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Optimization by parameters in the iterative methods for solving non-linear equations

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Abstract: *In this paper, we used the necessary optimality condition for parameters in a two-point iterations for solving nonlinear equations. Optimal values of these parameters fully coincide with those obtained in [6] and allow us to increase the convergence order of these iterative methods. Numerical experiments and the comparison of existing robust methods are included to confirm the theoretical results and high computational efficiency. In particular, we considered a variety of real life problems from different disciplines, e.g., Kepler's equation of motion, Planck's radiation law problem, in order to check the applicability and effectiveness of our proposed methods.*

Keywords: *Iterative methods; Optimal values of parameters; Iterations with memory;*

INTRODUCTION

Many iterative methods for solving nonlinear equations often include non-zero free parameters. Their suitable choices allow to increase the convergence order of methods. There are some choices of parameters based on the error analysis, see for example [1,2,3,4,5,8,9] and references therein. Recently, researchers have also proposed several biparametric two-step simple methods with and without memory [1,3,4,6]. On the other hand, optimization by parameters is one of the powerful techniques in science and engineering practice. The main goal of this paper is to find the optimal choices of parameters $\bar{\tau}_n$ and, γ in the two points iterative methods. We obtained analytic formulae for λ and γ without symbolic computation technique.

The paper is organized as follows. In section 2, we developed necessary optimality condition for parameters in the two-point method. In the last section, we present the results of numerical experiments that confirm the theoretical conclusion about the convergence order and made a comparison with well-known methods of the same order of convergence.

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In numerical analysis and engineering applications, it is often required to solve a non-linear equation $f(x) = 0$, where $f(x): D \subset R \to R$ is a scalar function defined on an open interval D. Assume that f is sufficiently smooth and has a simple zero $x^* \in D$ and f' is nonsingular in D.

In this paper, we consider optimization by parameters in the two-point iterative methods [6,7].

$$y_n = x_n - \frac{f(x_n)}{\phi_n + \lambda f(w_n)}, \ w_n = x_n + \gamma f(x_n),$$

$$x_{n+1} = y_n - \bar{\tau}_n \frac{f(y_n)}{\phi_n + \lambda f(w_n)}, \ \phi_n = \frac{f(w_n) - f(x_n)}{\gamma f_n}, \tag{1}$$

where γ, λ are non-zero parameters. It is easy to show that the minimization problem

$$\min_{\bar{\tau}_n, \lambda, \gamma} f(x_{n+1})^2 \tag{2}$$

is equivalent to

$$f(x_{n+1}) = 0. \tag{3}$$

Necessary optimality of condition for parameters in the two-point iterations

The Taylor expansion of $f(x_{n+1})$ at the point y_n gives

$$f(x_{n+1}) = (1 - \bar{\tau}_n \frac{f'(y_n)}{\phi_n + \lambda f(w_n)}) f(y_n) + O(f(y_n)^2). \tag{4}$$

From (4) it clear that the problem (3) leads to minimization of $f(y_n)$. If we choose $\bar{\tau}_n$ such that

$$\bar{\tau}_n = \frac{\phi_n + \lambda f(w_n)}{f'(y_n)}, \tag{5}$$

Then we have

$$f(x_{n+1}) = O(f(y_n)^2). \tag{6}$$

We find stationary point of $f(y_n(\lambda, \gamma))$ solving the system

$$\frac{\partial y_n}{\partial \lambda} = f'(y_n) \frac{\partial y_n}{\partial \lambda} = 0,$$

$$\frac{\partial y_n}{\partial \gamma} = f'(y_n) \frac{\partial y_n}{\partial \gamma} = 0, \tag{7}$$

Since $f'(x) \neq 0$ in the domain D containing the zero of the equation (3) the system (7) equivalent to the system:

$$\frac{\partial y_n}{\partial \lambda} = 0, \ \frac{\partial y_n}{\partial \gamma} = 0. \tag{8}$$

