);
Kepler, **Galileo**, **Newton** (1500-1700): modern era for astronomical sciences;
Nautical Almanac: from 1767 ephemeris annual publication of the main celestial bodies;
Halley (1718): discovery of proper stellar motions (Aldebaran, Arturo and Sirio);
Bradley (1725): discovery of the effect of stellar aberration due to terrestrial motion;
J. Lalande (1751): determination of the parallax of the Moon and Mars;
Herschel (1783): discovery of more than 800 double stars;
Bessel (1838): first trigonometric parallax of 61 Cygni (0.3");
Carte du Ciel (1887-1931): limit magnitude 11.5, not complete on the whole sky (20236 stars);
Astrographic Catalog (1895-1950): 3 exposures per plate, first photographic map of the sky, 20 observatories all over the globe, more than 22,000 photographic plates ($\sim 4.5 \times 10^6$ stars), visual measurements, limit magnitude 14, accuracy 0.5";
Schlesinger (1924): General Catalog of Stellar Parallaxes (from photographic plates);
VLBI technics (1965-1967): Matveenko-Kardashev-Sholomitsk and Clark-Kellermann-Cohen
Fricke et al. (1988): FK5 Fundamental Catalog (1535 stars, accuracy of some tens of mas);
Hipparcos satellite, ESA (1989): First astrometric space mission, 5 astrometry parameters accurate to the mas for 10^5 stars;
Thyco-2 catalog (2000): 2.5 million bright stars from Hipparcos (http:www.cosmos.esa.int/web/hipparcos/tycho-2);
van Altena et al. (1995): Yale Trigonometric Parallax Catalog (8112 stars, 10 mas);
Gaia satellite, ESA (2013): Second astrometry space mission, improvement of 2 orders of magnitude compared to Hipparcos, 10^9 Galactic and extragalactic objects.

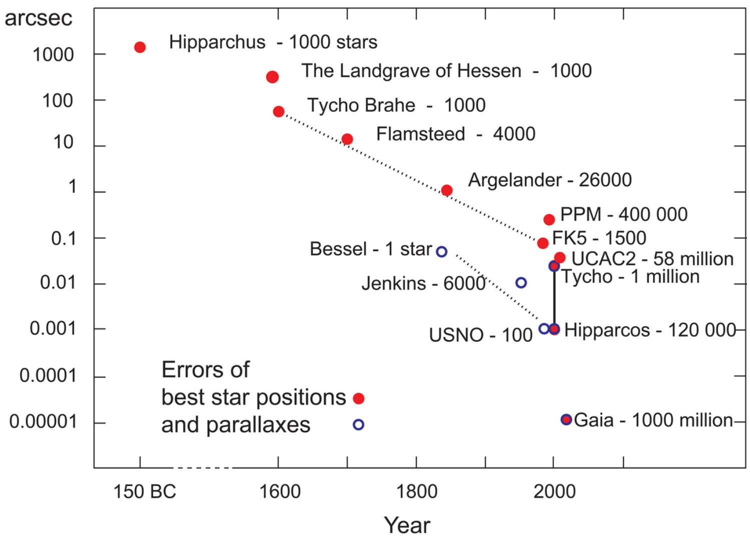

Fig. 1. – Progress in astrometric accuracy through ages. Image Credits: ESA (http://sci.esa.int/gaia/33840-progress-in-astrometric-accuracy/)

large angles. Observations depend on geographic latitude. Therefore, several instruments adequately distributed over the surface of the Earth are needed to survey the whole sky and to provide an overlap of the declinations zones. Interferometry at radio wavelengths is used both for small field and semi-global astrometry.

The accuracy of an astrometric measurement depends on the resolving power of the instrument and on the space-time properties of the incoming wave front, as well as on the accuracy of the model that links the astrometric parameters to the observations.

Astrometry from the ground has a certain number of limitations: (i) the atmospheric turbulence modifies the apparent direction of the source (the technique of adaptive optics works only within a few seconds of arc); (ii) the atmospheric refraction displaces the apparent direction; (iii) the Earth rotation must be considered; (iv) the instrumental (systematic) errors cannot be fully eliminated in a final combination of heterogeneous catalogs.

Global astrometry from space is the only way to obtain fully absolute parallax (distances). Ground based astrometry can only provide relative parallaxes and their transformation to absolute distance is a non-trivial problem.

The refinements of astrometric techniques improved the precision of star positioning from about 1500 arcsec at the time of Hipparchus to 0.2 arcsec or better by around 1980 [1].

With Hipparcos (High Precision PARallax COllecting Satellite), the first global astrometric satellite launched by ESA in 1989, the accuracy improved from 50 to 1 mas [8].

A scanning law was devised that permits the scanning of the whole sky, while maximizing the number of observations per star. Such a law is a combination of three independent motions of the satellite: the orbital motion, the rotation around its spin

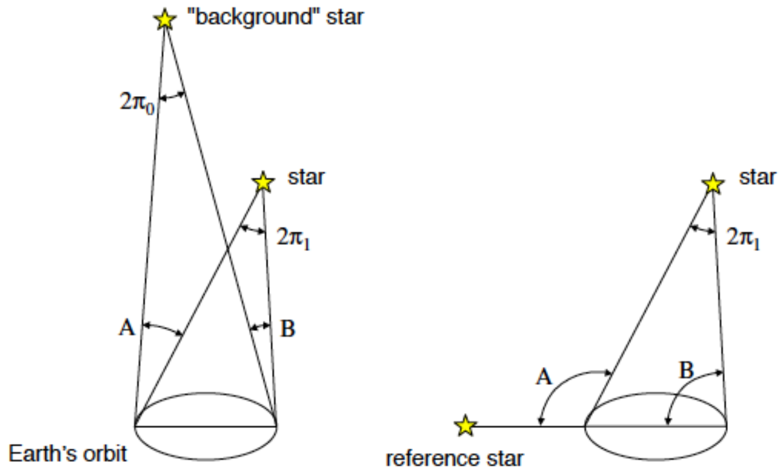

Fig. 2. – Principle of measurements of relative (left) and absolute parallaxes (right). Small angles A and B are related by $\pi_1 - \pi_2 = (A - B)/2$, whereas large angles allow to determine absolute parallax $\pi_1 = (A - B)/2$ not depending on the distance of the reference star [9].

axis, and the precession of this axis around the Sun-satellite direction. As the telescope rotated about its axis perpendicular to the plane formed by the directions to the centers of the two stellar field, it swept a circle in the sky.

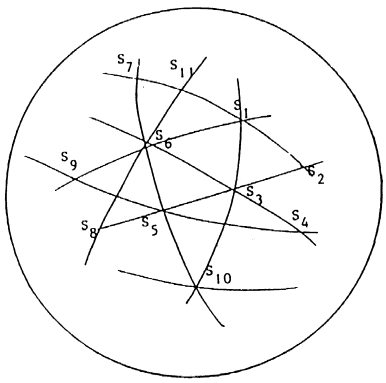

Fig. 3. – Sky grid obtained by repeatedly observing arcs via global astrometry.

In this way Hipparcos could measure large arcs and, therefore, absolute parallax. These angles were expressed as a function of the stellar unknowns. The large amount of observations performed during the mission lifetime, created a grid that covered the whole sky forming an over-determined system of equations, whose solution was found by means of a least-square method.

More details about the astrometric mission Hipparcos can be found in its dedicated web site [8].

With the advances in technology, namely CCDs, computers, and interferometry, global astrometry increases its accuracy to reach the μas level. Then, following Hipparcos, Gaia (European Space Agency, ESA) is the first astrometric mission of the twenty-first century dedicated to the study of the Milky It was successfully launched on 19th December 2013 from the European base of Kourou in French Guyana.

Exploiting the same scanning strategy as Hipparcos (figure 4), Gaia is surveying at the L2 point of the Earth-Sun gravitational system the entire sky down to the visual magnitude G=20, performing with the microarcsecond-level accuracy the deepest and most complete census of our Galaxy.

Gaia extends the astrometric accuracies by 2-3 orders of magnitudes over Hipparcos (table III), to better than 10 μas at 15 mag [2]. The limiting magnitude in the observations will be extended from 12 to 20 mag, and the completeness limit from $\sim 7-9$ mag down to a faint limit of 20 mag. On-board object detection will ensure that variable stars, supernovae, burst sources, micro-lensed events, and minor planets will all be detected and cataloged to this faint limit.

TABLE III. – *Overview of the Gaia performances in comparison to those of Hipparcos [2].*

	Hipparcos	**Gaia**
Magnitude limit	12	20
Completeness	7.3- 9.0 mag	$\sim$ 20 mag
Bright limit	~ 0	~ 3
Number of objects	120, 000	47 million to G=15 mag 360 million to G=18 mag 1192 million to G=20 mag
Effective distance limit	1 kpc	50 kpc
Quasars	1 (3C 273)	500,000
Galaxies	None	1.000,000
Accuracy	1 milliarcsec	7 μarcsec at G=10 mag 26 μarcsec at G=15 mag 600 μarcsec at G=20 mag
Photometry	2-color B and V	Low-res. spectra to G=20 mag
Radial velocity	None	15 km s^{-1} to G_{RSV}=16 mag
Observing program	Pre-selected	Complete and unbiased

The eclipse-free orbit around the L2 point of the Sun-Earth system offers a very stable thermal and low-radiation environment. An operational lifetime of 5 years was planned and an extension of two years has been decided.

The scan rate is 60 arcsec/s, equivalent to a 6-hour rotation period. The tilt of the spin axis with respect to the sun direction is 45° and represents a good compromise

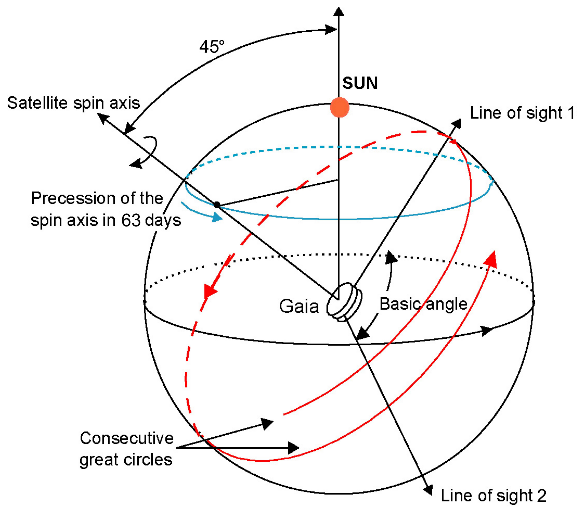

Fig. 4. – Gaia measurement principle. Credit: ESA

between astrometric performance and system aspects. Precession at such fixed angle to the Sun with an average period of 63.12 days ensures sky coverage.

The payload module has a toroidal structure made of Silicon Carbide which minimizes the thermal expansions and mechanical flexures. A set of instruments are mounted on the toroidal frame. The astrometric instrument consisting of two identical telescopes mounted on the same optical bench and combining at the same time two different stellar directions in one focal plane (FP), observing all objects that pass away in its two fields of view (FOV) separated by a controlled basic angle of 106.5 degrees and perpendicular to the spin axis (indispensable for absolute parallaxes).

The focal plane is a mosaic of CCDs divided into the astrometric sky mapper (A-SM), the astro field (AF), and the photometry field (continuous spectra in the band 330-1050 nm for astrophysics and chromaticity calibration of the astrometry). The ASM detects the objects passing through the FOV, filtering off for example cosmic rays, while the AF makes the astrometric measurements. The CCDs operate in time delay integration mode (TDI).

A Radial Velocity Spectrometer (RVS) is also sharing the same FP. RVS provides radial velocity and high resolution spectral data in the narrow band 847-874 nm.

Nearly one-two billion astronomical objects will be observed on about 80 times, leading to around 630 CCD transits, so a total of more than 150 billion measurements at the end of the mission.

Limiting the object selection to those with an astrometric parallax error of less than 10%, which is considered a safe threshold in the majority of astrophysical applications, implies that with microarcosecond-level accuracy we reach the Galactic scale. Then, Gaia's survey provides the detailed 3D distributions and space motion of some 1 billion individual stars in our Galaxy and beyond, extended to G=20.7 (i.e., ∼V=21), but not complete at this magnitude limit.

The primary topic for Gaia is to clarify the origin and history of our Galaxy, by providing tests of the various formation theories, and of star formation and evolution. All parts of the Hertzsprung-Russell diagram will be comprehensively calibrated, including all phases of stellar evolution, from pre-main sequence stars to white dwarfs and all existing transient phases, all possible masses, from brown dwarfs to the most massive O stars, all types of variable stars, all possible types of binary systems down to brown dwarf and planetary systems, all standard distance indicators (pulsating stars, cluster sequences, supergiants, central stars of planetary nebulae, etc.). A 1% parallax determination of Cepheides and RR Lyrae would turn in a significant improvement of the Period-Luminosity-Color relationship useful for the accurate determination of the distance scale.

Nevertheless, the improvement of the accuracy in the astrometric measurements conducted via space satellite requires rigorous relativistic models for the propagation of light in the framework of GR since several relativistic effects disturb light propagation already at level of the first post-Newtonian order (see table I and [10, 11, 12, 13] as precursor works). For this reason, *relativistic astrometry* has recently grown as a mature research field, providing, already at the microarcsecond level, a fully general-relativistic analysis of the inverse ray-tracing problem, from the observational data (e.g., stellar images on a digital detector) back to the position of the light-emitting star.

Gaia space mission constitutes a typical example where a fully GR theoretical framework is the necessary tool for the correct processing of the astrometric observations and their subsequent interpretation. Since null geodesics in the Einstein theory carry electromagnetic information, these tools extend their use to astronomical observations in all wavelengths.

3·2. *The relative positions and motions in records. Fundamental catalogs*. – All the worldwide efforts to observe star positions through ages converged in several catalogs, complied with the purpose of being a data base available to the users for astronomical studies and future observations.

A prerequisite to define classical celestial dynamics is the availability and maintenance of an ideal inertial reference system, indeed a conventional definition of a reference system, a conventional realization of the reference frame which materializes the system (by a number of points, objects or coordinates to be used for referencing any other point, object or coordinate), accompanied by constant improvements of the models for the observables, and determination of Earth's orientation on a regular basis. The term "system" includes the description of the physical environment as well as the theories used in the definition of the coordinates. On the other hand, the term *conventional* suggests that the choice to characterize both the system and the frame is not unique; usually it is based on a set of conventionally chosen parameters depending on the model used to define the relationship between the configuration of the basic structure and its coordinates.

In principle, to avoid rotation, only extragalactic objects, which are not influenced by the rotation of the Galaxy, can provide convenient fiducial "inertial" points. However, the implementation of an inertial reference frame may contain moving objects with known law of motion. The residual rotation of such a system in comparison with other reference frames will improve the description of the source models.

The construction of a modern catalogue is essentially based on observations obtained from: Earth-based radio astrometry and Space astrometry at optical wavelengths [1].

By mid eighties, the improvement in stellar positions and motions culminated in the fifth fundamental catalog (FK5), a series which began with the FC (Fundamental Catalog, Astronomische Gesellschaft) realized by A. Auwers in 1879.

The discovery of quasars in the early 1960s allowed the construction of catalogues of extragalactic radio sources with a theoretical position accuracy better than 1 mas, i.e.

about one hundred times smaller than uncertainties of the corresponding ground-based measurements in the visible domain. Since the late 1960s the technique of Very Long Baseline Interferometry (VLBI) is employed for position measurements of extragalactic radio sources and it became possible, by upgrading VLBI techniques up to the mas level, to establish a reference frame based on distant extragalactic objects. Initially, some 600 objects were selected as potential candidates [14]. They were recommended for regular observation by VLBI networks. A monolithic analysis of the original VLBI observables (time delay and fringe rate) from 979 through 1995 led to a fundamental catalogue of 608 extragalactic radio sources.

Thus, according to the International Astronomical Union resolution (IAU, 1992), the FK5, which constituted the IAU-recommended coordinate system until 1998, have been replaced (IAU 1999) by the fundamental catalogue of VLBI positions, the International Celestial Reference System (ICRS), the idealized barycentric coordinate system to which celestial positions are referred, kinematically non-rotating with respect to the ensemble of distant extragalactic objects ([1]) and the corresponding frame, the International Celestial Reference Frame (ICRF), aligned with the International Earth Rotation Service (IERS). The first ICRF resulted from 1 600 000 VLBI observations performed from 1979 to 1995, while the second version ICRF2 [15] resulted from 6 000 000 VLBI observations performed from 1979 to 2009 with a final accuracy about 40 μas. The set of extragalactic objects whose adopted positions and uncertainties realize the ICRS axes and give the uncertainties of the axes. The axis stability of ICRF2 is 10 μas, which is nearly twice as stable as ICRF1. It preserves, by construction, the same directions of the reference axes and contains 3414 radio sources, in which 295 are defining sources. The third revision on ICRF, ICRF3, was adopted by the IAU at the XXX IAU General Assembly on 30 August 2018 and become effective since January 1, 2019. It contains positions for 4536 extragalactic sources, with 303 identified as defining sources and incorporates the effect of the galactocentric acceleration of the solar system (correction of 5.8 μas/yr [16]).

In this context, since 1991 IAU General Assembly (GA) adopted GR as the fundamental theory, specifying the continuity with existing stellar and dynamic implementations [17]. Later, the IAU 2000 resolutions [18], adopted by the XXIVth IAU General Assembly (August 2000) and endorsed by the XXIIIrd International Union of Geodesy and Geophysics (IUGG, General Assembly, July 2003), have made important recommendations on space and time reference systems, the concepts, the parameters, and the models for Earth's rotation that should be consistent with General Relativity. In particular IAU 2006 Resolution B2 [19] recommends the Barycentric Celestial Reference System (BCRS) is assumed to be oriented according to the International Celestial Reference System (ICRS) axes.

Such a development would not have been possible if the Hipparcos catalog as counterpart of the ICRF would not have existed. The Hipparcos catalog of over 10^5 star coordinates defines the ICRS frame, materializing the ICRS in the visible [20]. Two major astrometric catalogs have been produced by the astrometric Hipparcos mission: the Hipparcos catalog comprising 118218 stars, and the Tycho Catalog containing 1058332 stars including 6301 stars of the previous one. These catalogs provide positions, annual proper motions and absolute trigonometric parallaxes free from regional distortions and

([1]) It has no intrinsic orientation but was aligned close to the mean equator and dynamical equinox of J2000.0 for continuity with previous fundamental reference systems. Its orientation is independent of epoch, ecliptic or equator.

with errors as good as 1 mas. The positional precision of the Hipparcos catalog was close to that of the extragalactic objects in the ICRF.

This has provided the means to extent the quasi-inertial radio frame, which does not rotate as the inertial one, but its origin may have acceleration, to the optical domain. As a result, the advantage is an optical reference frame that profits from the properties of the radio reference frame, but it is accessible to optical astrometry and sufficiently dense (2 to 3 stars per square degree on the average). Hipparcos frame was labeled the Hipparcos Celestial Reference Frame (HCRF), and used as a quasi-ideal reference system for measuring the angular motion of celestial bodies.

Nowadays, the first two releases of Gaia [21, 22] have replaced the Hipparcos catalogue.

The first general all-Gaia data delivery (DR1) has been released on 14th September 2016 (corresponding to 1000 days into science operations in Nominal Scanning Law). DR1 contains the five-parameter astrometric solution - positions, parallaxes, and proper motions - for stars in common between the Tycho-2 Catalogue and Gaia (TGAS), namely for 2 million stars complete to V=11.5 (solar neighborhood, open clusters and associations, moving groups) with sub-milliarcsec accuracy (10% at 300 pc). On April 25th 2018, the second release of the Gaia catalogue (DR2) became available to the scientific community worldwide. This time it contains the five-parameter astrometric solution for more than 1.3 billion sources, within the Gaia magnitude range $3 < G < 20.7$, and median radial velocities for more than 7.2 million stars. At the end-of-mission the astrometric accuracies are expected better than 5-10μas for the brighter stars and 130-600μas for faint targets and the final Gaia Reference Frame will contain 500,000 quasars. This optical alignment has the advantage to exclude astrophysical contaminations due the extremely variable environment in the active galactic nucleus at the center of the quasar.

With Gaia DR2, Gaia-CRF2 consitutes the first optical realization of a reference frame at sub-milliarcsecond precision, with a mean density of more than ten quasars per square degree. It represents a more than 100-fold increase in the number of objects from the current realization at radio wavelengths [23].

Despite these achievements in accuracy, refined definitions regarding the celestial reference systems are constantly adopted by IAU as well as new paradigms and high accuracy models to be used in the transformation from terrestrial to celestial systems. The accuracy of measurements provided by modern astronomy in space, the need of the realization of celestial reference frames at the μas level of accuracy implies also a self consistent set of units and numerical standards compliant with GR, as for example the use of the astronomical unit as constant, since its historical definition is not longer appropriate for being used with modern solar system ephemerides [17].

4. – General Relativistic Metrology for Astronomy

General Relativity (GR) is the theory in which geometry and physics are joined in order to explain how gravity works and the trajectory of a photon is traced by solving the null geodesic in a curved space-time.

The GR space-time structure has three main characteristics: (i) differential, which means that space-time is continuous and tell us how many dimensions we are dealing with; (ii) topological, that tells us how different parts of space-time are connected, which points are around others, i.e. "a rubber sheet" topology ([2]); and, from(ii), also (iii)

([2]) Property of space that does not vary due to arbitrary continuous space deformations.

geometric, which tells us how to construct parallel lines and defines the distance between space-time points ([24], section 2.1). Moreover, a Riemannian differential variety has a geometric structure distinguished into affine and metric geometry, corresponding to two different ways in which we can detect the curvature of space. The first implies how parallel lines are connected or correlated in different places; the second establishes how we measure lengths and areas.

Measurements performed from within a geometrical environment generated by a n-body distribution as it is that of our Solar System, are affected mainly by the mass monopole of the Sun and planets, which act as varying sources of its gravitational field and modify the incoming electromagnetic signals as well as the observer's motion.

Consequently, before modeling the astronomical observations, one has to write the field equations for the Solar System and find appropriate solutions with n-body sources, also in the case the former is gravitationally isolated from the rest of the Universe.

Since gravity according to the Einstein theory is the manifestation of a varied background geometry, solving the well known Einstein equation:

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta}, \tag{1}$$

requires basically to find a suitable metric, namely $g_{\alpha\beta}$. Equation (1) practically translates into a system of ten coupled, nonlinear, hyperbolic-elliptic partial differential equations with no straightforward solution because of the intrinsic non-linear coupling between the Ricci tensor $R_{\alpha\beta}$ (R is the Ricci scalar), depending on the first and second partial derivatives of the metric $g_{\alpha\beta}$, and the sources of the gravitational field $T_{\alpha\beta}$.

In point of fact, a variations of the background geometry may be induced by a choice of the coordinates or by the presence of matter and energy distribution. The first one originates inertial forces acting similarly but not equivalently to gravity. The second one, instead, generates gravity, whose effects can not be easily disentangled from the inertial force.

Denoting the coordinates generically as $x^\alpha = (t, x^i)$, as the Solar System generates a weak gravitational field, the metric can be linearized with respect to a small perturbation $h_{\alpha\beta}$ in the form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \sum_a h^{(a)}_{\alpha\beta}\,, \tag{2}$$

where $\eta_{\alpha\beta} = \mathrm{diag}[-1, 1, 1, 1]$ is the Minkowskian metric and a stands for the ath-source. The choice of the coordinates is actually a gauge one. Einstein equation determines 10 components of the metric, but the four Bianchi identities implies four degrees of freedom that are fixed by imposing a coordinate choice, basically the two conditions available in literature: harmonic (or de Donder) and ADM (Arnowitt-Deser-Misner, [25]).

