Relativistic light tracing in the Gaia era

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This contribution presents the analytical solution of the inverse ray tracing problem for photons emitted by a star and collected by an observer located in the gravitational field of the Solar System. This solution has been conceived to suit the accuracy achievable by the ESA Gaia satellite (launched on December 19, 2013) consistently with the measurement protocol in General Relativity adopted within the RAMOD framework. Aim of this study is to provide a general relativistic tool for the science exploitation of such a revolutionary mission, whose main goal is to trace back star directions from within our local curved space-time, therefore providing a three-dimensional map of our Galaxy. The calculations are performed assuming that the massive bodies of the Solar System move uniformly and have monopole and quadrupole structures. The results are useful for a thorough comparison and cross-checking validation of what already exists in the field of Relativistic Astrometry. Moreover, such an analytical solutions can be extended to model other measurements that require the same order of accuracy as that expected for Gaia.

Keywords: Light propagation; weak gravitational fields; relativistic astrometry; measurement protocol in general relativity; RAMOD.

1. Introduction

To fully exploit the science of the Gaia mission (ESA, Ref. 1), a relativistic astrometric model needs to be able to cope with an accuracy of few micro-arcseconds (μas) for observations within the Solar System.

Gaia acts as a celestial compass, measuring arcs among stars with the purpose to determine their position via the absolute parallax method. The main goal is to construct a three-dimensional map of the Milky Way and unravel its structure, dynamics, and evolutional history.

Since the satellite is positioned at Lagrangian point L2 of the Sun-Earth system, the measurements of Gaia are performed in a weak gravitational regime and the solution of Einstein equation, i.e the space-time metric, can take the general form

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \tag{1}$$

where $|h_{\alpha\beta}| \ll 1$ and $|\partial_i h_{\alpha\beta}| \ll 1$ can be treated as perturbations of a flat space-time and represent all the Solar System contributions to the gravitational field. Their explicit expression, however, can be described in different ways according to the physical situation we are considering. This means that, for the weak-field case, $h_{\alpha\beta}$ can always be expanded in powers of a given smallness parameter, expansion usually made in powers of the gravitational constant G (post-Minkowskian approximation, PM) or of the quantity $1/c$ (post-Newtonian approximation, PN); note that in the PM approach one can still make a further expansion in powers of $1/c$ signifying that a solution for light rays in the PM approximation is more general then the corresponding solution in the PN one. The estimates performed inside the near-zone of the Solar System are sufficiently well supported by an approximation to the required order in the small parameter $\epsilon \equiv (v/c)$, which amounts to about 10^{-4} for the typical velocities of our planets. Moreover, for the propagation of the light inside the Solar System, the sources of gravity should be considered together with their internal structure and geometrical shape. This is particularly true when the light passes close to the giant planets. In other circumstances it is an unnecessary complication to consider the planets different from point-like objects especially when the model is devoted to the reconstruction of *stellar positions* in a global sense. However, at the microarcsecond level of accuracy, *i.e.* $(v/c)^3 = \epsilon^3$, the contribution to the metric coefficients by motion and internal structure of the giant planets needs to be taken into account, in particular if one wants to measure specific light deflection effects, as for example, those due to the quadrupolar terms. For this purpose, calculations have been performed assuming that the massive bodies of the Solar System move uniformly and have monopole and quadrupole structures.

2. RAMOD and the inverse ray-tracing problem

The scope of this contribution is to present an analytical solution for a null geodesic of the metric (1) consistently with the requirements of the Gaia astrometric mission and according to the RAMOD framework (Ref. 2, Ref. 3). The Relativistic Astrometric MODel (RAMOD) is a mathematical tool conceived to model the astrometric measurements made by an observer in space. Since its original purpose was to address this problem for the ESA Gaia mission, whose final astrometric accuracy requires the physical model to be accurate at microarcsecond level, RAMOD had to take into account the general relativistic corrections due to the bodies of the Solar System. Nevertheless, the results can be extended to the light propagation in regime of weak field. RAMOD uses a 3+1 description of space-time in order to measure physical effects along the proper time and in the rest-space of a set of fiducial observers according to the following measurement protocol (Ref. 4):

i) specify the phenomenon under investigation;

ii) identify the covariant equations which describe the phenomenon;

iii) identify the observer making the measurements;

iv) chose a frame adapted to that observer allowing space-time splitting into the observer's space and time;

v) understand the locality properties of the measurement under consideration (namely, whether it is local or non-local with respect to the background curvature);

vi) identify the frame components of the quantities that are the observational targets;

vii) find a physical interpretation of the above components following a suitable criterion;

viii) verify the degree of the residual ambiguity, if any, in the interpretation of the measurements and decide the strategy to evaluate it (i.e. comparing to what already known).

