Critical mathematical thinking in young students

Monteleone, Chrissoula

CRITICAL MATHEMATICAL THINKING IN YOUNG STUDENTS

Chrissoula Monteleone

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Keywords

Applying; Assessing claims and arguments; Clarifying; Creating; Critical mathematical thinking; Critical thinking; Curriculum; Early years; Estimating; Evaluating; Explaining; Factual teacher questions; Guiding teacher questions; Interpreting; Justifying; Making judgements with criteria; Mathematics; Mathematical thinking; Non-algorithmic decision making; Noting relationships; Offering opinions and reasons; Presenting; Probing teacher questions; Querying; Self-regulating; Solving; Stating; Teacher questioning.

Abstract

The current expectation in curriculum is that as young students progress through primary school they are capable to think both critically and mathematically. Yet for these two types of thinking (mathematical and critical) there are a number of inconsistencies in and overlaps between their definition in the research literature and curriculum documents. Additionally, research identifies that not only do many teachers perceive that young students are incapable of these types of thinking, but also there is a paucity of research focussing on these two domains in the early years.

Thus, this study aims to investigate Critical Mathematical Thinking (CMT) in young students, and teaching actions/questions that help these young students exhibit their CMT. Two research questions informed the research design. They were:

- *What are the CMT capabilities young students exhibit as they begin formal schooling?*
- *What types of teacher questions help these young students exhibit their CMT?*

The term Critical Mathematical Thinking emerged from the synthesis and amalgamation of the sets of disparate literature pertaining mathematical thinking and critical thinking. As a result of this process the Critical Mathematical Thinking Conceptual Framework was also developed. This conceptual framework served as a tool to analyse the data and give a comprehensive understanding of CMT in young students.

Given that this study explored young students' ways to display CMT capabilities and the teaching actions/questions that help these young students exhibit their CMT, an interpretative research paradigm within a constructivism epistemology was adopted. An explanatory design using a mixed method approach informed the collection of the data. The quantitative data were used to narrow participant selection at intervals of the data collection and analysis process. The qualitative data supported the process of eliciting findings pertaining to the research aim. The research was conducted across five kindergarten classrooms from three schools (schools bounded

by their demographic data) located in New South Wales, Australia. The participants were all in their first six months of their first year of formal schooling (average age of six). Students were selected to participate in specific stages of the study, narrowing the student sample from 161 kindergarten students to sixteen students who exhibited high levels of CMT.

Findings from this study provide further insights into defining and understanding CMT capabilities in young students. First, a framework, titled the Critical Mathematical Thinking Framework for Young Students (CMTFYS) was a key finding as it assisted in defining the capabilities that young students exhibit. Second, while CMT cannot be measured by intelligence tests, there is a relationship between students' exhibiting CMT and their awareness of pattern and structure in mathematics. Third teacher questioning, in line with the CMTFYS, was found help these students to exhibit their CMT.

This study contributes to critical thinking and mathematical thinking for young students. Theoretical contributions to new knowledge include the development of a literature and data informed framework (CMTFYS). In addition, components of the CMTFYS can support teacher questioning to assist young students exhibit CMT.

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List of Abbreviations

ACARA	Australian Curriculum, Assessment, and Reporting Authority
ACECQA	Australian Children's Education and Care Quality Authority
ACER	Australian Council for Educational Research
AMPS	Awareness of Mathematical Pattern and Structure
CMT	Critical Mathematical Thinking
CMTFYS	Critical Mathematical Thinking For Young Students
CMTLE	Critical Mathematical Thinking Learning Experiences
DEEWR	Department of Education, Employment and Workforce Relations
EYLF	Early Years Learning Framework
GETQ	Guiding Explaining Teacher Questions
ICSEA	Index of Community Socio-Educational Advantage
LEQUE	Learning Experience Questions
MERGA	Mathematics Education Research Group Australasia
NCTM	National Council of Teachers of Mathematics
NSW	New South Wales
NAEYC	National Association for the Education of Young Children
NAPLAN	National Assessment Program Literacy and Numeracy
NQF	National Quality Framework
OECD	Organisation for Economic Co-operation and Development
PASA	Patterns and Structure Assessment
PAT	Progressive Achievements Test
PCK	Pedagogical Content Knowledge
PETQ	Probing Explaining Teacher Questions
PISA	Programme for International Student Assessment

SIT-P	Slosson Intelligence Test - Primary
STEM	Science, Technology, Engineering and Mathematics
TIMSS	Trends in International Mathematics and Science Study
UNESCO	United Nations Educational, Scientific and Cultural Organisation

Glossary

Critical Mathematical Thinking (CMT)	A term and acronym created by the Researcher to encapsulate the end of process of the amalgamation of the terms 'Critical' thinking and 'Mathematical' thinking.
Critical Mathematical Thinking For Young Students (CMTFYS)	A term and acronym created by the Researcher to describe the revised Conceptual Framework that has emerged as a finding in this research.
Critical Mathematical Thinking Learning Experiences (CMTLE)	A term and acronym created by the Researcher to describe the learning experiences the participants in this study engaged in.
Early Learner	The term used to describe children aged from birth to the time they transition into primary school. In some instances, the age range is up to 8 years.
Learning Experience Questions (LEQUE)	A term and acronym created by the Researcher for the teacher questions that were used in the initial setup for each learning experience.
Young Students	The young student participants included kindergarten (first formal year of schooling in NSW) students aged between 5 years and 1 month to 6 years and 8 months.

Statement of Original Authorship

The work contained in this thesis has not been previously submitted to meet requirements for an award at this or any other higher education institution. To the best of my knowledge and belief, the thesis contains no material previously published or written by another person except where due reference is made.

Signature: [redacted]

Date: 30/08/2021

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Publications

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Chapter 1: Introduction

1.1 INTRODUCTION

The purpose of this study was to investigate young students' critical thinking within the area of mathematics as they begin formal schooling. This was in a clear response to curriculum and policy directives, which have highlighted the importance of developing students to become critical thinkers across all learning areas, including mathematics. Despite these directives, there is a lack of research demonstrating how this looks in the early years of schooling, including what types of critical thinking young students exhibit, and how teachers can foster this type of thinking in the curriculum area of mathematics. After identifying a paucity of literature in relation to both critical thinking and mathematical thinking in young students, the term Critical Mathematical Thinking (CMT) was conceptualised to encapsulate pertinent features of both thinking domains. An interpretive research paradigm was adopted to assist in understanding how young students constructed their critical mathematical thinking experiences and the CMT Framework afforded the lens for the data collection and interpretation of the results.

The particular aims of the study were to investigate (a) Critical Mathematical Thinking (CMT) in young students, and (b) teaching actions/questions that help these young students exhibit their CMT. The study was situated in the early years context and involved young students (in their first formal year of school - Kindergarten in NSW Schools). The selection of Kindergarten students was to capture data just as the students entered school.

The chapter begins by presenting an overview of the study impetus, by positioning critical thinking and mathematical thinking within education. . Next the research problem, aims and questions are presented. Following this is a section that summarises the research design of the study. Next the study significance is outlined. Finally, an outline of the thesis is presented. Figure 1.1 presents an overview of Chapter 1.

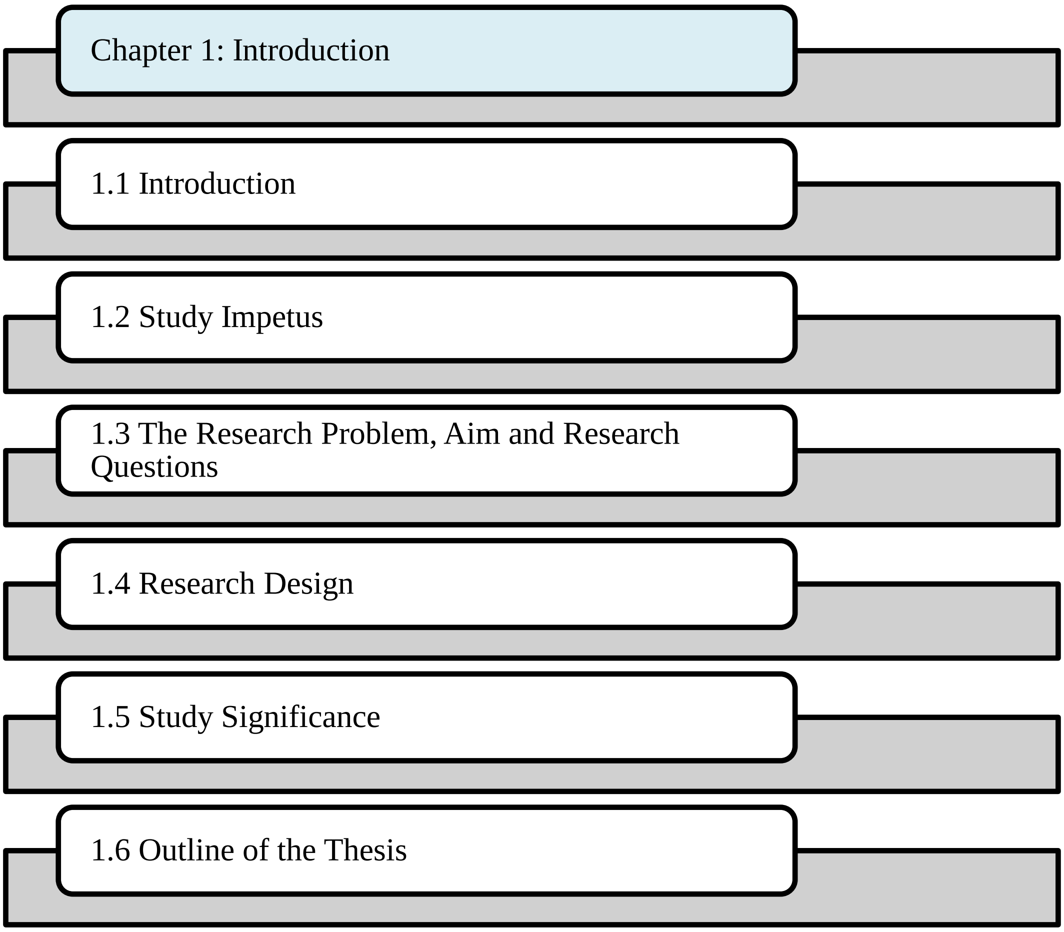


Figure 1.1. Overview of Chapter 1

1.2 STUDY IMPETUS

The impetus for this study emerged from my interest in young students' engagement in mathematics learning. I have worked in many classrooms, in various mathematics leadership roles and capacities, with a particular focus on young students. From my experience I noticed a range of teacher assumptions about young students' mathematical capabilities and their understanding of mathematics. For many teachers, the assumptions were that young students did not have prior mathematical abilities or understandings of mathematics as they began formal schooling. This was very evident in the teaching approaches adopted by Kindergarten teachers (teachers teaching in the first formal year of school in NSW) with many lessons focusing on understanding a new single digit number, one per week, or lessons focusing on naming two-dimensional shapes.

During this time, I was aware that mathematical thinking was a well-researched area in mathematics education, and that there were inclusions of mathematical thinking in the curriculum documents for mathematics. Working in many early years primary school classes, it was evident that there were limited mathematical thinking learning experiences occurring in the early years of schooling. Similarly, I knew that critical thinking skills were significant skills required in education for social empowerment and in employability. The distinct separation between these two types of thinking in the curriculum documents sparked my interest to review the research pertaining to both of these areas. Simultaneously, I reflected on my observations of my own classroom practice and other classes I had visited. It was evident that teachers were confused about both critical thinking and mathematical thinking, especially in the early years.

In addition, it was apparent that the approach I observed for mathematics learning for young students was problematic for four reasons: (1) the standardised measures focused mainly on mathematical content, for example the ACER Progressive Achievement Tests (PAT) in Mathematics; (2) the students that perform well in standardised measures may or may not be displaying their thinking capability (Critical Mathematical Thinking (CMT) – (term fully defined in Chapter 3); (3) as the current measures were not capturing the critical mathematical thinkers, these young thinkers were not identified, hence, were not encouraged to progress this type of thinking; and, (4) there was a perception by the teacher that young students begin formal schooling with an inability to think critically and mathematically.

1.2.1 Critical Thinking Position – for Education and Employability

Critical thinking is found to be a necessary requirement in both education and employability (Adams Becker et al., 2016; Alexander et al., 2019; Heard et al., 2020; National Council of Teachers of Mathematics [NCTM], 2000a; 2000b; Urib-Enciso et al., 2017). Internationally, curriculum and policy directions have embedded critical thinking as a key skill to support students to become prepared for the 21st century (Urib-Enciso et al., 2017). In addition, research evidences that 21st century skills that are underpinned by critical thinking are a requirement for workforce preparedness and long-term economic success (Burrus et al., 2013; Rios et al., 2020).

Presently, there are no generally accepted definitions of what constitutes critical thinking, especially for young students. Explanations for critical thinking exist (for example, Heard et al., 2020; Lipman, 1987; Paul & Elder, 2012; Siswono, 2010; Sternberg, 1986), however, in most cases the authors identify terms, such as non-algorithmic decision making and reasoning, rather than a clear definition of critical thinking. In addition, critical thinking literature is scarce with respect to young students. As critical thinking is not clearly defined, there is a value to investigate critical thinking within mathematics for young students. Section 3.3 in the literature review provides a synthesis of critical thinking literature.

Despite the lack of a clear definition, it is well acknowledged that the development of critical thinking is important for all learners. In a response to this, recent national and international policy and curriculum documents have emphasised the development of critical thinking skills as a way to prepare for the demands of the 21st century (Binkley et al., 2012; DEEWR, 2009; Dwyer et al., 2014; ACARA, 2014). The way critical thinking is embedded in these documents includes the association of skills, qualities, competencies and characteristics for critical thinking drawing from philosophy, psychology and education disciplines.

For the purpose of this study, critical thinking is identified as process skills, qualities, competencies and characteristics that support an individual to apply new knowledge and skills. Critical thinking can include processes and strategies for problem solving and decision making (Lai, 2011; Lipman, 1987; Paul & Elder, 2012; Sternberg, 1986) such as non-algorithmic approaches to decision making (Siswono, 2010) and self-regulating or self-evaluating of inferences (Facione, 1990; Resnick, 1987).

1.2.2 Mathematical Thinking Position – for Education and in Research

Like critical thinking, mathematical thinking definitions are not consistent. Mathematical thinking is often used in broad terms to define mathematical proficiencies or mathematical practices. The term mathematical thinking is broadly defined in the literature and as such there has been an expanding body of research with a focus on defining this term (for example, Breen & O'Shea, 2010; Carpenter et al., 2017; Fraivillig et al., 1999; Tall, 1991). Definitions include terms such as thinking styles (Karadag, 2009), a dynamic process (Mason et al., 1991), Advanced

Mathematical Thinking (Tall, 1995) and the importance of the use of mathematical thinking in real life contexts (Bal & Doganay, 2014).

For the purpose of this study, the definition of mathematical thinking refers to mathematical thinking as the practices individuals employ when engaging in mathematical tasks. These practices include experimentation, reasoning, generalising and the use of language to explain mathematics. Mathematical thinking is not about quick recall of facts, memorising formulas or applying procedures (Stein et al., 1996). Sections 3.2 and 3.3 in the literature review provide a synthesis of literature and the development of a Conceptual Framework that informs the study.

While there are similarities between the ways in which critical thinking and mathematical thinking are described in curriculum and the research literature, there are also discrepancies, suggesting they are not necessarily stand-alone constructs. Thus, it is necessary in this study to clarify the definition of mathematical thinking and critical thinking for young students, and to investigate the amalgamation of these constructs as CMT.

1.2.3 Critical Thinking within Mathematics

Critical thinking within mathematics is a major concern at both the international and national level. In comparison to other countries, Australian students' mathematical performance has at best plateaued and in some cases has declined over recent years (McGaw et al., 2020; Thomson et al., 2019). This decline is particularly prevalent in the area of mathematical thinking (McGaw et al., 2020; Thomson et al., 2019). Despite evidence to suggest the decline, many Australian teachers are not supporting students to develop the skills associated with mathematical thinking nor critical thinking in mathematical contexts (Dix et al., 2018). This may be a direct result of current assessment tools used within the Australian context which predominantly focus on assessing students' computational skills with little emphasis on assessing their ability to solve problems, reason and think critically or mathematically (Callingham et al., 2016). Hence, classroom teaching often reflects a narrow conception of mathematics (for example, a focus on computation only).

The ability to engage in both critical thinking and mathematical thinking is crucial to our students' future success in mathematics. This study focused on young students, specifically those aged between 5 years and 1 month to 6 years and 8 months of age. At present, there are no measures that are used for these young students in regards to their capabilities of critical thinking and mathematical thinking. While the Australian National Assessment Program for Literacy and Numeracy (NAPLAN), Trends in International Mathematics and Science Study (TIMSS), and Programme for International Student Assessment (PISA) do not specifically measure critical thinking and mathematical thinking, they do suggest that if we establish strong mathematical thinkers in the early years and build on these foundations effectively, the current negative trends we notice in NAPLAN, TIMSS and PISA may change.

Recently, there have been two significant reports published in relation to improving mathematics education in Australia that directly impact on mathematical thinking. The first report focused on a review of Australia's mathematics performance, *Improving mathematics performance of Australia's students* (Smith et al., 2018). The second report, *Nothing left to chance: Characteristics of schools successful in mathematics* (Callingham et al., 2016) reported on the findings of a national project aimed to identify factors that influence the learning and teaching of mathematics in Australian schools. Both of these reports provided key recommendations to support the improvement of mathematics in Australia. The following specific recommendations given in both reports align to this current study:

- Increasing opportunities for students to articulate their mathematical thinking and solution strategies (Callingham et al., 2016; Smith et al., 2018);
- Including investigative approaches in the learning to develop mathematical thinking (Callingham et al., 2016; Smith et al., 2018);
- Developing of mathematical proficiencies of problem solving, reasoning, understanding and fluency in students (Callingham et al., 2016; Smith et al., 2018).

While these key recommendations have been widely disseminated, there still remains a vast majority of teachers who fail to assist students to think critically in the context of mathematics. For example, Australian teachers' ability to facilitate teaching that assists students to engage in critical thinking tasks in mathematics was evaluated by the Australian Council for Educational Research (ACER) in 2018. Out of 304 teachers, only 155 stated that they help students to think critically in mathematics lessons (Dix et al., 2018).

Finally, in relation to assessing mathematical thinking, the *Nothing left to chance: Characteristics of schools successful in mathematics* report (Callingham et al., 2016) highlighted the narrow conceptual focus of mathematical assessment practices used to assess students' thinking. In particular, the current assessment tools used with Australian students for mathematics focus mainly on numbers, with minimum inclusion of the mathematics proficiencies (problem solving, reasoning, understanding and fluency), this key finding directly aligns with this present study. It was also found that the support offered to classroom teachers is not adequate to allow them to collect, analyse and infer data with respect to students' attainment of these mathematics proficiencies.

1.3 THE RESEARCH PROBLEM, AIM AND RESEARCH QUESTIONS

1.3.1 Research Problem

In spite of both critical thinking and mathematical thinking both being deemed important in education literature, Australian students continue to underperform in both International and National mathematical benchmarks. While it has been clearly stated in mathematics education curricula and frameworks the significance of embedding critical thinking and mathematical proficiencies/processes in learning experiences, there is empirical evidence indicating that this is frequently not occurring. Both constructs underpin one's ability to solve problems in mathematical contexts. There is a perception that young students cannot think mathematically nor critically. In addition, there is limited research identifying the commonalities of critical thinking and mathematical thinking, specifically for young students (identified as Critical Mathematical Thinking [CMT] in this present study). There is no framework in the literature or curricula that can support teachers of young students to identify and teach CMT. In particular, there is no framework or measure that can support teachers to identify young students that work beyond emergent levels and expectations. All of these issues are of major concern.

1.3.2 Research Aims

The main aims of the study are to investigate (a) Critical Mathematical Thinking (CMT) in young students, and (b) teaching actions/questions that help these young students exhibit their CMT. To achieve the first part of the study, *Critical Mathematical Thinking (CMT) in young students*, a synthesis of a wide range of literature and educational policies pertaining to mathematical thinking and critical thinking is to occur. The purpose of the literature review is to determine the status of mathematical thinking and critical thinking in mathematics teaching and learning, and to identify and synthesise key themes that emerge from the literature. The first approach is to explore what CMT is, specifically for young students. The second approach is to use and refine the Conceptual Framework developed through the literature review to identify CMT in young students. To support the second aim of the study, *teaching actions/questions that help these young students exhibit their CMT*, a review of literature pertaining to teacher questioning is to occur. Therefore, the third approach is to use components of the conceptual framework (Critical Mathematical Thinking For Young Students [CMTFYS]- further informed by the data) to review the types of teacher questions posed by the researcher that support young students to exhibit their CMT.

1.3.3 Research Questions

Two pertinent issues arising about young students' CMT relate to what it might look like and how can the existence of such thinking be exhibited. The two research questions generated after examining the literature (Chapters 2 and 3) to address these issues were:

1. *What are the CMT capabilities young students exhibit as they begin formal schooling?*
2. *What types of teacher questions help these young students exhibit their CMT?*

1.4 RESEARCH DESIGN

1.4.1 Epistemology

This study adopted a constructivist approach to highlight the way an individual creates or constructs their critical mathematical thinking through their interaction with the world (Lincoln & Guba, 1985; Lincoln & Guba, 2013). The study aligns with the constructivist approach as it is focused on young students' interactions with the world to discover relationships through exploration, experiences and experiment (Dennick, 2016). The researcher's role included interviewing students during mathematical learning experiences, taking note of student responses, posing questions and interpreting responses.

1.4.2 Theoretical Perspective

For this study, interpretivism was identified as the appropriate research paradigm. This paradigm allows the researcher to consider the critical mathematical thinking young students display and determine what they know, through the child's explanation, during CMT related learning experiences. The interpretative paradigm was used to understand and explain students' constructed experiences. The researcher's role is interactive and includes the researcher posing questions and note taking of observations.