Once chosen a form for the h terms, the background geometry constitutes the basic ingredient to model a 4-dimension relativistic sky and also the observer's motion from within the surrounding gravitational fields.

This implies, given a varying curvature, that in GR any observer carries on a proper clock and a proper meter. The act of measuring is, then, observer-dependent. Nevertheless a measurement should be invariant (namely a scalar quantity) regardless the coordinates system adopted. In technical language, all of the involved tensorial expressions in a measurement process should be expressed through covariant equations, with

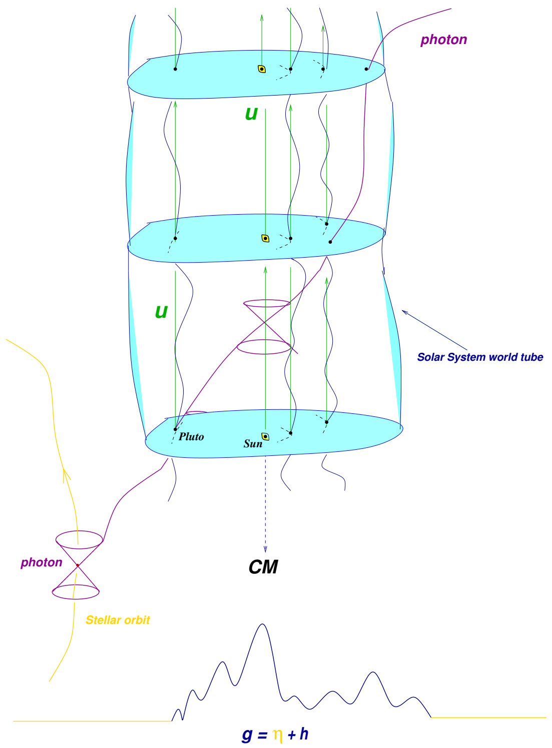

Fig. 5. – Photon emitted by a star crossing the Solar System world tube, perturbed by its n-sources.

respect to a chosen "global" coordinate system and with respect to a given observer's frame. Indeed, the act of measurement between different space-time events means to define the observer who performs the measurements and a proper procedure to compare her/his measurements with respect to another observer.

Any observer is termed time-like if $u_\alpha u^\alpha = -c^2$ (or $u_\alpha u^\alpha = -1$ in geometrized units). An observer's laboratory is usually represented as a world tube; in case of weak field and for not an extended body, the world tube can be restricted to a world line tracing the history of the observer's barycenter in the given geometry. A laboratory can be mathematically defined as a family of future-pointing time-like non-intersecting curves with tangent 4-vector $\mathbf{u}$, namely a congruence $C_\mathbf{u}$, each one representing the history of a point in the laboratory and parametrized by their proper time, thus determining the local time direction at each point of spacetime. The orthogonal complement of this local time direction (in the tangent space) is the local rest space of the observer at the same event.

All of the above can be realized by considering an infinitesimal normal neighborhood of $\mathbf{u}$ and a metric $g_{\alpha\beta}$. At any point along the worldline of $\mathbf{u}$ it is possible to define a local splitting of the space-time into 1-dimensional subspace plus a 3-dimensional one, each endowed with its own metric. Respectively, via the temporal operator:

$$T(u)_{\alpha\beta} = -u_\alpha u_\beta \,, \tag{3}$$

which projects parallel to **u**, and via the spatial projection operator

$$P(u)_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta \,, \tag{4}$$

which generates a local rest space spanned by vectors orthogonal to **u**, so that

$$g_{\alpha\beta} = P_{\alpha\beta} + T_{\alpha\beta} \,. \tag{5}$$

The family **u** constitutes a set of *fiducial* observes filling the space-time. A section orthogonal to the congruence describe the space-time evolution of the system as the time varies along the curves.

The covariant derivative generates the kinematics of such a congruence [26]. In particular,

$$\nabla_\beta u^\alpha = -a^\alpha(u) u_\beta - k^\alpha_\beta(u), \tag{6}$$

where the "four-acceleration" is defined as

$$a^\alpha(u) = P(u)^\alpha_\beta u^\gamma \nabla_\gamma u^\beta = P(u)^\alpha_\beta \dot{u}^\beta, \tag{7}$$

and the "kinematical tensor"

$$k^\alpha_\beta(u) = \omega^\alpha_\beta(u) - \theta^\alpha_\beta(u), \tag{8}$$

which depends on the "vorticity" $\omega^{\alpha\beta} = P(u)^\mu_\alpha P(u)^\nu_\beta \nabla_{[\mu} u_{\nu]}$ and the "expansion" $\theta^\alpha(u) = P(u)^\mu_\alpha P(u)^\nu_\beta \nabla_{(\mu} u_{\nu)}$, where square brackets mean anti-symmetrization and circular ones symmetrization. The expansion virtually measures the average expansion of the infinitesimally nearby surrounding geodesics and the vorticity measures how a world-line of an observer rotates around a neighboring one.

In general the splitting of the tangent space does not extend to the whole space-time manifold. From the Frobenius theorem, if an open set of space-time manifold admits a vorticity-free congruence of lines, then it can be *foliated* [26]. Only in this special case one has a family of orthogonal spacelike hypersurfaces which slice the spacetime.

This means that, given a general coordinate spacetime representation $(x^i, t)_{i=1,2,3}$, the metric being $g_{\alpha\beta}(x,t)$, there exists a family of three-dimensional hypersurfaces described by the equation $\tau(x,t) =$ constant, where $\tau(x,t)$ is a real, smooth and differentiable function of the coordinates. Let us denote each slice as $S(\tau)$. A spacelike foliation implies that a unitary one-form u_α exists which is everywhere proportional to the gradient of τ, namely:

$$u_\alpha = -(\partial_\alpha \tau) \mathrm{e}^\psi, \tag{9}$$

where

$$\mathrm{e}^{\psi'} = [-g^{\alpha\beta}(\partial_\alpha \tau)(\partial_\beta \tau)]^{-1/2} \tag{10}$$

is the normalization factor. Then the curves tangent to the vector field $u^\alpha = g^{\alpha\beta} u_\beta$, form a congruence of timelike lines $C_{\mathbf{u}}$, everywhere orthogonal to the slices $S(\tau)$. In order to understand the geometrical meaning of the one-form u_α, let us adapt the coordinate system to the spatial hypersurfaces $S(\tau)$ such that $x^0 = \tau(t, x^i)$ and $x^i = x^i(t, x^i)$. The new normals result:

$$
\begin{aligned}
u_\alpha(x^0, x^i) &= -\delta^0_\alpha \mathrm{e}^\psi \\
u^\alpha(x^0, x^i) &= -g^{0\alpha}\mathrm{e}^\psi = \frac{dx^\alpha}{d\sigma}
\end{aligned}
\tag{11}
$$

where σ is the parameter on the normals $\mathbf{u}$, $\psi = \psi(x^0, x^i)$ is a function that makes $u^\alpha u_\alpha = -1$. From the unitary condition then, it results

$$\mathrm{e}^\psi = d\sigma/d\tau = (-g^{00})^{-1/2}, \tag{12}$$

which means that the parameter σ runs uniformly with the coordinate time x^0 and depends on the metric term g^{00}.

But the congruence $C_{\mathbf{u}}$, however, does not preserve the spatial coordinates x^i. For example, denoting $N = 1/u^0$ and $N^i = u^i/u^0$ respectively the *lapse* and the *shift* functions (or factors, for similar definitions see [25, 27]), the line element can be cast into the form:

$$ds^2 = -(Ndx^0)^2 + g_{ij}(dx^i + N^i dx^0)(dx^j + N^j dx^0). \tag{13}$$

Thus, choosing at time τ_1, say, an arbitrary event $\mathcal{P}$, labeled by the coordinates (τ_1, x^i_1), the unique normal through that point will intersect the slice $S(\tau_1 + \Delta\tau)$ at a point $\mathcal{P}'$ with spatial coordinates that are shifted from the initial ones by the amount:

$$\Delta x^i = \int_0^{\Delta\tau} N^i(\tau')d\tau'. \tag{14}$$

The lapse factor N measures the rate of flow of proper time, with respect to the coordinate time as one moves normally to S, whereas the shift factor N^i measures the amount of "shift" tangential to S contained in the time flow vector field [28].

In order to vanish the shift factor a new spatial coordinate transformation has to be applied, i.e. $\hat{x}^i = x^i + N^i\Delta x^0$ and $\hat{x}^0 = x^0 = \tau$. Then, the vector field tangent to the congruence $C_{\hat{u}}$, which transports the spatial coordinate, results:

$$
\begin{aligned}
\hat{u}^\alpha(\hat{x}^0, \hat{x}^i) &= \frac{d\hat{x}^\alpha}{d\sigma} = \mathrm{e}^\phi \delta^\alpha_0 \\
\hat{u}_\alpha(\hat{x}^0, \hat{x}^i) &= \mathrm{e}^\phi g_{0\alpha} = -\mathrm{e}^\psi \frac{\partial\tau}{\partial\hat{x}^\alpha}.
\end{aligned}
\tag{15}
$$

and, in this form, is termed static (at rest with respect to the coordinate grid).

The intrinsic behavior of a general tensor field along a congruence $\mathbf{u}$ is specified with different space-time derivatives, namely absolute, Fermi-Walker, Lie ones or simply by parallel transport [25].

When the congruence is associated to an "isometry", i.e. the metric has null Lie-derivative, each of the spacelike hypersurfaces of the foliation has the same geometry. For instance, the worldlines of timelike "inertial" observers at rest with respect to the coordinates, implies that any observer moving along any of these worldlines will see an unchanging spacetime geometry in his vicinity.

Summarizing, the notions of time and space are complementary since a "time-like" means measuring the elapsing time at fixed point in space, while "constant time hypersurfaces" implies a synchronization of times at different points of space. For the former local time direction is fundamental, for the latter space is fundamental (non local correlation of local time, i.e. space at some moment of time) [27].

Despite of General Relativity unifies space and time into a unique scenario, a weakly gravitational field has the advantage to allow a spacetime foliation and accordingly one can define the observer's proper time and rest space, i.e. to reconstruct a sort of local "simultaneous observations" in the presence of a curved spacetime and to reproduce a scenario familiar to the Newtonian one in dealing with astronomical observations, but endowed with a fully intrinsic GR characterization. In many practical applications, spacetime is endowed at least locally with either a preferred congruence of integral curves of a timelike vector field or a preferred slicing by a family of spacelike hypersurfaces or both. In literature a future-pointing unit timelike tangent vector field is termed the *threading* congruence, while in the *slicing* point of view one has the timelike unit normal vector field (future-pointing). The two observers coincide in the special case of an orthogonal nonlinear reference frame, namely the frame imposed on spacetime by the given geometry and the one for which both the causality conditions hold and the slicing and threading are everywhere orthogonal [27].

Once equipped with these basic notions of classical GR, the fundamental steps to formulate an astronomical "measurements" in a weakly curved space-time consist in: (i) the definition of a suitable background geometry provided a suitable solution of the Einstein field equation in case of bound weak gravitational fields and (ii) a suitable global coordinate systems to define tensorial quantities (covariant expressions); (iii) a suitable definition of the observer's reference frame and (iv) an appropriate definition of the observable from the solution of null geodesic.

4·1. *The geometry from within the Solar System.* – Consider the background spacetime due to N gravitationally interacting bodies, each associated with its own world tube (or world line), with multipolar fields defined along it. One can define a global coordinate system $x^\alpha = (ct, x^i)$ with the origin at the center of mass and N local coordinate systems each attached to a single body. The general motion can be split into the external part for the motion of the centers of mass of the N bodies, and the internal one for the motion of each body around its center of mass [29].

In the global coordinate system, the center-of-mass of N world lines $\mathcal{L}_{(a)}$ $(a = 1 \dots N)$ is described by the curves $z^\alpha_{(a)} = z^\alpha_{(a)}(\tau_{(a)})$, where $\tau_{(a)}$ is the proper time parametrization along each line. Local coordinate systems, instead, can be represented by the attached set of coordinates, say, $X^\alpha_{(a)} = (cT_{(a)}, X^a_{(a)})$ along the lines $\mathcal{L}_{(a)}$. Then, a relationship between the global and local set of coordinates are given by the following mapping [29]

$$x^\mu(cT_{(a)}, X^i_{(a)}) = z^\mu_{(a)}(cT_{(a)}) + e^\mu_i(cT_{(a)})X^i_{(a)} + O[(X^i_{(a)})^2]\,, \tag{16}$$

where $e^\mu_i(cT_{(a)})$ denotes a triad of spacelike vectors $(i = 1, 2, 3)$ undergoing some geometric transport along the line $\mathcal{L}_{(a)}$ (e.g., Fermi-Walker, which defines locally non-rotating axes along the worldline). The last part $O(X^i_{(a)})^2$ stands for terms which are at least quadratic in the local space coordinate $X^i_{(a)}$ ([3]).

([3]) In virtue of the equivalence principle, the local reference systems for each source interfacing

In the gravitational weak regime where the local space-time domains do not overlap, the metric (2) can be expressed either in the global coordinates x^μ or in any of the N local coordinates $X^\mu_{(a)}$. The full four-dimensional coordinate transformation is just an extension of the usual Lorentz transformation.

Then, a priori equation (16) can be written in two equivalent forms, as $x^\mu(cT_{(a)}, X^i_{(a)})$ or $X^\mu(ct, x^i)$. The first transformation was developed in the Damour-Soffel-Xu formalism [29], the second one was described in the Brumberg-Kopeikin formalism [30].

For many applications it is advantageous to present the local gravitational potentials of a body-source as multipole series that usually converge outside the body. The metric outside shows two infinite sets of configuration parameters which are called the internal and external multipoles. The multipoles are purely spatial, 3-dimensional, symmetric trace-free (STF) Cartesian tensors residing on the hypersurface of constant coordinate time passing through the origin of the local coordinate chart. The internal multipoles characterize the gravitational field and the internal structure of the source itself. They consist of two types: the mass multipoles and the spin multipoles [31]. On the other hand, there are also two types of external multipoles: the gravitoelectric multipoles and the gravitomagnetic multipoles. Gravitoelectric dipole describes local acceleration of the origin of the local coordinates adapted to the source, whereas gravitomagnetic dipole are related to the angular velocity of rotation of the spatial axes of the local coordinates. Usually the external multipoles with rank ≥ 2 characterize tidal gravitational fields in the neighborhood of the source produced by other (external) bodies.

Given these premises about the sources, usually one adopts approximate schemes to get a useful solution of Einstein equation (1) in accordance with assumption (2) that linearizes the metric via perturbations. The purpose of the approximation schemes is indeed to expand the h term in power of small numerical parameters that allow to linearize equation (1). As far the Solar System is concerned, the so called post-Newtonian and the post-Minkowskian approximations are adopted (for a detailed description refer to [32]).

The post-Newtonian approximation method tries to model the solution of the Einstein equation as a Newtonian one plus small corrections to the prediction of the Newtonian theory, which represent the relativistic contributions. The correction term to $g_{\alpha\beta}$ is assumed to be everywhere small and admits an asymptotic expansion in power of $v^2/c^2 < GM/c^2R < GM/c^2L \ll 1$, where R is the distance from the barycenter, L the linear dimension, and v the internal or/and external velocity of the source. Thus the gravitational field is weak and of slow motion. In geometrized units where $G = M = L = 1$, one can say that the pN metric is a formal expansions in powers of $\epsilon \equiv 1/c$. This notation is kept for convention even when $G, M, L \neq 1$ [33]. In this sense all the correction terms, their time and space derivatives, and all the internal or/and external velocities become of the order of unity (or less) and the metric is built, with respect to a coordinate system (t, x^i) as:

$$g_{\alpha\beta}(t, x^i) = g_{(0)\alpha\beta}(t, x^i) + c^{-1}g_{(1)\alpha\beta}(t, x^i) + c^{-2}g_{(2)\alpha\beta}(t, x^i) + O(c^{-3}). \tag{17}$$

To the lowest pN order, with a linearized gravity and weak field slow motion assumption,

the external matter, should be given by tidal potentials whose expansions in powers of local spatial coordinates in the vicinity of the origin of the corresponding reference system start with the second order.

the corresponding spatial gravitational fields defined via threading or slicing are very closely related, although do not agree to full post-Newtonian order. However, the pN approximation provides crucial solutions to the problems of motion and gravitational radiation of systems of compact objects, in particular those for the motion of N point-like objects describing at the first pN approximation level the solar system dynamics.

If one needs to take into account the emission of gravitational waves (GWs), globally the equations of motion of a self-gravitating N-body system could be written in pN form, up to the 4th order, using one's favorite coordinate system in GR, as

$$\frac{d^2x_a}{dt^2} = A_a^N + \epsilon^2 A_a^{1PN} + \epsilon^4 A_a^{2PN} + \epsilon^5 A_a^{2.5PN} + \epsilon^6 A_a^{3PN} + \epsilon^7 A_a^{3.5PN} + \epsilon^8 A_a^{4PN} \tag{18}$$

The first term is the usual Newtonian acceleration of the a-source planet, beyond the Newtonian term is the historical Lorentz-Droste-Einstein-Infeld-Hoffmann (1pN correction). Up to the 2pN level the system is conservative, i.e., admits ten invariants associated with the symmetries of the Poincaré group. The first dissipative effect appears at the 2.5pN order representing the gravitational radiation-reaction force, tested by the observation of the secular acceleration of the orbital motion of the Hulse–Taylor binary pulsar PSR 1913+16. The 3pN expansion arises from the development of the LIGO-Virgo detectors and the need of accurate GW templates for inspiraling compact binaries. The dissipative 3.5pN term complete the 3pN one and so on [34]. Close to the merger event, the post-Newtonian expansion needs numerical-relativity computations.

As consequence of the above remarks, the metric tensor at 1pN order for the Solar System implicitly assumes that the perturbations from the gravitational-wave part of the metric are small and may be neglected in most cases. In such a case the construction above is valid only in the near zone, defined as a distance comparable to the wavelength of the gravitational radiation emitted by the system of sources generating a non stationary gravitational field ([4]). The use of the near-zone expanded retarded potentials may cause the appearance of divergent integrals, related to the fact that the expression of the metric represents only the first terms in the near zone expansion and says nothing about the behavior on the far-zone [33]. One may say that the pN approximation does not consider the hyperbolic property of the Einstein equations and does not provide any advanced and retarded solution of the field equations [35].

In order to match the requirements on the accuracy of the measurements and that of the chosen physical model in case of expansion (17), a minimal constraint is given by the virial theorem. As a matter of fact, all forms of energy density within the bound n-body system must not exceed the maximum amount of the gravitational potential in it, say, w. Then the energy balance requires that $|h_{\alpha\beta}| \leq w/c^2 \sim v^2/c^2$, where v is the characteristic relative velocity within the system ([5]). Since the latter is weakly relativistic ([6]), the $h_{\alpha\beta}'s$ are at least of the order of $(v/c)^2$ and the level of accuracy, to which it is expected to extend the calculations, is fixed by the order of the small quantity $\epsilon \sim (v/c)$. In practice, the perturbation tensor $h_{\alpha\beta}$ contributes with even terms in ϵ to g_{00} and g_{ij} (lowest order ϵ^2) and with odd terms in ϵ to g_{0i} (lowest order ϵ^3, [25, 26]);

([4]) The near zone of the Solar System, including only the Sun and Jupiter, is about 0.3 pc. All the visible objects are beyond these distance.
([5]) For a typical velocity ~ 30 km/s, $(v/c)^2 \sim 1$ milli-arcsec
([6]) This means that the length scale of the curvature is everywhere small compared to the typical size of the system; this implies also that $v^2/c^2 \ll 1$

its spatial variations are of the order of $|h_{\alpha\beta}|$, while its time variation is of the order of $\epsilon|h_{\alpha\beta}|$. Thus, the minimal requests in case of 1pN expansion are

$$h_{00} = O(c^{-2})\,, \qquad h_{0i} = O(c^{-3})\,, \qquad h_{00} = O(c^{-2})\,, \tag{19}$$

consistently in each of the local/global coordinate systems.

The pN equations (1) with gauge harmonic condition

$$g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu} = 0\,, \tag{20}$$

takes the form

$$\left(-\frac{1}{c}\frac{\partial^2}{\partial_t^2} + \nabla^2\right) w = -4\pi G\sigma + O(c^{-4}) \tag{21}$$

$$\nabla^2 w^i = -4\pi G\sigma^i + O(c^{-2}) \tag{22}$$

where w and w^i are the gravitational mass and mass current density, respectively. Mathematically they are related to the energy-momentum tensor $T^{\mu\nu}$ by the following integrals

$$\sigma = \frac{1}{c^2}\left(T^{00} + T^{ii}\right), \qquad \sigma^i = \frac{1}{c}T^{0i}. \tag{23}$$

The standard solution reads

$$w(t, x^i) = G\int \frac{\sigma(t, x')}{|x - x'|}d^3x' + \frac{1}{2c^2}G\frac{\partial^2}{\partial_t^2}\int \sigma(t, x')|x - x'|d^3x' \tag{24}$$

$$w^i(t, x^i) = G\int \frac{\sigma^i(t, x')}{|x - x'|}d^3x'. \tag{25}$$

In case of retarded time, we should solve the above integrals substituing t with $t_{ret} = t - |x - x'|/c$. However, within the first post-Newtonian approximation, these terms vanish as a consequence of the Newtonian mass conservation law.

Finally, the most suited 1pN form adopted for the metric in harmonic coordinates is

$$\begin{aligned} g_{00} &= -1 + 2\epsilon^2 w - 2\epsilon^4 w^2 + O(\epsilon^6) = -e^{-2w} + O(\epsilon^6) \\ g_{0i} &= -4\epsilon^3 w^i + O(\epsilon^5) \\ g_{ij} &= \delta_{ij}(1 + 2\epsilon^2 w) + O(\epsilon^4)\,, \end{aligned} \tag{26}$$

The exponential parametrization of the metric tensor has the effect of linearizing the field equations, as well as the transformation laws under a change of reference system. This linearity plays a crucial role in Damour-Soffel-Xu formalism, since in each frame there is a canonical and unique way to split locally the metric into (i) a part due to the gravitational action of a source itself (in the vicinity of the source), and (ii) an externally generated part, due to the action of the other bodies in the system (and that of the inertial forces in the accelerated local system).

Moreover, while standard gauges impose differential conditions to the metric, according to Damour-Soffel-Xu formalism, one can also require conformally Cartesian or isotropic spatial coordinates in all coordinate systems

$$h_{ij} = h_{00}\delta_{ij}\,, \tag{27}$$

which imposes restrictions to the various elements z^μ, e^μ_a, ξ^μ entering the map between the global coordinates and any of the local one. These conditions are purely algebraic and a convenient gauge flexibility is left open in the time coordinate. In this way one can easily deal either with harmonic space-time coordinates or "standard" post-Newtonian coordinates in all frames. The ADM gauge, instead, allows to eliminate the retarded-time dependence from h_{00} and h_{0i}, whereas it remains in h_{ij}.