The main procedure of the RAMOD approach is to express the null geodesic in terms of the physical quantities which enter the process of measurement, in order to entangle the entire light trajectory with the background geometry to the required approximations. Finally, the solution should be adapted to the relevant IAU resolutions considered for Gaia (Ref. 5).

Therefore the fundamental unknown of the RAMOD method is the space-like four-vector $\bar{\ell}^{\alpha}$, which is the projection of the tangent to the null geodesic onto the rest-space of the local barycentric observer, namely the one locally at rest with respect to the barycenter of the Solar System. Physically, such a four-vector identifies the *line-of- sight* of the incoming photon relative to that observer.

Once defined $\bar{\ell}^{\alpha}$, the equations of the null geodesic takes a form which we shall refer to as master-equations. Neglecting all the $O(h^2)$ terms, these read (Ref. 3, Ref. 6):

$$\frac{d\bar{\ell}^0}{d\sigma} - \bar{\ell}^i\bar{\ell}^j h_{0j,i} - \frac{1}{2}h_{00,0} = 0, \tag{2}$$

$$\begin{aligned}\frac{d\bar{\ell}^k}{d\sigma} &- \frac{1}{2}\bar{\ell}^k\bar{\ell}^i(\bar{\ell}^j h_{ij,0} - h_{00,i}) + \bar{\ell}^i\bar{\ell}^j\left(h_{kj,i} - \frac{1}{2}h_{ij,k}\right) \\ &+ \bar{\ell}^i\left(h_{k0,i} + h_{ki,0} - h_{0i,k}\right) - \frac{1}{2}h_{00,k} - \bar{\ell}^k\bar{\ell}^i h_{0i,0} + h_{k0,0} = 0.\end{aligned} \tag{3}$$

Here, σ is the affine parameter of the geodesic,

$$\mathrm{d}\sigma = \mathrm{d}t + \mathcal{O}\left(h\right), \tag{4}$$

and t is the coordinate time.

In order to solve the master equations one should define appropriate metric coefficients. To the order required for the accuracy targeted for Gaia, one has to take into account the distance between the points on the photon trajectory and the barycenter of the a-th gravity source at the appropriate retarded time together with the dynamical contribution to the background metric by the relative motion of the gravitational sources. In particular, in order to fully accomplish the precepts of the measurement protocol above, it is useful to isolate the contributions from the derivatives of the metric terms at the different retained orders . The choice of the space-time coordinates that justifies the form of the metric adopted allows one to think that the metric perturbations and their derivatives mainly carry information about the background gravity. The different analytical solutions obtained in Crosta et al. (Ref. 7) reflect consistently several orders of accuracy and are classified in different classes according to the contributions of the derivative of the metric.

Solving the astrometric problem implies practically the compilation of an astrometric catalog with the accuracy of the measurement model. Indeed there exist several models conceived for the above task and formulated in different and independent ways (Ref. 8–10, and references therein). Their availability must not be considered as an "oversized toolbox". Quite the contrary, they are needed to put the future experimental results on solid grounds, especially if one needs to implement gravitational source velocities and retarded time effects. Moreover, the classification that we have so far introduced turns out to be extremely useful for the implementation of RAMOD models and the testing of them through consistent internal checks at different levels of accuracy, allowing also a very simple procedure to identify where the possible discrepancies can arise.

As far as the light deflection is concerned, we expect that the velocity contributions become relevant in affecting light propagation in the case of close approach when general relativistic effects become of the order of Gaia's expected accuracy together with the multipolar structure of the source. In fact, when a photon approaches the weak gravitational field of the Solar System, heading to a Gaia-like observer, it will be subjected to the gravitational field generated by the mass of the bodies of the system while it will be rather insensitive to the contribution to the field due to their own motion. If one compares the scale of the Solar System and the photon crossing time through it - approximately 10 hours in total- the gravitational field of the Solar System cannot significantly change in a dynamical sense during such time, to the point that the source velocity can be considered constant all along the photon trajectory. This last remark facilitates the solution of the master equations.