1.4.2.1 Conceptual Framework

A conceptual framework emerged from a comprehensive literature review process that identifies concepts that are relevant and related with each other for the purpose of the study (van der Waldt, 2020). A conceptual framework was developed, applied and refined in this study. The Critical Mathematical Thinking (CMT) Conceptual Framework, later refined as Critical Mathematical Thinking Framework for Young Students (CMTFYS), was developed through an analysis of literature regarding mathematical thinking, critical thinking and early years students.

1.4.3 Methodology

Because the study explores young students' critical thinking, a case study approach was used to explore a phenomenon, within a particular context and with the use of various data sources (Baxter & Jack, 2008; Yin, 2018). The case study consisted of three schools, bounded by their demographic features. Case study also allows for the use of both qualitative and quantitative data sets to better understand a research problem (Yin, 2018).

An explanatory mixed methods approach was used to inform the data gathering phases and analysis of the study. For the purpose of this study, the quantitative data assisted in narrowing participant selection at intervals of the data analysis process. The analysis of the quantitative and qualitative data assisted to elicit findings connected to the research aim, the CMT capabilities of young students and teacher questions posed by the researcher that help these young students exhibit their CMT.

An explanatory design was used to explain significant or insignificant results, or unanticipated results (Morse, 1991). Two distinct phases of the explanatory design were used in this study:

1. Quantitative data – collected and analysed;
2. Qualitative data – builds on the quantitative data, is collected and analysed and elaborates on the results obtained in phase 1.

1.4.4 The Participants

The research was conducted in five Kindergarten classrooms from three schools in New South Wales, Australia. There were four groups of student participants which were selected across all three schools and were identified as*:*

- *All Kindergarten Students* (n=161);
- *Classroom Observation Students* (n=41);
- *Focus Students* (n=25); and
- *High CMT Students* (n=16).

Each group of students was a purposively selected sample of the previous group of students.

1.4.5 Data Gathering

Data gathering occurred at three phases across the study with the use of five instruments. The phases and instruments were:

- Phase 1: Qualitative: CMT Observation Guide with *All Kindergarten Students* (n=161) and *Classroom Observation Students* (n=41)
- Phase 2: Quantitative: Raven's Progressive Matrices, Slosson Intelligence Test, Patterns and Structure Assessment, Critical Mathematical Thinking Learning Experiences (CMTLE) with *Focus Students* (n=25)

- Phase 3: Qualitative: Critical Mathematical Thinking Learning Experiences (CMTLE) Clinical Interview with *High CMT Students* (n=16)

1.4.6 Data Analysis

The CMT Observation Guide, suite of quantitative instruments and the CMTLE Clinical Interview required three phases of data analysis. They were:

1. Analysis of the CMT Observation Guide at the conclusion of each lesson;

2 Analysis of the suite of quantitative instruments analysis to identify the *High CMT Students* for the study;

3. Analysis of the CMTLE Clinical Interview for the *High CMT Students* using the CMTFYS.

1.5 STUDY SIGNIFICANCE

This study aims to make a contribution to early years mathematics education research and teaching within the emerging field of Critical Mathematical Thinking. It is acknowledged that mathematical thinking and critical thinking are prevalent in education. As there are variations in the definitions and at times an overlap of the terms, a study into the definition of CMT for young students is required.

Additionally, the study intends to make a substantial contribution to research that provides a literature and research informed Conceptual Framework for CMT. The purpose of the Conceptual Framework is to support teachers to identify CMT in young students and use specific teaching actions/questions that help these young students exhibit their CMT.

1.6 OUTLINE OF THE THESIS

1.6.1 Chapter One: Introduction

In this chapter, the research aim of the study was described, the research problem was defined, the two research questions were identified, and the research design was proposed.

1.6.2 Chapter Two: Perspectives on Teaching and Learning in the Early Years

This chapter presents a review of the perspectives on teaching and learning for early learners, mathematics teaching and critical thinking and mathematical thinking related to young students internationally and locally.

1.6.3 Chapter Three: Mathematical Thinking

Chapter Three identifies literature relating to mathematical thinking and critical thinking in the early years context. The inclusion of the term ‘critical mathematical thinking’ will emerge from the literature and align with the context of this study.

1.6.4 Chapter Four: Research Design and Methodology

Chapter Four will describe and justify the research design for this study. The methodological approach will be explained. The data collection strategies will be aligned to inform the research questions.

1.6.5 Chapter Five: Results and Findings

Presented in this chapter are the results of the qualitative and quantitative data. These results comprise data collected over three phases. The summary of findings are presented under the following broad sections:

1. Young Students’ Critical Mathematical Thinking
2. Critical Mathematical Thinking Capabilities in Young Students
3. Exhibiting Critical Mathematical Thinking – The role of Teacher Questioning

1.6.6 Chapter Six: Discussion

A synthesis of the results from Chapter Five is presented in this chapter. The findings of the study are conversed and interpreted in light of the development of the CMT Conceptual Framework, critical thinking literature, and mathematical thinking literature. The role teacher questioning played to help young students exhibit their CMT is discussed. The chapter concludes with exemplars and indicators of CMT in classroom practice.

1.6.7 Chapter Seven: Conclusion and Implications

The final chapter addresses the research questions, the study’s contributions to research and the implications of the research. The study’s limitations are presented, and future research recommendations are made.

Chapter 2: Perspectives on Teaching and Learning in the Early Years

2.1 CHAPTER OVERVIEW

Chapter 2 provides background of the educational setting applicable for early learners, both in an early childhood and a formal school setting. In short, it presents the context of the study and establishes the environment in which the research problem is situated. The chapter begins by presenting an overview of early learners together with international and local Australian recommendations for these learners. Next is a review of international and Australian policies and curriculum frameworks applicable to early learners. Following this is a section that explores mathematical thinking and critical thinking. Finally, the chapter outlines mathematical teaching contexts for early learners. Figure 2.1 presents an overview of Chapter 2.

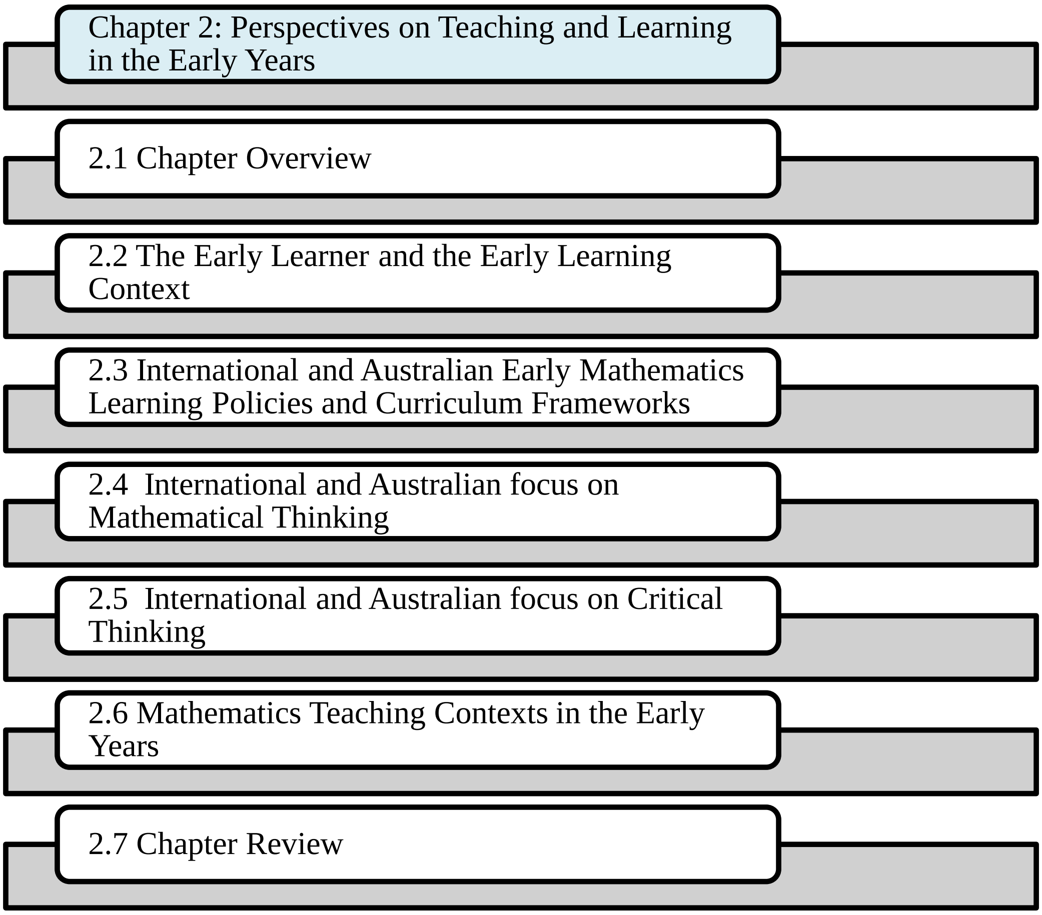


Figure 2.1. Overview of Chapter 2

2.2 THE EARLY LEARNER AND THE EARLY LEARNING CONTEXT

Internationally, the term early learners, young learners and the early years are used to describe children between the ages of birth to age eight (Organisation for Economic Co-operation and Development [OECD], 2015). Equally, Early Childhood Australia (ECA) has adopted the same definition and advocates that an early learner is aged between birth to eight years (Council on Early Childhood, 2014). It is during these first eight years that a positive and nurturing environment is to be fostered to enhance physical, social, emotional and cognitive abilities (Elliot, 2006; MacDonald & Carmichael, 2018; OECD, 2015).

As the early years is an influential period of time, studies have shown that a high-quality early learning experience positively impacts children's learning and development as they progress through schooling (Baroody et al., 2019; Elliot, 2006; Sammons et al., 2002). For example, as identified in a longitudinal study on the impact of pre-school on children’s later progress, a quality pre-school provision that promoted pre-reading, early number concepts and reasoning resulted in better cognitive outcomes for children as they enter the first year of formal schooling (Elliot, 2006). Therefore, the early education setting is an opportunity to include characteristics that provide solid foundational knowledge that influences later learning.

There are consistencies and variances in the way countries identify curriculum frameworks for children. Internationally, member countries of the OECD have contributed to a 2020 database that reports on the structures of curriculum or frameworks (OECD, 2020). It is noted that in many countries, there is often a different curriculum or framework for children under the age of five (prior to formal schooling) that then articulates into a more comprehensive curriculum in a formal school setting.

To clarify age classifications of the participants in the study, Figure 2.2 provides a contextualisation of the early years learner with regards to their age and the curricula that impacts on these children. There are three items to consider. The first item is the term early learner, learners from birth to age eight (shaded grey). The second item is the age children begin school. On average, countries in the OECD have identified that formal schooling begins at approximately age six (OECD, 2021). In Australia, although States and Territories have slightly different requirements for starting ages, on average, Australian children must commence formal school by age six, with many children beginning from age five (Australian Bureau of Statistics, 2016) (shaded green). The third item refers to the curriculum for these learners. Although it is agreed that an early years learner is up to age eight, it is clear that the there is a distinct change, approximately between the ages of five and six, where a transition to a comprehensive and formal curriculum is implemented. Shaded in purple are the early years frameworks/ curriculum frameworks and shaded in red are the formal schooling curriculum frameworks.

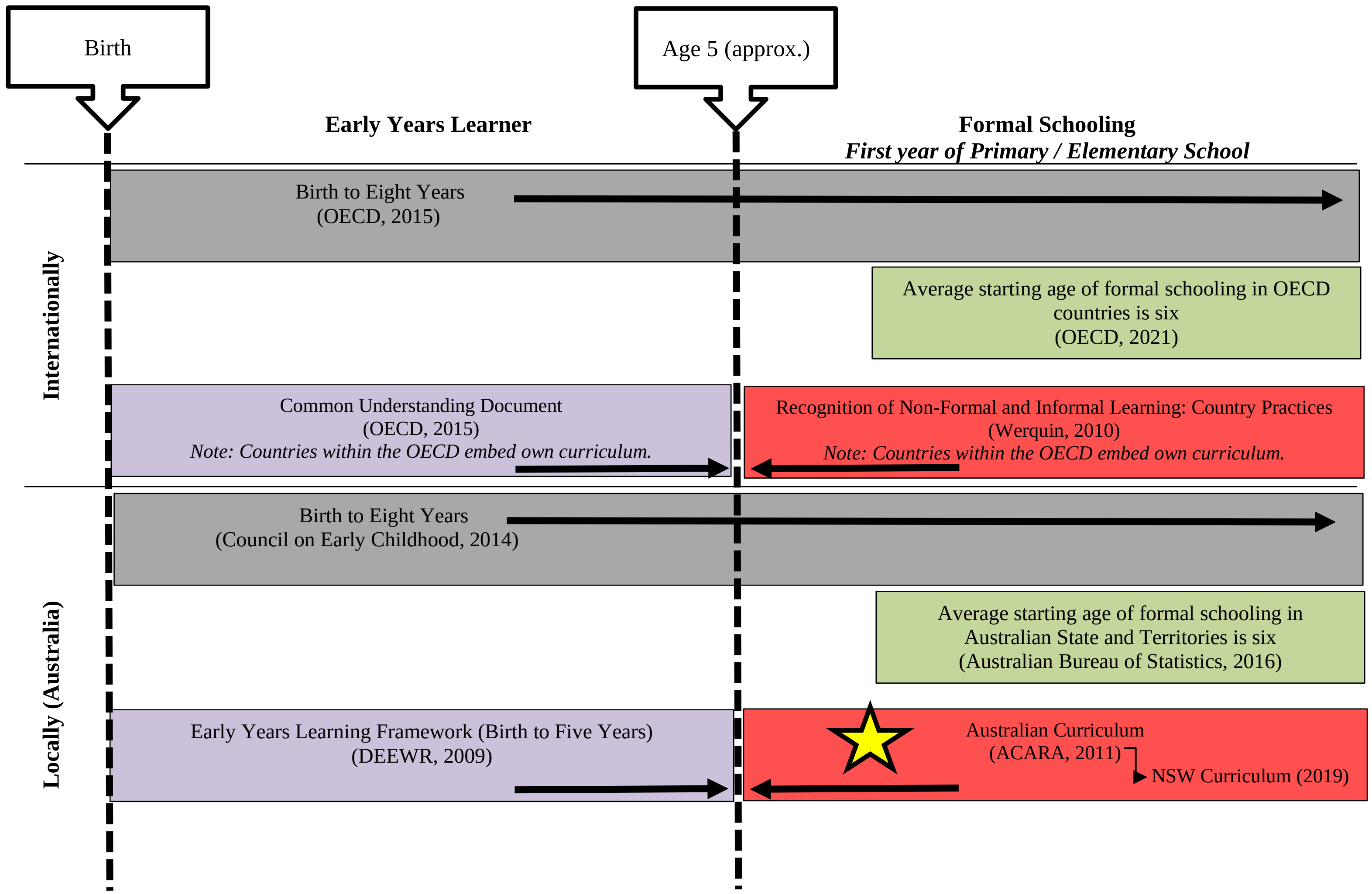


Figure 2.2. Contextualising the Early Years Learner

The participants in this study are at the crucial transitional period, identified with a yellow star in Figure 2.2, the average age that Australian students are as they transition into formal schooling. This is also the point where they transition from the Early Years Learning Framework to the Australian national curriculum. As this study focuses on the discipline of mathematics, a review of early years and formal schooling mathematics curriculum is considered in the next section.

2.3 INTERNATIONAL AND AUSTRALIAN MATHEMATICS LEARNING POLICIES AND CURRICULUM FRAMEWORKS

2.3.1 International Early Mathematics Learning and Curriculum Frameworks

Internationally, there is a diverse range of foci for early childhood practices, frameworks and curriculum documents (OECD, 2015; UNESCO, 2016). The UNESCO vision for education aims for nations to work collaboratively to identify standards and learning outcomes for all learners, beginning with early childhood (UNESCO, 2016). Internationally, early years guiding documents (frameworks or curriculum) have a shared language and common understandings about how children learn and develop. For example, countries within the OECD all agree that planning for learning "supports learning through active engagement, observation, experimentation, and social interaction and communication" (OECD, 2015, p. 10).

The European Commission has included early education and care as an essential priority since 1992. Within the Quality Framework presented by the European Commission (Working Group on Early Childhood Education and Care, 2014) partnerships, competent systems, quality processes, and the voice of the child are identified as priority areas. The Quality Framework for Early Childhood Education and Care (European Commission, 2014) uses action statements associated with the curriculum in the early years. Although not specific to mathematics, the action statements aligned with early education curriculum include:

- A curriculum based on pedagogical goals, values and approaches with enable children to reach their full potential in a holistic way;
- A curriculum that requires staff to collaborate with children, colleagues and parents and to reflect on their own practice.

(European Commission, 2014, p. 11).

Early childhood mathematics teaching is also represented in varying ways. For example, mathematics is presented in the UNESCO framework as a part of the science, technology, engineering and mathematics education (STEM) initiatives (UNESCO, 2016). Additionally, the UNESCO goal, “Ensure inclusive and equitable quality education and promote lifelong learning opportunities for all” (UNESCO, 2016, SD4), includes a proposed global indicator for learning in mathematics that states:

> Proportion of children and young people: (a) in grades 2/3; (b) at the end of primary; and (c) at the end of lower secondary achieving at least a minimum proficiency level in (i) reading and (ii) mathematics, by sex.
>
> (UNESCO, 2016, para. 1)

Internationally, policy statements and curriculum documents suggest perspectives for mathematics in the early years. The implementation of these policies varies from country to country. These differences and underlying principles are delineated in the following sections through a review of mathematical early learning curriculum documents and position statements for Italy, Sweden, France, Ireland, Asian Countries, the United States of America, England and New Zealand. The selected countries are representative of the international context.

Engaging in real life experiences in mathematics learning is a common theme identified internationally. For example, the Reggio Approach (Italy) (Edwards et al., 2011; Directorate for Education OECD, 2004) is firmly situated in the child’s environment and real life experiences. Similarly, the Singaporean Early Learning context promotes the teaching of mathematical concepts, skills and processes that are related to real life experiences (Ministry of Education Republic of Singapore, 2013). Sweden, France, Ireland, USA and New Zealand also state the importance of a real life approach to early mathematics learning.

Internationally, a focus on the concept of number is prevalent in early years approaches. In Sweden, their education policy highlights the importance of mathematics in the early years with a focus on big ideas, including children’s relationships with quantities, numbers and operations (Taguma et al., 2013). The Swedish national curriculum includes language and numbers as ways teachers can develop the child to their full potential (Directorate General for Schools, 2012). The English early years framework identifies counting, understanding and using numbers, calculating simple addition and subtraction problems as critical components in early mathematics learning (Department of Education United Kingdom, 2013). Almost all documents reviewed, identified number as a crucial part or early mathematics learning.

Internationally the integration of mathematics with other learning areas is encouraged. For American early childhood teachers, the National Council of Teachers of Mathematics (NCTM) includes a set of six principles for the pre-kindergarten year, namely, Equity, Curriculum, Teaching, Learning, Assessment, and Technology (2010). One key principle supports the integration of mathematics with other activities. Integrating learning is also presented in the Reggio Approach (Italy) where it is suggested that educators integrate the arts and play with mathematics teaching (Linder et al., 2011).

In addition to number, a range of other mathematics concepts are presented in international approaches to early years learning. In many international curriculum documents and policies for early mathematics there is the inclusion of patterns, shapes and measurement. For example, Kindergarten goals for Singaporean children include, "recognise and use simple relationships and patterns; use numbers in daily experiences; recognise and use basic shapes and simple spatial concepts in daily experiences" (Ministry of Education Singapore, 2012a, p. 21).

In summary, a focus on early childhood practices are recognised internationally. Nations are guided by frameworks and curriculum documents that guide learning and planning. These frameworks and curriculum documents do include mathematics; however, the approach varies in each country. What was found as a strong focus included real life mathematics experiences and number based content. The integration of mathematics with other learning areas was also found as a main feature in curriculum documents.

2.3.2 Australian Early Mathematics Learning and Curriculum Frameworks

In the Australian context, the growing recognition of early mathematical development has influenced policy reforms (Reid & Andrews, 2016; Australian Children's Education and Care Quality Authority [ACECQA], 2020). For early childhood educators within Australia, a national early childhood curriculum framework, Belonging, Being and Becoming: The Early Years Learning Framework for Australia (EYLF) (DEEWR, 2009) provided a heightened recognition of the sector. Simultaneously, other Australian documents relating to early years are the Australian National Quality Framework (NQF) for Early Childhood Education and Care (DEEWR, 2010) and the Australian Curriculum (ACARA, 2011). The EYLF (National approved learning framework of the NQF) and the Australian Curriculum both identify mathematics as a specific area related to a child's development.

The framework includes three inter-related elements: principles, practices and learning outcomes. The principles, practices and learning outcomes align with pedagogy specific to the way children learn in the early years. Mathematics teaching and learning, as such, is not clearly articulated but is seen as interrelated with the child's interest and experiences. There are five learning outcomes within the framework that provide points of description for educators to consider when planning for practice and assessing children. All five outcomes consider some aspects of mathematics. However, outcome four, "Children are confident and involved learners" (Australian Government Department of Education Employment and Workplace, 2009, p. 33) aligns with the way children can explore, collaborate and problem solve mathematically. In the EYLF, mathematics learning is embedded within broad learning outcomes. These outcomes focus on:

- children's sense of identity;
- children's connectedness with their world;
- children's sense of wellbeing;
- children as confident learners; and
- children's capacity to be effective communicators

(Department of Education, Employment and Workforce Relations [DEEWR], 2009).

In summary, the Australian framework predominantly focuses on the child and integrating the teaching and learning of mathematics with the child's interest and experience. The framework includes a generalised approach to mathematics teaching and learning. As stated in Section 2.2, the students in this study are positioned at the crucial point of transition between learning frameworks that exist in prior to school settings (summarised in Sections 2.3.1 and 2.3.2) and more formal curriculum frameworks located in schools (presented in the next two sections).