The gauge condition at 1 pN implies now

$$\partial_t w + \partial_i w^i = 0(\epsilon^4). \tag{28}$$

The "potentials" w and w_i are specified according to the context of applications. If we limit our considerations to bodies with only mass (M), spin (S) and time-dependent quadrupole moment $\mathcal{I}_{ij}(t-r)$, at rest with respect to chosen coordinate grid, the canonical metric tensor simplifies as ([29])

$$\begin{aligned} w &= \sum_A \frac{GM_A}{r_A} + \frac{3}{2}\sum_A \frac{GI^A_{ij}}{r_A^3}\left(n^i_A n^j_A - \frac{1}{3}\delta^{ij}\right) + \dots \\ w^i &= \sum_A \frac{GM_A}{r_A}\mathbf{v}^i_A - \frac{1}{2}\sum_A \frac{G[\mathbf{S}_A \times \mathbf{n}_A]^i}{r_A^2} + \dots \end{aligned} \tag{29}$$

where we have defined $n^i = x^i/r$ and $v^i = dx^i/dt$, with $(n^i_A n^j_A - \frac{1}{3}\delta_{ij})$ being the STF part of the tensor product $n_A \otimes n_A$, and

$$I^A_{ij} = \int_{\mathrm{vol}(A)} \rho_A(x^i - x^i_A)(x^j - x^j_A)d^3x \tag{30}$$

is the quadrupole moment of the object A of mass density ρ_A occupying a certain volume vol(A). The retarded time contributions appear in the quadrupole term.

Let us assume for the principal axes

$$I_1 = I_2\,, \qquad I_1 - I_3 = J_2 MR^2\,, \tag{31}$$

where R can be identified with the equatorial radius of the body. We have then

$$\begin{aligned} \frac{G}{r^3}\left(I_1(n_1^2+n_2^2) + I_3 n_3^2 - \frac{1}{3}(2I_1+I_3)\right) &= \frac{GJ_2MR^2}{r^3}\left(\frac{1}{3} - n_3^2\right) \\ &= \frac{GJ_2MR^2}{r^3}\frac{1}{3}\left(1 - 3\cos^2\theta\right) \end{aligned} \tag{32}$$

implying

$$w \approx \frac{GM}{r} - \frac{GJ_2MR^2}{r^3}P_2(\cos\theta). \tag{33}$$

In the unit system where $c = M = L = 1$ the numerical value of Newton's constant is $G \ll 1$, so one can also look for solutions of equation (1) by expanding $g_{\alpha\beta}$ in powers of G, which leads to a formal hierarchy of inhomogeneous wave equations for the correction terms $h_{\alpha\beta}$. This approach is called the post-Minkowskian (pM) formalism, which assumes only the weakness of the gravitational field and the metric is expanded in function of the parameter $r_s/r = GM/c^2r \ll 1$ (r_s stands for the Schwarzschild radius of the source), namely a non-linearity expansion around the Minkowski background with no restriction on the parameter ϵ. For this reason is sometimes called the weak-field fast-moving approximation. So far the equations of motion of systems of particles have been obtained up to 2pM order (quadratic in G) [34].

In this sense the pM scheme preserves the hyperbolic character of (1) and the positions of the bodies are functions of retarded time. The first step in the hierarchy represents the well known linearized approximation to General Relativity introduced by Einstein in 1916 and represents a solution linearized with respect to G, but that includes all powers of (v/c).

Quoting Damour ([33], pag.156), let us summarize the difference between the pN and pM formulations as follows:

- the pN approximation methods are based on the idea of staying as close as possible to the conceptual framework of Newtonian theory separating (absolute) time and (absolute) space, endowed with an (auxiliary) Euclidean metric, and using instantaneous potential;
- the pM approximation methods tries to stay as close as possible to the conceptual framework of the Minkowskian interpretation of special relativity, using an absolute spacetime endowed with an (auxiliary) Minkowskian metric and retarded potential.

The pM expansion is uniformly valid, if the source is weakly self-gravitating, over all space-time. PM formalism could in principle permit a smooth and unique global solution of the light propagation equation from a source at an arbitrary distance to an observer located within the Solar System and get a unique prediction of the moment of time at which coordinates and positions of gravitating bodies should be fixed. In this respect it is more powerful than the post-Newtonian one, because each coefficient of the post-Minkowskian series can in turn be re-expanded in a post-Newtonian fashion. Therefore, a way to take into account the boundary conditions at infinity in the post-Newtonian series is to control first the post-Minkowskian expansion.

In conclusion, it worth remarking that although the pN technique is the standard approach to define a metric source inside the Solar System according to IAU resolutions, if aiming at sub-μas astronomy, it would be more effective to use the pM formalism in order to trace back the photon trajectory. This might limit many approximations, that could generate cumbersome matching terms in the definition of astronomical observables.

4·2. *Global Reference Systems for Astronomy.* – Metric (26) is in accordance with IAU resolution B1.3 [18] for the definition of the *Barycentric Celestial Reference System*

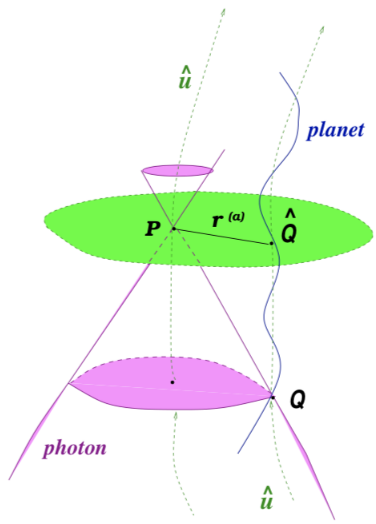

Fig. 6. – The metric terms at a general point P on the light trajectory depend on the a-th source of gravity when the latter was located at point Q of its trajectory at a retarded time $t' = t - r^{(a)}$, being $r^{(a)}$ the spatial distance between P and $\hat{Q}$ (event simultaneous to P).

(BCRS), which is the current reference system used by astronomers in order to refers the motion of the Sun with respect to the rest of the Universe, the ephemerides of planetary motions, ecliptic orbits and interplanetary spacecraft navigation.

As specified by the same resolution, BCRS should be used, with Barycentric Coordinate Time (TCB), as a global coordinate system. Instead, the Geocentric Celestial Reference System (GCRS) should be used, with Geocentric Coordinate Time (TCG), as a local coordinate system for the Earth, e.g. for the Earth's rotation and precession-nutation of the equator. Formally, the metric tensor of the BCRS does not fix the coordinates completely, leaving the final orientation of the spatial axes undefined. However, according to IAU 2006 Resolution B2, for all practical applications, the BCRS is assumed to be oriented according to ICRS axes, then kinematically non-rotating with respect to quasars. As remarked in the previous section, the BCRS-to-GCRS transformation was specified as an extension of the Lorentz transformation for the space and time coordinates that also contain acceleration terms and gravitational potentials [36]. A particular time-dependence of the alignment (namely a rotation) of the local spatial coordinate lines with respect to the global leads to an effacement of relativistic Coriolis effects in the local frame (a "dynamically non-rotating frame"). This effacement condition can simply be expressed as the vanishing of the central, external "gravitomagnetic field", or by the Fermi-transport condition of the vectorial basis with respect to the external metric. This means, for example, that the equations of motion of a satellite, with respect to the GCRS, will contain relativistic Coriolis forces that originates mainly from geodesic precession.

Thus, the actual astronomical reference systems are based on the approximate solutions of Einstein field equations for the Solar System. The BCRS metric allows to

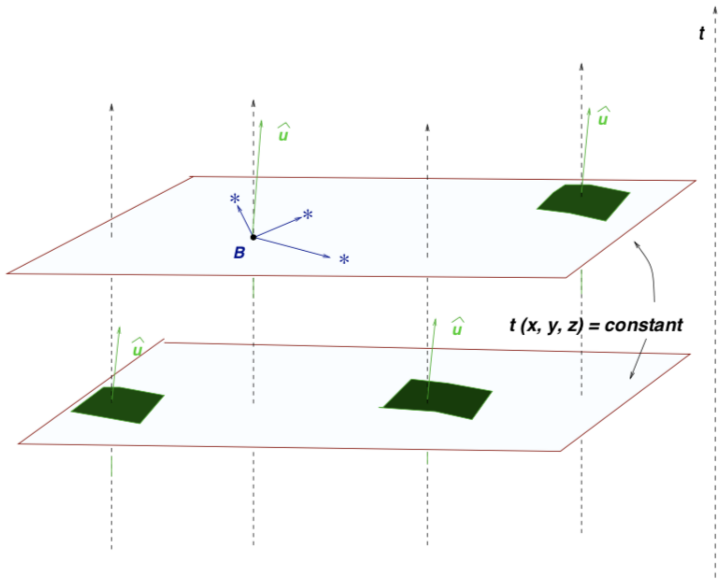

Fig. 7. – The local observer world line with respect to BCRS coordinates. The spatial axes of the BCRS point toward distant sources. The dashed lines are the world lines of observers at rest with respect to the barycenter. The rest space (green area) of $\hat{u}$ is locally superimposed to the space-like hypersurfaces with equation $t(x, y, z) = constant$.

configure the astrometric model and fix a reference frame with respect to which one describes the light trajectory, the "proper motion " of the stars and that of the observer.

As the Solar System can be considered an isolated weakly gravitational bound region of space, the BCRS can be identified requiring that a smooth family of space-like hypersurfaces exists with equation $t\,(x, y, z) = \text{constant}$ (see [37]). The function t can be taken as a time coordinate. On each of these $t\,(x, y, z) = \text{constant}$ hypersurfaces one can choose a set of Cartesian-like coordinates centered at the barycenter of the system and running smoothly as parameters along space-like curves which point to distant cosmic sources. The parameters x, y, z, together with the time coordinate t, provides a basic coordinate representation of the space-time according to IAU resolutions. Any tensorial quantity will be expressed in terms of coordinate components relative to coordinate bases induced by the BCRS.

The accuracy of the BCRS depends on the level of approximation of the metric used for its definition. This implies that for the future space missions aiming at the sub-μas or nano-arcsecond level of accuracy, a refinement of the pN approximations is needed accordingly.

4·3. *The observer's reference frames.* – Any observer performs her/his measurements by using a proper rest frame and a proper clock.

Consider a field of observers $\mathbf{u}$, a frame $\{e_\beta\}$ is termed adapted to $\mathbf{u}$ if $u^\alpha = e^\alpha_{\hat{0}}$ and $\{e_a\}$ $(a = 1, 2, 3)$ is orthogonal to $\mathbf{u}$, namely is the observer's rest frame. In literature one usually finds a Fermi-Walker frame, a generalization of the inertial Cartesian coordinates operationally fixed by a set of 3 mutually orthogonal gyroscopes. Comparing local rest frames requires non trivial mapping among them. The advantage to define Cartesian axes that match an instantaneous inertial frame, relies on the properties of the general Lorentz group connecting the transformation between them.

In general, a suitable proper reference frame of a given observer is a *tetrad* adapted to that observer. The tetrad is a set of four unitary mutually orthogonal four-vectors $\{\lambda_{\hat{\alpha}}\}$, which form a system of local Cartesian axes, and equips the observer with an *instantaneous inertial frame*; in fact (in virtue of the equivalence principle), it is defined in a point of the space-time as:

$$g_{\mu\nu}\lambda^{\mu}_{\hat{\alpha}}\lambda^{\nu}_{\hat{\beta}} = \eta_{\hat{\alpha}\hat{\beta}}. \tag{34}$$

The physical measurements made by the observer are obtained by projecting the appropriate tensorial quantities on the tetrad axes. The solution of a tetrad, however, is not trivial since it depends on the metric at each space-time point along the world line of the observer.

Concerning observes in the BCRS metric, at any space-time point, one can define a congruence of curves of unitary four-vector $\hat{\mathbf{u}}$ which is tangent to the world line of a physical observer at rest with respect to the spatial grid of the BCRS defined as:

$$\hat{u}^{\alpha}_{B} = (-g_{00})^{-1/2}\,\delta^{\alpha}_{0} = \left(1 + \frac{h_{00}}{2}\right)\delta^{\alpha}_{0} + \mathcal{O}\left(h^2\right). \tag{35}$$

This can be termed as the *barycentric static observer*, and its congruence defines a family of fiducial observers, namely those at rest with respect to the coordinates. The totality of these four-vectors over the space-time forms a vector field which is proportional to a time-like and asymptotically Killing vector field ([37]) (i.e. the metric asymptotically does not vary with time). To the order of accuracy required for Gaia-like measurements, for example, the rest space of $\mathbf{u}$ can be locally identified by a spatial triad of unitary and orthogonal vectors lying on a surface which differs from the t = constant one, but chosen in such a way that their spatial components point to the local coordinate directions as chosen by the BCRS. This frame will be called *local BCRS*, represented by a tetrad whose spatial axes (the triad) coincide with the local coordinate axes, but whose origin is the barycenter of the satellite ([38]):

$$\lambda^{\alpha}_{\hat{a}} = h_{0a}\delta^{\alpha}_{0} + \left(1 - \frac{h_{00}}{2}\right)\delta^{\alpha}_{a} + \mathcal{O}\left(h^2\right) \tag{36}$$

for $a = 1, 2, 3$. Then, any physical measurement refers to the local BCRS.

The world line of the center of mass of the Solar System belongs to this congruence while the world lines of the bodies of the Solar System would differ from the curves of the congruence by an amount which depends on the local spatial velocity relative to the center of mass.

Equivalently one can consider also the projection operators, especially when a tetrad solution is not an easy task. In this case the metric $P(u)_{\alpha\beta}$ projects on the rest-space of the observer $\mathbf{u}$ and allows to measure proper length, instead the metric $T(u)_{\alpha\beta}$ projects along the observer's world line and allows to measure proper time. Then the line element between two events in the space time can be reformulated as:

$$ds^2 = P(u)_{\alpha\beta}dx^\alpha dx^\beta + T(u)_{\alpha\beta}dx^\alpha dx^\beta \equiv dL_u^2 - c^2 dT_u^2, \tag{37}$$

from which we can extract the measurement performed by $\mathbf{u}$ of the infinitesimal spatial distance:

$$dL_u = \sqrt{P(u)_{\alpha\beta}dx^\alpha dx^\beta}, \tag{38}$$

and of the time interval

$$dT_u = -u_\alpha dx^\alpha. \tag{39}$$

Note that the above operators allow to define the observer's rest space and time dimension in neighborhood regions sufficiently small around a point of the observer's world-line, namely where the measurement domain does not involve explicitly space-time curvature. For example, the satellite instantaneous velocity with four-tangent vector $\mathbf{u}_s$ can be defined as

$$|\nu(u_s, u)| \equiv \lim_{\delta x \to 0} \frac{dL}{dT} = -\frac{(P(u)_{\alpha\beta}dx^\alpha dx^\beta)^{1/2}}{u_\alpha dx^\alpha}, \tag{40}$$

and the instantaneous velocity 4-vector can be written as:

$$\nu^\alpha(u_s, u) = |\nu(u_s, u)|\bar{\nu}^\alpha(u_s, u) \tag{41}$$

where $\nu^\alpha = P(u)^\alpha_\beta u_s^\beta$ and $\bar{\nu}(u_s, u)^\alpha$ is the unitary vector.

Then, the satellite four-velocity $\mathbf{u}_s$ can be expressed both in function of the projected spatial four-velocity as

$$u_s^\alpha = \gamma(u_s, u)[u^\alpha + \nu^\alpha(u_s, u)] \tag{42}$$

or in its coordinate form as

$$u_s^\alpha = \Gamma[\partial_0^\alpha + v_s^a \partial_a^\alpha], \tag{43}$$

where $v_s^a = dx^a/dt$, Γ is the normalization factor such as $u_s \cdot u_s = -1$ and γ is the Lorentz factor given by

$$\gamma(u_s, u) = (1 - |\nu(u_s, u)|^2)^{-1/2}, \tag{44}$$

so that the relative spatial four velocity of $\mathbf{u}_s$ with respect to an observer $\mathbf{u}$ results

$$\nu(u, u_s)^\alpha = \frac{1}{\gamma(u_s, u)}u^\alpha - u_s^\alpha. \tag{45}$$

At 1pN order the Γ factor of the satellite/observer 4-velocity (42) is

$$\Gamma = 1 + \left(\frac{1}{2}v_s^2 + h_{00}\right) + O(\epsilon^4), \tag{46}$$

whereas, the frame components $\nu(u_s, u)^{\hat{a}}$ and the associated Lorentz factor $\gamma(u_s, u)$ result

$$\begin{aligned} \nu(u_s, u)^{\hat{a}} &= (1 + 2h_{00})v_s^a\,, \\ \gamma(u_s, u) &= 1 + \frac{v_s^2}{2}\epsilon^2 + O(\epsilon^4)\,. \end{aligned} \tag{47}$$

An adapted tetrad frame to $\mathbf{u}_s$ is obtained by imposing

$$\lambda^{\alpha}_{\hat{0}} = u^{\alpha}_s\,, \tag{48}$$

and boosting the static tetard along $\mathbf{u}_s$, i.e. [38]

$$\lambda^{\alpha}_{b\hat{a}} = P(u_s)^{\alpha}_{\beta}\left[\lambda^{\beta}_{\hat{a}} - \frac{\gamma(u_s, u)}{\gamma(u_s, u) + 1}\nu(u, u_s)^{\beta}\left(\nu(u, u_s)^{\rho}\lambda_{\hat{a}\rho}\right)\right]. \tag{49}$$

The spatial triad (49) adapted to $\mathbf{u}_s$ at 1pN approximation is given, for example, by [39]

$$\lambda^{\alpha}_{\hat{a}} = \lambda^{\alpha}_{\hat{a}} + \epsilon v^a_s\left[1 + \epsilon^2\left(\frac{v_s^2}{2} + 3h\right)\right]\partial^{\alpha}_0 + \frac{1}{2}\epsilon^2 v^a_s v^b_s \partial^{\alpha}_b + O(\epsilon^4)\,. \tag{50}$$

As far as the time transformations is concerned, any observes in Solar System will carry out observations from a position in continuous motion around a point (e.g. L2), which is in turn in motion with respect to the Barycentric Celestial Reference System (BCRS).

Because of this motion and a non-zero gravitational potential at the observer location, his clock will deliver a time signal different from the BCRS coordinate time TCB that will be used as the time parameter in any data processing. As a consequence, one needs to determine the relationship between the two timescales ([7]).

Therefore, TCB = t is the time used nominally in the observation modeling. On the other hand the interval of time spanned between two events in the space-time, separated by the infinitesimal coordinate interval dx^{α}, along the worldline of the observer is defined via formula (39).

Then the relationship between the observer's proper time (T_u) and the BCRS coordinate time (t) is:

$$dT_u = -u_0\left[\left(g_{00} + g_{0i}v^i_s\right)dt + g_{0i}dx^i + g_{ij}v^i_s dx^j\right] \tag{51}$$

([7]) IAU/IUGG 2000 Resolution B1.3 has recommended as time coordinates TCB and TCG with origin in terms of International Atomic Time (TAI) in unit SI second. IAU 2006 Resolution B3 has clarified the definitions of Barycentric Dynamical Time (TDB) as a linear transformation of TCB, with constant coefficients, namely TCB - TDB = LB × (JD - 2443144.5003725)× 86400 s+ TDB_0 with LB = 1.550519768×10^{-8} and TDB_0 = 6.55 ×10^{-5} s . TCB is in use also for solar system ephemerides and pulsar timing.

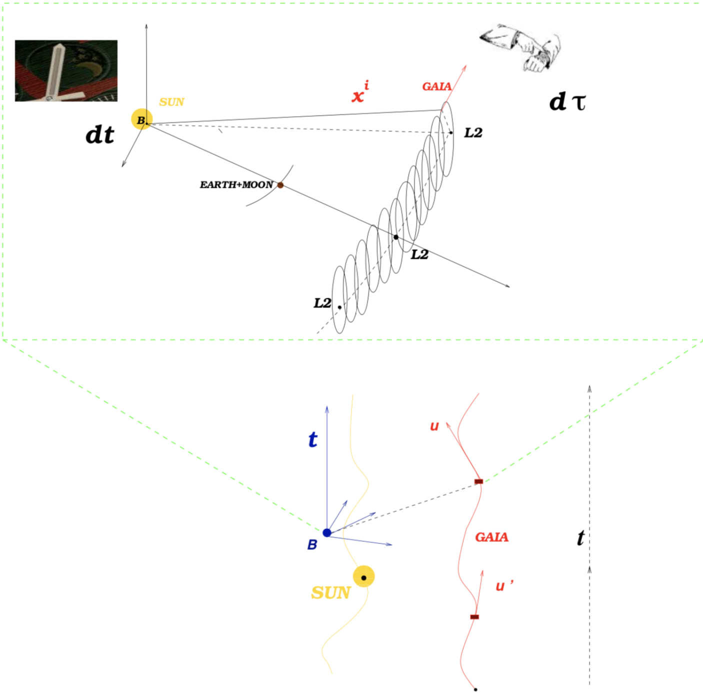

Fig. 8. – Worldline of an observer in L2 with respect to BCRS (Lissajous orbit).

where from the normalization condition ($u^\alpha u_\alpha = -1$)

$$u_0 = \left(- \frac{1}{g_{00} + 2g_{0i}v_s^i + g_{ij}v_s^i v_s^j} \right)^{1/2} \tag{52}$$

The specific form of this equation depends on the choice of the metric and the potentials defined in the metric terms. At 1pN approximation, with the IAU metric, equation (107) in subsection 5.3 gives an explicit example of a such time transformation.

4˙4. *The null geodesics*. – According to GR the light emitted by a star propagates via null geodesics, say, Υ_k with tangent null vector k^α.

The null geodesic with tangent vector field $k^\alpha = dx^\alpha / d\lambda$ satisfies the following equations:

$$\begin{aligned} k^\alpha k_\alpha &= 0 \\ \frac{dk^\alpha}{d\lambda} + \Gamma^\alpha_{\rho\sigma} k^\rho k^\sigma &= 0. \end{aligned} \tag{53}$$

Here λ is a real parameter on Υ_k and $\Gamma^\alpha_{\rho\sigma}$ are the affine connection coefficients of the given metric.

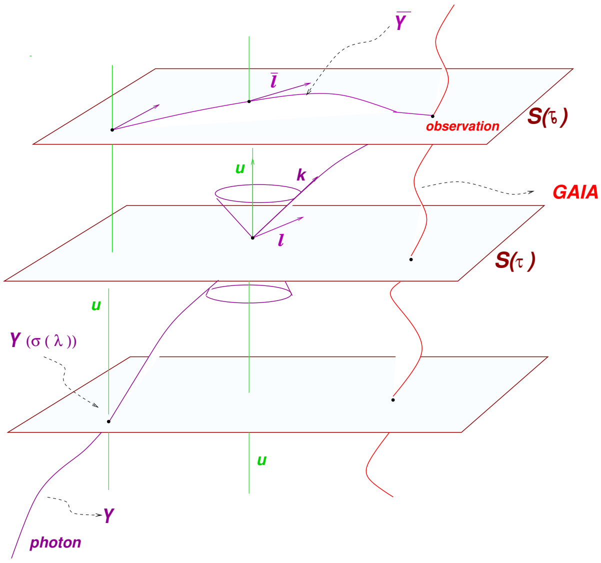

Fig. 9. – Space-like mapped geodesic on the slice at the time of observation.