From the first equation in (1) we find that

$$\begin{cases} \dfrac{\partial y_n}{\partial \lambda} = \dfrac{f(x_n)}{(\phi_n + \lambda f(w_n))^2} f(w_n), \\ \dfrac{\partial y_n}{\partial \gamma} = \dfrac{f(x_n)}{(\phi_n + \lambda f(w_n))^2} \left(\dfrac{\partial \phi_n}{\partial \gamma} + \lambda \dfrac{\partial f(w_n)}{\partial \gamma} \right). \end{cases} \tag{9}$$

Substituting (9) into (8) we get

$$\begin{cases} f(w_n) = 0, \\ \dfrac{\partial \phi_n}{\partial \gamma} + \lambda \dfrac{\partial f(w_n)}{\partial \gamma} = 0. \end{cases} \tag{10}$$

On the other hand, by definition of functions w_n and ϕ_n we have

$$f(w_n) = f(x_n)(1 + f'(x_n)\gamma + \frac{f''(x_n)}{2}\gamma^2 f(x_n) + \cdots), \tag{11}$$

and

$$\frac{\partial \phi_n}{\partial \gamma} + \lambda \frac{\partial f(w_n)}{\partial \gamma} = f(x_n)(\frac{f''(x_n)}{2}$$
$$+ \lambda f'(x_n) + \gamma f(x_n)(\frac{2f'''(x_n)}{3!} + \lambda f''(x_n)) + \cdots), \tag{12}$$

From (11) and (12) clear that the parameters γ and λ are determined, so that the system of equations (10) holds at least $f(y_n(\lambda, \gamma))$ with accuracy $O(f(x_n)^2)$. That is

$$\gamma = \gamma_n = -\frac{1}{f'(x_n)}, \lambda = \lambda_n = -\frac{f''(x_n)}{2f'(x_n)}. \tag{13}$$

Thus, we find the extremum point (13) of function $f(y_n(\lambda, \gamma))$ with accuracy $O(f(x_n)^3)$, because of (9), (11), (12).

The parameters given by the formulae (13) are naturally said to be optimal in the sense of necessary optimality condition (8). So, We can find approximations of γ_n, λ_n. They can be found using the information available from the current and previous iteration steps. The methods containig such parameters are called methods with memory.

Note that (13) fully coincides with those obtained in [3] and

$$f(y_n) = O(f(x_n)^4), \tag{14}$$

under (13). Now we shall find iteration parameter $\bar{\tau}_n$ with some accuracy. To this end, we approximate $f'(y_n)$ in (5) as:

$$f'(y_n) = \frac{f(y_n + \Delta_n) - f(y_n - \Delta_n)}{2\Delta_n} + O(\Delta_n^2), \tag{15}$$

where

$$\Delta_n = f(x_n)^p, \ p = 1,2. \tag{16}$$

As a result, we have

$$\bar{\tau}_n = \frac{2(\phi_n + \lambda f(x_n))\Delta_n}{f(y_n + \Delta_n) - f(y_n - \Delta_n)}. \tag{17}$$

From (4) and (15) we conclude that

$$f(x_{n+1}) = O(f(x_n)^{4+2p}), \ p = 1,2, \tag{18}$$

under choices (13), (17).

Thus, we can formulate the above considered results as:

Theorem 1. *Assume that $f(x_n)$ sufficiently smooth in D containing the localized zero of the equation (3) and the initial approximation x_0 is close enough to x^*. Then the two point iterative method (1) has $4+2p$, $p=1,2$ order of convergence under choices (13) and (17).*

Remarks. *Obviously, the optimal choices (13) also hold true for three-point iterative methods, the first two steps of which are the same as (1). Note that the stationary point of view is also applicable for solving system of nonlinear equations.*

Based on the choices (13), (17) one can construct new $4+2p$ order convergence iterations with memory:

$$x_0, \lambda_0, \gamma_0 \quad \text{are} \quad \text{given,} \quad \text{then} \quad w_0 = x_0 + \gamma_0 f(x_0),$$

$$\gamma_n = -\frac{2f(x_n)}{f(x_n + f(x_n)) - f(x_n - f(x_n))},$$

$$w_n = x_n + \gamma_n f(x_n), \qquad n = 1,2,\cdots,$$

$$\lambda_n = -\frac{f(x_n + f(x_n)) - 2f(x_n) + f(x_n - f(x_n))}{f(x_n)(f(x_n + f(x_n)) - f(x_n - f(x_n)))}, \qquad (19)$$

$$y_n = x_n - \frac{f(x_n)}{\phi_n + \lambda_n f(w_n)},$$

$$x_{n+1} = y_n - \bar{\tau}_n \frac{f(y_n)}{\phi_n + \lambda_n f(w_n)}, \quad n = 0,1,\cdots,$$

where $\bar{\tau}_n$ is given by (17).