2.3.3 International Formal Schooling: Mathematics Learning and Curriculum Frameworks

Mathematics curriculum is structured around mathematics content in alignment with ages or stages of formal schooling (Li & Lappan, 2014; Ruddock & Sainsbury, 2008). Educational reforms, research in mathematics teaching and learning and individual countries' initiatives inform changes that impact on the ways in which mathematics is taught. In addition, research has been conducted to compare and contrast the ways in which countries structure their mathematics curriculum (Werquin, 2010). Therefore, it is important to note that an individual country considers many factors when determining the best curriculum practices and approaches to adopt (Li & Lappan, 2014).

There are, however, commonalities in mathematics curriculum worldwide. For example, in the first-year formal schooling curriculum, there is a focus on foundational mathematics mainly in the area of arithmetic (Howe, 2014). Specifically, there is often a more pertinent focus on the four operations (addition, subtraction, multiplication and division). The inclusion of mathematical content focusing on operations is present in many international curriculum documents. For example, the Finnish curriculum comprises concepts and skills related to operations that begin from the first formal year. It is evident that the Finnish curriculum has a heavier emphasis on formal concepts such as multiplication (times tables) (Finnish National Board of Education, 2016; ACARA, 2018a). However, there is still evidence of foundational understandings that include concrete or pictorial stages (ACARA, 2018).

The Singaporean curriculum is explicit and includes content strands and elaborations (Ministry of Education Singapore, 2012a). It is noted that the Singaporean school curriculum includes limited use of concrete materials to develop early number skills (Ministry of Education Singapore, 2012b; ACARA, 2018b). The inclusion of formal mathematics, for example, using formal algorithms for addition and subtraction is presented to students earlier than other countries (Ministry of Education Singapore, 2012b; ACARA, 2018b).

The teaching of other mathematics concepts are also present in many international documents. In most cases, the content for mathematics is presented under the broad concepts of number, algebra, geometry, measurement, data and probability (Li & Lappan, 2014; Ruddock & Sainsbury, 2008). Each country determines the topics within those broad concepts and identifies the positioning of the learning within a grade or age structure.

In summary, international curriculum for school aged students is organised according to ages or stages. The content is also organised to include strands and elaborations. Most curriculum documents include a focus on number (arithmetic) for students in their first year of formal schooling. A comparison of the *Australian Curriculum: Mathematics* with curriculum documents in other countries will be presented in Section 2.3.4.

2.3.4 Australian Formal Schooling: Mathematics Learning and Curriculum Frameworks

2.3.4.1 The Australian Curriculum: Mathematics

The *Australian Curriculum: Mathematics* covers the first formal year of schooling up until the 11^{th} year (Year 10 in Australia). The document was introduced in 2010 and was the first time the national government agreed on one approach (Stephens, 2014). Prior to that, individual States and Territories were responsible for creating and implementing their own curriculum. Current practice includes States and Territories either adopting the entire Australian Curriculum or an adapted version of the document.

The *Australian Curriculum: Mathematics* for the first formal year of schooling includes content and proficiency (process) strands. The three content strands (and sub-strands and four proficiency strands (further discussed in Section 2.4.4) are as follows:

Content Strands and Sub-Strands:

- Number and Algebra
    - Number and Place Value
    - Patterns and Algebra
- Measurement and Geometry

- Using units of measurement
- Shape
- Location and Transformation

- Statistics and Probability
  - Data representation and interpretation

Proficiency Strands:

- Understanding
- Fluency
- Problem Solving
- Reasoning

(ACARA, 2015)

For students in their first year of schooling, the *Australian Curriculum: Mathematics* end of year achievement is brief and includes the following goals:

- make connections between number names, numerals and quantities up to 10
- compare objects using mass, length and capacity
- connect events and the days of the week
- explain the order and duration of events
- use appropriate language to describe location
- count to and from 20 and order small collections
- group objects based on common characteristics and sort shapes and objects
- answer simple questions to collect information and make simple inferences

(ACARA, 2015, p.1)

The Australian Curriculum, Assessment and Reporting Authority (ACARA) undertook studies comparing the *Australian Curriculum: Mathematics* with the curricula from British Columbia, Finland, Singapore and New Zealand. The proceeding Section (2.3.4.2) provides the comparison.

2.3.4.2 The Australian Curriculum: Mathematics Content Compared with other Curricula

In 2018 and 2019, the Australian Curriculum, Assessment and Reporting Authority (ACARA) released four reports that compare the Australian Curriculum with four other curricula from British Columbia, Finland, Singapore and New Zealand. The documents report on all key learning areas. The following discussion looks specifically at the mathematics comparisons.

The British Columbia new curriculum was found to be structured differently to the Australian Curriculum: Mathematics. The *Australian Curriculum: Mathematics* is organised firstly by content, whereas the British Columbia new curriculum is organised on competencies (reasoning, analysing, understanding, solving, communicating, representing, connecting and reflecting) (ACARA, 2018c). An in-depth comparison on grade six level learning found that the content for this grade level is very different. The Australian Curriculum focuses on a verb that outlines what a student is to do; whereas, the British Columbia new curriculum uses statements for teachers to guide what is to be taught.

The Finnish National Core Curriculum is found to be similar to the *Australian Curriculum: Mathematics* in regard to the development of mathematical understandings and skills in the early years (ACARA, 2018a). Differences are evident in the content coverage; however, both documents identify that concrete and pictorial states are required to ensure sound early understandings of mathematics.

The comparison of the Singapore Curriculum found that a focus on early years (pre-school) mastery of basic processes supports students as they progress in formal schooling (ACARA, 2018b). This was evident when the Singapore Curriculum was compared with the *Australian Curriculum: Mathematics* grade six level. Findings show that Singapore students have acquired a greater depth and breadth due to early consolidation of mathematics.

The New Zealand curriculum was found to be similar in breadth, depth and rigour. The study focused on grade level two and found that the discrepancies included that the *Australian Curriculum: Mathematics* is more rigid in terms of developmental levels; however, the New Zealand curriculum provided more time to consolidate early number sense (ACARA, 2019). In regard to proficiencies, problem solving is lacking in the grade two Australian Curriculum in comparison with the New Zealand curriculum.

In summary, it is noticeable that curriculum frameworks (formal schooling) for mathematics are very similar around the world. Number based approaches are emphasised with a more rigid and descriptive outcome-based curriculum provided to teachers of these students. As stated in Section 2.2, the students in this study have recently moved to a more formal approach to mathematics learning. The following section discusses the ways in which the *Australian Curriculum: Mathematics* is used by teachers.

2.3.4.3 Australian Curriculum: Mathematics Implementation

The *Australian Curriculum: Mathematics* provides teachers with ‘what’ to teach and not ‘how’ to teach. There are many sources that guide the pedagogical practice of Australian teachers. For example, national and local reviews with recommendations, professional associations, local research on practice, State/Territory curriculum support documents and system approaches (for example, Catholic, Independent, Department) provide pedagogical guidance to teachers.

With regard to ‘how’ to teach mathematics, reviews have taken place in Australia that identify areas that require improvement. The *National Numeracy Review Report* (Stanley, 2008) identified clear recommendations for schools, systems or sectors to consider for mathematics teaching and learning. The following two recommendations align with this study:

> Recommendation 3: That from the earliest years, greater emphasis be given to providing students with frequent exposure to higher-level mathematical problems rather than routine procedural tasks, in contexts of relevance to them, with increased opportunities for students to discuss alternative solutions and explain their thinking. (Stanley, 2008, p. xiii)

> Recommendation 5: That the necessary resources be directed to support teachers to use diagnostic tools including interviews to understand and monitor their individual students’ developing

> strategies and particular learning needs. These diagnostic tools should not be restricted to school-entry assessments. (Stanley, 2008, p. xiii)

Recommendation three highlights the demand for classroom teachers to embed high-level mathematical problems in their teaching. Although the term ‘high-level mathematical problem’ is not directly defined in the *National Numeracy Review Report* (Stanley, 2008), statements embracing characteristics of effective teaching are described as: “cooperative learning” (p. 27); “mastery learning and direct instruction” (p. 28); “collaborative learning” (p. 28); “explicit instruction” (p. 28); “connectionist” (p. 29) and “provided a challenging curriculum” (p. 29).

Recommendation five states teachers need to ensure diagnostic tools are used regularly to identify where individual students are in their mathematics learning. This recommendation considers using a variety of assessment approaches to determine the level of students in mathematics. The recommendation also considers moving away from standardised measures that are in use for students in their first year of formal school.

More recently, a review of Australia’s mathematics performance, commissioned by the Office of the Australian Chief Scientist (Smith et al., 2018), identified a need for the performance of Australian students to improve when comparing Australian students’ mathematics performances against other countries. Findings from this report indicate an increase in students’ mathematics performance occurs when: there is support from senior leadership; professional learning communities are in-situ; and, teachers are enthusiastic about teaching mathematics, use data to inform their practice, and there is a focus on the development of conceptual understanding.

Similarities exist in the above mentioned 2008 and 2018 reviews. One key similarity includes moving beyond procedural tasks for students (Stanley, 2008). The use of data to inform practice is another similarity (Smith et al., 2018). Providing tasks that challenge students is also suggested in both reviews (Smith et al., 2018; Stanley, 2008). Thus, these reviews identify a continued focus on improving mathematics pedagogical practices in Australian schools.

Australian researchers have contributed a plethora of mathematics pedagogical approaches to support Australian teachers. For example, the *Australian Primary*

Mathematics Classroom Journal, published by the Association of Mathematics Teachers Inc, has 9933 publications to support primary classroom teachers. Some examples of articles within this publication that align with the recommendations listed above include:

- Beyond getting answers: Promoting conceptual understanding of multiplication (Tzur et al., 2020)
- Going further with reasoning: using and developing student logic as evidence of mathematical reasoning (Symons & Holton, 2020)
- What do they understand? (Mitten et al., 2017)

In addition, individual systems have delineated specific pedagogical approaches required in their schools. The NSW Department of Education identify deep content knowledge as a focus area and provide suggested approaches to support teacher instruction. For example, the pedagogy of direct instruction should include students learning from "watching clear, complete demonstrations of how to solve problems with accompanying explanations and accurate definitions" (Stevens et al., 2019, para. 27).

In summary, there are similar inclusions both internationally and locally when it comes to mathematics education in school curriculum. Such similarities include a focus on arithmetic, the four operations, geometry and early measurement ideas. The structure of the curriculum both internationally and locally is organised according to age or grade. There are similarities to the type of content taught to students in their first year of schooling. It mainly consists of number based concepts, geometry and early measurement ideas. Locally the Australian curriculum has undergone reviews that have identified that teachers need to provide high-level mathematical problems in their teaching (Smith et al., 2018; Stanley, 2008) as well as early identification of student needs (Stanley, 2008).

2.4 INTERNATIONAL AND AUSTRALIAN FOCUS ON MATHEMATICAL THINKING

2.4.1 International Early Years Focus on Mathematical Thinking

Mathematical thinking is presented in international early years policies and documents for early learners. Although some documents or policies do not explicitly use the terms mathematical thinking, references are made. For example, mathematical reasoning is emphasised strongly in various international curricula (Herbert & Williams, 2021). On the whole, there is no clear definition across many of these documents as to what mathematical thinking entails. In this section, the identified mathematical thinking references that do exist in Italy, New Zealand, Sweden, Ireland and the USA are considered.

The Reggio Approach (Italy) identifies the mathematical thinking processes of exploring, discussing, conjecturing and explaining (Directorate for Education OECD, 2004). Within the associated documents, statements are provided to educators related to mathematical thinking. There is an expectation that teachers provide children with opportunities to exhibit mathematical thinking (Directorate for Education OECD, 2004; Linder et al., 2011).

In the New Zealand mathematics curriculum for early learners, students are encouraged to solve problems or model situations (Ministry of Education, 2017). Hence, mathematical thinking is presented as a key principle teachers need to include in their teaching practices.

Swedish approaches state particular skills that could be associated with mathematical thinking. They include "cooperative skills, responsibility, initiative, flexibility, reflexivity, active attitudes, communicative skills, problem solving skills, critical stance, creativity, as well as an ability to learn" (Directorate for Education OECD, 2004, p. 23).

In Ireland the teaching practice is to focus on children's in-depth mathematical understandings and the use of problem solving skills (Dunphy et al., 2014). Although mathematical thinking is not explicitly mentioned, the term problem solving is, and can be considered a strategy students engage in when they think mathematically.

In the USA, curriculum and teaching documents focus on children's problem-solving and reasoning processes. Additional processes listed include representing, communicating, and connecting mathematical ideas (National Association for the Education of Young Children [NAEYC] and the NCTM, 2010).

In summary, mathematical thinking does appear in international early years frameworks, policies or procedures in varying ways. The common thread appears to be that mathematical thinking in these documents is closely aligned with solving problems. In many cases, mathematical thinking is also closely aligned with the teacher's role to guide, support and challenge young children. These trends are also apparent within the Australian Framework, EYLF.

2.4.2 Australian Early Years Focus on Mathematical Thinking

The elaborations of the outcomes within the Australian Framework, EYLF, suggests that thinking mathematically may occur when children problem solve. Mathematics and mathematical thinking can be identified in outcome four of the EYLF (discussed in Section 2.3.2), and include:

- create and use representation to organise, record and communicate mathematical ideas and concepts (p. 38);
- make predictions and generalisations about their daily activities, aspects of the natural world and environments, using patterns they generate or identify and communicate these using mathematical language and symbols (p. 38);
- contribute constructively to mathematical discussions and arguments (p. 38).

(DEEWR, 2009)

Specific strategies pertinent to the teacher are also presented in the EYLF. Suggestions are provided to educators on how they can promote mathematical thinking in the early years. These strategies include:

- recognise mathematical understandings that children bring to learning and build on these in ways that are relevant to each child (p. 38);
- model mathematical and scientific language and language associated with the arts (p. 38).

(DEEWR, 2009)

Teaching practices are emphasised in the EYLF. For example, there is a focus on the importance of children engaging in new mathematical understandings when problem solving. In addition, the EYLF highlights the educator's role in supporting children's mathematical vocabulary and their understandings of mathematical ideas

(DEEWR, 2009). These ideas are evident in children when they "use language to communicate thinking about quantities to describe attributes of objects and collections, and to explain mathematical ideas" (DEEWR, 2009, p. 38). The educator's way of promoting this mathematical learning could "include real-life resources to promote children's use of mathematical language" (DEEWR, 2009, p. 38). Hence, children's use of their own language appears to play an important role in identifying their own understanding.

In summary, mathematical thinking varies in how it is described for early years learners. Some international approaches are more transparent, for example, New Zealand includes mathematical thinking as a principle teachers are to embed in teaching and learning. Whereas, the Australian EYLF includes general descriptions that could be interpreted as elements of mathematical thinking, for example, the teacher builds on children's understanding of mathematical ideas.

2.4.3 International Formal Schooling Focus on Mathematical Thinking

Mathematical thinking is not easily identified in formal school-based curriculum frameworks. In addition, the approach to include mathematical thinking in curriculum frameworks varies worldwide. There is an emphasis in relation to mathematical thinking in international assessment programs. For example, the Programme for International Student Assessment (PISA) uses the term mathematical literacy which refers to the "capacities of students to analyse, reason, and communicate ideas as they pose, formulate, solve, and interpret mathematical problems in a variety of situations" (Thomson et al., 2013, p. 42).

While it is common for curriculum documents to list terms associated with mathematical thinking, the nomenclature often differs. Additionally, mathematical thinking is often labelled as mathematical processes or proficiencies. The terms to describe and define proficiencies and processes also differ. For example, the American National Council of Teachers of Mathematics (2000a) includes the processes of problem solving, reasoning and proof, communication, connections and representations. Whereas, the Singaporean Mathematics Framework identifies five areas of mathematical proficiency. These are conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (Groves, 2012). The Singaporean approach identifies a range of terms associated with problem solving. Some terms include thinking, reasoning, communicating, and applying. These terms are also often linked to mathematical thinking. The thinking skills are listed as one of three core aims of the syllabus (Ministry of Education, Singapore, 2012a). The Irish documents include mathematical thinking as a way students articulate their ideas to “peers, teachers and others verbally, and in written form using diagrams, graphs, tables and mathematical symbols” (Government of Ireland, 2017, p. 6). Finally, American National Research Council (2001) includes five interrelated strands that describe mathematical proficiencies (flexibility, solve mathematical problems, logical thought, explaining and justifying).

Thus, in summary, the embedding of mathematical thinking as a theme in curriculum documents varies. Not all curriculum frameworks include mathematical thinking for primary aged students as a distinct theme. Additionally, there are some curriculum frameworks that do provide components of mathematical thinking either within the mathematics content descriptors or as processes teachers are to embed in their teaching.

2.4.4 Australian Formal Schooling Focus on Mathematics Teaching and Mathematical Thinking

It was in an *Australian Curriculum: Mathematics* shaping paper (ACARA, 2009) that proficiencies were presented to educators in Australia. Although not specifically titled, mathematical thinking, components of the proficiencies align with skills associated with mathematical thinking. The proficiencies appeared as interrelated strands, graphically displayed as interwoven ropes (Burrows et al., 2020; Kilpatrick et al., 2001). The elements presented in the graphic included adaptive reasoning, strategic competence, conceptual understanding and productive disposition and procedural fluency.

The approved *Australian Curriculum: Mathematics* displays the proficiencies as: understanding, fluency, problem solving and reasoning. For students in their first year of formal schooling, the elaborations for each proficiency identify what is expected as an end of year goal. These goals are:

- **understanding** includes connecting names, numerals and quantities
- **fluency** includes readily counting numbers in sequences, continuing patterns and comparing the lengths of objects

- **problem-solving** includes using materials to model authentic problems, sorting objects, using familiar counting sequences to solve unfamiliar problems and discussing the reasonableness of the answer
- **reasoning** includes explaining comparisons of quantities, creating patterns and explaining processes for indirect comparison of length.

(ACARA, 2010, para. 3)

It is well established that open-ended learning opportunities are a platform for mathematical thinking within the curriculum. Specifically, curriculum support documents promote open-ended learning experiences as ways classroom teachers can plan learning that still covers the knowledge and skill requirements of the curriculum and provides students with the opportunities to demonstrate a range of solutions, techniques and tools (Stanley, 2008). In NSW Education Standards Authority (2019), the context in which this present study is situated, open-ended questioning is encouraged as a way to differentiate student learning

When comparing the formal school curriculum frameworks to early years learning frameworks in the Australian context, the distinct difference is that mathematical thinking, referred to as processes and proficiencies, is very prominent in the formal school curriculum frameworks and also most absent in the latter.

2.4.4.1 The Australian Curriculum: Mathematics Proficiencies Compared with other Curricula

The British Columbia new curriculum includes five Big Ideas and competencies within their content strands. The curriculum is organised to include the competencies of reasoning, analysing, understanding, solving, communicating, representing, connecting and reflecting (ACARA, 2018c).

The Finnish National Core Curriculum includes thinking skills (thinking and learning to think) and specific development of process. The development of process includes “comparing, classifying, ordering and identifying causal relationships in their observations” (ACARA, 2018a, p. 62) .

The comparison of the Singapore Curriculum identified that broader aims and proficiencies are presented in their document. This includes “acquisition and application of mathematical concepts and skills, the development of the ability to solve problems by the application of metacognitive skills and the development of positive attitudes to Mathematics" (ACARA, 2018b, p. 67).

The New Zealand curriculum for levels 1 and 2 (New Zealand Ministry of Education [NZME], 2014), statements of learning begin with "In a range of meaningful contexts, students will be engaged in thinking mathematically and statistically. They will solve problems and model situations that require them to…" (NZME, 2014).

In summary, it is noticeable that inclusion of proficiencies or process appear in school-based curriculum framework internationally. However, some terms differ and the ways in which they are presented in the documents vary. The *Australian Curriculum: Mathematics* provides the proficiencies in the core of the curriculum assuming that teachers will see this as a core component in the document. That is a distinct difference to what appears in early years learning frameworks for children in prior to formal school settings.

2.5 INTERNATIONAL AND AUSTRALIAN FOCUS ON CRITICAL THINKING

2.5.1 International Focus on Critical Thinking

Although critical thinking is not clearly defined, there are many ways in which critical thinking is presented in education. It is suggested that critical thinking is a learnable skill (Snyder & Snyder, 2008) and is required for lifelong learning (Urib-Enciso et al., 2017). To obtain critical thinking skills in students, the curriculum is to include opportunities for students to develop, practice and integrate their learning (Snyder & Snyder, 2008).

Internationally, critical thinking is seen as an important skill set required for success across a range of contexts and domains. Globally, critical thinking is a skill assessed in international mathematics testing regimes. For example, The Programme for International Students Assessment (PISA) (OECD, 2016, 2018; Vincent-Lancrin et al., 2019) includes critical thinking as an assessed skill, together with creative skills. The criteria used to assess critical thinking include:

- Inquiring:
    - understand context/home and boundaries of the problem;

- identify and question assumptions, check accuracy of facts and interpretations, analyse gaps in knowledge;

- Imagining:
    - identify and review alternative theories and opinions and compare or imagine different perspectives on the problem;
    - identify strengths and weaknesses of evidence, arguments, claims and benefits;
- Doing:
    - Justify a solution or reasoning on logical, ethical or aesthetic criteria/reasoning;
- Reflecting:
    - Evaluate and acknowledge the uncertainty or limits of the endorsed solution or position;
    - Reflect on the possible bias of one's own perspective compared to other perspectives.

(Vincent-Lancrin et al., 2019, p. 27)

In most international curriculum documents, critical thinking is a component that is embedded across all learning areas. The Singapore curriculum combines critical thinking with creative thinking. This is also evident in the New Zealand Education and Training Act (2020) where one key objective includes, “promoting the development of: resilience, determination, confidence, and creative and critical thinking” (p. 28). This is also the presentation in the Australian Curriculum discussed in Section 2.5.2.