A photon traveling from a distant star to the observer who lives in weakly bound gravitational fields, would see the spacetime as a time development of the $S(\tau(x^i, t))$ slices. Because of the unitary condition, the parameter σ, running along the normals $u^\alpha \equiv dx^\alpha/d\sigma$, is not constant on the slices $S(\tau)$ but it varies differentially with the position as $\sigma = \sigma(x^i, \tau)$; if one assumes that the spatial coordinates x^i are Lie-transported along the unique normal going through the point with those coordinates, the parameter along it will be function of τ only for each set x^i, i.e. $\sigma = \sigma_{(x^i)}(\tau)$. Assume that the trajectory starts at a point p_* on a slice $S(\tau_*)$ (say) and with spatial coordinates x^i_*. The light trajectory will *end* at the observer on a slice $S(\tau_o)$ and at a point with spatial coordinates $x^i_{(o)}$.

Although the slicing allows us to define a coordinate system which is centered at the barycenter of the bound system, each slice is not the local rest-space of a barycentric observer **u** ([8]). Any of these observers which are at rest with respect to the BCRS coordinates x^i, once intersected by the null ray will *see* the light signal along a spatial direction ℓ in her or his rest space given by

$$\ell^\alpha = \frac{\partial x^\alpha(\sigma(\tau_0))}{\partial x^\beta(\sigma(\tau))} k^\beta(\tau) = P^\alpha_\beta(\hat{u}_B) k^\beta(\tau) \tag{54}$$

where P is the operator which projects orthogonally to the $\hat{\mathbf{u}}_\mathbf{B}$'s. We can parameterize the curve Υ with the parameter σ which makes unitary the locally projected vector field ℓ which we term as $\bar{\ell}$, so $\bar{\ell}_\alpha \bar{\ell}^\alpha = 1$.

Then, relationship (54) acts as a map on a 4-dimensional manifold with image in a

([8]) We drop any symbol related to the barycentric observer for convenience.

3-dimensional one hence the coordinates of the target points ($\Upsilon \cap S(\tau_o)$) and those of the image points on $S(\tau_0)$ are, respectively, $\ell^\alpha = P^\alpha_\beta k^\beta$. The spatial projection of the null geodesic on the slice $S(\tau_o)$ at the time of observation is a curve denoted as $\bar{\Upsilon}_\ell$ and is naturally parameterized by λ. From $u^\alpha u_\alpha = -1$, it follows that:

$$\ell^\alpha = k^\alpha + u^\alpha (u_\beta k^\beta)\,. \tag{55}$$

Hence, being

$$d\sigma = -(u_\alpha k^\alpha) d\lambda\,, \tag{56}$$

one defines the new tangent vector field:

$$\bar{\ell}^\alpha \equiv -\frac{\ell^\alpha}{\left(u^\beta k_\beta\right)}. \tag{57}$$

In the same way one denotes

$$\bar{k}^\alpha \equiv -\frac{k^\alpha}{(u_\beta k^\beta)} \tag{58}$$

so that $\bar{k}^\alpha = \bar{\ell}^\alpha + u^\alpha$, which implies $\bar{\ell}^\alpha \bar{\ell}_\alpha = 1$.

The projection of $\mathbf{k}$ turns out into writing the null geodesic in function of the four-spatial unknown $\bar{\ell}^\alpha$. From (56) and (58), the second of equations (53) writes:

$$\begin{aligned} &\frac{d\bar{\ell}^\alpha}{d\sigma} + \frac{du^\alpha}{d\sigma} - (\bar{\ell}^\alpha + u^\alpha)(\bar{\ell}^\beta \dot{u}_\beta + \bar{\ell}^\beta \bar{\ell}^\tau \nabla_\tau u_\beta) \\ &\quad + \Gamma^\alpha_{\beta\gamma}(\bar{\ell}^\beta + u^\beta)(\bar{\ell}^\gamma + u^\gamma) = 0\,. \end{aligned} \tag{59}$$

In this equation the quantity $\bar{\ell}^\beta \bar{\ell}^\tau \nabla_\tau u_\beta$ can be written explicitly in terms of the expansion $\Theta_{\rho\sigma}$ of the congruence $C_{\mathbf{u}}$ [26], as:

$$\begin{aligned} &\frac{d\bar{\ell}^\alpha}{d\sigma} + \frac{du^\alpha}{d\sigma} - (\bar{\ell}^\alpha + u^\alpha)(\bar{\ell}^\beta \dot{u}_\beta + \Theta_{\rho\sigma} \bar{\ell}^\rho \bar{\ell}^\sigma) \\ &\quad + \Gamma^\alpha_{\beta\gamma}(\bar{\ell}^\beta + u^\beta)(\bar{\ell}^\gamma + u^\gamma) = 0 \end{aligned} \tag{60}$$

where $\Theta_{\rho\sigma} = \gamma^\alpha_\rho \gamma^\beta_\sigma \nabla_{(\alpha} u_{\beta)}$, with

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2} g^{\alpha\rho} \left(g_{\rho\beta,\alpha} + g_{\rho\alpha,\beta} - g_{\alpha\beta,\rho} \right)\,. \tag{61}$$

Note that this procedure is non local insofar as the measurement domain is finite and one can neglect the curvature contributions far from the local domain (as the case of an isolated Solar System). Some tensorial components may differ if written in terms of a vector $\bar{\ell}$ parallel propagated along a space-like geodesic (i.e. from the projected coordinate of the stars to the observer) and not along a null ray (see [26], section 9.7, for example). For instance, assuming a variation $du^\alpha/d\sigma = (\partial u^\alpha/\partial x^\beta)(dx^\beta/d\sigma)$, with respect to the (spatial) local line-of-sight $\bar{\ell}^\beta$, namely only along the space-like geodesic connecting to the observer the projected star position onto the hypersurface $S(\tau_o)$ of simultaneity at the time of observation, one gets

$$\frac{d\bar{\ell}^0}{d\sigma} = \bar{\ell}^i \bar{\ell}^j h_{0j,i} + \frac{1}{2} h_{00,0} \tag{62}$$

$$\begin{aligned}\frac{d\bar{\ell}^k}{d\sigma} &= \frac{1}{2}\bar{\ell}^k \bar{\ell}^i \bar{\ell}^j h_{ij,0} - \bar{\ell}^i \bar{\ell}^j \left(h_{kj,i} - \frac{1}{2} h_{ij,k} \right) \\ &- \frac{1}{2}\bar{\ell}^k \bar{\ell}^i h_{00,i} - \bar{\ell}^i \left(h_{k0,i} + h_{ki,0} - h_{0i,k} \right) \\ &+ \frac{1}{2} h_{00,k} + \bar{\ell}^k \bar{\ell}^i h_{0i,0}.\end{aligned} \tag{63}$$

Alternatively, it is possibile define the projected null geodesic in term of its energy and four-momentum via the kinematics of the congruence (1+ 3 splitting, timelike congruence).

The decomposition of the photon 4-momentum is given by the same formula (55)

$$k^\alpha = -(u \cdot k) u^\alpha + \ell^\alpha(u) \equiv E(k,u) u^\alpha + \ell^\alpha(u)\,, \tag{64}$$

with $\ell(u)^\alpha = P(u)^\alpha{}_\beta k^\beta$ orthogonal to u^α as previously defined. The trajectory is parametrized by σ such that

$$\bar{k}^\alpha = \frac{k^\alpha}{E(k,u)} = \frac{\mathrm{d}x^\alpha}{\mathrm{d}\sigma}\,, \tag{65}$$

and

$$\bar{\ell}(u)^\alpha = \frac{\ell(u)^\alpha}{E(k,u)} = \bar{k}^\alpha - u^\alpha\,, \tag{66}$$

which is related to the affine parameter λ by $\mathrm{d}\sigma = E(k,u)\mathrm{d}\lambda$, implying that

$$\bar{k}^\alpha \nabla_\alpha \bar{k}^\beta = -\frac{\mathrm{d} \ln E(k,u)}{\mathrm{d}\sigma} \bar{k}^\beta\,. \tag{67}$$

Then

$$\begin{aligned}\frac{\mathrm{d} \ln E(k,u)}{\mathrm{d}\sigma} &= \bar{\ell}(u)^\alpha \bar{\ell}(u)^\beta k(u)_{\alpha\beta} - \bar{\ell}(u)^\alpha a(u)_\alpha \\ &= -\bar{\ell}(u)^\alpha \bar{\ell}(u)^\beta \theta(u)_{\alpha\beta} - \bar{\ell}(u)^\alpha a(u)_\alpha\,,\end{aligned} \tag{68}$$

where the kinematical tensor $k(u)^\alpha{}_\beta = \omega(u)^\alpha{}_\beta - \theta(u)^\alpha{}_\beta$ and the acceleration vector $a(u)^\alpha$ are two spatial fields coming from the splitting of the covariant derivative of u^α, as presented in the previous section.

Null geodesic thus transforms into

$$\frac{\mathrm{d}\bar{k}^\alpha}{\mathrm{d}\sigma} + \Gamma^\alpha{}_{\mu\nu} \bar{k}^\mu \bar{k}^\nu - \left[\bar{\ell}(u)^\mu \bar{\ell}(u)^\nu \theta(u)_{\mu\nu} + \bar{\ell}(u)^\mu a(u)_\mu \right] \bar{k}^\alpha = 0\,, \tag{69}$$

The fact that $\bar{k}^\alpha = \bar{\ell}(u)^\alpha + u^\alpha$ implies again

$$\begin{aligned}(70)\qquad &\frac{\mathrm{d}\bar{\ell}(u)^\alpha}{\mathrm{d}\sigma} + \Gamma^\alpha{}_{\mu\nu}\bar{\ell}(u)^\mu(\bar{\ell}(u)^\nu + u^\nu) + a(u)^\alpha - k(u)^\alpha{}_\sigma\bar{\ell}(u)^\sigma \\ &\quad - \left[\bar{\ell}(u)^\mu\bar{\ell}(u)^\nu\theta(u)_{\mu\nu} + \bar{\ell}(u)^\mu a(u)_\mu\right](\bar{\ell}(u)^\alpha + u^\alpha) = 0\,.\end{aligned}$$

For the static observers with 4-velocity (35) the kinematical fields results

$$(71)\qquad a(\hat{u})^i = \partial_0 h_{0i} - \frac{1}{2}\partial_i h_{00}\,, \qquad \theta(\hat{u})_{ij} = \frac{1}{2}\partial_0 h_{ij}, \qquad \omega(\hat{u})_{ij} = \partial_{[i}h_{|0|j]}$$

so that equation (70) becomes

$$\begin{aligned}\frac{\mathrm{d}\bar{\ell}(\hat{u})^0}{\mathrm{d}\sigma} &= \bar{\ell}(\hat{u})^i\bar{\ell}(\hat{u})^j h_{0i,j} + \bar{\ell}(\hat{u})^i h_{0i,0}\,, \\ \frac{\mathrm{d}\bar{\ell}(\hat{u})^k}{\mathrm{d}\sigma} &= -\bar{\ell}(\hat{u})^i\bar{\ell}(\hat{u})^j\left(h_{ki,j} - \frac{1}{2}h_{ij,k}\right) - \bar{\ell}(\hat{u})^i(h_{k0,i} + h_{ki,0} - h_{0i,k}) - h_{k0,0} + \frac{1}{2}h_{00,k} \\ (72)\qquad &\quad + \bar{\ell}(\hat{u})^k\left[\frac{1}{2}\bar{\ell}(\hat{u})^i\bar{\ell}(\hat{u})^j h_{ij,0} + \bar{\ell}(\hat{u})^i\left(h_{0i,0} - \frac{1}{2}h_{00,i}\right)\right]\,.\end{aligned}$$

For zero angular momentum observers (called ZAMOs, 3+1 splitting) with 4-velocity [40]

$$(73)\qquad z^\alpha = \left(1 + \frac{1}{2}h_{00}\right)\partial_0^\alpha - h_{0i}\partial_i^\alpha$$

the kinematical fields is

$$(74)\qquad a(z)^i = -\frac{1}{2}h_{00,i}\,, \qquad \theta(z)_{ij} = \frac{1}{2}h_{ij,0} - h_{0(i,j)}\,, \qquad \omega(z)_{ij} = 0\,,$$

so that Eq. 70 becomes

$$\begin{aligned}\frac{\mathrm{d}\bar{\ell}(z)^0}{\mathrm{d}\sigma} &= 0\,, \\ \frac{\mathrm{d}\bar{\ell}(z)^k}{\mathrm{d}\sigma} &= -\bar{\ell}(z)^i\bar{\ell}(z^j)\left(h_{ki,j} - \frac{1}{2}h_{ij,k}\right) - \bar{\ell}(z)^i(h_{ki,0} - h_{0i,k}) + \frac{1}{2}h_{00,k} \\ (75)\qquad &\quad + \bar{\ell}(z)^k\left[\bar{\ell}(z)^i\bar{\ell}(z)^j\left(\frac{1}{2}h_{ij,0} - h_{0i,j}\right) - \frac{1}{2}\bar{\ell}(z)^i h_{00,i}\right]\,.\end{aligned}$$

The projection parallel and perpendicular to $\mathbf{u}$ (static or ZAMO) at a fixed point along the photon world line of the above formula allows to define a relative gravitational force as follows [40]

$$(76)\qquad F^{(\mathrm{rGf})}(k,u)^\alpha = E(k,u)\left[-a(u) + V(u)_{\rho\nu}\ell^\rho(k,u)\omega^\nu(u) - \theta(u)_{\rho\nu}\ell(k,u)^\nu\right]^\alpha\,,$$

where $V(u)^\alpha_{\rho\nu}$ is the spatial four volume related to $\mathbf{u}$. From the 1+3 form of the equations of motion one gets the work theorem and the Newton second law

If, instead, **u** is the four-velocity field supporting the world line of a single observer, the "relative gravitational force" cannot be defined in terms of acceleration, vorticity and expansion because u does not belong to a congruence. One may use, for example, a Frenet-Serret like procedure as indicated in [40].

Let us now summarize the above expressions for the pN form (26) of the metric and focus on formula useful for the actual astronomy. We consider as fiducial observers those at rest with respect to the BCRS coordinates:

$$\hat{u}_B^\alpha = \frac{1}{\sqrt{-g_{00}}}\partial_0^\alpha \approx \left(1+\epsilon^2 w\right)\partial_0^\alpha \,. \tag{77}$$

They form a congruence which one can characterize in terms of their acceleration, vorticity and expansion:

$$\begin{aligned} a(\hat{u}_B)^i &= -\epsilon^2\partial_i w + 2\epsilon^4[w\partial_i w - 2\partial_t w^i] + O(5) \\ a(\hat{u}_B)_i &= -\epsilon^2\partial_i w - 4\epsilon^4\partial_t w^i + O(5) \\ \omega(\hat{u}_B)^i &= 2\epsilon^3[\mathrm{curl}\,\mathbf{w}]^i + O(5) = 2\epsilon^3\epsilon^{ijk}\partial_{[j}w_{k]} + O(5) \\ \theta(\hat{u}_B)^{ij} &= \epsilon^3\delta^{ij}\partial_t w + O(5) = \theta(m)_{ij}\,, \\ \mathrm{Tr}\Theta(m) &= 3\epsilon^3\partial_t w + O(5) \end{aligned} \tag{78}$$

where $\partial_0 = \epsilon\partial_t$. The projector orthogonal to $\hat{u}_B$ results in

$$P(\hat{u}_B)^{ij} = (1-2\epsilon^2 w + 4\epsilon^4 w^2)\delta^{ij}\,,\quad P(\hat{u}_B)^{0i} = P(u_B)^{i0} = -4\epsilon^3 w^i\,. \tag{79}$$

The tracefree part of $\theta(\hat{u}_B)$, usually denoted by $\sigma(\hat{u}_B)^{ij}$, vanishes identically

$$\begin{aligned} \sigma(\hat{u}_B)^{0i} &= \sigma(\hat{u}_B)^{i0} = O(5) \\ \sigma(\hat{u}_B)^{ij} &= \theta(\hat{u}_B)^{ij} - \frac{1}{3}\mathrm{Tr}\Theta(\hat{u}_B)\,P(u_B)^{ij} = O(5) \end{aligned} \tag{80}$$

Furthermore, one often uses a "gravitomagnetic language," i.e. introduces of a gravitoelectric vector field

$$G^\alpha(\hat{u}_B) = a^\alpha(\hat{u}_B)\,, \tag{81}$$

and a gravitomagnetic vector field

$$M^\alpha(\hat{u}_B) = 2\omega^\alpha(\hat{u}_B)\,, \tag{82}$$

which are associated with the $\mathbf{u}_B$ family of observers and play a role similar to the electromagnetic corresponding quantities when a gravitational force is reintroduced in analogy with the Lorentz force.

An observer-adapted orthonormal spatial frame results in the following three vectors

$$\lambda(\hat{u}_B)_a^\alpha = \left(1-\epsilon^2 w + \frac{3}{2}\epsilon^4 w^2\right)\partial_a^\alpha - 4\epsilon^3 w_a\partial_0^\alpha \tag{83}$$

with $\hat{u}_B \cdot \lambda(\hat{u}_B)_a = 0$ and $\lambda(\hat{u}_B)_a \cdot \lambda(\hat{u}_B)_b = \delta_{ab}$.

If one refers to the family of ZAMOs, which is vorticity free, the associated 4-velocity is

$$z^\alpha \approx \left(1 + \epsilon^2 w + \epsilon^4 \frac{1}{2} w^2\right) \partial_0^\alpha + 4\epsilon^3 w^a \partial_a^\alpha \,. \tag{84}$$

The acceleration and expansion tensor are given by

$$\begin{aligned} a(z)^i &= -\epsilon^2(1 - 2\epsilon^2 w)\partial_i^\alpha w + O(5) \\ a(z)_i &= -\epsilon^2 \partial_i^\alpha w + O(5) \\ \theta(z)_{ij} &= \epsilon^3[\delta_{ij}\partial_t^\alpha w + 4\partial_{(i}^\alpha w_{j)}] + O(5) = \theta(Z)^{ij} \,, \\ \mathrm{Tr}\Theta(Z) &= \epsilon^3[3\partial_t w + 4\partial_a^\alpha w^a] + O(5) \,, \end{aligned} \tag{85}$$

whereas the vorticity tensor vanishes identically. The gravitoelectric and gravitomagnetic vector fields are given by

$$G(z)_a^\alpha = \epsilon^2 \partial_a^\alpha w + O(6) \,, \tag{86}$$

and

$$M(z)_a^\alpha = -4\epsilon^3 \epsilon^{abc} \partial_b^\alpha w_c + O(5) \,, \tag{87}$$

respectively.

The adapted spatial triad in such a case is given by

$$\lambda^\alpha(Z)_{\hat{a}} = \frac{1}{\sqrt{g_{aa}}} \partial_a^\alpha \approx \left(1 - \epsilon^2 w + \epsilon^4 \frac{3}{2} w^2\right) \partial_a^\alpha \,. \tag{88}$$

4˙5. *The GR measurements protocol.*. – A measurement process can be considered "local " when its space-time domain does not explicitly involve curvature.

Quoting de Felice and Bini ([40, 41]), a measurements protocol can be defined following these steps: i) specify the phenomenon under investigation; ii) identify the covariant equations which describe the phenomenon; iii) identify the observer making the measurements; iv) chose a frame adapted to that observer allowing space-time splitting into the observer's space and time; v) understand the locality properties of the measurement under consideration (namely, whether it is local or non-local with respect to the background curvature); vi) identify the frame components of the quantities that are the observational targets; vii) find a physical interpretation of the above components following a suitable criterion; viii) verify the degree of the residual ambiguity, if any, in the interpretation of the measurements and decide the strategy to evaluate it (for example, comparing to what already is known in special relativity or in non-relativistic theories).

Requirements v-viii lead us to investigate the different techniques to solve null geodesic and how to implement its unknown, i.e. the four tangent null vector, in astronomical observations for different observes immersing in the local varying gravitational fields. Next section will focus only on the astrometric observable.

5. – Relativistic Astrometry

Heron of Alexandria had the first idea, later taken up by Fermat for the refractivity, of measuring the time that light takes to go from one point to another and to be reflected by a flat surface. He deduced that light travels at a constant speed and uses the path that requires minimal time [6].

The first measurement from Earth of the time-of-flight of light was made by Ole Römer, while working in Paris: he estimated the finiteness of the light velocity by observing the eclipses of Jupiter's moons.

General Relativity today establishes that light passing close to a spherical mass M, is deviated in space-time by an angle:

$$\delta\phi = \frac{4GM}{c^2 b} = 1.75'' \left(\frac{M}{M_\odot}\right), \tag{89}$$

where b is the distance of closest approach.

The number that factorizes the deviation is due to the perturbative term h_{00} of the Sun; such a deviation influences the finite light velocity and increases with mass, acting like a lens and decreases farther away from the lens. Alternatively equation (89) can be written as:

$$\delta\phi = \frac{2(1+\gamma)GM}{c^2 b} \tag{90}$$

where the parameter γ is equal to one for GR and it indicates how much the curvature deviates a unit rest mass.

Newton himself wrote in his 1704 opus Opticks: "Do not Bodies act upon Light at a distance, and by their action bend its Rays" but there is no prove that he estimated the magnitude of the effect [42]. The first calculation was published by German mathematician Johann Georg von Soldner in 1804, who predicted the value qualitatively just using Newton's theory of gravity.

Initially Einstein derived the above quantity with a factor 2 in order to be consistent with Newtonian theory. Thus Einstein asked to Hale if a deviation of less than an arcsecond, i.e. ~ 0.86", could be detected close to the Sun. The answer was affirmative in case of an eclipse to avoid Sun's starlight. Fortunately the first attempt to measure the effect amounting to ~ 0.86" failed. Freudelinch, the astronomer in charge to measure it, did the expedition in Crimean Penisula on August 1914. He was imprisoned and released after the ecplise. Then, just by chance Einstein's mistake did not become a proof against his theory. After the publication in 1915 of the GR theory, Einstein corrected the wrong factor into 4 and Dyson proposed to test the effect with the total solar eclipse in 1919. The eclipse's path of totality passed from northern Brazil across the Atlantic to West Africa. At totality, many stars would have been visible near the eclipsed disk at different angles by the Sun.

In May 29, 1919, Dyson and Eddington tested the deflection due to the solar mass on Hyades cluster in the island of Principe off the coast of West Africa. Another team was in Sobral, in Brazil, lead by Andrew Crommelin (Royal Greenwich Observatory, London) to provide an alternative experiment in case of bad weather. The more stars measured, the better corrections for systematic errors and reduction of random ones obtained.

The chronicle tells that there was poor weather in Principe and Eddington managed to make fewer measurements than expected. Since he could not stay in Principe long enough to measure the star positions on his plates on site, he did the analysis after in England and obtained $\delta\phi = 1,16 \pm 0,40$ arcsecond. Crommelin, instead, had much better conditions in Brazil. Despite of technical issues with instrumentation that left many plates blurred, the actual measurements were $\delta\phi = 1,98 \pm 0,16$ arcsecond, noticeably closer to the Einstein prediction than to the Newtonian one [24]. All of the 1919 eclipse measurements are tabulated in [43].

Therefore the year 2019 marks a century since the famous experiment that confirmed worldwide the Einstein theory. Light deflection measurements due to the Sun has beard a great scientific impact for astrophysics even if it was performed with a 20% relative error because of the weakness of the gravitational field of the Sun and few timing events available (i.e. few stars close to the Sun during the solar eclipse). It has been repeated only six times, the last one in 1973. Photographs plates were used and the accuracy of star position was never greater than 10%. So far, VLBI light deflection measurements have reached the 0,01%, while in space Hipparcos reached 0.1% followed by the best measurement performed by the Cassini spacecraft, which provided a value in agreement with GR to 0,001% [44].