Let us make explicit the vorticity of the congruence **u**:

$$\begin{aligned}\omega_{\rho\sigma} &= P^{\alpha}_{\rho}(u)P^{\beta}_{\sigma}(u)\nabla_{[\alpha}u_{\beta]} \\ &= \nabla_{[\rho}u_{\sigma]} + u_{[\rho}\dot{u}_{\sigma]},\end{aligned} \tag{5}$$

where $P^{\alpha}_{\rho}(u) = \delta^{\alpha}_{\rho} + u^{\alpha}u_{\rho}$ is the metric induced on each hypersurface of simultaneity of $\mathbf{u}$. Considering that $u^{\alpha}u_{\alpha} = -1$, and $u^{\alpha}\nabla_{\alpha}u^{\beta} = \dot{u}^{\beta}$, we deduce:

$$\omega_{\rho\sigma} = -\eta_{0[\rho}\partial_{\sigma]}h_{00} + \partial_{[\rho}h_{\sigma]0} + \partial_0(\eta_{0[\rho}h_{\sigma]0})$$

which implies

$$\begin{aligned}\omega_{00} &= 0, \\ \omega_{0i} &= 0, \\ \omega_{ij} &= \partial_{[i}h_{j]0}.\end{aligned} \tag{6}$$

Taking into account, for example, the IAU metric, equations (6) show that if we want a vanishing vorticity we have to choose $h\vec{\nabla} \times \vec{\bar{v}} + \vec{\nabla}(h) \times \vec{\bar{v}} = 0$, which is satisfied if the velocity of the source is zero, i.e. a static case, or is constant, which corresponds to the case of our Solar System as mentioned above. Now, within the scale of a vorticity-free geometry, from the Frobenius theorem, the space-time can be foliated and one can always map the whole geodesic onto the hypersurface of simultaneity of the local barycentric observer at the time of observation. In this case the mapped trajectory can be expressed in a parametrized form with respect to the centre-of-mass (CM) of the gravitationally bounded system and easily solved by integration with respect to such a parameter. For details see Crosta et al. (Ref. 7).

3. Conclusion

The theory of light propagation within the RAMOD framework have been improved by the analytical solutions developed by Crosta et al. (Ref. 7). Despite the apparent straightforwardness of the task and the linearity of the metric given the weak gravitational field regime inside the Solar System, the solution of the inverse ray tracing problem, which allows us to reach the aim above, is rather intractable unless treated numerically, particularly if local gravitational fields and retarded time contributions need to be accounted for (Ref. 3). As far as RAMOD is concerned, the reason lies mainly in the fact that the main unknown of the differential equations is the observed direction as projected on the rest space of the local barycentric observer and represents *locally* what the observer measures of the incoming photons in his/her gravitational environment. This aspect transforms the geodesic equation into a set of nonlinear coupled differential equations which comprises also that for the time component. The original version of the RAMOD model was therefore numerical and although successful in its applications with the inclusion of the relativistic satellite attitude (Ref. 11), it was hard to control and compare with similar astrometric models even with a comprehensive error budget for stellar positions (Ref. 12). While the retarded time approximation adopted and the solution for the static cases recover the results obtained by similar astrometric models (as proved also in Crosta, Ref. 6), the solutions including the constant velocity of the source give rise to different expressions that deserve to be carefully evaluated in a separated work as were done, for example, in Crosta et al., Ref. 5, and Bertone et al., Ref. 13, especially in consideration of the recent applications done with the Time Transfer Function approach (Hees, Ref. 10).

RAMOD was originated to satisfy the Gaia validation requirements; however, the procedure developed can, indeed, be extended to all physical measurements implying light propagation. The analytical solution is general enough to be applicable under observing conditions more demanding than those of the Gaia mission, i.e. to other missions conceived to exploit photon trajectories and extends within the RAMOD formalism (since other similar solutions are already known from the literature) the analysis of the trajectory perturbations due to gravitating sources with a non negligible quadrupole structure.

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