2.5.2 Australian Focus on Critical Thinking

Within an Australian context, critical thinking is a term referred to in the National Australian Curriculum (ACARA, 2014) and the Early Years Learning Framework (DEEWR, 2009) with an expectation that it is included in students' learning (Ab Kadir, 2018; Heard et al., 2020). In Australia, critical thinking is not seen as discipline specific but rather a general capability that can be embedded in all subject areas.

In the EYLF, critical thinking is identified as a component of how children learn through play. The strategies in the EYLF encourage educators to provide supportive play environments for learners that allows children to “ask questions, solve problems and engage in critical thinking” (DEEWR, 2009, p. 17). As the EYLF is not a discipline specific framework, there is no specific identification of critical thinking occurring in mathematics. Critical thinking is only referred to once in the framework.

For formal schooling, critical thinking is partnered with creative thinking (critical and creative thinking) to serve as one of the general capabilities (ACARA, 2016). The intent of the general capabilities is to build on the individual to be a “successful learner, confident and creative individual, and active and informed citizen” (ACARA, 2016). The link between critical and creative thinking in the curriculum are a way to bring “complementary dimensions to thinking and learning” (ACARA, 2014d.). It is intended that embedding critical thinking within subject areas broadens the learning and highlights the ways the subject knowledge can be applied in varying contexts.

The description of critical thinking within education documents in Australia focuses on elements that can be embedded in learning experiences. The Australian Curriculum (ACARA, 2014) does not give one clear definition of the combined and interchangeable capabilities of critical and creative thinking. Provided for educators is a critical and creative thinking learning continuum and a description of elements of learning that can support the development of critical and creative thinking in education settings. The elements include: inquiring – identifying exploring and organising information and ideas; generating ideas, possibilities and actions; reflecting on thinking and processes; analysing, synthesising and evaluating reasoning and procedures (ACARA, 2015, para.1). Each element has an accompanied description and sub-element. Table 2.1 provides the list all elements, descriptions and sub-elements as presented on the ACARA website (ACARA, n.d.).

Table 2.1
Critical and Creative Thinking in the Australian Curriculum

Elements of Learning Critical and Creative Thinking	Description	Sub-elements
Inquiring – identifying exploring and organising information and ideas	Students pose questions and identify and clarify information and ideas, and then organise and process information. They use questioning to investigate and analyse ideas and issues, make sense of and assess information and ideas, and collect, compare and evaluate information from a range of sources. (para.11)	• Pose questions • Identify and clarify information and ideas • Organise and process information.
Generate ideas, possibilities and actions	Students imagine possibilities and connect ideas through considering alternatives, seeking solutions and putting ideas into action. They explore situations and generate alternatives to guide actions and experiment with and assess options and actions when seeking solutions. (para.12)	• Imagine possibilities and connect ideas • Consider alternatives • Seek solutions and put ideas into action.
Reflecting on thinking and processes	Students describe their thinking and provide reasons for their thinking. They justify their selections and strategies and make connections to other areas. (para.13)	• Thinking about thinking (metacognition) • Reflect on processes • Transfer knowledge into new contexts.
Analysing, synthesising and evaluating reasoning and procedures element	Students identify reasons for particular outcomes. They use their understandings to analyse and synthesise information and apply it to a new course of action. Students evaluate the effectiveness of their ideas. (para.14)	• Apply logic and reasoning • Draw conclusions and design a course of action • Evaluate procedures and outcomes.

(ACARA, 2015)

Locally, within NSW education settings, critical thinking is present in curriculum/syllabus documents and support materials (NSW Education Standards Authority, 2019; Willingham, 2019). NSW teachers are required to implement strategies that help students develop their ability to analyse and thinking critically. The types of skills suggested by the NSW Education Standards Authority (2019) include:

- classifying, categorising and sorting information;

- comparing and contrasting two events, theories and/or processes;
- analysing, evaluating and expressing opinions;
- investigating, collecting and analysing data;
- conducting interviews;
- researching;
- predicting the outcome of an event and testing theories;
- debating;
- participating in group discussions;
- rewriting stories from a different perspective or point of view.

(NSW Education Standards Authority, 2019, para. 4)

In summary, critical thinking is presented in international and Australian approaches to curriculum. International assessments (PISA) include critical thinking as a critical component for testing (OECD, 2016, 2018). Both internationally and locally, critical thinking is included to address all learning areas, and is not discipline specific. In many cases, critical thinking is paired with creative thinking in curriculum documents. Thus, it can be stated that critical thinking is not only required for mathematics, it is also a term that curricula writers believe is associated with creativity.

2.6 MATHEMATICS TEACHING CONTEXTS IN THE EARLY YEARS

Both the EYLF and the Australian Curriculum allow for the implementation of a range of curriculum content and processes aligned with outcomes (Perry & Dockett, 2011). It is evident that both the EYLF and the Australian Curriculum were prepared for different purposes. However, an early years educator in a NSW school, for example kindergarten, grade one and grade two, can work across both documents to ensure consistency of learning (Perry & Dockett, 2011), improved child development, increased school readiness and improved educational attainment (Robinson et al., 2011). There are differences between prior to formal school and formal school pedagogical approaches, the status of mathematical thinking and elements of critical thinking. Preschool educators focus primarily on play based and child-centred pedagogies (DEEWR, 2009). Whereas, primary schools tend to focus on a formal approach that include structured lessons and teacher accountability (Stanley, 2008).

The influence the teaching of mathematics has on students learning mathematics in the early years is a well-recognised area of research (Bobis et al., 2005; Clements & Sarama, 2011; Papic & Mulligan, 2005; Perry et al., 2015). Studies indicate a strong foundation in mathematics can predict later achievement (MacDonald & Carmichael, 2018). Frameworks (EYLF) and Curriculum documents (Australian Curriculum) (discussed in Sections 2.3.1 and 2.3.2) exist to support the teacher in framing the learning. Recent studies in mathematics teaching contexts have also identified that all teachers of mathematics are influenced by (a) their beliefs about mathematics teaching and learning (Beswick, 2007; Stipek et al., 2001) and (b) by their own mathematical knowledge (Anthony & Walshaw, 2009; Ball et al., 2008; Depaepe et al., 2013; Sarama & DiBiase 2004; Shulman, 1986).

Teachers' beliefs and values towards mathematics influence their adopted approaches (Stipek et al., 2001). For example, traditionally, within the school context mathematics is known as a discipline that encompasses rules, formulas and procedures and often one correct answer (Stipek et al., 2001). Within the more traditional approach, the teacher is the controller of the learning and transmits information to the students. Research in the space has identified that a traditional style is commonly present in mathematics lessons (Lessani et al., 2017).

Findings from studies have identified that more traditional beliefs about teaching and learning in mathematics are associated with more traditional practices (Stipek et al., 2001). A study conducted by Stipek et al. (2001) found that teachers of sixth grade students who identified as having traditional teaching beliefs were more interested in students obtaining a correct answer or a high grade. These teachers also emphasised speed and rote learning instead of providing opportunities for students to learn, understand and engage in the mathematics.

Contemporary approaches to teaching mathematics are often associated with an inquiry style of learning. Inquiry approaches are more dynamic, include problem solving, allow for construction of ideas, include a variety of mathematical concepts, provide platforms for reasoning and creativity and allow students to communicate their ideas (Cobb et al., 1995; Lessani et al., 2017; Wood et al., 1995). While each approach to inquiry may vary, there is a strong emphasis on including 'mathematical talk' as a way to justify and reason the students' selected approaches and solutions (Downton et al., 2020; Kalogeropoulos et al., 2021). 'Mathematical talk' is beneficial to both teachers and students as it helps teachers to better understand how students think mathematically and how teachers can support students to engage in this thinking process (Russo & Russo, 2019).

The mathematical knowledge that teachers possess also has a substantive influence on their teaching. Studies have found a reluctance from early years teachers to involve explicit mathematics teaching (Lee & Ginsburg, 2009; MacDonald, 2020). Reasons for this vary, however, in many cases it may be due to teachers' limited awareness of the mathematics content (Cohrssen et al., 2013). A study by Lee (2014) examined early years' teachers' ability to identify and interpret children's mathematical thinking. The findings from this study indicated that teachers of pre-kindergarten children held varied mathematical content knowledge. The problem resulted in inconsistencies of the way the teachers interpreted the mathematics displayed by the young learners. While both studies indicate how teacher mathematical knowledge is an important factor in children's learning, however, the results provided different accounts of the teacher's mathematical knowledge level.

Early years teachers' knowledge of teaching mathematics is often associated with the work of Shulman (1986, 1987) that focuses on the concept of Pedagogical Content Knowledge (PCK). PCK is a term used to describe how a teacher relates their teaching knowledge (pedagogical knowledge) to their subject knowledge (content knowledge) (Anthony & Walshaw, 2009; Depaepe et al., 2013; Shulman, 1986).

Additionally, the quality of teaching mathematics in the early years seems to be impacted by the amount of time teachers have been in the profession and their pedagogical content knowledge of mathematics. For example, a study conducted by Lee (2010) in the United States with 81 participants focused on Kindergarten (5 year olds) teachers' PCK. The study identified the qualifications the teachers possessed and their years of teaching experience. It was noted in the findings that new teachers to the profession have less PCK compared with those who had more than 10 years of teaching experienced. Additionally, the teachers displayed the highest scores of PCK in the area of "number sense" and lowest PCK for "spatial sense". This indicates that number-based learning is often a more comfortable content area as opposed to other concepts in mathematics.

Also, in some cases, early years teachers experience difficulty in identifying children’s mathematical thinking. A study conducted by Lee (2017) investigated PCK of pre-school teachers in South Korea; however, this study included how pre-school teachers can enhance children’s mathematical thinking. The participants in the study were 36 teachers with various levels of teaching experience who taught pre-schoolers aged three to six-years. In South Korea, a curriculum is in place for these learners. A finding included the lack of capacity of the teachers to identify children’s mathematical thinking. Lee (2017) identified this finding in line with the teacher’s pedagogical content knowledge. A lack of the broad range of mathematical knowledge required for teaching was identified by Ginsburg et al. (2008) as the reason teachers do not provide mathematical learning experiences that are beyond the basics.

The recognition of young children’s mathematical thinking during play can be difficult to identify for some early years teachers. A study with early childhood teachers in Hong Kong and the United States focused on the ways teachers identified mathematics in children’s play, specifically block play (Hsieh & McCollum, 2018). Findings indicate that a teacher’s lack of mathematical knowledge limited their ability to identify how students were progressing and how to respond to their learning (Hsieh & McCollum, 2018). Ginsburg et al. (2008) found similar results where teachers were often providing learning experiences that include narrow or low level mathematical content. Examples of low level learning experiences include students identifying the names of common shapes or counting to 20 (Ginsburg et al., 2008).

Finally, it seems that many early years teachers are unable to identify the mathematical content that is associated with the learning that is occurring. Similar to the previous studies discussed, a study that included 161 kindergarten teachers in Shanghai focused on teacher ability to identify mathematical knowledge of the students (Tian & Huang, 2019). It was found that most teachers were unable to adequately explain the learning in relation to mathematical concepts the students were engaging in.

In summary, research findings identify the influence that teachers' mathematical beliefs about teaching and their mathematical content knowledge have on the teaching practices they provide to young students. As both beliefs and content knowledge are identified as factors that influence teaching practices in the early years, the problem lies in the inability of a traditional approach allowing for young students to display different ways of expressing their mathematical learning. While creativity, critical thinking, problem solving and inquiry approaches to learning are positioned as contemporary requirements in mathematics teaching and learning, there are inconsistencies in these approaches in the early years.

As both beliefs and in particular mathematical content knowledge are identified as factors that influence teaching practices in the early years, the way a constraining traditional approach affects the opportunity for young students to display different ways of expressing their mathematical learning could be questioned.

With regard to mathematical thinking, many early years teachers experience difficulties in identifying students' mathematical thinking, particularly in the context of play. Further, within the play context, teachers also experience difficulties in identifying the particular mathematic content that students' are engaging with.

2.7 CHAPTER REVIEW

The first part of the chapter identified the early learner, internationally recognised as a child between the ages of 0 – 8. Next a review of early learners and the international and local Australian contexts was discussed. It was at this point that the participants in this study were positioned at the crucial period of transition and in between two curriculum approaches. As this study focuses on the discipline of mathematics, a review of early years and formal schooling mathematics curriculum is considered in the next section.

A review of international and Australian curriculum frameworks for early learners followed. Internationally, curriculum frameworks have a strong focus on real life mathematics experiences, number based concepts and the integration of mathematics content. The Australian curriculum framework was found to be more generalised with respect to mathematics.

Next, a review of international and Australian formal school curricula was reviewed. Similarities were found internationally and locally. They include the organisation of content according to stage or age. The first formal year of schooling includes similar content, which is predominately number based, geometry and early measurement ideas.

An in-depth review of mathematical thinking in curricula followed. It was found that mathematical thinking varied in how it was presented in early years frameworks or curricula. The main inclusion of mathematical thinking was found when statements appeared that were related to problem solving in mathematics. Mathematical thinking in formal schooling curricula found that not all documents include mathematical thinking. The approach in which mathematical thinking is presented differs. Some countries are more explicit and including mathematical thinking across all content areas, whereas others include elements of mathematical thinking in their content elaborations.

A review of critical thinking in curricula followed. It was found that critical thinking is presented in international and Australian curricula as overarching skills that go across other subject or learning areas. It was also found that critical thinking was often paired with creative thinking.

To contextualise the review of early learning and formal schooling mathematic curricula, a review of the mathematics teaching context in the early years followed. The early mathematics teaching practices, including content and pedagogical knowledge of teachers identified inconsistencies in early years teaching approaches.

The next chapter presents the review of the literature pertaining to mathematical thinking and critical thinking and teacher actions that enhance critical mathematical thinking (CMT). Sub-themes from the literature review are also developed to describe CMT in young students.

Chapter 3: Critical Mathematical Thinking

3.1 CHAPTER OVERVIEW

The purpose of this chapter is to present the review of the literature related to delineating critical mathematical thinking (CMT) in young students as they begin formal schooling. In addition, the review of the literature focuses on the teaching actions that help young students exhibit their CMT. The literature that justifies the research aim is presented through four themes: mathematical thinking, critical thinking, critical mathematical thinking, and teacher actions that help students exhibit their CMT. Specific themes and sub-themes are formed from the literature to begin to describe CMT in young learners.

A search of the literature pertaining to the broad area of mathematical thinking was conducted. The purpose of this approach was to determine the importance mathematical thinking has in mathematics teaching. The review process allowed for the differentiation between Advanced Mathematical Thinking and the mathematical thinking pertaining to young students. The identification and synthesis of key themes emerged from the literature search.

While mathematical thinking is a commonly used term to describe the actions students engage in during mathematics lessons (see Section 3.2), former research pertaining to the way students engage in mathematical thinking prior to school and at kindergarten age suggest that there is no clear definition of the term (see Section 3.2). In addition, there appears to be no clear framework on which to anchor the various understanding and terms relating to this construct. In order to begin to address these concerns, the literature relating to critical thinking was examined.

Critical thinking was found to be described as a way to represent problem solving and the decision-making process (see Section 3.3). In most cases, the critical thinking literature was found in journals that were not related to mathematics.

The review and synthesis of literature allowed for the development of a framework to situate the construct of Critical Mathematical Thinking (CMT). In

Section 3.4 of the chapter the literature review supported the amalgamation of terms mathematical thinking and critical thinking to create the term CMT. With little literature addressing the term CMT, an opportunity was provided to blend critical and mathematical thinking and develop a literature based conceptual framework for CMT. The conceptual framework themes (interpreting, analysing, evaluating, explaining and creating) and associated sub-themes emerged from the review of the literature.

It was evident that literature focusing on the teacher's role in helping students to exhibit their CMT needed to be considered. The review of research literature that highlighted the role of a teacher during mathematics lessons occurred, which in turn supported the identification of teacher actions to elicit CMT. Figure 3.1 presents an overview of Chapter 3.

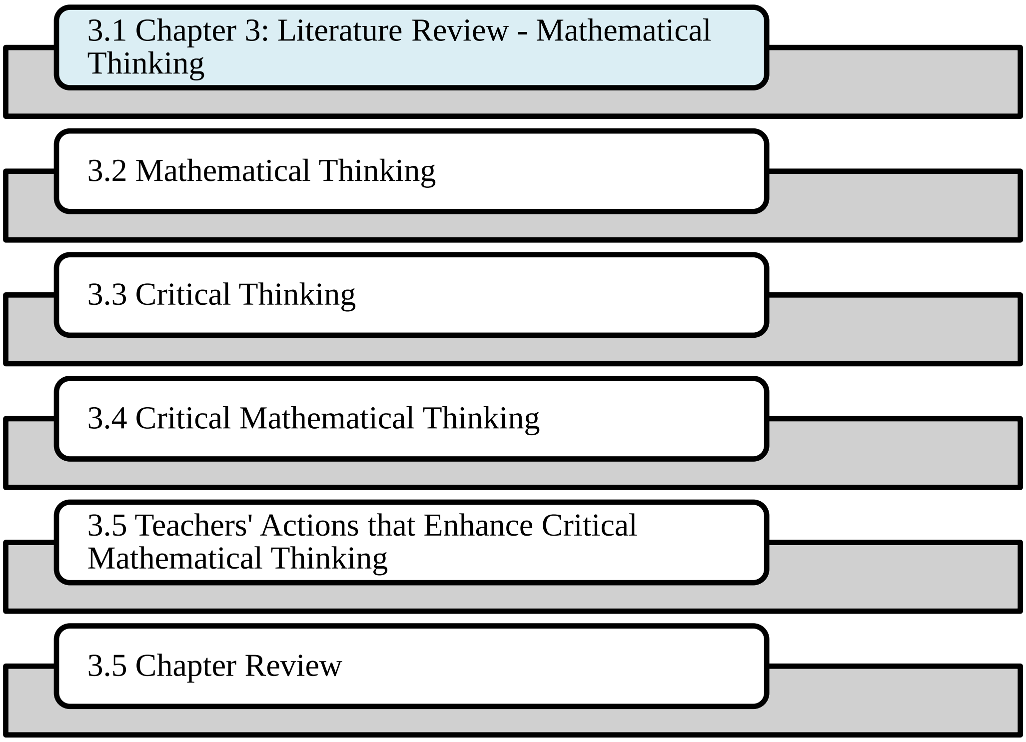


Figure 3.1. Overview of Chapter 3

3.2 MATHEMATICAL THINKING

The use of the term mathematical thinking is widespread; however, in the literature there appears to be no agreed upon definition. For example, Burton (1984) states that there is a “clear distinction between mathematical thinking and the body of knowledge (i.e., content and techniques)” (p. 36). Schoenfeld (1992) elaborates on mathematical thinking as: (1) “developing a mathematical point of view”; and (2) “developing competence with the tools of the trade and using those tools in the service of the goal of understanding structure—mathematical sense-making” (p. 335). Clements and Sarama (2014) take a more hierarchical approach to classifying how children think within developmental progressions or trajectories. Alternatively, Suzuki (1998) refers to mathematical thinking as “global concepts including all mathematical activities and traditional meanings, such as routinely solving mathematics problems” (p. 1). Suzuki further explains that mathematical thinking includes “doing, understanding, and ... problem solving” (p. 1). Mathematical thinking has been also described as involving mental activity focusing on the abstraction and generality of mathematical ideas (Wood et al., 2006). Consistent with this approach, the term Advanced Mathematical Thinking has gained currency (Tall, 1991). Hence, mathematical thinking deals with one’s ability to engage with different levels of mathematical content, sophistication of thinking processes and approaches such as problems solving. Additionally, mathematical thinking is often classified according to the age or level of development of those being investigated.

3.2.1 Mathematical Thinking - Advanced Mathematical Thinking

There is a difficulty in understanding what is meant by the term Advanced Mathematical Thinking because the use of the word ‘advanced’ can differ. One approach is to attach ‘advanced’ to the ‘mathematics’ involved, usually referring to advanced topics found in upper secondary or tertiary. Alternatively, ‘advanced’ may be attached to ‘mathematical thinking' and, hence understanding. Advanced mathematics is mostly associated with secondary and tertiary mathematics where the mathematics is highly symbolic and involves structures, which historically have been invented by mathematicians. Formal definitions, logical deductions, and rigorous reasoning are dominant in advanced mathematics (Edwards et al., 2005; Tall, 1991). Establishing a set of constructs that characterise such thinking is problematic. There is also difficulty in identifying advanced mathematics as compared to identifying that, which is not advanced (Dreyfus, 1990). The use of the term ‘advanced’ can refer to the content or the processes of “abstracting … and reasoning under hypothesis” (Dreyfus, 1990, p. 113) or both, and that there is an increase in frequency of the use of these processes when students are engaging in more complex mathematics.

However, there have been studies that demonstrate that abstracting and reasoning are evident in younger learners (for example, Hawes et al., 2015; Papic et al., 2011; Uttal et al., 2013; Van Oers, 1996, 2001). Such abstraction has been referred to as *empirical abstraction* (White & Mitchelmore, 2010) and is based on the seminal work of Skemp (1986). Empirical abstraction focuses on students looking at different contexts or examples of more fundamental mathematical concepts (such as percentages, angles), searching for similarities, and generalising to some abstract description or definition of the concept. *Theoretical abstraction*, initially promoted by Davydov (1990), on the other hand, consists of the creation of concepts that fit into some structure or theory. Davydov's approach is consistent with Piaget's reflective abstraction and has been employed by other researchers including Dubinsky (1991) and Hershkowitz et al. (2007).

In summary, Advanced Mathematical Thinking, known as the development in mathematical thinking, is a hierarchy being built on as an individual engages with more complex mathematics (Edwards et al., 2005; Harel & Sowder, 2005; Hawes et al., 2015; Papic et al., 2011; Tall, 1991; Uttal et al., 2013; Van Oers, 1996, 2001). This hierarchy means that, even though evidence of reasonably sophisticated mathematical thinking is evident in young children (Edwards et al., 2005; Tall, 1995; White & Mitchelmore, 2010), the development of Advanced Mathematical Thinking is regularly aligned to stages of schooling and higher levels seen as predominantly occurring in secondary and university contexts.