Conversely, it took almost sixty years (in 1979) to get a first direct evidence of a lensing event, i.e. a pair of quasars separated by $\sim 6''$ [45], as the effect was too small to be detected via the available instrumentation. Curiously gravitational lensing was not devised by Einstein, but by an amateur engineer, Rudi W. Mandl, as Einstein himself reported [46].

Lensing determination requires to implement the positions of the source and the lens in order to evidence the deviation effects. Such deviations are essentially in modern astrometry since they become relevant locally in angle measurements accurate up to μas (see table I).

Therefore, rigorous models of Gaia-like observables consistent with the precepts of GR and the theory of measurements (like, e.g., the implementation of a local relativistic observer), and relativistic consistency of the whole data processing chain become all indispensable prerequisites for a physically correct determination of directions (coordinates), distances (parallaxes), and radial and proper motions.

The relativistic consistency is needed for any mission which aims to achieve high accurate measurements, making Gaia a benchmark case in the field of relativistic astrometry.

5˙1. *The astrometric observable.* – As emphasized many times through the text, thanks to the launch of the Gaia mission, GR is now at very core of Astrometry. The challenging astrometric goal of the mission is the census of about 2 billion individual stars comprising the Milky Way (MW) to be materialized in the form of a catalog listing, from a prescribed set of observables, the the five fundamental astrometric parameters: coordinates (α, δ), parallax ϖ , and the proper motions $(\mu_\alpha \ , \mu_\delta)$.

Once solved the system, in order to transform the BCRS coordinates to those of the astrometric catalogue (α, δ) at a given epoch t_0, it is enough to apply the well-known transformations

$$\mathbf{x}_* = \frac{1}{\varpi}(\cos\alpha\cos\delta, \sin\alpha\cos\delta, \sin\delta), \tag{91}$$

and the proper motion can be deduced by a Taylor expansion of the first order

$$\alpha(t) = \alpha(t_0) + \mu_\alpha(t - t_0) + O(\Delta t^2), \qquad \delta(t) = \delta(t_0) + \mu_\delta(t - t_0) + O(\Delta t^2). \tag{92}$$

While in classical astrometry these quantities are well defined, in GR they must be interpreted in order to be completely compliant with the relativistic framework.

As described in the previous sections, the first step of the relativistic modeling is to identify the gravitational sources and then fix the background geometry compatible with the basic physical assumptions, consequently choose a suitable coordinate system to label spacetime points. These steps allow us to identify a reference frame to which one refers light trajectory, stellar motion, and motion of the observer.

The localization of a star is determined through the collection of the light emitted by the star in different observation times. In the previous section we discuss how to handle light propagation through the Solar System, writing the differential equations which would enable us to reconstruct the light trajectory from a distant star to the observer. A mathematical boundary condition, which links the observation data to the above star parameters, is the prerequisite to solve uniquely such differential equations where the measured local line of sight, $\bar{\ell}^\alpha$, is the unknown. For this step we need both the satellite frame to define the observations and the barycentric one to express all the coordinate tensorial components.

The required formula which relates the observables and the data to the astrometric parameters, namely the *observables* $\mathbf{e}_{\hat{a}}$, at time t_0, is the angles between the directions of the incoming photon, when it hits the telescope, and the three spatial directions of a satellite adapted frame.

Let $\mathbf{u}_s$ be the vector field tangent to the satellite's world line and $\{\lambda_{\hat{a}}\}$ (where $\hat{a} = 1, 2, 3$) a spacelike triad carried by the observer. The angles $\psi_{(\lambda_{\hat{a}},\bar{k})}$ that the incoming light ray forms with each of the triad direction, is given by the following direction cosine [26]:

$$\cos\psi_{(\lambda_{\hat{a}},\bar{k})} \equiv \mathbf{e}_{\hat{a}} = \frac{P(u_s)_{\alpha\beta}\bar{k}^\alpha\lambda^\beta_{\hat{a}}}{(P(u_s)_{\alpha\beta}\bar{k}^\alpha\bar{k}^\beta)^{1/2}(P(u_s)_{\alpha\beta}\lambda^\alpha_{\hat{a}}\lambda^\beta_{\hat{a}})^{1/2}} \tag{93}$$

where no sum is meant over $\hat{a}$ and it generalizes the "newtonian" angle between two straight direction by introducing the tensorial quantities and the scalar product with respect to the metric. The above equation can be written more conveniently as

$$\mathbf{e}_{\hat{a}} = \frac{\bar{\ell}^\alpha_{(o)}\lambda_{\alpha\hat{a}} + u^\alpha\lambda_{\alpha\hat{a}}}{u_{s\alpha}\bar{\ell}^\alpha_{(o)} + u_\alpha u^\alpha_s} \tag{94}$$

where $\bar{\ell}_{(o)} \equiv \bar{\ell}(\tau_o)$ and considering that $P(u_s)_{\alpha\beta}\lambda^\alpha_{\hat{a}}\lambda^\beta_{\hat{a}} = 1$; expression (94) is a matrix equation where the unknowns are the spatial directions $\bar{\ell}^k_{(o)}$ of the photon at the time of observation; they can be singled out as

$$\bar{\ell}^i_{(o)}[u_{si}\mathbf{e}_{\hat{a}} - \lambda_{i\hat{a}}] = u^\alpha\lambda_{\hat{a}\alpha} - \mathbf{e}_{\hat{a}}(u^\alpha_s u_\alpha). \tag{95}$$

The direction of the light ray, as it is observed from within the satellite, depends on the motion of the latter relative to the center of mass which we take as origin of the spatial coordinates on each slice. This dependence causes the stellar aberration. Did the satellite not move with respect to the spatial coordinate grid, namely if $u^\alpha_s = \hat{u}^\alpha_B$, then (95) would become $\bar{\ell}^i_{(0)}\lambda_{i\hat{a}} = \mathbf{e}_{\hat{\mathbf{a}}}$ as expected.

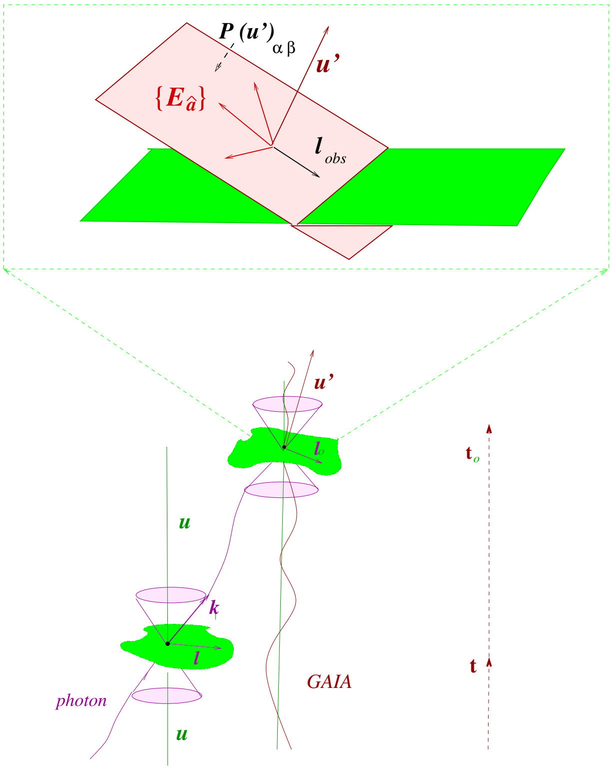

Fig. 10. – Projected light-of-sight with respect to local barycentric observer and the attitude frame of an observer.

The spatial axes of the tetrad, $\lambda^{\alpha}_{\hat{a}}$, are used to model the observer's rest frame. The Gaia's attitude frame is specified by the following spatial rotation of the adapted triad

(96) $$E^{\alpha}_{\hat{a}} = \mathcal{R}(\alpha_i)_{\hat{a}}{}^{\hat{b}} \lambda^{\alpha}_{\hat{b}} \,,$$

then, in term of such a set of three orthonormal spacelike vectors, comoving with the satellite, the cosine is given as follows:

(97) $$\cos\psi_{(\hat{a},\bar{k})} = \frac{P(u_s)_{\alpha\beta}\bar{k}^{\alpha}E^{\beta}_{\hat{a}}}{[P(u_s)_{\alpha\beta}\bar{k}^{\alpha}\bar{k}^{\beta}]^{1/2}}$$

Actually, each astrometric observation can be translated into the measured abscissa along the scanning direction on the x-y focal plane which can be modeled as a function $f(x_*, x_C, x_I, x_G)$ of the stellar coordinates x^i_*, as well as of those of the satellite attitude x^i_C, instrumental x^i_I, and of another kind called global x^i_G which include the PPN parameter γ [47]. Moreover, the attitude reconstruction requires the inclusion of a certain number of across-scan measurements, namely those of the coordinate orthogonal to the same plane.

Then, in the IAU framework, the actual observables of Gaia in power of ϵ (i.e. assuming a pN metric) are along and across the scanning direction rispectively:

$$\cos\phi = \frac{\cos\psi_{(\hat{1},k)}}{\sqrt{1-\cos^2\psi_{(\hat{3},k)}}} = \cos\psi^{(0)} + \epsilon\cos\psi^{(1)} + \epsilon^2\cos\psi^{(2)} + \epsilon^3\cos\phi^{(3)} \tag{98}$$

$$\sin\zeta = \cos\psi_{(\hat{3},k)} = \cos\psi^{(0)}_{(\hat{3},k)} + \epsilon\cos\psi^{(1)}_{(\hat{3},k)} + \epsilon^2\cos\psi^{(2)}_{(\hat{3},k)} + \epsilon^3\cos\psi^{(3)}_{(\hat{3},k)}. \tag{99}$$

For their explicit expressions refer to [39]. Merging repeated observations of the same objects from different satellite orientations and on different times allows to estimate their angular positions, parallaxes, and proper motions, *i.e.* the actual materialization of an absolute Reference Frame. This process is conventionally called Astrometric Sphere Reconstruction.

Nevertherless, since Gaia collects a large number of observations, a large overdetermined system is produced that can be linearized around a given starting point x^i_o and solved with the least square method. Then, in order to determine the unknowns, we have to consider the solution of a linear system of equations such as

$$\mathbf{b} = \mathbf{A}\delta\mathbf{x} \tag{100}$$

where $\mathbf{b} = \{-\sin\phi_j\delta\phi^j\}^T$ ($j = 1, ..., n_{\rm obs}$), $\delta\mathbf{x} = \mathbf{x}_{\rm un} - \mathbf{x}_o$ is the unknowns vector, and $\mathbf{A}$ is the $n_{\rm un} \times n_{\rm obs}$ design matrix of the system whose coefficients are $a_{ji} = \partial f/\partial x^i$. The variation of (97) with respect to any parameters p_i is

$$\delta\left(\cos\psi_{(\hat{a},k)}\big|_{\sigma=0}\right) = \sum_j \frac{\partial f_{\hat{a}}(p_i)}{\partial p_j}\delta p_j\,, \tag{101}$$

which has been computed in the case of IAU metric in [39]. Direct solution, no block-adjustment, of such a system via an iterative method provides estimates of variances [48, 49, 50].

The direction cosines being physical quantities not depending on the coordinates, are a powerful tool to compare the astrometric relativistic models.

It is worth stressing that the direction cosines (i.e. the astrometric measurements strictly dependent on the mathematical characterization of the attitude) taken as a function of the *physical* local line-of-sight (i.e. the quantity defined in the rest space of the observer), at the time of observation, allow to fix the boundary conditions needed to solve the null geodesic and to determine uniquely the star coordinates.

5˙2. *Relativistic astrometric models at work.* – Having a control on the error budget at the level of μas for Gaia is even more critical if one considers that the solar system generates several varying perturbations of the order of the measurement accuracy in different observation times and for different satellite positions.

Since high astrometric accuracy necessarily translates into the need of using GR, a fully self consistent astrometric relativistic models suitable to describe correctly such observations should be available possibly according the GR measurement protocol.

Thanks to Gaia, nowadays there exist several models conceived for the above task and formulated in different and independent ways ([37, 39, 51, 52, 53, 54, 55, 56, 57], and references therein). Their multiple and simultaneous availability is needed in order to rule out possible spurious contributions (especially systematic errors due to, e.g., insufficient instrumental modeling) and helps to consolidate the results via an independent mutual cross-checking. From the experimental point of view, in fact, global relativistic astrometry opens a largely uncharted domain and cross-checked theoretical models are of capital importance to interpret the same data.

As a Parameterized Post Newtonian (PPN) extension of a seminal study conducted in the framework of post-Newtonian (pN) approximation of GR [53], Klioner produced a PPN model for relativistic astrometry, accurate to 1 μas, in which the finite distance and the angular momentum of the gravitational deflector are included, linked to the motion of the observer and the source in order to consider the effects of parallax, aberration, and proper motion [58]. This model reproduces in a relativistic framework the classical approach of astrometry, where the quantities which ultimately enter the catalogue are referred to a global inertial reference system, taking into account, one by one and independently from each other the astrometric parameters. This approach uses harmonic coordinates and assumes at 1pN order all the necessary specifications of background spacetime, photon and satellite orbit which pertain to the Gaia mission. This model, GREM, is considered the baseline for the Gaia data reduction.

On the other hand, since the pM formalism has the ability to make calculations at any desired order of approximations in (v/c) and the same IAU resolutions B1.3 [17] can be adopted, Kopeikin&Schafer [59] solved the metric tensor in function of the retarded Lienard-Weichert potentials and later Kopeikin&Mashhoon [55] included all relativistic effects related to the gravitomagnetic field, produced by the translational velocity/spin-dependent metric terms. Both studies rewrite the null geodesic as a function of two independent parameters and solve the light trajectory as a straight line (Euclidean geometry) plus corrections in the form of integrals, containing the perturbations encountered, from a source at an arbitrary distance to an observer located within the Solar System.

Another model is based on the Time Transfer Functions (TTF), which conceptually stands as an application of Synge World Function Ω [60], an integral approach based on the principle of minimal action (see [61] and references therein). While the World Function is an implicit equation of the trajectory of the photon through the metric, related to the spatial length between two simultaneous spacetime events, the TTF formalism, based on a solution of an eikonal equation adapetd to a perturbative expansion, focus on the light propagation between two points at finite distance, the so-called coordinate time of flight $\mathcal{T}_{e/r}$ of an electromagntic signal between the event of emission $x_A = (ct_A, x_A)$ $(\mathcal{A})$ and the event of reception $x_B = (ct_B, x_B)$ $(\mathcal{B})$. In doing so the TTF method does not provide the full trajectory of light, however the evaluation of $\mathcal{T}_{e/r}$ is important in various fields of astronomy and space science, such as the positioning of space probes or the lunar laser ranging. The main useful formula is the ratio of the spatial and temporal covariant components of the tangent vector to the null-geodesic at its emission and reception point, i.e.

$$(102)\qquad \left(\widehat{k}_i\right)_B \equiv \left(\frac{k_i}{k_0}\right)_B = -c\,\frac{\partial \mathcal{T}_e}{\partial x^i_B} = -c\,\frac{\partial \mathcal{T}_r}{\partial x^i_B}\left[1-\frac{\partial \mathcal{T}_r}{\partial t_B}\right]^{-1}, \quad \frac{k_0^B}{k_0^A} = 1-\frac{\partial \mathcal{T}_r}{\partial t_B},$$

where $\left(\widehat{k}_i\right)_B$ is termed the light direction triple at reception event. Similarly, one can define the light direction triple $\left(\widehat{k}_i\right)_A$ at emission event. Given $\mathcal{T}_e$ and $\mathcal{T}_r$ as two distinct (coordinate) Time Transfer Function, the time of-flight is defined as

$$t_B - t_A = \mathcal{T}_e(t_A, x_A, x_B) = \mathcal{T}_r(t_B, x_A, x_B)\,. \tag{103}$$

The TTF have been formulated as a general pM series of ascending powers of G [61]. Nevertheless, this formalism has been only recently applied to light propagation in a time dependent space-time, in particular in comparison with the RAMOD approach, which relies on the GR formalism presented in the previous sections.

RAMOD stands originally for Relativistic Astrometric MODel, conceived to solve the inverse ray-tracing problem through a family of models of increasing intrinsic accuracy in a general relativistic framework ([62, 63, 37, 51, 39]), not constrained by a priori approximation and not necessarily applicable only to astrometry. In fact, RAMOD approach to modeling tries to be as close as possible to the concept of curved geometry of GR and to foliate the spacetime in order to define the satellite's proper time and a rest space, i.e. to reconstruct all the *simultaneous observations* in a curved spacetime as exposed in section 4. The model works according to the measurement protocol in GR and with any metric solution in the form of (1), the only constraint is that it must represent a weakly relativistic gravitational field. Indeed RAMOD does not specify neither the coordinates nor the perturbed metric and mainly develops the so called observer's point of view (having in mind that whatever measurement is performed it involves a given observer). Hence the results provided by RAMOD are general enough to be easily adapted to special astrophysical situations (light deflection by a single or more bodies, satellite orbit geodesic or not, doppler observable, global and differential astrometric observations and so on). In RAMOD any physical measurement could refer to the local BCRS or ZAMOs.

Note that the World function is defined as the measurement of length of space-like geodesic, between an event P and the observer A , which strikes the observer's worldline orthogonally at the point of observation A. Then it connects simultaneous events and it is easy to check [26] that the projection operator $P(u)$ and $T(u)$ can be recovered from the Synge function definition. In this respect TTF application is very close conceptually to RAMOD framework, based on projected (or mapped) tensorial quantities in order to define a measurement.

The classification based on increasing levels of accuracy introduced in RAMOD turns out to be extremely useful for the implementation of the relativistic models inside the same framework and the testing of them through consistent internal checks at different levels of accuracy, allowing also a very simple procedure to identify where the possible discrepancies could arise.

In RAMOD the major difficulty is to integrate the set of non linear coupled differential equations (70), which are obtained once one projects the four-momentum of the photon on the rest space of the observer implying a new four dimensional spatial tangent vector as unknown, to trace back the star positions and which include, a priori by definition, any varying background property of a curved space-time.

One can proceed to solve such a set numerically if the form of the metric is an implicit function of the retarded time and no suitable expansion of the metric can be applied, or analytically adopting the same perturbative and iterative solutions of the alternative approaches indicated above. Indeed, whether analytical or numerical, the solution of those equations aims to contain "globally" all relativistic perturbations affecting a photon moving along its trajectory.

Therefore different models of the family can be adopted according to the needs of a specific problem or measurement in the Solar System. For example, in principle

RAMOD3a [52], a static model at the ϵ^2 level, should be sufficient for the sphere reconstruction at the Gaia accuracy, and RAMOD3b/4a ($O(\epsilon^3)$) will be used if the retarded distances need to be included appropriately, because it is able to consider them in a rigorous way. As far as the light deflection is concerned, one expects indeed that the velocity contributions become relevant in affecting light propagation in the case of close approach when general relativistic effects become of the order of Gaia's expected accuracy together with the multipolar structure of the source. Dynamical models like RAMOD4 ($O(\epsilon^4)$) could be suitable, for example, for taking into account gravitomagnetic relativistic effects. IAU model solutions suitable for Gaia are reported in [39], developed for the data validation and comparison with GREM approach. Recent developments to include retarded distance effects in a pM scenario, due to the moving bodies of solar system, are in progress and in comparison with the Time Transfer approach.

Then, the advantage of RAMOD is that the solutions interface numerical and analytical relativity at different levels of accuracies. This turns out quite necessary, since in the Gaia context the practical realization of the celestial sphere is an extremely challenging problem as the high quality of the observations and the large number of unknowns involved. The huge number of unknowns and observations puts this task at the forefront of High-Performance Computing problems [49].

For this reason inside the Consortium constituted for the Gaia data reduction (Gaia CU3, Core Processing, DPAC), two procedure have been developed: i) the GREM one baselined for the Astrometric Global Iterative Solution for Gaia (AGIS, [64]), and ii) RAMOD implemented in the Global Sphere Reconstruction (GSR) of the Astrometric Verification Unit (AVU) at the Italian data center (DPCT). GSR acts as an independent verification unit for AGIS and is now part of the European excellence in the field of modeling photon paths in low gravity and of the Gaia effort of building, with two independent methodologies, the first fully relativistic reconstruction, both differential and absolute, of the celestial sphere. Specifically for this purpose, and thanks to the continued support of INAF and the Italian Space Agency, the Italian DPCT, as part of the European-wide effort for the reduction of the Gaia data was established through a specific ASI contract via a partnership between OATo (for INAF) and ALTEC S.p.A. in Turin. This is the only Data Processing Center, within the network of 6 DPCs dedicated to Gaia, which specializes in the treatment of the satellite astrometric data [2]. The DPCT will also provide the necessary training and support for everything related to Gaia data access and processing, including access to the MARCONI supercomputer at CINECA (through a specific INAF/CINECA MOU in effect since 2013) for the most demanding astrometric tasks, like, e.g., the all-sky sphere reconstructions. The modular structure of GSR, moreover, allows one to implement different astrometric models, thus transforming this pipeline in a sort of machinery for the numerical testing of different relativistic models.

Both GREM and RAMOD models are internally consistent at the μas level required by Gaia. However, due to their different conception, a good reciprocal understanding is a complex task; for example, they differ in the definition and use of the unknown, the local-line-of-sight, already at the level of the aberration contribution [65]. Evidence of the importance of comparing relativistic models for the appropriate definition of the astrometric observable and the interpretation of proper light direction measurements is given also in [66, 67].

Since both models are used for the Gaia data reduction, any inconsistency in the relativistic model(s) would invalidate the quality and reliability of the scientific outputs. The comparison of reciprocal results is mandatory in order to validate the final astrometric catalog.

The next subsections show how GREM and RAMOD model the local-line-of sight (up to the mass monopole deflections term) and how important is to take into account the local masses for a correct estimate of the parallaxes.

5˙3. *The algorithms for coordinate and proper local-sight-of-light deflection*. – In GREM the observed direction to the source (s^i) with respect to the local inertial frame ($\mathcal{X}^\alpha$) of the observer (SRS) is defined as

$$s^i = -\frac{\mathrm{d}\mathcal{X}^i}{\mathrm{d}\mathcal{X}^0}. \tag{104}$$

GREM observed direction converts into a coordinate one via several steps which divide the effects of the aberration, the gravitational deflection, the parallax, and proper motion. In order to be directly compared with the global expression derived in the RAMOD-like models it should be expressed as a sum of a set of relativistic astrometric parameters for the Gaia-like catalogue. The astrometric parameters should simultaneously link all possible "astrometric" relativistic effects related to the light propagation.

The GREM coordinate direction to the light source at the satellite location $x^i_{\rm s}$ is defined by the four-vector $p^\alpha = (1, p^i)$, where $p^i = c^{-1}dx^i/dt$, x^i and t being the BCRS coordinates. In order to account for stellar aberration in the algorithm for the reduction of the astrometric observation, one has to transform the Satellite Reference System observed direction to the source (s^i) into the BCRS coordinate unit direction of the light ray (n^i) at the point of observation.