RESULTS AND DISCUSSION

We employed an iterative method (1) with (13), (17) and method (19) to solve nonlinear equations. We compared the new methods (1) and method (19) to X. Wang, T. Zhang [8] method (*WZ*), Dzunic [3] method (*Dz*), Cordero et al. [1] method (*Co*).

The numerical results are displayed in Tables 1-4. The comparison with existing methods is included as well. For numerical tests we considered the following problems:

$$f_1 = e^{x^3 - x} - \cos(x^2 - 1) + x^3 + 1, \quad x^* = -1$$

$$f_2 = e^{x^3 - 3x} \sin x + \log(x^2 + 1), \quad x^* = 0,$$

$$f_3 = M - E + \varepsilon \sin E, \quad 0 < \varepsilon < 1, x^* = 0.38$$

$$f_4 = e^{-x} + \frac{x}{5} - 1, \quad x^* = 4.96$$

which has been selected from [5] and [8,9,10]. For f_1, f_2, we consider the initial guesses -0.6 and 1.3, respectively. In particular, f_3 is Kepler's equation which relates the eccentric anomaly E, the mean anomaly M and the eccentricity ε in an elliptic orbit.

One of the classical laws of planetary motion due to Kepler says that a planet revolves around the sun in an elliptic orbit.

Suppose one needs to find the position (x, y) of the planet at time t. This can be determined by the following formula:

$$x = a\cos(E - \varepsilon)$$

$$y = a\sqrt{1 - \varepsilon^2} \sin E.$$

To determine the position (x, y), one must know how to compute E, which can be computed from Kepler's equation of motion:

$$M - E + \varepsilon \sin E = 0, \quad 0 < \varepsilon < 1.$$

The solution of the above equation can be obtained by implementing various iterative methods with different values of M and ε.

The approximate root of f_3 is 0.3899777749463621.

In [10] we have considered one of the famous classical physics problem, which is known as the Planck's radiation law problem. Fourth non-linear function f_4 arises from this problem. Planck's radiation law problem calculates the energy density within an isothermal blackboard and is given by the following formula:

$$\varphi(\lambda) = \frac{8\pi cp\,\lambda^{-5}}{e^{\frac{cp}{\lambda\,\mathrm{BT}}}-1},$$

where λ is the wavelength of the radiation, T is the absolute temperature of the blackbody, B is the Boltzmann constant, p is the Planck constant and c is the speed of light. We are interested in determining wavelength λ which corresponds to maximum energy density $\varphi(\lambda)$.

Further, $\varphi'(\lambda) = 0.$ implies that the maximum value of φ occurs when

$$\frac{\frac{cp}{\lambda\,\mathrm{BT}}e^{\frac{cp}{\lambda\,\mathrm{BT}}}}{e^{\frac{cp}{\lambda\,\mathrm{BT}}}-1} = 5.$$

If $x = \frac{cp}{\lambda\,\mathrm{BT}}$, then the equation is satisfied when

$$f_4 = e^{-x} + \frac{x}{5} - 1.$$

$f_4(x) = 0$ has two zeros. Obviously, one of the roots $x = 0$ is not taken for discussion. Another root is $x^* = 4.965114231744276303699$. Moreover, the initial guesses are chosen that 0 and 6 for those problems.