3.2.2 Mathematical Thinking: 5 -12 year old Students (Primary students)

Similar to Advanced Mathematical Thinking, mathematical thinking for primary students has also been portrayed as hierarchical in nature. A mathematical thinking framework presented by Williams (2000) extended by Wood and colleagues (2006) presents categorisations which increase in complexity. The mathematical thinking framework includes: (a) recognising comprehending, (b) recognising applying, (c) building - with analysing, (d) building with - synthetic analysing, (e) building with - evaluating analysing, (f) constructing synthesising, and (g) constructing evaluating (Wood et al., 2006).

While these levels portray the notion of levels of thinking, Wood and colleagues also suggest that the types of tasks primary students are engaging with affects the thinking they are likely to exhibit. Aligned with this, Krute͡tskiĭ (1976) proposed a framework that provides direction for teachers in selecting tasks that promote ways of thinking beyond common recall. The framework consists of three levels: (i) Level 1: formalising mathematical information; (ii) Level 2: processing, including logical thought, flexibility in mental processes, striving for clarity and simplicity of solutions; and (iii) Level 3: solving authentic mathematical problems (Krute͡tskiĭ, 1976).

Across the literature there appears to be five areas that help guide the formulation of tasks and encapsulate mathematical thinking frameworks. These are:

1. Students engaging with strategies that support sense making (Ball, 1993; Fennema et al., 1996; Fennema et al., 1998; Ginsburg & Seo, 1999; Papandreou & Tsiouli, 2020; Steinberg et al., 2004; Wood et al., 2006).

2. Students displaying reasoning and justifying during mathematical learning experiences (Aliseda, 2003; Anthony et al., 2015; Carpenter et al., 2003; Diezmann et al., 2001; Henningsen & Stein, 1997; Herbert et al., 2015; Hufferd-Ackles et al., 2004; Hunter & Anthony, 2011; Melhuish et al., 2020; Papandreou & Tsiouli, 2020; Papic et al., 2009; Stein et al., 1996; Vale et al., 2017; Warren et al., 2013; Watters & English, 1995; Wood & McNeal, 2003).

3. Students making connections to known mathematical ideas and transferring their thinking (Alexander et al., 1989; Carpenter et al., 1990; Clements & Sarama, 2007; House & Coxford, 1995; Fraivillig et al., 1999; Kinnear et al., 2018; Mulligan et al., 2015; Mulligan et al., 2006; Papic et al., 2011; Raven et al., 1998a; Stein et al., 1996; Warren & Cooper, 2005; White, 1998).

4. Students progressing in trajectories and in turn displaying mathematical thinking (Blanton et al., 2015; Bobis et al., 2005; Clements & Sarama, 2014; DiSessa, 2000; Empson, 2011; Fraivillig et al., 1999; Siemon et al., 2017; Simon & Tzur, 2004; Wilson et al., 2013).

5. Students engaging in problem solving tasks (Boaler, 1998; Briars & Larkin, 1984; Cheeseman, 2008; Copley & Oto, 2000; Fennema et al., 1998; Franke & Carey, 1997; Freiman, 2018; Hashimoto, 1987; Mueller et al., 2014; Schoenfeld, 1987, 1988; Sullivan et al., 2015; Sullivan & Lilburn, 1997; van Bommel & Palmer, 2018; Wood et al., 2006).

These five areas are considered in more detail in the following sections.

3.2.2.1 Students Engaging with Strategies that Support Sense Making

Young learners need opportunities to engage in several strategies to make sense of the mathematics learning presented to them (Fennema et al., 1996; Fennema et al., 1998; Ginsburg & Seo, 1999; Papandreou & Tsiouli, 2020; Wood et al., 2006). Research indicates that invented strategies are one way students make sense of the mathematics learning. For example, providing no paper or pencil to students when posing addition and subtraction word problems (Fennema et al., 1996) or displaying an addition number sentence horizontally rather than vertically (Wood et al., 2006) discouraged students from using a written algorithm. Wood and colleagues identified that students in their study responded with invented strategies when the learning experience was presented with no paper or pencil. This response aligns with Woods et al. (2006) mathematical thinking framework component constructing – synthesising (category (f) of the framework presented in Section 3.2.2).

Additionally, allowing students to invent their own mathematical strategies establishes a foundation for them to engage in more complex mathematical ideas as they progress through their learning. A study by Papandreou and Tsiouli (2020) found that three to five year olds used representations during mathematics free time to connect mathematical understandings. The representations included using blocks for patterns, illustrating their responses and using gestures to represent their findings. A study by Fennema and colleagues (1996) found that students who invented strategies during problem solving tasks displayed deeper mathematical thinking. Another study with grade one to three students by Fennema and colleagues (1998) showed that students who used invented strategies displayed the ability to tackle more complex mathematical problems as they progressed through the grades. Invented strategies were also evident in seven to eight year old students in a study by Wood and colleagues (2006). In this study, students used generalisations to compare and contrast two different subtraction methods when solving 72 subtract 39. The strategies compared during student discussions included 72 – 30 – 9 and 70 – 40 + 1. The mathematical thinking of comparing and contrasting was classified as *build-with synthetic analysing* (category (d) in the framework presented in Section 3.2.2).

Moreover, classrooms that are inclusive of inquiry and argument allow for students to demonstrate thinking that exhibit reasoning, justifying, evaluating, and considering mathematics from different perspectives. This allows these students to express their invented methods (Ball, 1993; Steinberg et al., 2004; Wood et al., 2006). The area of thinking pertaining to reasoning and justifying is explored further in the next section.

3.2.2.2 Students Displaying Reasoning and Justifying during Mathematical Learning Experiences

Although reasoning and justifying are prominent elements of mathematical thinking within the literature, the terms are often seen as interchangeable, and at other times they are separated with distinct features. For example, there are four key terms referred to in the literature to describe mathematical reasoning. These are inductive reasoning, deductive reasoning, abductive reasoning, and adaptive reasoning (Kilpatrick et al., 2001). Although these terms are often defined as distinct there is some overlap in their use. Students, who logically use existing knowledge to generate new ideas, or knowledge, are displaying inductive reasoning (Watters & English, 1995). Deductive reasoning supports inductive reasoning but goes beyond to include known or already hypothesised understandings and mathematical proofs (Reid, 2002) to find new conclusions (Herbert et al., 2015). Abductive reasoning includes the ability to find the best explanation with the use of additional information to build on new learning (Aliseda, 2003). Thus, deductive reasoning appears to be about making predictions, and inductive reasoning appears to be about verifying those predictions. Finally, according to Carpenter et al. (2003) students who use a collection of findings, processes, concepts and solutions, to make sense of the mathematical problems are using adaptive reasoning. Adaptive reasoning is about logical thinking to formulate mathematical problems and solve the problems (Kilpatrick et al., 2001).

While many studies have endeavoured to define mathematical reasoning, the definitions utilised are specific to each study's aims and methodology. Reasoning, as defined by Diezmann and colleagues (2001) is the process that includes the use of "facts, properties, and relationships to make and test conjectures and to follow and develop logical arguments" (p. 5). Giving reasons, as explained by Wood and McNeal (2003) is when an individual can justify or defend solutions. Further, Wood and McNeal (2003) state that there are unclear lines between reasoning and justification. It is also noted by Melhuish and colleagues (2020), reasoning is evident in an individual's justification and generalisation. However, across the definitions, there are common features with respect to reasoning and justifying. These include: defending solutions, providing evidence, finding relationships, making conjectures and having logical arguments (Melhuish et al., 2020). For the purpose of this present study, both reasoning and justifying are explored when considering mathematical thinking in young students.

There are two types of studies that have examined reasoning and justifying. First, studies that have focused on students' reasoning or justifying have also explored mathematical learning experiences, including discussions, that can be used to elicit mathematical thinking and reasoning in students (Anthony et al., 2015; Hunter & Anthony, 2011; Papic et al., 2009; Papandrewou & Tsiouli, 2020; Vale et al., 2017). Second, studies have focused on the teacher's role to promote, identify or notice reasoning or justifying in student work (Diezmann et al., 2002; Hunter & Anthony, 2011; Papic et al., 2009; Vale et al., 2017).

Encouraging student talk in learning experiences provides opportunities for upper primary students to display their reasoning and justifying as they construct mathematical knowledge. For example, incorporating opportunities in the lessons for students to explain and discuss their ideas in small group activities involving mathematical inquiry, resulted in students aged 11 - 12 exhibiting a propensity to use representations and justifications as they explained and discussed their ideas (Hunter & Anthony, 2011). Findings from these studies identified that a focus on mathematical practices over time that promoted and scaffolded student participation, in particular student talk, resulted in students communicating their mathematical ideas more regularly in class (Hufferd-Ackles et al., 2004; Hunter & Anthony, 2011).

Incorporating student talk in learning experiences helps students to evaluate their reasoning. During classroom discussions, individuals evaluate if a mathematical approach is reasonable or efficient (Cengiz et al., 2011; Franke & Kazemi, 2001;

Wood et al., 2006). Cengiz and colleagues (2011) studied purposefully including actions that extend student thinking and explaining. Findings indicate that the most common approach that supported mathematical thinking with 8 – 12 year olds was when these students were asked to evaluate or explain their own or other students' ideas or solutions. Additionally, appropriately planned mathematical content selection, and learning experiences that include opportunities for student talk, provide greater opportunities for students to evaluate their mathematical thinking. A study of 256 third grade students focused on the types of learning experiences that allow for students to evaluate information and evaluate solutions (Maričić & Špijunović, 2015). Findings from these studies indicated that the skill of evaluating was possible when the tasks included mathematical problems that allowed students to identify mathematical language and make connections with other mathematical concepts.

Thus, well-planned learning experiences that allow for students to validate their thinking and discuss their explanations provide a platform for reasoning. Vale and colleagues (2017) found that seven to ten year olds displayed reasoning during open-ended learning experiences. The learning experiences focused on comparing and contrasting the numbers 30, 12 and 18. The teacher posed prompting and probing open questions. For example, "I wonder could these numbers belong together; what is your reason…?" (p. 878). Further components of the learning experiences asked for students to justify and generalise their responses. This was followed by a plenary session for sharing and discussion. Results from this study noted that open-ended learning experiences provided a platform for students to demonstrate the reasoning actions of comparing, contrasting, justifying and generalising. Ensuring the teacher is engaging in the learning with the use of probing and prompting questions enhanced the students' ability to demonstrate reasoning actions.

In an early learning context, learning experiences that include student talk promote reasoning in young students. In a study in rural and regional Australia by Papic and colleagues (2009), young children engaged in number, algebraic thinking, spatial thinking and measurement learning experiences. The learning experiences were planned and structured and occurred during play situations. The role of the teacher involved the facilitation of children's discussions during play. This role included asking open questions such as, "why is it a pattern?" (p. 652). These types of questions promoted the use of justification and problem solving strategies in the children's explanations. Papandreou and Tsiouli (2020) in a study of three to five year olds found that reasoning was expressed during children's play in the form of explaining why items are symmetrical and explaining how patterns are made. Diezmann and colleagues (2002) in a study with seven to eight year old students found similar results. Evidence of reasoning in the students occurred when the teacher planned learning experiences that promoted reasoning possibilities. An example of supporting students' reasoning included a teacher reading a picture book, and posing specific questions such as "Have they got the same amount of cookies? How do we know? How else could we know?" (p. 292). Thus, with young children the types of questions asked in discussions also seem to be related to the level of reasoning young students exhibit.

In summary, research findings about reasoning and justifying during learning experiences have identified four key conclusions for consideration. First, unlike Advanced Mathematical Thinking (see Section 3.2.1), eliciting reasoning and justifying in primary students appears to be linked to the types of learning experiences primary students engage with rather than the complexity of the mathematics being taught. Second, the structure of the learning experience can make a significant contribution to student learning throughout the lesson. Third, the addition of probing or prompting questions can lead to students exhibiting deeper levels of reasoning and provide opportunities for evaluating their reasoning. Finally, the studies reviewed have also highlighted the importance of supporting student talk or discussions within the learning experience, that is, the contribution that classroom discourse makes to the learning.

3.2.2.3 Students Making Connections to known Mathematical Ideas and Transferring their Thinking

The ability to find interrelationships within mathematical concepts is considered a key element of reasoning and mathematical thinking (Fraivillig et al., 1999; Kinnear et al., 2018). The process of making mathematical connections is cognitive and includes identifying links within and between mathematical ideas. The ability to move from conceptualisation of a mathematical idea to recognising links with other mathematical ideas and topics is known as an indicator of high levels of mathematical thinking (Fraivillig et al., 1999; Kinnear et al., 2018). The way in that

House & Coxford (1995) categorised mathematical connections included: unifying themes, mathematical processes, and mathematical connectors. Learning experiences that are purposely constructed to build on prior knowledge, and make conceptual connections, engage students in higher levels of mathematical thinking (Stein et al., 1996).

Fundamental to making connections between mathematical concepts is one's ability to recognise the pattern and structure inherent in and between the concepts. There is some debate with regards to the relationship between the terms of pattern and structure. A mathematical pattern is described as the predicable regularity within mathematical relationships (Mulligan et al., 2020; Mulligan et al., 2006; Orton & Orton, 1999; Warren & Cooper, 2005). For example, patterns consisting of repeating terms (for example, ABBABBABB) and patterns that grow by a constant amount (2, 4, 6, 8…) all have predictable regularity. Mulligan and Mitchelmore (2009) define structure as the way in which the pattern is organised. It seems that the two terms are somewhat interchangeable. By contrast, Kellman and colleagues (2010) suggest that recognising a pattern and extracting its structure are different processes. Structural extraction entails the ability to extract the information from a task, see it as independent of the task, and recognise this structure across different representations (and tasks) (Mason & Scrivani, 2004). This distinction is further exemplified in the spatial domain and students' capability of constructing rectangular arrays within the structure of rows and columns (Battista et al., 1998; Mulligan et al., 2020; Outhred & Mitchelmore, 2000). In this instance there appears to be an absence of predicable regularity within mathematical relationships (the notion of a distinct pattern).

Research in pattern and structure in Australia and internationally evidences that primary students who make a connection between two or more structural mathematical understandings demonstrate higher levels of mathematical ability (Mulligan et al., 2006; Warren & Cooper, 2005) and achievement (Booth & Thomas, 1999). For example, high achieving students tend to use well-developed structures and notations when working with a task. By contrast, low achieving students tend to produce poorly organised pictorial and symbolic representations of the tasks (Mulligan et al., 2012). Additionally, English and Watters (2004) reported that the ability to recognise structural relationships within tasks is related to one's ability to reason analogically, that is to map the structure of one problem (representation) into a structurally similar problem (representation).

An understanding of pattern and structure is important in the early years of mathematical learning. Findings from a study of students aged 5 to 12 years identified that early years exposure to patterns and structure tasks allowed students to transfer their understandings to other mathematical concepts (Mulligan et al., 2015). The study incorporated specific patterns and structure, teaching and learning strategies, and results from a post-assessment using the Patterns and Structure Assessment (PASA) instrument (Mulligan & Mitchelmore, 2009; Mulligan et al., 2004). Findings included an increase in the participating students' acquisition of specific mathematical concepts and processes including the transition from a concrete mathematical thinking approach to a more abstract approach.

The ability of young students to identify patterns and structures can impact on their mathematical achievement in later years (Papic et al., 2009; Papic et al., 2011). This was recognised in a study by Papic and colleagues (2009) that identified students' ability to identify sophisticated patterning concepts and strategies. One year after the intervention entailing a focus on repeating patterns and spatial structure patterns, participating children's counting and arithmetic skills were considerably more advanced than those children who had not participated in the intervention (Papic et al., 2011). Initially, many students provided random responses to patterning tasks, prior to the intervention. These responses included copying and repeating patterns using a direct comparison strategy. By contrast, one year later young students who engaged in the patterning intervention could engage with complex repeating patterns. The post-test interview revealed that these students were able to recognise the unit of repeat structure of repeating patterns. Overall, the study identified that the intervention strategies increased students' understanding of simple repetition, spatial patterns, and units of repeat. The researchers concluded that the teaching strategy, where teachers "repeatedly encouraged children to look for structural similarities and differences between the given pattern and their copy of it" (Papic et al., 2011, p. 261) had the most impact on the young students' attention to structure.

Finally, students' ability to see different ways to represent mathematical understandings and make connections between and across representation draws on a range of reasoning processes including analogical reasoning, and spatial and quantitative reasoning (English & Watters, 2004; Genter et al., 2001). Analogical reasoning is when an individual reasons with relational patterns, that is, detects patterns, identifies recurrences of a pattern, and abstracts from patterns (Genter et al., 2001). For example, when students recognise the common structure in a problem and use this structure to solve related problems of similar structure then they are reasoning analogically. By contrast, spatial reasoning involves students recognising the spatial properties of an object, and the spatial relations between objects. Previous studies have included measures such as the Raven's Progressive Matrices (Raven et al., 1998b), a non-verbal problem solving test, and the Test for Analogical Reasoning in Children (Alexander et al., 1989; White, 1998), a test using blocks to pair and / or eliminate possible answers, to identify students' capability to display analogical reasoning and abstract visual spatial ability (Carpenter et al., 1990). Such measures have proven to help identify young students' cognitive ability (Raven, 2000), and have also identified the need for mathematical learning to include richer learning opportunities as a means to help uncover and interpret young children's mathematical thinking, including their ability to think analogically and spatially.

3.2.2.4 Students Progressing in Trajectories and in turn Displaying Mathematical Thinking

Developmental progressions in learning mathematics concepts are referred to as learning trajectories, stepping stones, or growth points (Bobis et al., 2005; Clements & Sarama, 2004, 2014; Siemon et al., 2017; Simon & Tzur, 2004). According to Simon and Tzur (2004), learning trajectories assist a teacher to conjecture where students' conceptual development is at and what learning needs to be considered for the next task. It is a term that appears in studies related to the development of young students' thinking. Similarly, Siemon and colleagues (2017) consider learning trajectories as a teacher guide and use the term 'learning sequences' as a way to identify students' learning journey. The learning journey is utilised by the teacher when determining what learning experiences support students' progression to the next step. Sarama and Clements (2009) define learning trajectories as multidimensional and include "the simultaneous consideration of mathematics goals, models of children's thinking, teachers' and researchers' models of children's thinking, sequences of instructional tasks, and the interaction of these at a detailed level of analysis of processes” (p. 87). Further, Sarama and Clements (2009) identify three learning trajectories components, namely, a goal, a developmental progression, and instructional activities. It is during the process of developmental progression that mathematical thinking is known to occur. Researchers have also used other terms to describe these learning pathways. Bobis and colleagues (2005) and Clarke and colleagues (2001) refer to trajectories as ‘stepping stones’ or ‘growth points’, and that students tend to progress along a common developmental path in areas of mathematics, as they progress through school.

As the term trajectory implies a sequential path of learning, discussion is required about the complex and non-linear ways individuals learn concepts in mathematics. It is noted by Empson (2011) and DiSessa (2000) that there are varying factors on how individuals come to understand mathematics. As such, the term learning trajectory is considered a hypothesis of how an individual develops an understanding of mathematical ideas rather than a guide for teaching mathematics (Simon & Tzur, 2004). Thus, a learning trajectory is somewhat helpful in giving insights into what students do know and do not know. However, from a conceptual stance, it is not necessarily a path students need to follow in order to reach conceptual understanding or develop their mathematical thinking (Siemon et al., 2017). There are alternate routes. In addition, the alignment of the advancement in trajectories to advancement in displaying mathematical thinking seems to mirror the theories underpinning the notion of Advanced Mathematical Thinking, theories that are not necessarily identified as applicable to the primary or early years contexts.

3.2.2.5 Students Engaging in Problem Solving Tasks

The problem-solving process is central in the discipline of mathematics and is a platform for mathematical thinking. The integration of problem solving within mathematics learning experiences is known to be an important part of the learning experience (Schoenfeld, 1987; Schoenfeld & Sloane, 2016; Sullivan et al., 2015). Problems used often include language-based problems with links to real world contexts (Briars & Larkin, 1984; Hunting et al., 2012; Papadakis et al., 2017). The dispositions required to problem solve include: (a) preserving; (b) focusing attention on the problem; (c) testing the hypothesis; (d) taking risks; (e) displaying flexibility; (f) attempting to solve the problem in a different way; and, (g) displaying to self-regulate (Copley & Oto, 2000; Intaros et al., 2014; Muir et al., 2008). Within an early years setting, students engage in problem solving by manipulating “physical objects, progress to various counting strategies, and then move on to more abstract strategies such as derived facts or invented algorithms” (Fennema et al., 1998, p. 6). Young students learn the processes of problem solving and the mathematical ideas that underpin that learning through exploring mathematical problems (Lesh & Zawojewski, 2007; van Bommel & Palmér, 2016).

Young students often have a perceived way or sets of rules they follow to solve mathematical problems (Cheeseman, 2008; Franke & Carey, 1997). Franke and Carey reviewed studies pertaining to young students’ perceptions of problem solving environments and the varying ways to problem solving strategies they used. The research focused primarily on the young students’ perception and beliefs about doing mathematics. They found many students believed all solution strategies used during problem solving were equally valuable. Although such a finding is welcomed, it raises concerns with respect to their future learning of new mathematics and working with new, more efficient, problem solving strategies. Hence the teacher’s role is crucial to assisting students to reflect on various strategies as these learners build up their own set of rules to solve problems (Boaler & Sengupta-Irving, 2016). However, another finding from Franke and Carey’s review indicated that when problem solving is valued in the classroom, young students are more inclined to share their ways of thinking. Thus, problem solving appears to be a platform for young students to engage in and exhibit their mathematical thinking.