Then, considering the metric tensor which defines the BCRS and from the property of the null light ray ($g_{\alpha\beta}p^\alpha p^\beta = 0$), the normalization factor results

$$p \equiv (\delta_{ij}p^ip^j)^{1/2} = \left(1 - \frac{4w(t,x)}{c^2} - \frac{8w^i(t,x)p_i}{c^3} + O(c^{-4})\right)^{1/2}. \tag{105}$$

The infinitesimal transformation $\mathcal{X}^\alpha(x^\beta)$ between SRS and BCRS is given by the formula the Lorentz matrix between the coordinate bases on the spacetime coordinates of the point of observation, i.e $\mathcal{X}^\alpha = \Lambda^\alpha_\beta dx^\beta$, namely

$$s^i = \frac{\Lambda^i_j p^j - \Lambda^i_0}{\Lambda^0_0 - \Lambda^0_j p^j}. \tag{106}$$

Adopting the IAU resolutions, the transformations between the proper and the coordinate time or the spatial coordinates read:

$$\begin{aligned} \mathcal{X}^0 &= t - c^{-2}[A(t) + \delta_{ij}v^i_s R^j_s] \\ &\quad + c^{-4}[B + \delta_{ij}B^i R^j_s + \delta_{im}\delta_{jk}B^{ij}R^m_s R^k_s + C(t,x^i)] + O(c^{-5}) \end{aligned} \tag{107}$$

$$\mathcal{X}^i = \left[\delta^i_j + c^{-2}\left(\frac{1}{2}v^i_s v_{sj} + qF^i_j(t) + D^i_j(t)\right)\right] R^j_s + c^{-2}D^i_{jk}(t)R^j_s R^k_s + O(c^{-4}), \tag{108}$$

where: $R^i_s = x^i - x^i_s$ are the coordinate displacements with respect to the center of mass of the satellite (CoMs) in the BCRS, $v_s = dx^i_s/dt$ and $a^i_s = dv^i_s/dt$ are the velocity and acceleration of the CoMs of the satellite in the BCRS. Moreover:

$$\frac{dA}{dt} = \frac{1}{2}v_s^2 + w(\mathbf{x}_s), \tag{109}$$

is the contribution of the Doppler effect of the second order plus the gravitational red shift;

$$\frac{dB}{dt} = -\frac{v_s^4}{8} - \frac{3w(\mathbf{x}_s)}{2}v_s^2 + 4w^i(\mathbf{x}_s)v_{is} + \frac{1}{2}w^2(\mathbf{x}_s), \tag{110}$$

where B is a function of t and represents the post-Newtonian correction to the function A;

$$B^i = -\frac{1}{2}v_s^i v_s^2 + 4w^i(\mathbf{x}_s) - 3v_s^i w(\mathbf{x}_s), \tag{111}$$

$$B^{ij} = -v_s^i \delta_{aj} Q^a + 2w^{i,j}(\mathbf{x}_s) - v_s^i w^{,j}(\mathbf{x}_s) + \frac{1}{2}\delta_{ij}\dot{w}(\mathbf{x}_s), \tag{112}$$

and $C(t, \mathbf{x}) = -(1/10)R_s^2(\delta_{ij}\dot{a}_s^i R_s^j)$ are the post-Newtonian corrections to eq. (107) due to the displacement of the observer from the SRS origin. In particular, Q_a characterizes the deviation of the actual world line of the origin of the SRS from the geodesic motion in the external gravitational field;

$$\frac{dF^{ij}}{dt} = \frac{3}{2}v_s^i w^{,j}(\mathbf{x}_s) - v_s^j w^{,i}(\mathbf{x}_s)] + \frac{1}{2}[v_s^i Q^j - v_s^j Q^i] - 2[w^{i,j}(\mathbf{x}_s) - w^{j,i}(\mathbf{x}_s)] \tag{113}$$

expresses the geodesic, Thomas, and Lense-Thirring precession (originating from the velocity of the SRS origin orbiting around the BCRS origin) respectively. The numerical parameter q allows to distinguish between dynamically ($q = 1$) and kinematically ($q = 0$) non rotating reference systems [53].

Finally, $D^{ij} = \delta^{ij} w(\mathbf{x}_s)$ is the relativistic contraction (or dilation) of the spatial coordinates of SRS relative to the BCRS due to the gravitational potential of the Solar System and $D^{ijk} = (1/2)(\delta^{ij}a_s^k + \delta^{ik}a_s^j - \delta^{jk}a_s^i)$ represents the post-Newtonian correction to the spatial transformation of the SRS coordinates with respect to the BCRS due to the barycentric acceleration of the SRS.

By means of these formulas, we obtain the elements of matrix of the above Lorentz transformation (106). Terms like $(\delta_{ij}a^i R_s^j)$ can be estimated as $(v_s^2/c^2)\cdot(R_s/r) \approx \epsilon^4 \cdot 10^{-3}$(9), therefore they can be neglected.

If one keeps all of the terms up to the order 0.1μas and neglect terms of the order $O(\epsilon^3 \cdot 10^{-1}) \sim 0.01\mu$as equation (106) turns out:

$$\begin{aligned} s^i &= n^i + c^{-1}[n \times (v_s \times n)]^i \\ &+ c^{-2}\left\{\frac{1}{2}(n\cdot v_s)[n \times (n \times v_s)]^i - \frac{1}{2}n^i(n \times v_s)^2 + qF^{ij}n^j\right\} \end{aligned} \tag{114}$$

(9) r represents the BCRS distance of the satellite from the origin of BCRS, $r = 151.5$ AU and $R_s \approx 5$ m in the case of Gaia.

$$+c^{-3}\left\{\frac{1}{2}(n\cdot v_s)^2[n\times(v_s\times n)]^i+\frac{1}{2}n^i(n\times v_s)^2(n\cdot v_s)\right.$$
$$\left.+2w(x_s)[n\times(v_s\times n)]^i+qF^{ij}[n\times(v_s\times n)]^j\right\}+O(\epsilon^3\cdot 10^{-1}),$$

where $\delta_{ij}v_s^i n^j \equiv\ v_s\cdot n$ to ease the notation. Note the aberrational contributions up to the μas order, which depends also on the local gravitational fields. The unit vector n^i is defined at the point of observation and still includes relativistic light deflection due to the Solar System gravitational field. The explicit deflection is obtained mapping n^i into the asymptotic flat spacetime using the solution of the equations of light propagation. This procedure transforms n^i into the vector k^i lying in the flat spacetime and attached to the point of observation.

All of the perturbations, say $\Xi^i(t,x^i)$, in pN or pM approximation can be included in the trajectory of a photon as [35]:

$$x^i(t)=x^i_N(t)+\Xi^i(t,x^i)-\Xi^i(t_o,x^i) \tag{115}$$

where $x^i_N(t)=x^i_0+ck^i$ represents the unperturbed trajectories of the light ray in the Minkowskian flat spacetime, i.e. a straight line and k^i is the velocity at past-null infinity. The term $\Xi^i(t,x^i)$ is obtained through an analytical procedure for the integration of the equations of the geodesic, since the retarded time has been made explicit in the metric and the geodesic is linearly parametrized.

The component of vector n^i becomes:

$$n^i\equiv-\frac{\dot{x}^i(t_o)}{|\dot{\vec{x}}(t_o)|}=-\frac{\dot{x}^i_N(t_o)+\dot{\Xi}^i(t_o,x^i)}{|\dot{\vec{x}}_N(t_o)+\dot{\Xi}^i(t_o,x^i)|}, \tag{116}$$

and its expansion will be in function of the relativistic correction $\dot{\Xi}^i(t_o,x^i)$. The latter can be evaluated by considering that the maximum contribution to the light deflections is due to the Sun, which amounts, for grazing ray, to ≈ 2 arcsec $\sim 10^{-1}\cdot\epsilon$.

To achieve a unique expression for s^i, let us define the following dimensionless vector:

$$\Psi^i\equiv c^{-1}[k\times(k\times\dot{\Xi})]^i \tag{117}$$

as the total angle of deflection of the light ray at the point of the observer and calculated with respect to vector k^i at past null infinity. Finally,

$$s^i=-k^i+\psi^i+\frac{1}{2}k^i\Psi^2+c^{-1}\left\{[k\times(k\times v_s)]^i+k^i(\psi\cdot v_s)+\Psi^i(k\times v_s)\right.$$
$$\left.+\Psi^i(k\cdot\dot{\Xi})\right\}+c^{-2}\left\{-\frac{1}{2}(k\cdot v_s)[k\times(k\times v_s)]^i+\frac{1}{2}(k\times v_s)^2k^i-qF^{ij}k^j\right\}$$
$$+c^{-3}\left\{\frac{1}{2}(k\cdot v_s)^2[k\times(v_s\times k)]^i+\frac{1}{2}k^i(k\times v_s)^2(k\cdot v_s)\right.$$
$$\left.+2w(x_s)[k\times(v_s\times k)]^i+qF^{ij}[k\times(v_s\times k)]^j\right\}+O(\epsilon^3\cdot 10^{-1}), \tag{118}$$

which takes into account the aberration and deflection contributions.

The RAMOD formulation, on the other hand, in principle naturally entangles every GR "effect" in the observables, i.e. the three direction cosines which identify the local line-of-sight to the observed object, relative to a spatial triad associated to the satellite.

Therefore, to retrieve a similar expression of equation (118) in RAMOD, one needs to specialize eq. (93) to the case of a tetrad adapted to the CoMRs of the satellite. If one assumes no attitude parameters, the observation equation gives a relation between the "aberrated" direction represented by the cosines as measured by the satellite and the "aberration-free" direction. The latter is given by the quantity $\bar{\ell}^\alpha = P^\alpha_\beta(u_B)k^\beta$ referred to the local BCRS frame $\{\lambda_{\hat{a}}\}$.

Considering the IAU metric, one obtains

$$
\begin{aligned}
\tilde{\mathbf{e}}_{\hat{a}} \approx\ & \bar{\ell}^a + \frac{1}{c}\left[-v_s^a + \left(v_s\cdot\bar{\ell}\right)\bar{\ell}^a\right] + \frac{1}{c^2}\left\{w(x_s)\bar{\ell}^a - \frac{1}{2}\left(v_s\cdot\bar{\ell}\right)v_s^a + \left[\left(v_s\cdot\bar{\ell}\right)^2 - \frac{1}{2}v_s^2\right]\bar{\ell}^a\right\} \\
& + \frac{1}{c^3}\left\{-2w(x_s)v_s^a - \frac{1}{2}\left(v_s\cdot\bar{\ell}\right)^2 v_s^a + \bar{\ell}^a\left[3w(x_s)\left(v_s\cdot\bar{\ell}\right)\right.\right. \\
& \left.\left. + \left(v_s\bar{\ell}\right)^3 - \frac{1}{2}v_s^2\left(v_s\cdot\bar{\ell}\right) + w(x_s)\left(v_s\cdot\bar{\ell}\right)\right]\right\} + \mathcal{O}\left(v_s^4/c^4\right)
\end{aligned}
\tag{119}
$$

where $\tilde{\mathbf{e}}_{\hat{a}}$ are the cosines related to the tetrad without the attitude parameters.

At first glance, differences show up at the level of ϵ^2 (in particular the term $w\bar{\ell}^a$) and ϵ^3 order which cannot allow to compare straightforwardly, as expected, to the GREM formula, where the aberration is expressed in terms of a vector n^i. From the physical point of view n^i and $\bar{\ell}^\alpha$ have the same meaning, as the *observed* "aberration free" direction to the star. However, expanding $(u|k)$ with the IAU metric, the projection of $\bar{\ell}^\alpha$ with respect to $\hat{\mathbf{u}}_B$ is

$$
\ell^i = n^i\left(1 - \frac{h_{00}}{2}\right) + \mathcal{O}\left(\frac{v^4}{c^4}\right), \tag{120}
$$

which clearly puts in evidence how $\bar{\ell}^\alpha$ contains the "gravitational" aberration. Combining eq. (119) with (121), one obtains

$$
\begin{aligned}
\tilde{\mathbf{e}}_{\hat{a}} =\ & n^a + \frac{1}{c}\left[-v^a + \left(v_s\cdot n\right)n^a\right] + \frac{1}{c^2}\left\{-\frac{1}{2}\left(v_s\cdot n\right)v_s^a + \left[\left(v_s\cdot n\right)^2 - \frac{1}{2}v_s^2\right]n^a\right\} \\
& + \frac{1}{c^3}\left\{-2w(x_s)v_s^a - \frac{1}{2}\left(v_s\cdot n\right)^2 v_s^a + \left(v_s\cdot n_s\right)n^a\left[2w + \left(v_s\cdot n\right)^2 - \frac{1}{2}v_s^2\right]\right\} \\
& + \mathcal{O}\left(\frac{v_s^4}{c^4}\right).
\end{aligned}
\tag{121}
$$

In this way the right-hand side of the aberration expression in RAMOD is rewritten with the GREM quantities at the $(v/c)^3$ order. The final step is to find a relation between $\tilde{\mathbf{e}}_{\hat{a}}$ and s^i. Using the definition of the projection operator and the tetrad property ($\lambda^{\hat{\mu}}_\alpha\lambda_{\hat{\mu}\beta} = g_{\alpha\beta}$), it results:

$$
\tilde{\mathbf{e}}_{\hat{a}} \equiv \frac{P(u)_{\alpha\beta}k^\alpha\tilde{\lambda}^\beta_{\hat{a}}}{\left(P(u)_{\alpha\beta}k^\alpha k^\beta\right)^{1/2}} = \frac{k^\alpha\tilde{\lambda}^{\hat{a}}_\alpha}{(u|k)} = -\frac{k^\alpha\tilde{\lambda}^{\hat{a}}_\alpha}{g_{\alpha\beta}\tilde{\lambda}^\alpha_{\hat{0}}k^\beta} = \frac{k^\alpha\tilde{\lambda}^{\hat{a}}_\alpha}{k^\beta\tilde{\lambda}^{\hat{0}}_\beta} = \frac{\mathrm{d}\tilde{x}^{\hat{a}}}{\mathrm{d}\tilde{x}^{\hat{0}}}. \tag{122}
$$

This justifies the conversion of the physical stellar proper direction of RAMOD into its analogous Euclidean coordinate counterpart, which ultimately leads to the derivation

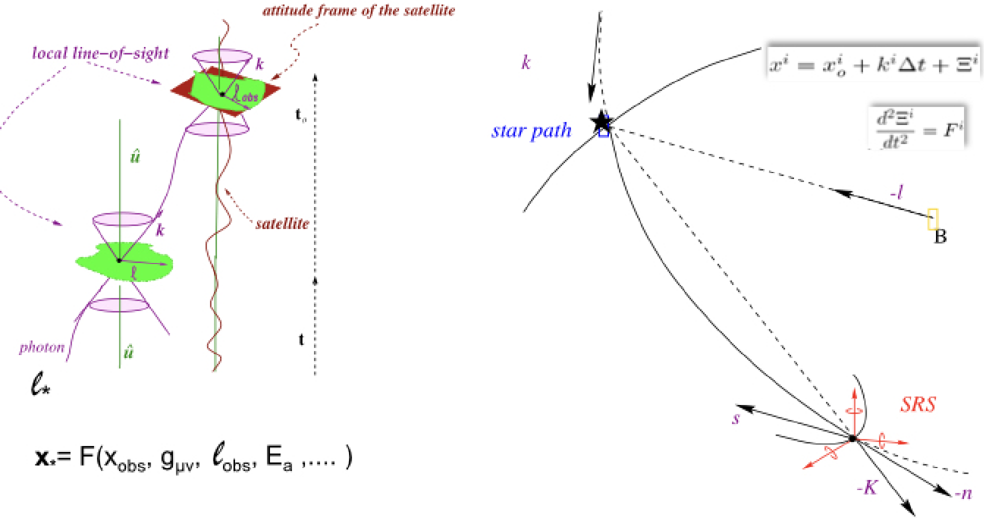

Fig. 11. – Comparison of the RAMOD approach (left) and the pN/pM (right) in the inverse ray-tracing problem.

of a GREM-style aberration formula (115). Note, also, that the observables of RAMOD can be matched with components of the observed s^i of GREM only if the origins of the boosted local BCRS tetrad in RAMOD and of the CoMRS in GREM coincide (and only locally, since the tetrad are not in general holonomic).

The last step for the comparison is the inclusion of the deflection. The RAMOD solution for the deflection contributions comparable with the GREM approach, namely in the IAU framework and for uniform moving body([10]), is available in [39]. Denoting $x^i - x^i_{(a)} \equiv r^i_{(a)}$, $x^i_{\rm obs} - x^i_{(a)} \equiv r^i_{\rm obs}$, $n^i = r^i/r$, $\bar{\ell}_\emptyset$ the unperturbed local direction, and dropping the summation symbol and the subscript (a), the local-deflections due to n-mass monopoles with constant velocities $\tilde{v}^i$ to be implemented in the astrometric problem in the RAMOD/IAU framework are:

$$(123) \qquad \bar{\ell}^0 = -4GM\epsilon^3 \frac{\bar{\ell}_\emptyset \cdot \tilde{v}}{r}\,,$$

([10]) A relaxing hypothesis that matches the IAU adopted accuracy.

$$
\begin{aligned}
\bar{\ell}^a - \bar{\ell}^a_{\rm obs} = & -GM\epsilon^2 \left\{ \left(\frac{1}{r} - \frac{1}{r_{\rm obs}}\right) \left[\bar{\ell}^a_{\emptyset} - 2\epsilon \left(2d^a_v - \frac{d^a}{d^2}(\tilde{v}\cdot r_{\rm obs})\right)\right] \right. \\
& \left. +2\frac{d^a}{d^2}[1-2\epsilon(\tilde{v}\cdot\bar{\ell}_{\emptyset})]\left(n\cdot\bar{\ell}_{\emptyset} - n_{\rm obs}\cdot\bar{\ell}_{\emptyset}\right)\right\} \\
& +2GM\epsilon^3 \frac{r_{\rm obs}}{d^2 r}\left[d^a_v - 2\frac{d^a}{d^2}(\tilde{v}\cdot d)\right]\left[r - r_{\rm obs} - (n_{\rm obs}\cdot\bar{\ell}_{\emptyset})\sigma\right]
\end{aligned} \tag{124}
$$

where

$$
\sigma = (r\cdot\bar{\ell}_{\emptyset}) - (r_{\rm obs}\cdot\bar{\ell}_{\emptyset}) + O(\epsilon^2) = (x - x_{\rm obs})\cdot\bar{\ell}_{\emptyset} + O(\epsilon^2), \tag{125}
$$

and

$$
d^a = [\bar{\ell}_{\emptyset}\times(r_{\rm obs}\times\bar{\ell}_{\emptyset})]^a = r^a_{\rm obs} - \bar{\ell}^a_{\emptyset}(r_{\rm obs}\cdot\bar{\ell}_{\emptyset}) \equiv P(\bar{\ell}_{\emptyset})^a{}_b r^b_{\rm obs}\,, \tag{126}
$$

$$
d^2 = r^2_{\rm obs} - (r_{\rm obs}\cdot\bar{\ell}_{\emptyset})^2\,, \tag{127}
$$

represents the impact parameter with respect to the (a)-source (being $P(\bar{\ell}_{\emptyset})^a{}_b$ the projector orthogonal to $\bar{\ell}_{\emptyset}$), whereas

$$
d^a_v = [\bar{\ell}_{\emptyset}\times(\tilde{v}\times\bar{\ell}_{\emptyset})]^a = P(\bar{\ell}_{\emptyset})^a{}_b\tilde{v}^b\,, \tag{128}
$$

is the projected source velocity with respect to $\bar{\ell}_{\emptyset}$. Setting $\tilde{v}_{(a)} = 0$ the static solution is easily obtained.

5˙4. *Parallaxes from deflection due to a variable number of aligned bodies.* – In order to understand how planet masses alterate parallax computation, consider formula (124) in the simplest case of two observers at opposite positions on the Earth orbit around the Sun (i.e. $r_1 = r_2 = r_\oplus$, $\theta_1 = \theta_2 = \pi/2$ and $\phi_2 = \phi_1 + \pi$) and the stars also lying on the orbital plane ($\theta^* = \pi/2$). Then the two observers are simmetrically placed with respect to the Sun ($\phi^* = (\phi_1 + \phi_2)/2$). Assume that their distances r^* goes from 1 pc to 10 kpc.

With this configuration, it is easy to calculate the parallax of a star as $p^* = 1/r^*$ rad (the distances being in AU), and so it is equally easy to determine the numerical accuracy of the position in terms of angles by simply subtracting the parallax p^*_e of the exact model (h_{00} contributions from Schwarzschild-like sources) and the approximate one reconstructed from the numerical integrations p^*_a. The geometrical configuration consider the planets not moving and aligned behind the Sun with respect to the observer; the light ray always grazes the solar limb. The expectation is that the parallax enlarges when adding more planets, since they increase the total amount of deflection.

The the total deflections due to the Sun plus a variable number of planets are reported in Table IV and in figure 12.

For each set of 4 rows, the first one is the derived distance d in AU (i.e. that obtained by the intersection of the two geodesics), the second is the corresponding parallax $p = 206265/d$ arcsec; each column gives the results for a given number of planets (i.e. the single Sun in first column, the Sun plus Jupiter in the second, and so on), we will refer to the quantities in the first row as $d_\odot$, $d_♃$, …and $p_\odot$, $p_♃$, …for those in the second row.

The third row contains the difference between $p_\odot$ and the parallax of the corresponding column. For instance, in the case of Jupiter, the value is $p_\odot - p_♃ \equiv \Delta p_♃$.

Table IV. – *Results for the n-body tests of the stellar distances*

	⊙	⊙+♃	⊙+♃+♄	⊙+♃+♄+⛢	⊙+♃+♄+⛢+♆
			distance $\sim$ 1 pc		
d (AU)	206261.338	206259.698	206259.216	206259.142	206259.054
p_i ($''$)	1.00001775	1.00002571	1.00002804	1.00002840	1.00002883
$p_\odot - p_i$ (μarcsec)		-7.95	-10.29	-10.65	-11.07
$\Delta\psi_i$ (μarcsec)		-8.00	-10.30	-10.60	-11.10
			distance $\sim$ 10 pc		
d (AU)	2062283.55	2062120.14	2062072.27	2062064.94	2062056.3
p_i ($''$)	0.10001777	0.10002569	0.10002802	0.10002837	0.10002879
$p_\odot - p_i$ (μarcsec)		-7.93	-10.25	-10.60	-11.02
$\Delta\psi_i$ (μarcsec)		-8.00	-10.30	-10.60	-11.00
			distance $\sim$ 100pc		
d (AU)	20589921.6	20573645.4	20568881.5	20568152.5	20567292.3
p_i ($''$)	0.01001777	0.01002569	0.01002801	0.01002837	0.01002879
$p_\odot - p_i$ (μarcsec)		-7.93	-10.25	-10.60	-11.02
$\Delta\psi_i$ (μarcsec)		-7.99	-10.30	-10.60	-11.00
			distance $\sim$ 1000 pc		
d (AU)	202664764	201098829	200644596	200575248	200493473
p_i ($''$)	0.00101776	0.00102569	0.00102801	0.00102837	0.00102879
$p_\odot - p_i$ (μarcsec)		-7.93	-10.25	-10.60	-11.02
$\Delta\psi_i$ (μarcsec)		-7.99	-10.30	-10.60	-11.00
			distance $\sim$ 10000 pc		
d (AU)	1751489760	1641052370	1611285280	1609051510	1603803810
p_i ($''$)	0.00011777	0.00012569	0.00012801	0.00012819	0.00012861
$p_\odot - p_i$ (μarcsec)		-7.93	-10.25	-10.42	-10.84
$\Delta\psi_i$ (μarcsec)		-7.99	-10.30	-10.60	-11.00

Finally, the fourth row gives the contribution $\Delta\psi_i$, ($i =$ ♃, ♄, ⛢, ♆) of the planets to the total deflection.