The results, for methods with memory, are computed with the same initial values of γ and λ. The calculations have been performed in the MATHEMATICA 12 using multi-precision arithmetic with 1000 digits and we used the following stopping criterion:

$$|x_n - x^*| < 10^{-60}.$$

The number of iterations (n), the absolute errors $|x_n - x^*|$ and computational order of convergence (ρ) are also shown in Tables 1-2. Here, (ρ) is calculated by the following formula [2,9]:

$$\rho \approx \frac{\ln(|x_n - x^*|/|x_{n-1} - x^*|)}{\ln(|x_{n-1} - x^*|/|x_{n-2} - x^*|)},$$

where x_n, x_{n-1}, x_{n-2} are three consecutive approximations of iterations.

From Tables 1- 4, we see that the computed results completely support the theory of convergence discussed in previous sections. The convergence order of the method (19) with $p = 2$ is eight which is higher than other two-step methods with memory.

Table 1. Comparison of methods for example 1

Methods	$\overline{\tau}_n$	n	$\|x_n - x^*\|$	ρ
(1), (p=1, $\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	(17)	3	0.37e-96	6
1), (p=2, $\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	(17)	3	0.51e-168	8
(19), (p=1, $\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	(17)	3	0.69e-69	6
(19), (p=2, $\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	(17)	3	0.71e-126	**8**
WZ [8], ($\lambda_0 = -0.1, \lambda_n = -H''_n/2f'$)	-	3	0.45e-98	7
Dz [3], (p=-0.1, $\gamma_0 = -0.01$, $g = 1 + \theta_n$)	-	3	0.24e-168	7
Co [1], ($\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	-	3	0.11e-167	7

Table 2. Comparison of methods for example 2

Methods	$\overline{\tau}_n$	n	$\|x_n - x^*\|$	ρ
(1), (p=1, $\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	(17)	3	0.26e-94	6
(1), (p=2, $\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	(17)	3	0.79e-145	8
(19), (p=1, $\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	(17)	3	0.71e-91	6
(19), (p=2, $\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	(17)	3	0.88e-161	**8**
WZ[8], ($\lambda_0 = -0.1, \lambda_n = -H''_n/2f'$)	-	4	0.40e-129	7
Dz [3], (p=-0.1, $\gamma_0 = -0.01$, $g = 1 + \theta_n$)	-	4	0.42e-360	7
Co[1], ($\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	-	4	0.67e-245	7

Table 3. Comparison of methods for example 3

Methods	$\bar{\tau}_n$	n	$\lvert x_n - x^* \rvert$	ρ
(1), (p=1, $\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	(17)	3	0.54e-105	6
(1), (p=2, $\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	(17)	3	0.61e-153	8
(19), (p=1, $\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	(17)	3	0.13e-102	6
(19), (p=2, $\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	(17)	3	0.68e-193	**8**
WZ[8], ($\lambda_0 = -0.1$, $\lambda_n = -H''_n/2f'$)	-	4	0.39e-133	7
Dz [3], (p=-0.1, $\gamma_0 = -0.01$, $g = 1 + \theta_n$)	-	4	0.52e-177	7
Co[1], ($\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	-	4	0.17e-192	7

Table 4. Comparison of methods for example 4

Methods	$\bar{\tau}_n$	n	$\lvert x_n - x^* \rvert$	ρ
(1), (p=1, $\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	(17)	3	0.18e-112	6
(1), (p=2, $\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	(17)	3	0.12e-172	8
(19), (p=1, $\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	(17)	3	0.26e-181	6
(19), (p=2, $\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	(17)	3	0.63e-198	**8**
WZ[8], ($\lambda_0 = -0.1$, $\lambda_n = -H''_n/2f'$)	-	4	0.33e-162	7
Dz [3], (p=-0.1, $\gamma_0 = -0.01$, $g = 1 + \theta_n$)	-	4	0.32e-165	7
Co[1], ($\lambda_0 = -0.1$, $\gamma_0 = -0.01$)	-	4	0.33e-197	7

CONCLUSIONS

In this paper, a new two-point derivative-free iterative method with memory for solving nonlinear equations was introduced and studied. Using optimal values of parameters, the higher order method with memory is obtained. Exact analytical formulas for the optimal values of the parameters have been found for the first time. The convergence order increased from four to eight without any additional computations. Finally, numerical experiments have shown that the new method is eight-order and effective. We also solved some real world applications of different nature to show the effectiveness of the proposed methods.

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