In conclusion, the above sections have focused on examining research findings pertaining to mathematical thinking across all primary students (i.e., students in the age range of 5 – 12 years) with the aim of giving guidance to this present research that is situated in the first year of school. The applicably of these specific findings to five year old students is yet unknown. The next section specifically explores four to five year old students’ ability to engage in mathematical thinking.

3.2.3 Mathematical Thinking in the Early Years

Research in the early years has established that mathematical thinking begins from a young age (Bobis et al., 2005; Doig & Ompok, 2010; Sarama & Clements, 2009). The results of research in the early years have indicated that young students have well-developed mathematical competence prior to formal schooling (Clarke et al., 2006; Clements et al., 2003; Hunting & Pearn, 2009; McMullen et al., 2020). The foundation for learning mathematics lies in the ways in which students internalise representations (Cobb, 1995; Cobb et al., 1997), build new mental constructs (Bruner, 1966), and then externalise new mathematical understandings (Bruner, 1996). Students' self-created invented strategies, reasoning and justifying, making connections with other mathematical ideas, and problem solving give teachers a window into their mathematical thinking.

A review of the 19 studies identified in Section 3.2.2 has identified a lack of research validating mathematical thinking in prior to school children and students in their first formal year of schooling. Thus, to date literature pertaining to the mathematical thinking of young students who have entered the first year of school is very limited. Table 3.1 presents a summary of the 19 studies, identifying the age/grade range of the participating students together with the sector of schooling they are situated in. For the purpose of this present study, the studies have been split into four groups delineated by the age of the participant students. These are: prior to school, first year of formal school, grades one - three and grades four - six.

Table 3.1

Summary of Studies that focus on Mathematical Thinking

Section	Researchers	Age/grade range	Prior to school	First year of formal school	Grades one to three of formal school	Grades – four to six of formal school
3.2.2.1 Student invented mathematical strategies	Fennema et al. (1996)	Grade one to three students				
	Fennema et al. (1998)	Grade one to three students				
	Wood et al. (2006)	Seven to eight year old students				
3.2.2.2 Reasoning and justifying during mathematical learning experiences	Hunter & Anthony (2011)	Eleven to twelve year old students				
	Vale et al. (2017)	Seven to ten year old students				
	Papic et al. (2009)	Two to six year old children				
	Papic et al. (2011)	Three to five year old children				
	Diezmann et al. (2002)	Seven to eight year old students				
	Cengiz et at. (2011)	Eight to twelve year old students				
3.2.2.3 Students making connections to known mathematical ideas and transferring their thinking	Mulligan et al. (2012)	Five to Twelve year old students				
	English and Watters (2004)	Eight year old students				
	Mulligan et al. (2004)	Grade two to three students				
	Fraivillig et al. (1998)	First grade students				
	Clements & Sarama (2014)	Birth to eight year old students				
	Empson (2011)	Birth to eight year old students				
3.2.2.5 Students engaging in problem solving tasks where mathematical thinking opportunities may exist	Copley & Oto (2000)	Two to six year old children				
	Fennema et al. (1998)	Grade one to three students				
	Franke & Carey (1997)	Grade one students				
	Cheeseman (2008)	Five to seven year old students				
	Mueller et al. (2014)	Grade six students				
	Freiman (2018)	Kindergarten students				

Five studies were predominately situated in prior to school settings and an additional five studies were situated in the first formal year of schooling. Thus, it is evident that research focusing on high levels of mathematical thinking in very young learners is limited.

Emerging from studies of students in prior to school settings and primary aged students are a range of ways of thinking mathematically. These ways of thinking mathematically are presented in Table 3.2. Each dimension of mathematical thinking is defined and the research from which it has emerged is listed.

Table 3.2
Overview of Dimensions of Mathematical Thinking for Young Students

	Dimensions of Mathematical Thinking in Young Students	Literature
1	Connecting understandings and procedures: noting relationships	(English & Watters, 2004; Fraivillig, et al., 1999; Mulligan et al., 2015)
2	Tackling complex problems in novel ways	(Cheeseman, 2008; Franke & Carey, 1997; Freiman, 2018; Mueller et al., 2014)
3	Reasoning and justifying strategies	(Diezmann et al., 2001; Hunter & Anthony, 2011; Papic et al., 2009; Papic et al., 2011; Vale et al., 2017)
4	Clarifying concepts and ideas: making 'sense' of mathematics	(Fennema et al., 1996; Fennema et al., 1998; Wood et al., 2006)
5	Evaluating the strategies being used	(Cengiz et al., 2011; Jeannotte & Kieran, 2017; Franke & Kazemi, 2001; Wood et al., 2006)
6	Considering other methods/strategies/ alternate solutions	(Cengiz et al., 2011; Fraivillig et al., 1999; Diezmann et al., 2001; Hunter & Anthony, 2011; Papic et al., 2009; Papic et al., 2011; Vale et al., 2017)
7	Describing solutions/ clarifying solutions solutions/Elaborating on ideas	(Clements & Sarama, 2014; Empson, 2011; English & Watters, 2004; Fraivillig et al., 1999; Mulligan et al., 2012; Mulligan et al., 2004)

Thus, it appears to date that there are seven dimensions of mathematical thinking that prior to school and kindergarten aged students are capable of exhibiting.

3.2.4 Concluding Comments on Mathematical Thinking

The following presents an overview of the literature review in relation to mathematical thinking.

Research pertaining to mathematical thinking and students in the primary school shows that:

1. encouraging students to invent their own mathematical strategies allows them to exhibit their mathematical thinking and helps build a foundation for more complex thinking;
2. reasoning and justifying, though hard to define, together with connecting and transferring within and between mathematical ideas are fundamental dimensions of mathematical thinking;
3. including student talk in learning experiences provides opportunities for students to display their reasoning and justifying as they construct mathematical knowledge;
4. incorporating student talk in learning experiences helps students evaluate their reasoning;
5. learning trajectories serve a dual role in thinking mathematically. They can be helpful and limiting. They can help by giving guidance with regard to the progression of learning experience, but also be limiting in tying the progression of mathematical thinking to learning progressions; and
6. engaging in problem solving context in conjunction with a discourse that encourages student talk helps students to exhibit their mathematical thinking.

Previous research pertaining to mathematical thinking of students in the prior to school and kindergarten age group suggests these students have the capability of exhibiting seven different dimensions of mathematical thinking (see Table 3.2).

There is no “absolute way of thinking about mathematics” (Tall, 1991, p. 6), however there are approaches in which mathematical thinking is considered in literature that guides education. These approaches include: a) mathematical knowledge and conceptual understanding is required to be able to think mathematically (Burton, 1984; Tall, 1991); b) mathematical thinking occurs before children enter school (Ginsburg et al., 1998; Song & Ginsburg, 1987); c) mathematical thinking emerges when a learning environment fosters mathematical development (Ginsburg et al., 2008); and, d) mathematical thinking is more than memorising formulas or application of procedures (Stein et al., 1996).

Finally, even though there have been a large number of studies situated in the primary school context, there is a lacuna in the literature that examines high levels of mathematical thinking of the five to six year old students as they begin formal schooling.

The review of literature related to mathematical thinking has provided insights into capabilities that emerge when an individual engages in mathematical tasks that allow for mathematical thinking processes to occur. Thus, reviewing the literature pertaining to mathematical thinking within this chapter has begun to allow for the development of the definition of young students' Critical Mathematical Thinking for this thesis. In order to further clarify this definition, the exploration of what makes such thinking *critical* is examined in the next section.

3.3 CRITICAL THINKING

The term, *critical thinking*, has been explored by many researchers, for example, Davies (2006), Ennis (1992) and Facione (1990), and primarily within the 'critical thinking movement' in the United States. The definition is complicated as critical thinking often encapsulates: a judgement; a sceptical and interim view of knowledge; a simple innovation; a way to make sense of texts; rationality; the adoption of an ethical and activist stance; and, self-reflexivity (Moore, 2013).

Critical thinking is a term used to identify skills, qualities, competencies and characteristics in philosophy, psychology and education (Heard et al., 2020). Sternberg (1986) defined critical thinking as "a mental process, strategies, and representations people use to solve problems, make decisions, and learn new concepts" (p. 1). Refinement of the definition by Lipman (1987) concludes that mental processes employed in decision-making are not employed only in decision-making. Thus, there are instances in which decision making is not critical, as the skill of decision making is also dependent on the context. A definition that further expands on the term critical thinking includes "analysing arguments, making inferences using inductive or deductive reasoning, judging or evaluating, and making decisions or solving problems" (Lai, 2011, p. 2). Siswono (2010, p. 19) states that, "critical thinking is thinking that examines, relates, and evaluates all aspects of a situation or problem". He distinguishes creative thinking in mathematics as "thinking that consists of non-algorithmic decision making" (p. 548). A more colloquial approach is provided by Elder and Paul (2020) where critical thinking is a process one engages in by deconstructing and reconstructing your own thinking. Non-algorithmic decision making is considered a complex approach to thinking where there are many possible solutions (Miri et al., 2007; Resnick, 1987). Self-regulating is an action, similar to non-algorithm decision making, whereby an individual self-evaluates their own inferences (Facione, 1990; Resnick, 1987).

Metacognition, thinking about thinking, is closely linked to critical thinking (Elder, 2001). Domain or situation specific metacognition is consistent with domain specific critical thinking where a person makes a judgement about a strategy in a specific problem-solving situation (Kim, 2011). The relationship between metacognition and critical thinking has significant value in future academic fields as successful adults use metacognitive thinking and problem solving skills in many ways. In addition, the American Philosophical Association (APA) through a systematic review with experts reached a consensus on the disposition that constituted critical thinking, and the themes and sub-themes that align with critical thinking (Ellerton, 2018; Facione, 1990). In summary, these are:

- Interpretation (categorising, decoding, clarifying);
- Analysis (examining, identifying);
- Evaluation (assessing claims and arguments);
- Inference (querying evidence, conjecturing alternatives, drawing conclusions);
- Explanation (stating, justifying, presenting);
- Self-regulation (self-examination and correction).

(Facione, 1990, p. 5)

Since the APA systematic review, scholars have highlighted the inclusion of metacognition. Ellerton (2018) contests that one requires the cognitive ability to self-regulate and self-examine in order to think critically.

There are also differing views as to whether critical thinking is a standalone subject or is discipline specific. It has been argued that critical thinking is heavily reliant on domain knowledge and that, at times, critical thinking is easier to learn within a subject or content specific domain (Bailin, 2002; Willingham, 2007). On the contrary Kuhn (1999) believes critical thinking is best presented when an individual demonstrates or applies critical thinking across a range of learning areas. Similarly, Facione (1990) identifies dispositions of an ideal critical thinker, as not domain specific, but as an individual that has a range of skills including:

> "…habitually inquisitive, well-informed, trustful of reason, open-minded, flexible, fair-minded in evaluation, honest in facing personal biases, prudent in making judgements, willing to reconsider, clear about issues, orderly in complex matters, diligent in seeking relevant information, reasonable in the selection of criteria, focused in inquiry, and persistent in seeking results…" (p. 3)

Additionally, in the literature higher order thinking is often associated or interchangeable with critical thinking (for example, Ennis, 1985; King et al., 2013). Ennis identifies critical thinking as the "practical higher order thinking enterprise" (1985, p. 47) that supports decision making. Lewis and Smith (1993) highlight higher-order thinking as a broader term that encapsulates "problem solving, critical thinking, creative thinking, and decision making" (p. 136). Further, Lewis and Smith identify the definition of higher order thinking as when "a person takes new information and information stored in memory and interrelates and/or rearranges and extends this information to achieve a purpose or find possible answers in perplexing situations" (p. 136). Thus, there is a body of literature that identifies higher order thinking as critical thinking.

Another term often associated with critical thinking is good thinking. Good thinking is different from ordinary thinking (Lipman, 1987). In his seminal work Lipman compares and contrasts ordinary thinking with *good thinking.* He believes the transition from ordinary to good thinking is dependent on the individual being able to sufficiently reason to support their opinions. Thus, *good* thinking requires an individual to demonstrate critical thinking and reasoning through the following ways of thinking:

- estimating;
- evaluating;
- classifying;
- assuming;

- inferring logically;
- grasping principles;
- noting relations among other relationships;
- hypothesising;
- offering opinions with reasons;
- making judgement with criteria.

(Lipman, 1987, p. 40)

Therefore, good thinking includes a range of capabilities that an individual applies in education settings. Literature pertaining to higher order thinking, good thinking and critical thinking has identified specific themes and sub-themes that allow for an individual to critically think. Table 3.3 presents an overview of these themes and sub-themes together with the sources from which each was evidenced.

Table 3.3
Critical Thinking – Themes and Sub-themes together with the related References from the Literature

Critical thinking		
Themes	Sub-themes	Reference
Interpreting Facione (1990)	• Assuming • Clarifying • Categorising • Classifying • Decoding • Estimating • Examining • Identifying • Remembering • Understanding	· Lipman (1987) · Facione (1990) · Facione (1990) · Lipman (1987) · Facione (1990) · Lipman (1987) · Facione (1990); Siswono (2010) · Facione (1990) · Siswono (2010)
Analysing Facione (1990) Lai (2011)	• Applying • Conjecturing • Drawing conclusions • Grasping principles • Hypothesising • Inferring • Noting relationships	· Facione (1990) · Facione (1990) · Lipman (1987) · Facione (1990); Lipman (1990) · Siswono (2010); Facione (1990); Lipman (1987); Lai (2011) · Lipman (1987)
Evaluating Facione (1990) Lai (2011)	• Assessing claims and arguments • Offering opinions and reasons • Making judgements with criteria • Querying evidence • Solving	· Facione (1990) · Lipman (1987) · Lipman (1987); Lai (2011) · Facione (1990) · Lai (2011); Kim (2011)
Explaining Facione (1990)	• Stating • Presenting • Justifying • Conjecturing alternatives	· Facione (1990) · Facione (1990) · Facione (1990) · Facione (1990)
Creating (Wilson, 2016; Anderson, 1999; Siswono (2010)	• Self-regulating • Non-algorithmic decision making	· Facione, (1990); Resnick, (1987) · Siswono (2010); Resnick (1987); Miri et al. (2007).

Thus, while the themes together with the sub-themes of critical thinking appear to incorporate some of the terms identified as mathematical thinking (see Section 3.2), namely, estimating, evaluating, noting relationships, transferring, reasoning and analysing, there are additional sub-elements that contribute to the understanding of what constitutes Critical Mathematical Thinking in an educational context.

As already stated, mathematical thinking is a term that is widely used to refer to mathematical knowledge, mathematical processes and mental activity when engaging in mathematical tasks (Burton, 1984; Wood et al., 2006). The actions an individual engages in to think mathematically include: connecting procedures/ noting relationships; tackling complex problems in novel ways; reasoning; sense-making; evaluating; considering other methods/strategies/ alternate solutions; describing solutions/ clarification of solutions/ elaborating on ideas (Cengiz et al., 2011; Fraivillig et al., 1999; Jacobs et al., 2010; Monteleone et al., 2018; Stein et al., 1996).

Critical thinking refers to the process an individual engages in when they problem solve, engage in decision making and engage in new content (Sternberg, 1986). Specifically, critical thinking is about mental process and strategies used to solve problems. For an individual to critically think, they would display understanding by providing reasoning, consider alternative solutions and display similarities and differences to their learning (Lai, 2011; Lipman, 1987).

3.4 CRITICAL MATHEMATICAL THINKING

The review of both terms *mathematical thinking* and *critical thinking* have led to the amalgamation of the terms as *critical mathematical thinking*. In the review of the literature, there is little to suggest that the term *critical mathematical thinking* exists as a notion. The term *critical mathematical thinking* is not *critical mathematical education*, that is, it is not a movement to broaden the use of mathematics for social justice (Skovsmose, 1994). Further, the term critical mathematical thinking differs from Palinussa's (2013) term 'critical mathematical thinking'. Palinussa's (2013) definition refers to an individual's character development during culture based mathematics education in a secondary context. In addition, there is some reference to the term critical 'in' mathematical thinking as a way to assess pre-service teachers' numeracy skills in a research project by Furness and colleagues (2017). Thus, there is a lack of clarity around the notion of critical thinking, mathematical thinking, and inconsistency in the use of the term critical mathematical thinking. The review of literature provided an opportunity to blend *critical* and *mathematical* thinking to provide a framework for defining the term *critical mathematical thinking.*

Themes and sub-themes that emerged from the literature review with regards to mathematical thinking and critical thinking display some commonality. For example, from the mathematical thinking literature, *reasoning and justifying* strategies align with critical thinking terms of, *inferring logically, offering opinions with reasons, making judgements with criteria /reasoning, making decisions and drawing conclusions.*

The merging of mathematical thinking and critical thinking is presented in Table 3.4. Literature pertaining to critical thinking is colour coded in yellow. Literature associated with mathematical thinking is colour coded in green. The themes and sub-themes that appear in both critical thinking and mathematical thinking literature are colour coded in blue. The first two columns of Table 3.4 (Column 1 and Column 2) are the first two columns presented in Table 3.3 from Section 3.3, namely, the themes and sub-themes identified in the critical thinking literature review. Column 3 was created by mapping the *Dimensions of Mathematical Thinking for Young Students* delineated in Table 3.2 in Section 3.2.3 onto the particular CMT theme and sub-theme presented in Column 1 and Column 2 in Table 3.4 .

Table 3.4

A Combined Overview of Critical Thinking and the links with Mathematical Thinking in the Early Years

Critical Mathematical Thinking		Mathematical Thinking in the Early Years
Themes	Sub-themes	
Interpreting	Assessing Assuming Clarifying Categorising Classifying Decoding Estimating ◆ Examining Identifying Remembering ◆ Understanding ◆	4. Clarifying concepts and ideas = making sense of mathematics
Analysing	Applying Conjecturing Drawing conclusions Grasping principles Hypothesising Inferring Noting relationships◆ Querying	1. Connecting understandings and procedures: noting relationships
Evaluating	Assessing claims and arguments Judging Offering opinions and reasons ◆ Making judgements with criteria ◆ Querying evidence Solving	5. Evaluating the strategies being used 3. Reasoning and justifying strategies
Explaining ◆	Stating Presenting Justifying ◆	6. Considering other methods/strategies/ alternate solutions 7. Describing solutions/clarifying solutions/ Elaborating on ideas
Creating	Self regulating Non-algorithmic decision making	2. Tackling complex problems in novel ways

Key:

Critical thinking literature	Mathematical thinking literature	Both critical thinking and mathematical thinking literature ◆

As evidenced in Table 3.4, all the identified dimensions of mathematical thinking for young students mapped onto all themes, sub-themes. Thus Table 3.4

provided a conceptual framework for CMT. Therefore, this initial conceptual framework formulated the first research question:

- *What are the CMT capabilities young students exhibit as they begin formal schooling?*

When exploring CMT in young students, a major consideration is the teacher actions that help students to exhibit their CMT. These actions include the types of learning experiences and mathematical representations teachers choose to present in their classrooms, and the questions teachers ask. Thus, the final section of the literature review pertains to teachers' actions that help students engage in and exhibit critical mathematical thinking.

3.5 TEACHERS' ACTIONS THAT ENHANCE CRITICAL MATHEMATICAL THINKING

The identification of CMT in young learners requires practical components that rely on teacher knowledge, planning and facilitation. The essential aspects required to support and identify CMT in young students include: the type of learning experience and mathematical representations they engage with, the way young students exhibit their thinking and the teacher questioning techniques that help elicit CMT.

3.5.1 Learning Experiences that Enhance Critical Mathematical Thinking

Purposeful learning experiences are a vital component in mathematics education for early years students (Giardini, 2016). There are a range of strategies used to develop learning experiences that encourage students to think in a mathematical way. For example, the selection of learning experiences needs to consider the students' mathematical conceptual and procedural understandings. Additionally, the way in which the learning experience unfolds in action relies on the teacher to take pedagogical actions that develop, deepen and enhance students' learning.

Within mathematical learning, there is a range of learning experiences that a teacher can plan and utilise with early years students. Such learning experiences can include: closed tasks that have a single answer or method (Boaler, 1998); open tasks that include more than one possible answer or many ways to work out the solution

(Kwon et al., 2006; Nicol & Bragg, 2009; Small, 2012; Sullivan & Lilburn, 1997; Sullivan, 2008); exercises from books or worksheets; rich tasks (Foster, 2018; Henningsen & Stein, 1997); tasks that enhance fluency; practical tasks; real-life language-based problems (Clarke et al., 2001); and investigations. Within one learning experience, the implementation of one or more than one component listed above can be used. For example, a real-life language-based problem might be open-ended as well as an investigation. This present study used open-ended styles of learning experiences.

The way problems are framed can promote mathematical thinking. The open-ended approach includes posing problems in which a range of solutions and strategies can be demonstrated or obtained (Hashimoto, 1987; Sullivan & Lilburn, 1997; Wood et al., 2006). The pedagogical strategies used in open-ended problems allow a student to consider many possible solutions (Kwon et al., 2006). Generally, open-ended style learning experiences are carefully considered so that they are not just about prompting students to remember factors or apply learnt skills, such as, algorithms. Bobis and colleagues (2011) highlight the value of open-ended tasks as a way for students to use their current knowledge and understanding to explore rich mathematical situations. The development of open-ended tasks tend to take on one or more of the following features:

- a single question or task that allows for different processes or strategies to be used (Kwon et al., 2006; Small, 2012);
- tasks that have a variety of possible responses (Kwon et al., 2006; Nicol & Bragg, 2009; Small, 2012; Sullivan & Lilburn, 1997; Sullivan, 2008).
- Open-ended tasks can be developed by:
    - Turning a question around (Small, 2017)
    - Teacher provides the answer and asks the students for the question
    - Asking for similarities or differences between differing representations (for example, two different graphs) (Small, 2017)
    - Adding a blank line to replace a number (for example, ___ +___ = 7 (Small, 2017)

- Changing the question (Small, 2017) or adapting a standard question) (Sullivan & Lilburn, 1997)
- Working backwards (for example, if the answer is 23 what are three numbers) (Sullivan & Lilburn, 1997)

Students display higher levels of thinking and understanding when engaging in open-ended learning experiences. Findings from studies that explore using open-ended problem solving tasks have found that students display higher understandings of mathematical thinking when the tasks developed by the teacher “encourage the use of multiple solution strategies, multiple representations and explanations” (Stein & Lane, 1996, p. 50). Likewise, well-designed open-ended tasks encourage students to discuss their mathematical thinking during discussions (Mueller et al., 2014). In contrast, it is noted that closed text book style tasks limit students’ ability to (a) apply their knowledge (Schoenfeld, 1988), and (b) display problem solving strategies (Boaler, 1998).