The results show that the distances get lower when adding more planets, confirming our qualitative expectations. Moreover the numerical residuals are $\Delta p_i - \Delta\psi_i \sim 10^{-1}\mu$arcsec ($i =$ ♃, ♄, ⛢, ♆), and this is compatible with a Gaia-like accuracy.

If one translates these numbers into a distance estimation via the distance modulus ($m - M = 5\log r_{pc} - 5$), only considering the Sun contribution for d=10 kpc, gives a relative error in magnitude of the order

$$\sigma_{m-M} \approx 2\frac{\sigma_d}{d} \approx 0.2\text{mag} \tag{129}$$

meaning that if we do not include the relativistic perturbations properly this translates a 20% error in the distance module (i.e. for a distance of 10 kpc) with obvious implications for the distance scale. Indeed, since the main Solar System curvature perturbation amounts approximately to 100 μas at L2, this will cause the individual parallaxes to

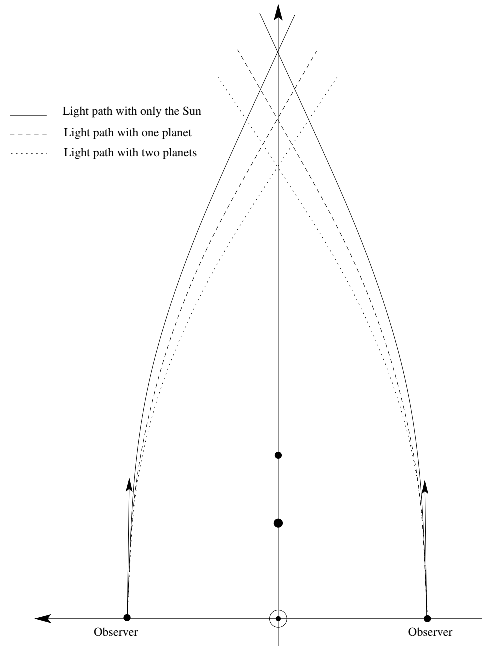

Fig. 12. – Distance versus light deflection produced by n-bodies

fast degrade beyond 1 kpc, while completely invalidating the most accurate calibration of, e.g., the primary distance calibrators. This alone is sufficient reason for allowing the existence and making a theoretical comparison of different approaches a necessity. It is of the utmost importance to use different accurate formulas to test with independent procedures the reliability of the data, as formula (121) for the local gravitational aberrated direction already proves.

6. – The Universe from within the local gravitational fields. Relativistic Astrometry in action.

The weak gravitational regime can play a pivotal role and provide a complementary perspective in understanding gravity especially in light of the recent discovery of gravitational waves and the subsequent beginning of the Gravitational Astronomy era. Most of the physical information about the astrophysical sources is carried on by light signal. Astrometric observations collect photons, which have interacted with different time-dependent gravitational fields and record perturbations along their propagation. This could imply a new detection window of many subtle relativistic effects naturally enfolded in the light while it propagates through the geometry of space-time from the source up to the local observer. In this respect, relativistic astrometry fills the gaps by properly matching the gravitational fields at the source to those where the measurements are performed and potentially proving properties of gravity to solve basic questions in fundamental physics.

As a matter of fact, by providing an homogenous all-sky survey of high precision parallaxes, space motions (proper motions and radial velocities) and astrophysical characterization for more than one billion stars throughout the volume of the Galaxy, Gaia will have a huge impact across many fields, including many branches of stellar astrophysics, exoplanets, solar system objects, the cosmic distance ladder and fundamental physics tests, for example, on dilaton run-away scenario, i.e., on the level of validity of GR as the theory of gravitation. Gaia will not only greatly enhance our knowledge of the Milky Way (MW) astrophysics, but it will also provide precise information allowing astronomers to frame a much more detailed picture of its kinematics than what presently available. New "accurate" distances and motions of the stars within our Galaxy will provide access to the cosmological signatures left in the disk and halo offering independent, direct and detailed comparisons the predictions of the most advanced cosmological simulations.

Nonetheless, it is worth stressing again and again that all of the above goals will not be achieved without the correct characterization and exploitation of the "relativistic", i.e. highly accurate, astrometric data. The independent astrometric solution underway at the Italian data processing center in Turin (DPCT), for verification purposes, is based on a general relativistic treatment of the data that implements, in a sophisticated high-performance computing infrastructure, theoretical models for the observables and the observer. Thus the five-parameter global astrometric solution, made available with each release of the Gaia catalog, must be understood as providing relativistic kinematics demanding in turn, at least for consistency, a relativistic representation of the Galaxy's dynamics.

In this regard, the GR tools of gravitational astrometry should also to apply to other future missions, as Euclid for instance.

This will guarantee, at least, a GR coherent space-phase picture of the Milky Way against which theories, simulations, predicting dark components or possible deviations from GR (and not only from Newtonian mechanics) can be tested. The situation is somewhat similar to what was done to explain the advancement of Mercury's perihelion: GR cured inconsistencies by accounting for the non-linear overlapping of the weak gravitational fields in the Solar System which amounts only 43"/century because of the curvature of the Sun but "strong" enough to justify a modification of Newtonian theory. We should bear in mind that general relativistic weak field regime is not necessarily equivalent to the Newtonian one.

Therefore taking the probing power of gravitational astrometry to its fullest, one could possibly gauge breakdowns of GR.

6·1. *Absolute reference frame.* – The fundamental step toward the realization of the Gaia catalogue is the global astrometric sphere reconstruction, which determines the celestial reference frame using the observations of a selected subset of up to 100 million stars (primary sources) among those observed by Gaia. Afterwards, the catalog is completed with the reduction of the Gaia measurements of the remaining stellar objects (secondary stars). Basically, a star is included in the primaries subset when its astrometric model can be described by the classical 5 parameters. Binary/multiple stars with a relatively short period, or stars with a sufficiently large variability are examples of objects that cannot belong to the primary source list.

Given the absolute character of the Gaia catalogs, as mentioned, the Consortium constituted by ESA for the Gaia data reduction (DPAC) agreed to set up two independent astrometric sphere solutions: AGIS and GSR. The process described above, including the definition of the two subsets, the global sphere reconstruction, and the reduction of the secondary stars, is realized within CU3 (DPAC) by the AGIS pipeline.

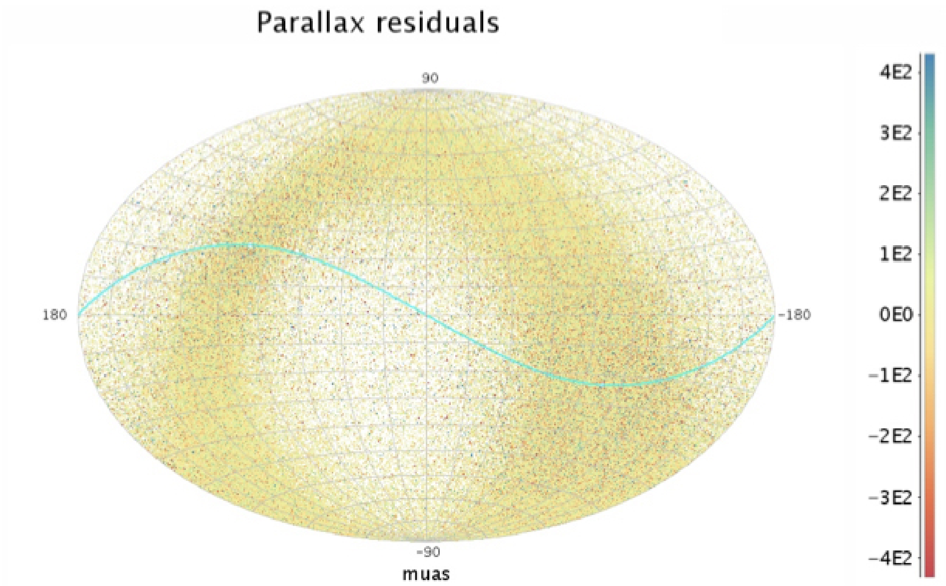

Fig. 13. – The AVU-GSR sphere reconstruction. Demonstration Run with 1 million primary sources, whole mag range, 5 years, blind simulation. Credits: A. Vecchiato et al. [68].

Focusing on RAMOD-like models, the realization of the Gaia catalogue is performed by the GSR, which determines the celestial reference frame using the observations of the selected subset of the primary sources, among those observed by Gaia, in order to validate the baseline method adopted for Gaia. Recent blind simulations show that GSR works as expected in the range of accuracy required for Gaia [68].

In figure (13) are plotted parallaxes "residuals", namely the differences between the true and reconstructed parallaxes. It is the result of the testing (from the most realistic situation) both wrong trigger points and measurement errors. In the latter case, in fact, the residual systematic error is not clearly visibile because it is drowned in the Gaussian residue which is much larger, as it is clear by comparing the color scales.

Beside the determination of the most fundamental PPN parameter (see subsection below), which enters as unknown the global reduction process, in order to make the comparison useful, the Gaia observable relies on completely different relativistic observation equations and least-squares solution methods. In a nutshell AGIS and GSR present: independent relativistic astrometric model; independent relativistic attitude model; independent (iterative) least-squares solution method (all-unknowns solved).

This guarantees the largest degree of independence between the two solutions and in itself represents a powerful test of General Relativity thanks to the billions of observation equations delivered by Gaia. Any discrepancy between the relativistic models, indeed, if it can not be attributed to errors of different nature, will mean either a limit in the modeling/interpretation - that a correct application of GR should fix, therefore validating GR - or provide a new stringent limit on GR validity.

6˙2. *Testing gravity in the Solar System.* – General Relativity governs the Universe up to 60 orders of magnitude, from its early stage to the present, fixing its dynamics and constraining the evolution of its physical constituents. The formation of its structures like cluster of galaxies, the morphology of individual galaxies and the type of stars they are made of, are all dictated by the initial conditions of the Universe. Moreover, the primordial density ripples that eventually gave origin to the galaxies (the small variations in the CMB temperature) are thought to be the result of fluctuations in a scalar field that couples with gravity and drove inflation in the earliest evolution of the Universe after the Big Bang. Gravity theories alternative to GR require the existence of this scalar field and predict it fades with time, so that this residue would manifest itself through very small deviations from Einstein's GR. Several alternative theories of Gravity were proposed during the years, and there are usually confronted by using the PPN formalism, probably the most powerful tool for modeling gravity tests within the Solar System. In PPN formalism, each gravity theory is characterized by specific values of ten parameters, and the remnant of the primordial scalar field would be detected through the measurements of one of them, γ. This parameter, by measuring the amount of curvature produced by unit rest mass, is strictly related to the deflection effect, and is equal to one in GR. It was constrained to the GR value to $1+(2.1\pm2.3)\times10^{-5}$ by the Cassini experiment [70]. Although impressively accurate, this result is not sufficient to test the dilaton-runaway scenario (for instance [69]), which predicts a present-day variation from the GR value in the range $10^{-6}-10^{-7}$.

Very accurate global astrometry is a very powerful and independent tool to unveil the presence of the scalar field. The estimation of the PPN parameter γ comes naturally as a by-product of the sphere reconstruction. A mission like Gaia-like, repeatedly observing over years a billion light directions uniformly scattered across the sky to a precision of μas, could be potentially able to determine γ to 10^{-7} at the 3σ level [47], thus reaching the sensitivity level of the dilaton-runaway scenario.

Then, nearly a century later the experiment of Eddington and Dyson, astrometry remains one of the most fundamental and sensitive methods to test the validity of GR in the weak-field regime. Gaia global astrometry will provide a massive repetition of the Eddington astrometric test of GR with 21st century technology, and this thanks to a combination of analytical and numerical relativistic methods [39]. However, as the systematic errors in DR2 [22] are still relatively large, the expectation is to estimate a deviation, from the GR predicted value of 1, for the PPN γ, at the level of 10^{-6} with the final calibrations after DR3, at the end of the mission when the astrometric accuracies will be better than 5-10 μas for the brighter stars and 130-600μas for fainter targets. Continuously observation from space, but based on pointing stellar fields, are more promising and could reach the level of 10^{-8} (for example [71]).

Observations from global astrometry can be used also to create small stellar reference frames against which tiny relativistic light deflection effects due to a single source can be tested, as that due to the quadrupole shape of the sources.

While the global tests will be done at mission's end only after all the observations are

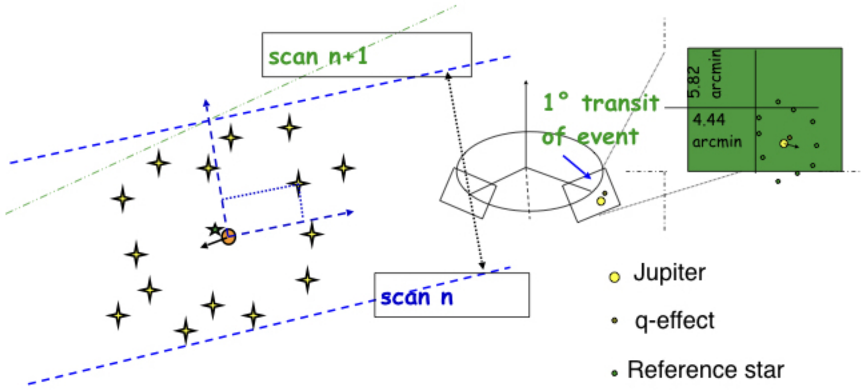

Fig. 14. – Principle of differential astrometry applied to GAREQ from global astrometric observations. Credit M.G. Lattanzi.

taken, differential experiments, exploiting the precision of the elementary measurements, can be implemented in the form of repeated Eddington-like experiments by comparing the evolution of angular distances in bright stellar asterisms consecutively observed by the satellite within a few planet's radii from the limb of a giant planet like Jupiter (figure 14). Indeed, Jupiter offers an optimal target for second order light deflection experiments, thanks to its precisely known mass, relatively large deflection, and the ability to observe a target very close to the limb without the difficulties posed by the Sun. For Jupiter the magnitude of the monopole deflection for a grazing ray is ∼16 milli-arcsecond (mas), to which a component from the quadrupole moment is superimposed with an amplitude of $\sim 240\mu$as [72]. The actual GAREQ (for GAia Relativistic Experiment on Quadrupole [73]) observation was carried out by the satellite on February 22th, 2017. Gaia's spin axis orientation was intentionally optimized to catch a star close to the limb of Jupiter in 2017. The initial spin phase axis orientation was decided in 2014 to maximize the measurement success on Feb 2017. At the beginning of 2017, and towards the end of February 2017, Gaia provided measurements for 31 bright reference stars (G <13 mag) all lying within a field of 0.8×1.3 degree surrounding the target star (G= 12.68 mag).

The target star was seen a total of 26 times over a 2-month period out of which 15 transits over a time interval of a couple of days surrounding the observation at closest approach were used (figure 15). The observation epochs were executed successfully and are under reduction.

Uncertainties of the DR2 astrometry are still too high to detect clearly the varying relativistic effects associated with the received null geodesic from within the multi-gravitational fields of the Solar System. However, thanks to the multiple observations over a few consecutive scans and the appropriate statistical analysis of the local coordinates on the two Gaia fields of view (FOVs), differential astrometry is used to adjust all the frames to a common frame by means of translations, rotations and possible distortion terms if necessary [74].

Beside the fact that such GAREQ-like experiments could provide an important science case for assessing the health of the instruments during the mission, the precise

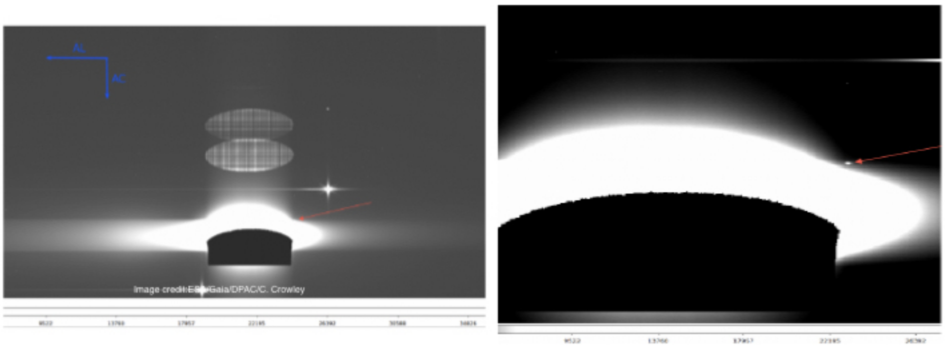

Fig. 15. – Observations for the closest transit of the Target star at an angular separation of 6.73” from Jupiter’s limb seen on 2017-02-23 T02:55:01.694. Left: section of the SM image around Jupiter for the closest transit. The location of the target star is shown by the red arrow. Right: A zoom-in to the top image for the closest transit. The target star is visible just outside the saturated region. Image credit:ESA/Gaia/DPAC/C. Crowley.

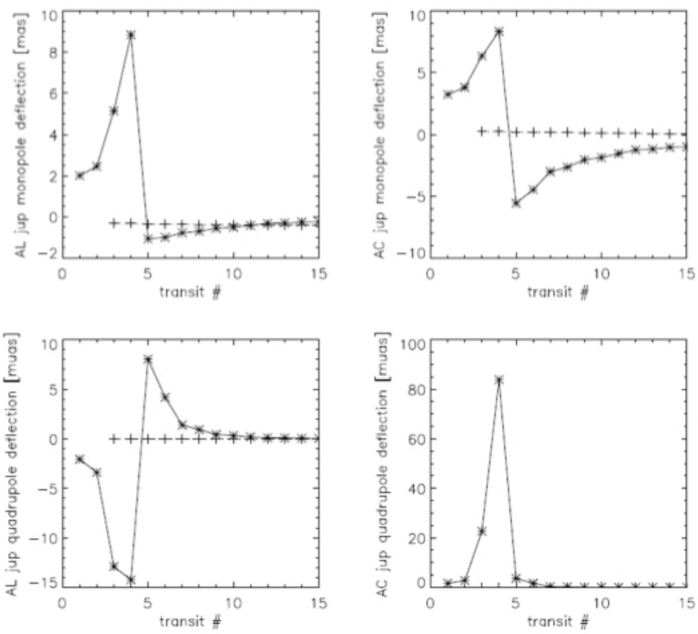

Fig. 16. – Simulated light deflections effects around Jupiter via differential procedure [75]. Upper panels show the simulated AL/AC monopole deflection signal and lower panels the quadrupole deflection signal. Asterisks denote the target star and plus signs a randomly chosen reference star.

estimate of contributions to light deflection due to non-spherical mass in the Solar System will constitute an independent verifications of alternative theory of gravity and a first quantitative measurement of the gravitational potential due to a non-spherical lens (for example, [76]).

Moreover, the consolidation of a such a tiny effect will allow to confirm extra weak field contributions to the light deflection produced by planets which, once rescaled to other eso-systems, could contribute to refine more parameters in the characterization and indirect discovery of eso-planets, via gravitational perturbation. Furthermore this extra term of weak lensing might also contribute to better investigate, for example, Euclid-like measurements. Indeed, the gravitational pull of the clusters of large masse distributions is usually deduced from a multipolar patterns of the gravitational lens generated by the cluster provided that GR works, plausibly, as expected.

6˙3. *Testing gravity in the Milky Way: Astrometric Cosmology*. – Gaia accurate space-phase structure can fully probe the Milky Way outer halo (*i.e.* mass content and distribution) and compare the prediction of ΛCDM models *in situ*. By considering just Newtonian models, the line pursued is to search for new kinematic streams in the local halo and redefine membership of known streams. Cold Dark Matter (CDM) models predict that structures grow by hierarchical merging, mainly driven by dynamical friction and tidal disruption, leaving streams and substructures as relicts of this process considered as tracers of the distribution of dark matter (DM) [77]. Cosmological effects can be detected also via the secular aberration drift of the ICRF sources [78].

However, in a complementary way, provided that the Galaxy is not a point source but an extended one, a first attempt should be to apply the relativistic kinematics delivered by Gaia to trace the observed MW rotation curves - from where the hypothesis of DM originated - without any a priori assumption on the origin of its observed flatness at large radii from the galactic center. In this respect DM is actually explained as a deviation from the Newtonian velocity profile possibly because of the presence of DM in the halo (however at 200 kpc the MW cannot be considered as an isolated galaxy any longer) or of a modified gravity law.

The challenge is to establish to what extent a weak curvature due a Galactic metric could fill the gap in baryons-only galactic rotation curves once a correct relativistic kinematic/dynamic is provided, thus extending a correct GR Galaxy model also to other galaxies. In fact, the derivation of the model for the GR-only rotation curve is not well known in the literature. Besides, the "classical" models with a DM halo are well studied since many years to the point that, indeed, we can call them "classical".

The ansatz for the first attempt to trace a GR rotational curve assumes an axially symmetric, stationary and asymptotically flat Galaxy-scale metric and, in parallel, the mass inside a large portion of the Galaxy, far away from the central bulk, is simplified as a pressure-less perfect fluid (i.e. "dust" for GR) avoiding the bulge where resides the axis of symmetry. A co-rotating dust is defined to be a continuous distribution of matter with stress-energy tensor $T_{\alpha\beta}$ in the form of (in geometrized units): $T_{\alpha\beta} = \rho u^\alpha u^\beta$, where the time-like vector field u^α represents the 4-velocity of the co-rotating fluid proportional to the killing vector k^α (namely a static observer), which in virtue of the definition of $T_{\alpha\beta}$, and in the limit of small density (ρ) results geodetic.

The considerations above constitute the basis of the tailored solution adopted by Balasin and Grumiller (BG) [79] in order to trace the velocity profiles for galactic curves in a weakly relativistic scenario. In this context the spatial velocity turns out to be proportional to the off-diagonal term of the chosen metric, which implies frame dragging once one projects the static observer (the asymptotically killing BCRS) with respect to the rest frame of the ZAMO, actually the observer at rest with respect the axis of rotation of the Galaxy.

Then, the question before us is: the flat observed curve is due to dark matter or geometry driven?

To accomplish the study of the rotation curve profile of our Galaxy requires to select only DR2 source for an highly accurate 6-dimensional reconstruction of the phase-space location occupied by each individual star as derived by the same observer, namely: (i) availability of the complete astrometric set, and of its corresponding error (covariance) matrix; (ii) availability of the Gaia-measured velocity along the line of sight, RV, and its error; (iii) parallaxes good to 20%, i.e., $p/\sigma_p \geq 5$; (iv) avalaibility of a cross-matched entry in the 2MASS catalog for the materialization of the sample [80].

Of course this sample is not complete (see, e.g., the 20% error selection criteria adopted), nevertheless it closely traces the kinematic of the young disk population and we are left with mostly late type A stars.

As final step, the BG fit to the MW rotational data has been compared with well-studied classical models for the MW (MWC), which is comprised of a bulge, a stellar thin and thick disk and a Navarro-Frenk-White (NFW) DM halo.