In summary, there are many ways to organise learning experiences. There are, however, particular elements that support students to engage in high levels of mathematical thinking. Learning experiences that are text book in style or closed, often do not provide a platform for learners to exhibit their mathematical thinking capability. Finally, studies have found that the pedagogical approaches that deepen and enhance the student’s learning often occur during learning experiences that are open-ended in design, and for young students, include appropriate concrete representations and manipulatives (for example, dice, counters, grids, number-lines).

3.5.2 The ways Students Represent their Mathematical Thinking

The ways in which students communicate their mathematical thinking is through modes of representations. Goldin and Kaput (1996) state that representations “do not occur in isolation” (p. 398) but include highly structured systems that Goldin and Kaput have termed “representational acts” (p. 399). The types of representational acts are a challenge for teachers to identify (Kendrick & McKay, 2004).

Internal and external representations are the ways mathematical ideas are presented. Use of manipulatives, illustrations, language and written symbols are the ways mathematical ideas are presented externally (Cooper & Warren, 2008; Lingefjärd, & Ghosh, 2016; MacDonald & Lowrie, 2011). Lesh and colleagues

(1987) also identify use of manipulatives along with static pictures, written symbols, spoken language and real scripts as external representations. According to Lesh and colleagues (2000), a student can reason effectively if they can shift in and out of the many forms of external representations. Internal representations are not directly observable and refer to the learner's mental configurations they display (Goldin & Kaput, 1996). As internal representations are not directly visible to the teacher, often a teacher will request additional information by asking students to discuss their thinking (Goldin & Kaput, 1996). The purpose of asking for additional information is for teachers or researchers to attempt to infer a student's internal representation from external representations they present. For example, a student might illustrate their understanding (external representation) and the teacher or researcher might infer what the student was thinking (internal representation) (Goldin, 2014).

As mentioned, use of manipulatives or concrete materials (examples of external representations) can support the ways students represent their mathematical thinking (Miller & Hudson, 2006; Osana & Pitsolantis, 2019). Manipulatives are hands-on materials that support a learner to represent ideas in mathematics that are abstract (Moyer, 2001). Studies have identified the use of manipulatives are beneficial to learners as they support sense making of the mathematics (McDonough & Clarke, 2003). However, other studies have found that using manipulatives for rote learning does not guarantee success (Baroody, 1989; Hiebert & Wearne, 1992). Thus, it is important to consider that the use of manipulatives does not necessarily result in supporting student learning. The epistemic fidelity of the manipulatives (their alignment with the mathematics being explored) is crucial to promoting understanding (Moyer, 2001). Thus, there needs to be a strong analogical mapping between the physical manipulatives and the target knowledge for their use to be effective (Stacey et al., 2001). In fact, in some instances prior understanding of a concept needs to occur for the manipulative to be understood (Stacey et al., 2001).

Early years teachers and researchers have found that manipulatives can facilitate mathematical thinking (Osana & Pitsolantis, 2019). The Early Numeracy Research Project research found that foundation (kindergarten) teachers who developed learning experiences that included concrete materials and teacher questioning enhanced students to think mathematically by explaining and reasoning (McDonough & Clarke, 2003; Way, 2008).

Students can connect their internal and external representations by illustrating or drawing their mathematical ideas. As teachers and researchers, we can infer student's mathematical understandings from the ways in which they display their external representations (Goldin & Shteingold, 2001; Hebert & Powell, 2016; Thomas & Mulligan, 1995). One approach to encourage students to discuss their mathematical ideas is the drawing-telling approach (MacDonald, 2013; MacDonald & Murphy, 2019). A study focusing on the drawing-telling approach showed that students in their first year of formal school were able to share their understanding of measurement (MacDonald & Murphy, 2019). This included students providing a detailed explanation of their thinking, which included the use of measurement specific language and reasoning.

Given the age of the students in this present study and the focus of critical mathematical thinking, the open-ended tasks presented to the students included an array of purposely chosen external representations. The purpose of these external representations was to allow these young students to make sense of the learning experiences (McDonough & Clarke, 2003), reason effectively (Lesh et al., 1987), and explain their thinking (MacDonald & Murphy, 2019).

3.5.3 The Role of Teacher Questioning

Teacher questioning plays an important role in promoting students' mathematical thinking (Fraivillig et al., 1999; Franke et al., 2009; Jacobs et al., 2010; Martino & Maher, 1999; McCullough & Findley, 1983; Rigelman, 2007; Sahin & Kulm, 2008; Sukmadewi, 2014; Wood & McNeal, 2003). A range of teacher questioning used in the classroom supports student engagement in building mathematical understandings (Martino & Maher, 1999). Yet, research with grades two and three students first identified that over 76% of teacher questions were surface style questions requiring yes/no answers (Di Teodoro et al., 2011). There are many types of terms used to describe the questioning used by teachers. Some examples are: pre-prepared questions; open-ended questions; higher order questioning; adapting questions according to student needs; probing questions; leading questions; questions to prompt further explanations; extending questions; and, questions to clarify student thinking.

Factual, probing and guided questions are commonly used by teachers to support students to exhibit mathematical thinking (Sahin & Kulm, 2008). Factual questions are a common style of questioning utilised by many teachers. These questions are often used to obtain mathematical knowledge rather than to elicit mathematical thinking. An example of a factual question is asking a student to provide a definition of a term. Research with grade two and three students first identified that over 76% of teacher questions were surface style questions requiring yes/no answers (Di Teodoro et al., 2011). By contrast, probing questions provide teachers with opportunities to seek students' explanations of their thinking. Students are asked to justify or prove their thinking, elaborate their ideas and make links to prior learning. The purpose of probing questions is to "extend students' knowledge, encourage student explanations, and promote deeper thinking" (Sahin & Kulm, 2008, p. 224). Guiding questions are used when the teacher is supporting students to discuss their solutions, strategies, or procedures they used in the learning experience. A teacher might adopt a guiding question when a student is unsure of how to proceed or the teacher would like to direct the student.

Factual questions are the most commonly used by teachers during mathematics lessons. These types of questions are aligned with recall of mathematical facts (King, 1989; Sahin & Kulm, 2008; Vacc, 1993) as they include lower order or lower cognitive questions (Shahrill, 2013). Some examples of factual questions include:

- What is the name of figure 0?
- Which figure has five sides?
- How many figures have an acute angle?
- Is 76 larger than 67?

(Vacc, 1993)

Factual questions provide very little information regarding students' understanding of a concept or particular content. It is also noted that factual style questions generally do not allow an individual to collaborate or discuss possible strategies or solutions with others.

Probing questions are known as ways to extend student's understanding, knowledge and mathematical thinking (Barnes, 1976; Franke et al., 2009; Martino &

Maher 1999; Rigelman, 2007; Sahin & Kulm, 2008). In many cases, the purpose of probing questions is to move students from low level thinking, often factual recall, to higher order thinking, that includes explaining their thinking. According to Barnes (1976) probing questions tend to fall into two categories. They are used (a) as a response to students' answers, and (b) as an assessment check on how students are progressing in their learning. In addition, Franke et al. (2009) identify the need for probing questions as a way for teachers to gain further clarity about a student's explanation. A study by Martino and Maher (1999) with grades three, four and five students considered the types of questions teachers used when students were engaging in open-ended mathematics questions. The findings indicate that the probing questions asked by the teachers encouraged students to explain, justify and generalise. In addition, a study by Franke et al. (2009) with grades one - three students found that the inclusion of probing questions provided support to students who had an incorrect or incomplete explanation of their mathematical thinking.

The review of literature on probing questions has identified the types of questions that can be used by teachers to support young students to exhibit mathematical thinking. These include:

- asking students for another way to solve a problem (for example, Is there another way to do this; what do you mean by that?)
- guiding the student to find interrelatedness with other mathematical concepts (for example, How did you work that out; do you think it matters?)
- encouraging students to justify, clarify, interpret or represent their mathematical ideas (for example, What would happen if…?: Okay, why?)

(Martino & Maher, 1999; Rigelman, 2007)

Guiding questions are characterised as the types of questions that direct students to derive concepts or procedures to solve problems (Funahashi & Hino, 2014; Kojo et al., 2018; Mata-Pereira & da Ponte, 2017; Ortenzi, 2002; Sahin & Kulm, 2008; Sukmadewi, 2014). Thus, guiding questions can help to promote students' mathematical thinking (Sahin & Kulm, 2008). The process of guiding or supporting a student requires the teacher to know the direction of the mathematics lesson and the processes required for learning to occur (Mata-Pereira & da Ponte, 2017). Including guiding questions helps students to interpret their own and others thinking, reasoning, and to evaluate the thinking that occurs (Mata-Pereira & da Ponte, 2017). A study of fourth grade students and their teachers found that teachers' use of guiding questions enhanced mathematics learning (Kojo et al., 2018). The types of guiding questions noted in the literature that support students in their thinking to include questions such as:

- Which method do you need to use now? (Ortenzi, 2002);
- Have you developed a strategy to …? (Kojo et al., 2018);
- Are there any other options…? (Kojo et al., 2018);
- What do you notice about this problem? (Sukmadewi, 2014);
- What facts does the problem tell you? (Sukmadewi, 2014);
- What are you trying to find out? (What is the problem asking for?) (Sukmadewi, 2014);
- What tool might help you? Why? (Sukmadewi, 2014);
- How could drawing, table or diagram help you? (Sukmadewi, 2014).

In summary, the teacher's role during a mathematical learning experience needs to be carefully considered. The types of questions a teacher poses, at all intervals of the learning, need to have purpose in guiding students to exhibit high levels of mathematical thinking. In most cases, guided and probing questions best support students to display higher levels of mathematical thinking.

The critique of the literature on teacher actions that enhance (critical) mathematical thinking identified specific strategies for teachers raised the need to focus teacher questioning as a key component when working with young students. This formulated a second research question:

- *What types of teacher questions help these young students exhibit their CMT?*

3.6 CHAPTER REVIEW

The first part of the chapter explored the term *mathematical thinking.* Further exploration of the term identified a body of literature pertaining to *Advanced Mathematical Thinking,* which aligns with a hierarchical nature an individual engages in sophisticated mathematical tasks at a secondary schooling or at university level. The next section examined mathematical thinking specific to 5 – 12 year old primary students. Five key areas emerged and include: (1) students engaging with strategies that support sense making (primary students); (2) students displaying reasoning and justifying during mathematical learning experiences; (3) students making connections to known mathematical ideas and transferring their thinking; (4) students progressing in trajectories and in turn displaying mathematical thinking; and, (5) students engaging in problem solving tasks. An identified gap in the literature identified that very few studies were about five to six year old students. Thus, the exploration of these five areas identified some understanding about young students' mathematical thinking, however, the relevance of the research findings with five to six year old students as they begin formal schooling is explored within this present study.

Some studies relevant to five to six year old students were reviewed to identify mathematical thinking that could emerge from those learners. The review provided an indication of mathematical thinking that is relevant and appropriate for students who are in their first formal year of school, with the ages of five to six. However, the literature pertaining to these students within mathematical thinking is limited.

A key development in the synthesis of the literature pertains to the two terms, *mathematical thinking* and *critical thinking.* The literature review of *mathematical thinking* and *critical thinking* led to the amalgamation of the terms to provide a framework for *critical mathematical thinking.* Thus, the exploration of critical thinking literature was conducted to expand the understanding on young children's thinking in educational contexts. Emerging from the synthesised literature were critical thinking themes and sub-themes that align with the literature. The themes identified were as a result of the literature pertaining to mathematical thinking, critical thinking and the early years. The commonality amongst the literature sources resulted in the identification of 5 themes and 31 sub-themes.

The literature review also identified that a key component of enhancing or identifying a student's ability to be a critical mathematical thinker included teacher actions and the types of tasks that students engage in. One of the main actions identified was the use of teacher questioning. Further, the ways in which five to six year old students represent their mathematical thinking was explored. The literature review process has provided a lens in which to examine CMT in five to six year old students.

The critique of the literature, mathematical thinking, critical thinking and teacher actions, led the focus on the following two relevant research questions:

1. *What are the CMT capabilities young students exhibit as they begin formal schooling?*
2. *What types of teacher questions help these young students exhibit their CMT?*

In the next chapter, the findings related to students' CMT is presented, reviewed and analysed in alignment with the two research questions. The findings align students' performance on standardised assessment measures, association with the CMT themes and elements, learning experiences that enhance CMT, the role of teacher questioning, and the ways students represent their mathematical thinking.

Chapter 4: Research Design and Methodology

4.1 CHAPTER OVERVIEW

The purpose of this chapter is to present and justify the research design that supported this study. The study investigated (a) Critical Mathematical Thinking (CMT) in young students, and (b) teaching actions/questions that help these young students exhibit their CMT.

The following research questions informed the overall research design employed for this investigation:

1. *What are the CMT capabilities young students exhibit as they begin formal schooling?*
2. *What types of teacher questions help these young students exhibit their CMT?*

Figure 4.1 presents an overview of the sections that comprise this chapter.

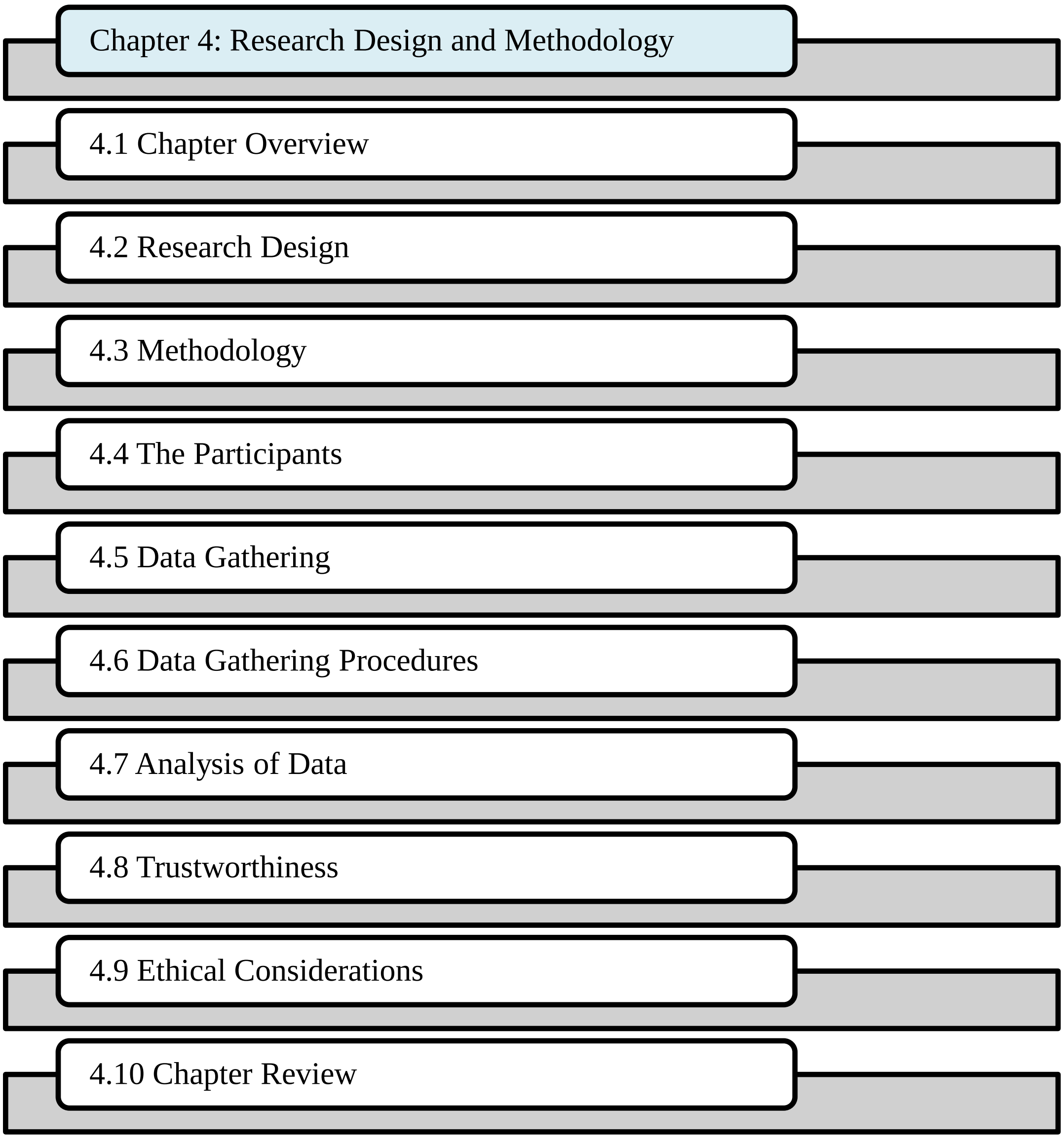


Figure 4.1. Overview of Chapter 4

4.2 RESEARCH DESIGN

The research process adopted for this study sought to identify CMT capabilities of young students as they begin formal schooling. Underpinning this inquiry is the position that these CMT capabilities are based on young students' previous experiences and background knowledge. In order to develop this type of thinking young students need to actively engage in the sense making process (Levers, 2013; Lincoln & Guba, 2013; Ultanir, 2012). Thus, a constructivist epistemology was adopted for the study. The analysis of the data collected required an interpretative approach to consider student representations and teacher actions in relation to CMT. Thus, the principle perspective utilised for this study was an interpretative paradigm (Denzin & Lincoln, 2011). This paradigm allowed the researcher to interpret or gain in-depth insight through the use of multiple forms of data.

The three participating schools were bounded by their socio-economic profiles, academic profiles and geographical location; thus, the methodology considered these three schools as a single case study. In total, five kindergarten classes were involved. As the research aimed to gather a better understanding of the CMT capabilities in young students, a mixed method explanatory design was adopted. A mixed method design includes the collection of both qualitative and quantitative data. When these types of data are used together, a more robust analysis occurs (Creswell 2013; Tashakkori & Teddlie, 2009). The mixed method explanatory design included data collection using one classroom observation guide, three quantitative instruments and a clinical interview. As students progressed through various data collection phases of the study, the number of participants was narrowed to allow for a more in-depth focus on the types of CMT young students who exhibited high levels of CMT displayed as they entered school. Table 4.1 presents the components of the research design that underpinned this study.

Table 4.1
Research Design of the Study

Epistemology	Constructivism
Theoretical Perspective	Interpretative Research Paradigm *CMT Conceptual Framework as presented in Section 4.5.3*
Methodology	Case study Mixed Methods Explanatory design
Data collection methods	Class observations using the CMT Observation Guide (*All Kindergarten Students* n = 161)
	Student observation using the CMT Observation Guide *(Classroom Observation Students* n = 41)
	Focus group quantitative instruments (one-on-one) (n=25) • Raven's Progressive Matrices (Raven et al., 1998); • Slosson Intelligence Test (SIT-P) (Erford et al., 2008b); • Patterns and Structure Assessment (Mulligan et al., 2015)
	Focus group Clinical interview (one-on-one) (n=25) • The Critical Mathematical Thinking Learning Experiences (CMTLE)

4.2.1 Epistemology: Constructivism

This study adopted a constructivist approach to make sense of the meaning of intentional actions (Candy, 1989). Constructivism highlights that truth and meaning is created or constructed by an individual throughout their interaction with the world (Lincoln & Guba, 1986). As such, an individual constructs their meaning in differing ways; through interacting with the interpreter and through the way meaning is interpreted (Crotty, 1998). Thus, constructivism in education is twofold: "how learners construe (or interpret) events and ideas, and how they construct (build or assemble) structures of meaning" (Candy, 1989, p. 108). This approach was critical to the study as the researcher observed and interacted with young students and interpreted their construction of knowledge.

As constructivism is the overarching epistemology for the study, a constructivist pedagogy was adopted to inform the instructional strategies. A constructivist pedagogy is fundamentally about interacting with the world to discover relationships. This can include opportunities to explore, experience and experiment (Dennick, 2016). Within this study, the students interacted with learning experiences to gain sense making. Listening to a child's responses (Lerman, 1989), and investigating the individual's "perceptions, purposes, premises, and ways of working things out" (Noddings, 1990, p. 14) assisted the researcher to gain sense making of the way these young students were constructing understanding through the learning experiences. Thus, the researcher had an essential role which was to work with the participants to gain sense making and meaning of the observations the participants displayed during activities (Levers, 2013; Lincoln & Guba, 2013). The researcher's interpretation considered that the observations were shaped and influenced by the phenomena and society. Therefore, in this study, the researcher presented the findings as a result of how these students constructed meaning (Levers, 2013).

4.2.2 Theoretical Perspective: Interpretative Research Paradigm

The research paradigm provides a theoretical lens that informs both the design of the study and the interpretation of the findings. For this study, interpretivism was identified as the appropriate research paradigm. Interpretative research is related to hermeneutics, which considers the theory of meaning (Schwandt, 1994). This is particularly relevant for this study as one aim was to investigate CMT in young students. Thus, the interpretative paradigm allowed the researcher to consider the mathematical thinking young students display and determine what they know, through the child's explanation, during CMT related learning experiences. Interpretivism, provided a lens for the researcher to identify the child's CMT through the child's representations (Denzin & Lincoln, 2011).

Interpretivists believe that knowledge is socially constructed (Kelliher, 2011). Manning (1997) identifies that the interpretive paradigm is interactive, and the researcher supports the understanding of the construction of knowledge with "questions, observations, and comments" (p. 96). This study used an interpretive research paradigm as an approach to understand students' constructed experiences. Thus, the researcher's role with each student was interactive and included the researcher posing questions and conducting note taking of what was observed.