To quantitatively asses this, a Monte Carlo Markov Chain (MCMC) analysis was implemented. For the likelihood analysis the BG and MWC models appear almost identically consistent with the data as shown in figure 17 (for details [80]). For the MWC model, the estimated parameters are, within the errors, compatible with the very latest literature values. This is important in itself, proving that the 11 kpc range in (galactocentric) cylindrical radius covered by the selected DR2 sample of disk stars is sufficiently large already for the task.

Differently from the BG analysis, the estimated local baryonic matter density, via the 00-term of the Einstein field equation, results $\rho_\odot(R = R_\odot, z = 0) = 0.090 \pm 0.006 M_\odot pc^{-3}$ that is perfectly in line with current independent estimates (for example, [81, 82, 83]). Then, it appears that no extra-mass is required for the GR rotational curve once one takes into account the off-diagonal term of the metric in the GR weak field approximation.

In conclusion, relativistic astrometry provides a new outstanding observed stellar rotational curve of the MW from a highly performing space mission, a derived classical rotational velocity curve with a DM-halo compatible with recent literature data, and a new GR-only (no DM) rotational velocity model that the Gaia-derived observed velocities make statistically viable for the very first time.

This is a very promising result that urges to refine and improve GR Galaxy models in occurrence of the next Gaia data releases.

Details on the presented study are under publication. References and full text are available in Crosta et al. [80].

6˙4. *Testing space-time: Gravitational waves*. – Passing gravitational waves alterate the background geometry and can produce an extra deflection on the light ray. This direction shift has been estimated at the nanoarsecond level [84, 85], therefore beyond the capabilities of the Gaia astrometry, but possible observable with future, specifically designed, space-born astrometric instruments

As a matter of fact, such an extra-deflection effect on star direction at a given time due to a passing gravitational wave cannot be modeled in the same fashion as that of the astrometric sphere reconstruction, i.e., the determination of the astrometric parameters of those objects in the sky accessible to a Gaia-like observer throughout its multi-year

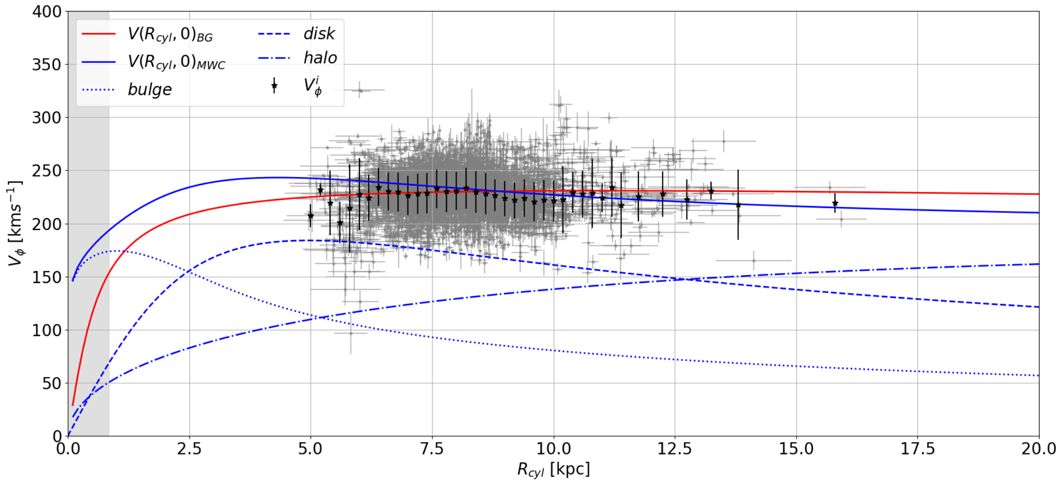

Fig. 17. – The Gaia observed rotation curve of the Milky Way up to 16 kpc, once transformed properly the Gaia catalog data into galactocentric azimuthal velocities. Both best fits to them, the GR compliant profile (red line) and the classical one (blue lines), are exactly the same. The black starred symbols represent the medians of the velocity V_ϕ derived from Gaia DR2 with the corresponding error bars that take into account the measurements uncertainties and systematic errors for each radial bin. The other blue curves represent the kinematical substructures that contribute to the MWC model: the dotted line is the bulge contribution, the dashed and dot-dashed lines that of the thin and thick disk, and the solid line is for the NFW halo. The gray vertical band represents twice the value of internal radius estimated with the BG model. Credit: Crosta, Giammaria, Lattanzi, and Poggio.

mission lifetime. GWs, at the sub-μas levels, could be most transient phenomena (i.e., merging of two black holes), then one needs to extract them from the residuals of an astrometric observable once reduced via an appropriate modeling, that has taken into account the instrumental effects (calibrations) as well.

The assumption of adding a corrective direction cosine due only to the passing GW signals with respect to the satellite attitude is completely impracticable, has it would impose an impossible requirement on the knowlegde accuracy of the attitude itself. Moreover, naively assuming that the astrometric observable (96) can be approximated as the sum of the cosine determined by the metric of the Solar System and that due to the GW metric, would imply considering twice the flat Minkowskian spacetime. Beside that, the background metric should contain all terms of the same level of accuracy of the GWs in order to model properly all of the background effects due to the Solar System bodies. Conversely, in case of a GW with period larger than the time of observation, one should completely change approach. For example, adopt spherical harmonics after the global reduction process [86, 85, 87].

The primordial gravitational wave density can be detected considering the secular aberration drift of the extragalatic radio source proper motions caused by the rotation of the Solar System barycenter around the Galactic center [88, 90] or the radio source velocity field [89].

Nevertheless, the astrometric direction cosine could be simplified if one takes into

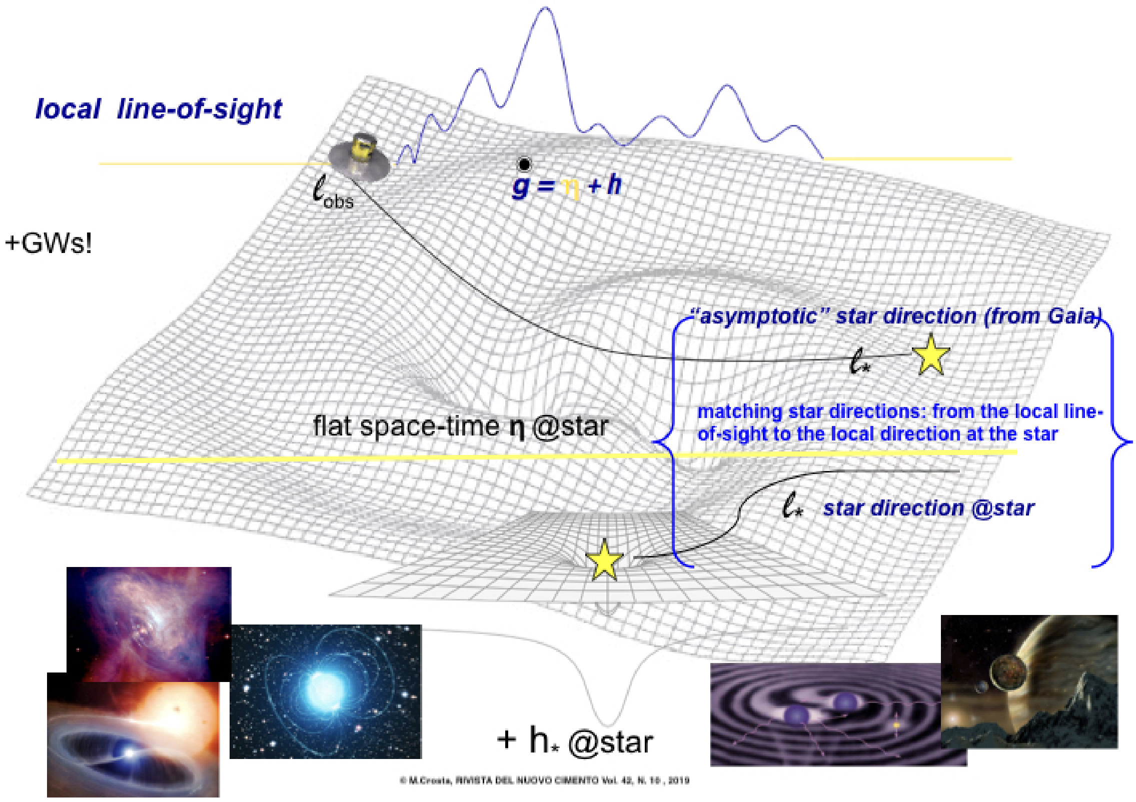

Fig. 18. – Relativistic astrometry links the local gravitational fields at the observers with those at the source through a varying space-time, which includes also passing gravitational waves.

account the angle between two space-like directions of light $\bar{\ell}_1$ and $\bar{\ell}_2$, namely:

$$\cos\psi_{(\bar{\ell}_1,\bar{\ell}_2)} = g_{\alpha\beta}\bar{\ell}_1^\alpha\bar{\ell}_2^\beta \tag{130}$$

where in principle the metric contains the perturbations due to the gravitational waves in the linearized form:

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}^{IAU} + h_{\alpha\beta}^{GW}. \tag{131}$$

In this case we are not obliged to deal with the satellite's attitude, but consider only the local barycentric observer (35) with respect to which the four null tangent vector k^α is projected. One should takes care of the appropriate order, considering that according to equation (124): $\bar{\ell}^\alpha = \bar{\ell}^\alpha_{obs} + \epsilon^2\bar{\ell}^\alpha_{(2)} + \epsilon^3\bar{\ell}^\alpha_{(3)} + O(\epsilon^4)$, and, on the other hand, gravitational waves are at least of the order of ϵ^4 (see equation (18)).

Now, let us consider a weak plane, elliptically polarized, gravitational wave propagating along the x direction with a wave vector ω and two amplitudes, A_{yy} and A_{yy}, such that

$$h_{yy} = A_{yy} \sin[\omega(t-x)], \qquad h_{yz} = A_{yz} \cos[\omega(t-x)], \tag{132}$$

corresponding to two polarization states (linear if A_{yy} or A_{yz} equal to 0, whereas circular if $A_{yy} = \pm A_{yz}$). This generates, far away from the Solar System (where $\hat{u}_B^\alpha = \delta_0^\alpha$), the following line element:

$$ds^2 = -dt^2 + dx^2 + (1-h_{yy})dy^2 + (1+h_{yy})dz^2 - 2h_{yz}dydz. \tag{133}$$

In order to give an idea of the potential of formula (130), let us assume for sake of simplicity that all the perturbations due to the IAU metric sources have been removed in the reduction process (i.e. one is evaluating residuals in the local proper-line-of-sight), so that $\bar{\ell}_{obs} = \bar{\ell}_{\mathcal{G}} + \bar{\ell}_{GW}$. With metric (132) the component of the four-vector of the null geodesic can be expressed as [91]

$$k^0 = (f+E^2)/2E, \qquad k^x = (f-E^2)/2E \tag{134}$$

$$k^y = (\alpha(1+h_{yy}) + \beta h_{yz})/D, \quad k^z = (\alpha h_{yz} + \beta(1-h_{yy}))/D \tag{135}$$

where

$$D = 1 - h_{yy}^2 - h_{yz}^2, \quad f = \alpha^2(1+h_{yy}) + \beta^2(1-h_{yy}) + 2\alpha\beta h_{yz} \tag{136}$$

and α, β, E are the constants of the motion (conserved Killing quantities) for the photon.

In such a case expression (130) results in ([11]):

$$\begin{aligned} \cos\psi_{(1,2)} = {} & \delta_{ij}(k_1^i/k_1^0)(k_2^j/k_2^0) - h_{yy}(\cos\delta_1\cos\delta_2 - \cos\theta_1\cos\theta_2) \\ & - h_{yz}(\cos\delta_1\cos\theta_2 + \cos\delta_2\cos\theta_1) \end{aligned} \tag{137}$$

being $E = -(u_B|k) = -k_0$, $\cos\phi = k^x/k^0$, $\cos\delta = k^y/k^0$, and $\cos\theta = k^z/k^0$. The last expression gives all the relative configurations of the components of the two null directions with respect to the passing GWs, or, alternatively, how the euclidian angle is modified by the passing GWs. Thus, it constrains the properties of the GWs (polarization states, energy and angular momentum) to the observed angle between two collected local-line-of-sights, which have interacted with the GW.

Let us assume that $\cos(\psi) = \cos(\psi_0 + \delta\psi_{GW})$, where ψ_0 represents the unperturbed angle between the two light directions. Then,

$$\cos(\psi_o + \delta\psi_{GW}) = \cos\psi_0\cos(\delta\psi_{GW}) - \sin\psi_0\sin(\delta\psi_{GW}), \tag{138}$$

which for $\delta\psi_{GW} \ll 1$ implies that equation (137) reduces to

$$\delta\psi_{GW} \approx \frac{h_{yy}(\cos\delta_1\cos\delta_2 - \cos\theta_1\cos\theta_2) + h_{yz}(\cos\delta_1\cos\theta_2 + \cos\delta_2\cos\theta_1)}{\sin\psi_0}. \tag{139}$$

([11]) The formulas anticipated here for the first time are part of a work in submission by Crosta et al. which includes all the details of the calculations and the observability estimates.

Then, if one chooses an appropriate separation angle ψ_0 (for example about few arcseconds, but it could be smaller depending only on the resolution power of the astrometric telescope!), the term in the denominator can mitigate the smallness of the terms h_{yy}, h_{yz} in the numerator. Formula (139) provides promising possibilities for the astrometric detection of $\delta\psi_{GW}$. In fact the signal could range between subμas and tens of nanoarcoseconds for a plane GWs, with characteristic strain of about $10^{-18} - 10^{-21}$ (assuming random values for k^α compatible with formulas (134), (135) and observed light directions taken from published or upcoming Gaia catalogue).

In addition, considering the same plane gravitational wave, one can reformulate the direction cosine in terms of the Synge function. The world function is defined as:

$$\Omega(\sigma_1, \sigma_2, \Upsilon) = \frac{1}{2}(\sigma_2 - \sigma_1)^2(\zeta|\zeta) \tag{140}$$

being $\zeta = \dot{\Upsilon}$ the tangent vector of the unique space-like geodesic Υ joining two points P_1, P_2 parametrized by σ. So it is strictly related to the length of the curve: $\Omega = L^2/2$. One of its mathematical proprieties gives:

$$\Omega_{\alpha 1} = -(\sigma_2 - \sigma_1)\zeta_{\alpha 1} \quad \Omega_{\alpha 2} = (\sigma_2 - \sigma_1)\zeta_{\alpha 2}, \tag{141}$$

and if the space-time admits Killing vectors ξ^α, the world function connecting points on geodesics satisfies the relation:

$$\xi^{\alpha 1}\Omega_{\alpha 1} + \xi^{\alpha 2}\Omega_{\alpha 2} = 0. \tag{142}$$

Let us assume that the stars are in geodesic motions at rest with respect to the local barycentric observer. Since

$$\Omega(\sigma_1, \sigma_2, \Upsilon)_{\text{flat}} = \frac{1}{2}\eta_{\alpha\beta}(x_1 - x_2)^\alpha(x_1 - x_2)^\beta, \tag{143}$$

provided the solutions above, by integrating equations (134)-(136) in order to get the trajectory of the photon, it results [40]

$$\begin{aligned} \Omega(\sigma_1, \sigma_2, \Upsilon) = {} & \Omega(\sigma_1, \sigma_2, \Upsilon)_{\text{flat}} \\ & + \frac{A_{yy}}{2\omega}\left[(y_2 - y_1)^2 - (z_2 - z_1)^2\right] \frac{\cos\omega(t_2 - x_2) - \cos\omega(t_1 - x_1)}{t_2 - x_2 - t_1 + x_1} \\ & - \frac{A_{yz}}{\omega}\left[(y_2 - y_1)(z_2 - z_1)\right] \frac{\sin\omega(t_2 - x_2) - \sin\omega(t_1 - x_1)}{t_2 - x_2 - t_1 + x_1}, \end{aligned} \tag{144}$$

which means that we can monitor the spatial distance between two star coordinates when a plane GW is passing. Then, $\sqrt{\Delta\Omega}/\sqrt{\Omega_{\text{flat}}} = \Delta L/L_{\text{flat}}$.

In term of direction cosines, suppose that the star is moving on its geodesics and sending a signal from a point A_1 along its trajectory to an observer in a space-time point P. The same point will receive the signal by the same star from the location on its trajectory A_2 affected by a passing GWs. In this case ζ^α coincides with ℓ^α and the direction cosine, for example, is given by:

$$\cos\psi_{1,2} = \frac{\Omega_{\alpha 1}\Omega_2^\alpha}{(\Omega_{\alpha 1}u_B^\alpha)(\Omega_{\alpha 2}u_B^\alpha)} \tag{145}$$

that can be computed according to the given definitions and solutions. In particular one can compute also the variation of the direction cosine in passing from A_1 to A_2 as [26]:

$$\Delta(\cos\psi) = -\frac{2}{3}(R_{\alpha\beta\gamma\delta}E^{\alpha}_{\hat{a}}\zeta^{\beta}u^{\gamma}\zeta^{\delta})_A L^2_u(A,P), \tag{146}$$

where A is the point on the stellar geodesic simultaneous to P along ζ, $R_{\alpha\beta\gamma\delta}$ is the Riemann tensor, $E^{\alpha}_{\hat{a}}$ is a Fermi tetrad carried on by the observer $\mathbf{u}$, which can be coupled with the interval of proper time from A_1 and A_2 (for detailed calculations see [26], section 9.2):

$$dT_u(A_1,A_2) = \sigma_{A_2} - \sigma_{A_1} \approx 2L(A,P) - \frac{1}{3}(R_{\alpha\beta\gamma\delta}u^{\alpha}\zeta^{\beta}u^{\gamma}\zeta^{\delta})_A L^3_u(A,P) \tag{147}$$

leading to an angle-proper-time relation to the first order in the curvature. World function allows, therefore, to measure the "true" variation of curvature among configuration of star positions on the sky in case of a passing gravitational waves, if monitored continuously by a pointing telescope.

The advantage of the above discussed scenarios is to link with extensive statistics the properties of a GW source to a large number of null geodesics (for example those from a cluster), so that to realize a sort of gravitational astrometric antenna. Proposals for such advanced astrometric missions are already in place [92].

In conclusion, the constant development of methods and techniques for relativistic astrometry to exploit the Gaia potential to its fullest suggests to look into the "future" of nanoarcsec regimes after Gaia. Insofar, the differential astrometric technique developed for the GAREQ experiment on Gaia could be extended to consider passing gravitational waves that affect photon propagation. In particular, the GAREQ method can be utilized to detect collective astrometric shifts on the four spatial light direction over angularly narrow but dense stellar fields. The critical aspect in this case is the implementation of an appropriate retrieval and calibration procedure at DPCT, which is on-going as a study case.

6˙5. *Relativistic metrology and spacetime travel*. – Astrometry has provided through ages the means to measure time and its variations. Nowadays, space-time astrometry recovers this old tradition as the celestial relativistic reference systems establish space-time coordinate transformations. But the application of the relativistic metrology to astrometry tells us more.

Imagine that the Gaia catalogue will contain a list of habitable planets and we would like to reach one, equipped by the Gaia stellar map. First of all we need to compute our local gravitational fields in order to get a proper "target" direction free from the local gravitational aberrations, otherwise we will risk to never reach our final destination, since missing few microarseconds translate in a considerable error on parallaxes.

Now, once arrived at the esoplanet spacetime coordinates, our "astrometric" tool give us the possibility to compute the new local gravitational fields generated by the esosystem (hosting, for example, more bodies than our solar system and perhaps a planet of about 3-4 Jupiter mass) and to correct the proper direction in order to return back to home, i.e. "our" spacetime coordinate. Then, we need also to take into account of equation (51), namely to compute the esosystem weak gravitational fields in order to reach the Earth at the right "terrestrial" proper time, trying to avoid in the way back possibly naked singularities...In brief, relativistic astrometry will be the necessary tool to navigate through a varying spacetime and avoiding space-time consuming!

Even though these are speculative ideas about the future use of relativistic astrometry and the correlated relativistic charting, however inevitably they open new frontiers and potential application on spacetime navigation. All this involves not only general relativistic metrology but quantum metrology as well [93], especially in view of a possible interaction between gravitational waves and photons via astrometry.

7. – Final remarks

Special and General Relativity, both theories developed by Albert Einstein a little more than a century ago, have involved a change in our conceptions of space and time of such an impact that they have not yet been fully applied, both in the field of fundamental physics and in that of astrophysics. In fact, only relatively recently with the advent of microarcsecond astrometry is mandatory to shift from the classical approach of fundamental astronomy to a relativistic one.

The implementation and availability of methods of experimental gravity, both theoretical and numerical, are more necessary than ever today, especially in the weak gravity regime, whose effects influence the electromagnetic propagation on a much wider domain than the strong one of which it constitutes, nonetheless, an important and complementary part. This paradigm shift is required both from the point of view of data processing and from that of their interpretation for the subsequent scientific exploitation. This "know-how" will be necessarily part of the fundamental baggage to define the high precision astronomical measurements made from within the gravitational fields of the solar system (inevitably!), as well as the correct definition of the celestial reference systems used in astronomy both from Earth and from Space.

The future potential of these kind of measurements relies on the development of useful tools for the treatment of systematic errors in the data analysis of precise experiment in space. Fundamental physics in space employs a very wide range of technologies to achieve the instrumental performance required for each dedicated mission. This it turns out challenging for scientists and engineers but also very fruitful as test bed for emerging technologies, which have immense potential in other fields of application.

Gaia-like missions are offering, then, the unique possibility of being a multi laboratory for extensively testing weak gravitational fields either at local scale, such as the Solar System, either at that of the Milky Way. In particular for: (i) the validity of General Relativity (GR) at all scales, and therefore the nature of dark matter and dark energy; (ii) multiple detections and analysis of relativistic effects, being mainly light deflections/retarded time travel overlapping in any direction and in any time, due to the varying gravitational local sources measured from within the system, and possibly extrapolated with appropriate rescaling to other systems and other measurements of weak lensing effects; (ii) tests on cosmological models that predict the cosmological evolution shaped by gravity as we observe it at Galaxy scale; (iii) the relations among baryonic structures (and their evolution) and the dark components of the Universe, the formation of these structures, from primordial galaxies to the Milky Way (MW) and its stellar populations; (iv) future observations of GWs; finally, (v) developing new technology.

Given the number of celestial objects (a real Galilean method applied on the sky!) and directions involved (the whole celestial sphere!), the realization of the relativistic celestial sphere is not only a scientific validation of the absolute parallax and proper motions obtained with Gaia. Reaching 10-20 μas accuracy on individual parallax and annual proper motions for bright stars (V $<$ 16) is also the key possibly to perform the largest GR experiment ever attempted from space with astrometric methods since May 29, 1919.

And beyond the microarcsecond? Gaia represents only a ground step, increasing the level of accuracy requires to refine consistently the metric of the solar system, the solutions for the null geodesic, the observables, the attitude, and so on.. As a matter of fact, once a relativistic model for the data reduction has been implemented, any subsequent scientific exploitation should be consistent with the precepts of the theory underlying such a model. From Gaia and Relativistic Astrometry we have learnt that Astronomers need to be ready to exploit all of the scientific potential of the local measurements entangled to the varying gravitational fields from within the Solar System and to maximize its impact: *"One day, our actual knowledge of the composition of the fixed stars sky, the apparent motion of the fixed stars, and the position of the spectral lines as a function of the distance will probably have come far enough for us to be able to decide empirically the question whether or not Λ vanishes"* (Einstein, 1917, letter to de Sitter).

* * *

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