The interpretivist's paradigm considers the many differing ways of knowing including multiple realities (Denzin & Lincoln, 2005). This approach emphasises the role of the researcher as the vehicle to reveal the reality studied. An interpretivist acknowledges that "objective reality can never be captured" (Denzin & Lincoln, 2005, p. 5). Therefore, the researcher holds a significant role allowing for "such subjectivity to the fore, backed with quality arguments rather than statistical exactness" (Garcia & Quek, 1997, p. 459). In order for the researcher to ensure that the many different responses were captured in the data collection process, a variety of data collection methods were used.

Participants' responses guided the researcher's interpretation of the data. Thus, the researcher's relations and interactions, together with an analytical (and conceptual) review of data supported the formulation of generalisations (Howell, 2013; Willis et al., 2007). In order to understand the complex and intricate ways young students display CMT, it was necessary to develop a conceptual framework to support the researcher's data analysis and interpretation.

4.2.2.1 Conceptual Framework for Thick Description and Thick Interpretation

A conceptual framework is generally developed from an in-depth literature review process that provides a visual map to support the context of the study (Marshall & Rossman, 2014). Thus, a conceptual framework incorporates items from the literature, however, the presentation or structure is flexible and can be modified. As for this study the construction of the conceptual framework was guided by the literature review process, the researcher considered "relevant and related concepts" (p. 4) and the inter-relatedness of the concepts (van der Waldt, 2020).

A conceptual framework is not the same as a theoretical framework. A theoretical framework considers previously trialled and tested knowledge to support the data analysis and interpretation process. The theories considered in a theoretical framework are often derived from leaders in the field of research who have already investigated the topic (Collins & Stockton, 2018; Kivunja, 2018). Included in a theoretical framework are tested ways to plan, investigate, solve a problem and interpret findings. A theoretical framework differs from a conceptual framework as a conceptual framework not only considers theories, but also includes "underlying thinking, structures, plans and practices and implementation of your entire research project" (Kivunja, 2018, p. 47). As the notion of CMT represented a lacuna in the existing literature, particularly in the early years, this research relied on the use of a conceptual framework.

The conceptual framework applied to this study was a CMT Conceptual Framework developed through an analysis of literature concerning mathematical thinking, critical thinking and early years students (see Figure 4.2).

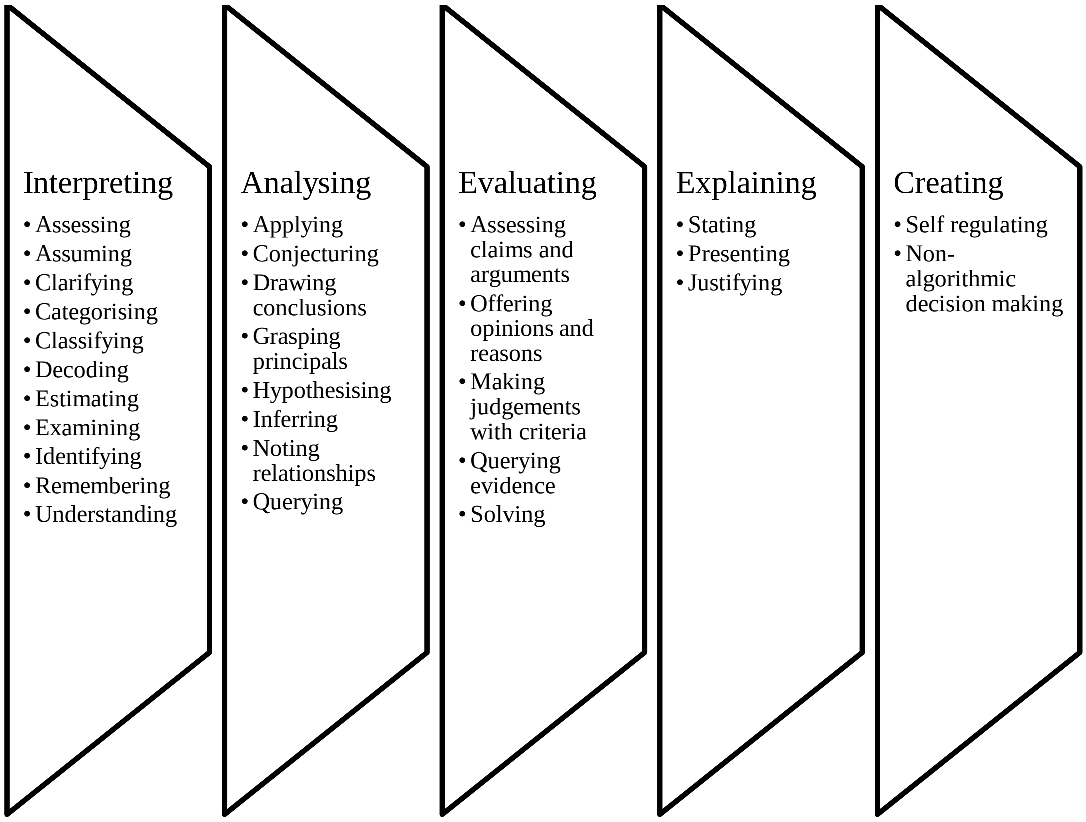


Figure 4.2. CMT Conceptual Framework

The skills, qualities, competencies and capabilities supported the development of the CMT conceptual framework. Five overarching interrelated themes of CMT were identified from the literature, namely, interpreting, analysing, evaluating, explaining and creating. Within each CMT theme were several sub-themes. For example, within the theme of Explaining are the sub-themes of stating, presenting and justifying.

The role of the CMT Conceptual Framework was to aid the investigation of the problem and the interpretation of research findings so that the findings were significant and generalisable. Thus, the conceptual framework was used as a methodological lens to (a) analyse the student data and the teacher questioning data (gathered during the interviews of the *16 Focus Students*), and (b) support the development of research findings. The use of a conceptual framework helped to frame the findings and structure the analysis to provide meaningful statements (Imenda, 2014). Using the conceptual framework, the qualitative data were interrogated so that a thick description (Ponterotto, 2006), with conceptual density (Strauss & Corbin, 1990), supported the development of findings, conclusions and recommendations.

4.3 METHODOLOGY: Case Study with Mixed Methods Explanatory Design

4.3.1 Case Study Methodology

Case study is an approach that is used to explore a phenomenon, within a particular context and with the use of various data sources (Baxter & Jack, 2008; Yin, 2018). The purpose of engaging in case study research is to make sense of real-world ideas (Yin, 2018). It provides a space where both qualitative and quantitative data sets can be used to build a deep understanding of the research problem (Yin, 2018). Thus, in a case study approach, the topic of interest is well explored as the phenomenon is viewed by many lenses (Baxter & Jack, 2008).

A defining feature of case study research is the bounded system (Merriam & Tisdell, 2015). According to Merriam (2009) the "what" is the bounded system. Case study is applicable to this research as it allowed for the in-depth investigation of young students in their first formal year of schooling (kindergarten) within their real world context (school environment). Therefore, the three schools from the same local area form the bounded case for this study (see Section 4.4.1).

Bounded systems provide opportunity for the use of multiple sources of information, for example, observations, interviews and illustrations (Creswell et al., 2007). Typically, case studies are not limited to one evidence or data type, and both qualitative and quantitative data can support the process, outcomes and description of the case (Tellis, 1997; Yin, 2009). Within case study, a researcher can use the qualitative results to understand the quantitative elements (Zainal, 2007). Specifically, a researcher can include qualitative and quantitative data to support observation, reconstruction and analysis of the investigation (Tellis, 1997). Within this study, the quantitative data assisted in narrowing participant selection at particular intervals of the data analysis process. Both the analysis of the quantitative and qualitative data helped to elicit findings pertaining to the research aim, CMT in young students and teaching actions/questions that help these young students exhibit their CMT.

A limitation of a case study methodology is that the researcher takes on the role of both data collector and the data analyst (Flyvbjerg, 2006). Pre-conceived ideas can influence the research findings. The researcher attempted to address the validity and trustworthiness concerns with the inclusion of member checking by experienced mathematics education personnel as the data collection and analysis progressed (see Section 4.8).

4.3.2 Mixed Method Approach – Explanatory Design

A mixed methods approach was used to inform the data gathering and analysis of the study. This approach supports the researcher to follow a sequence that guides the aim of the project. The sequence includes: (i) the development of research questions; (ii) consideration of the mixed method design, collection of the data, analysis of the data; and, (iii) integration of the findings, determining inferences all with the use of both qualitative and quantitative methods (Tashakkori & Teddlie, 2009). Educational research theorists have identified that there are four main types of mixed method designs (Creswell et al., 2006). They include: (i) Triangulation Design; (ii) Embedded Design; (iii) Explanatory Design; and, (iv) Exploratory Design.

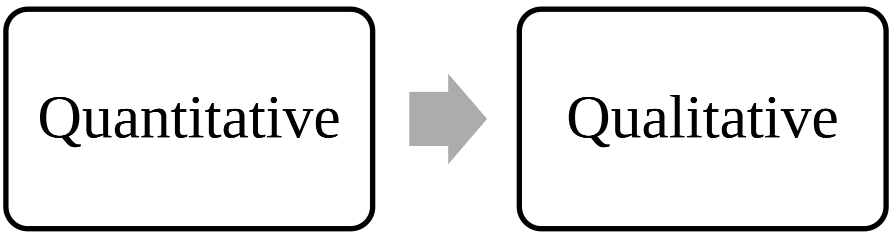
For the purpose of this study the explanatory design was used, as the quantitative data provided a widespread understanding of the research aim and problem, and the addition of qualitative data explained the statistical analysis in more depth (Creswell et al., 2006). It has been described that this approach is best used when the researcher is required to explain significant or insignificant results, outlier results or unanticipated results (Morse, 1991).

As indicated, there are two distinct phases in explanatory design. This entails iterative phases of data gathering and analysis. The phases include:

1. Quantitative data – collected and analysed;
2. Qualitative data – builds on the quantitative data, is collected and analysed and elaborates on the results obtained in phase 1.

(Hesse-Biber, 2010; Ivankova et al., 2006)

Figure 4.3 provides an overview of the explanatory design, including the way in which the quantitative data is used initially, followed by the qualitative data to interpret the results.



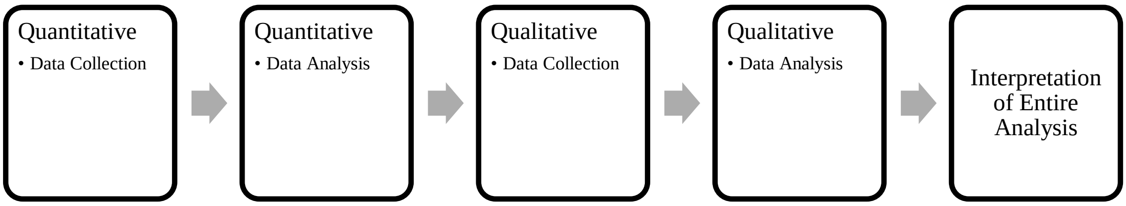


Figure 4.3. The Explanatory Design. (Adapted from Creswell et al., 2003)

Both quantitative and qualitative data were purposely introduced during distinct times in the explanatory design (see Section 4.6).

4.3.3 Summary of Research Methodology: Case Study with an Explanatory Mixed Methods Design

Case study methodology formed the basis for data collection and analysis in this study. It is known that case studies follow a qualitative or quantitative approach or a mix of both qualitative and quantitative (Yin, 2003). The aim of the study was to understand a phenomenon of CMT, therefore, an interpretative approach to the analysis of the case study data was utilised (Section 4.3). An explanatory design (Section 4.3.2) framed the qualitative and quantitative approach of this study. The researcher reviewed the qualitative data (clinical interview, further discussed in Section 4.6.3), and used quantitative measures (standardised assessments, further discussed in Section 4.6.2.1 and 4.6.2.2) to interpret the results.

4.4 THE PARTICIPANTS

The research was conducted across five kindergarten classrooms from three schools located in New South Wales, Australia. Participants included young students who were in their first six months of their first year of formal schooling. *All Kindergarten Students* from the three schools participated in the study (n=161). Consent was obtained from each school principal and each classroom teacher before inviting students to participate (see Section 4.9). The following section presents an overview of the school contexts, student selection at each data stage of the study, and the role of the researcher as teacher.

4.4.1 Overview of the School Contexts

Three Catholic primary schools located in New South Wales Australia took part in the study. All three schools were located in an urban setting in southern Sydney and were from the same educational sector. The researcher was guided by the school sector, in determining schools that may be willing to participate in the research.

School A and School B each consisted of two kindergarten classes, while School C had one kindergarten class. Table 4.2 displays school demographic data that is made publicly available on the Australian *My School* website published by the Australian Curriculum Assessment and Reporting Authority (ACARA) (2020). These data include the Index of Community Socio-Educational Advantage (ICSEA) value, and the benchmark used to determine the level of the school's educational advantage (the Australian national average for ICSEA is 1000). In addition to this, the enrolment data including the number of Indigenous students and students with Language Background other than English (LBOTE) are provided. In addition, the Numeracy component of the National Assessment Program for Literacy and Numeracy (NAPLAN) data are reported. ACARA provide a 5-point scale to interpret the NAPLAN data, namely, well above, above, close to, below, well below and no comparison available. The NAPLAN average school result is also compared with the average Australian result. The items presented in Table 4.2 display the data available to the public for the year that the data was collected for this study.

Table 4.2
School Demographics, NAPLAN Numeracy Achievement and Average Student Results compared to the Australian Average

School	School Demographics				Numeracy NAPLAN achievements (2015) Average school result (Australian average result)	
	Sector	ICSEA	Indigenous	LBOTE	Grade Three 2015	Grade Five 2015
School A	Catholic Primary School Southern Sydney	1112	5%	9%	447 (398)	551 (493)
School B	Catholic Primary School Southern Sydney	1092	2%	87%	410 (398)	521 (493)
School C	Catholic Primary School Southern Sydney	1102	4%	25%	427 (398)	503 (493)

NB. NAPLAN data retrieved from *myschool* (ACARA, 2020).

In summary, all three schools were from the same sector (Catholic Education) and had similar demographics and ranges in ICSEA level (1092-1112). An ICSEA score above 1000 means that these schools would be considered socio-economically advantaged. There were similar enrolment numbers for Indigenous students (2-5%). Finally, it appeared that all average NAPLAN results are above the Australian average results. Grade three average results are between 12 – 49 points higher than the Australian average. Grade five average results are between 10 – 58 points higher than the Australian average. School A appears to the highest performing school overall for Numeracy in Grade three and five. This is followed by School C and then School B.

The only significant difference between the schools is the data pertaining to the number of students who come from a Language Background Other Than English (LBOTE). LBOTE students are those who speak another language other than English in their home as well as students whose parents speak another language other than English in their home. The LBOTE data in 2015 for the three schools were: School A 9%; School B 87 %; School C 25%.

4.4.2 Students

As one aim of the study was to investigate CMT in young students, kindergarten students who had just entered their first formal year of schooling were invited to participate in the study. The age range for the students at the time of the data collection was between 5 years and 1 month to 6 years and 8 months of age.

The following section presents the participants with reference to the order of selection and refinement that occurred in the study.

As the study progressed, students were selected to participate in each stage of the study. As the study was examining the CMT in young students, and teaching actions/questions that help these young students exhibit their CMT, it was necessary to work with students that were considered to exhibit high levels of CMT. The four groups of student participants were selected across the three schools and were identified as*: All Kindergarten Students* (n=161); *Classroom Observation Students* (n=41); *Focus Students* (n=25) and *High CMT Students* (n=16). Each group of students was a purposive selected sample of the previous group of students. The selection of the students was informed by the analysis of the data collected from the qualitative and quantitative instruments administered throughout the study. Figure 4.4 presents an overview of the student participants.

All Kindergarten Students (five classes and three schools)
n=161 students

↓

Classroom Observation Students
n=41 students

↓

Focus Students
n=25 students

↓

High CMT Students
n=16 students

Figure 4.4. Overview of Kindergarten Students that Participated in the Study

4.4.2.1 All Kindergarten Students

As mentioned, five kindergarten classes participated in the study. This included two classes from School A (64 students in total), two classes from School B (65 students in total) and, one class from School C (32 students in total). In total, 161 students were included in the study. Figure 4.4 identifies these students as *All Kindergarten Students.*

The Kindergarten class organisation and setting for each of the five classes were similar. Each school had an individual classroom space for each kindergarten class and the teachers taught their class independently. The researcher visited the classes to observe the students engaging in teacher-led mathematics lessons. This occurred four times over a two-week period for each class. The researcher used the CMT Observation Guide (see Appendix A) to identify CMT capabilities that the students exhibited during their teacher-led mathematics lessons. The frequency of CMT observations was tallied using the CMT Observation Guide.

4.4.2.2 Classroom Observation Students

Classroom Observation Students were selected from the 161 *All Kindergarten Student* participants (see Figure 4.4) across the five kindergarten classes. The selection process was based on the data collected using the CMT Observation Guide (see Section 4.5).

The selection process resulted in 41 *Classroom Observation Students* being selected from a possible 161 (25% of the total cohort sample). The participants included 18 students from School A (28% of the sample from School A), 15 students from School B (23% of the sample from School B), and eight students from School C (25% of the sample from School C). Thus, the proportion of *Classroom Observation Students* selected was similar across the three schools. Table 4.3 presents the number of *Classroom Observation Students* from each school, their gender and class.

Table 4.3
Demographics of the Classroom Observation Students (n=41)

School	Total number of students	Gender		Class	
		Male	Female	K1	K2
A	18	10	8	11	7
B	15	9	6	7	8
C	8	5	3	8	-
Total	41	24	17		

NB: School C had only one kindergarten class

In summary of the 41 *Classroom Observation students*, 24 were male and 17 were female. The number of students from each class ranged between 7 – 11 students.

4.4.2.3 Focus Students

Twenty-five students were identified as *Focus Students*. These students were selected from the 41 *Classroom Observation Students.* The 41 students were observed for an additional two-week period during teacher-led mathematics lessons (four times over a two-week period per class). The researcher re-used the CMT Observation Guide (see Appendix A) to identify CMT capabilities that the students exhibited (see Section 4.5). The 25 *Focus Students* were selected from the 41 *Classroom Observation Students* on the basis of the number of times each displayed CMT capabilities in this additional two-week period.

The participants included eight students from School A (32% of the sample from School A), nine students from School B (36% of the sample from School B), and eight students from School C (32% of the sample from School C). Therefore, the average age of these 25 *Focus Students* was 6 years of age. Table 4.4 presents the number of *Focus Students* from each school together with their gender and the class that they were enrolled.

Table 4.4
Demographics of Focus Students (n=25)

School	Total number of students	Gender		Class	
		Male	Female	K1	K2
A	9	5	4	4	5
B	8	5	3	5	3
C	8	5	3	8	-
Total	25	15	10		

NB: School C had only one kindergarten class

All of the students who were identified as *Classroom Observation Students* from School C were selected as *Focus Students*. In summary, of the 25 *Focus Students* 15 were male, and 10 were female. The number of students from each class ranged between 4 – 8 students.

4.4.2.4 High CMT Students

All 25 *Focus Students* participated in three quantitative instruments (see Section 4.6.2) and the Critical Mathematical Thinking Learning Experiences (CMTLE) (see Section 4.6.3). After analysis of these data, 16 students from the 25 *Focus Students* were selected as *High CMT Students* (see Section 4.7.3*)*. With regards to the CMTLE, only the students' total scores for the learning experiences were utilised in this selection process.

The 16 *High CMT Students* included students from all five kindergarten classrooms. The participants included four students from School A (25% of the sample from School A), five students from School B (32% of the sample from School B), and seven students from School C (43% of the sample from School C) (16 students in total). The age range for *High CMT Students* at the time of the interviews were between 5 years and 3 months to 6 years and 5 months of age (average age 6 years). Table 4.5 presents the number of *High CMT Students* from each school together with their gender and the class that they were enrolled.

Table 4.5
Demographics of High CMT Students (n=16)

School	Total number of students	Gender		Class	
		Male	Female	K1	K2
A	4	2	2	2	2
B	5	3	2	3	2
C	7	4	3	7	-
Total	16	9	7		

NB: School C had only one kindergarten class

In summary of the 16 *High CMT Students*, nine were male and seven were female. The number of students from each class ranged between 4 - 7 students. The data analysis pertaining to these 16 *High CMT Students* involved the analysis of the transcripts of their response to the CMTLE, and in particular how they exhibited their solutions for each Learning Experience.

In summary, in undertaking the research, instruments were included to narrow the student participants in the study. The process supported the goal of identifying the *High CMT Students* for further in-depth analysis. The CMT Observation Guide was implemented with *All Kindergarten Students* from three schools (n=161) and the *Classroom Observation Students* (n=41). The three quantitative instruments and the one Clinical interview were implemented with the *Focus Students* (n=25). A detailed in-depth data analysis process of the *High CMT Students* (n=16) followed.

4.4.3 Researcher

The researcher held a pivotal role in the delivery of the CMTLE. The researcher is an experienced early childhood and primary school educator and teacher. The researcher assumed the role of teacher/interviewer with the students. This ensured that the data collected from each student was reliable and consistent. Key to the interviews were the questions asked that supported students to demonstrate their CMT. The researcher was able to encourage students to elaborate on their responses while refraining from steering students towards a desired answer (Opper, 1977). Research by Warren et al., (2012) and Miller (2014) has adopted this approach with the inclusion of probing questions. Such questions are a common interviewing technique and encourage the participant to clarify and justify their responses.

4.5 DATA GATHERING

Data gathering occurred at three phases across the study. As described in the preceding sections, the data analysis at each point informed the selection of the *Classroom Observation Students*, the *Focus Group Students* and the *High CMT Students*. Figure 4.5 presents the relationship between the selection of different groups of kindergarten students and the three phases where various quantitative and the clinical interview were administered.

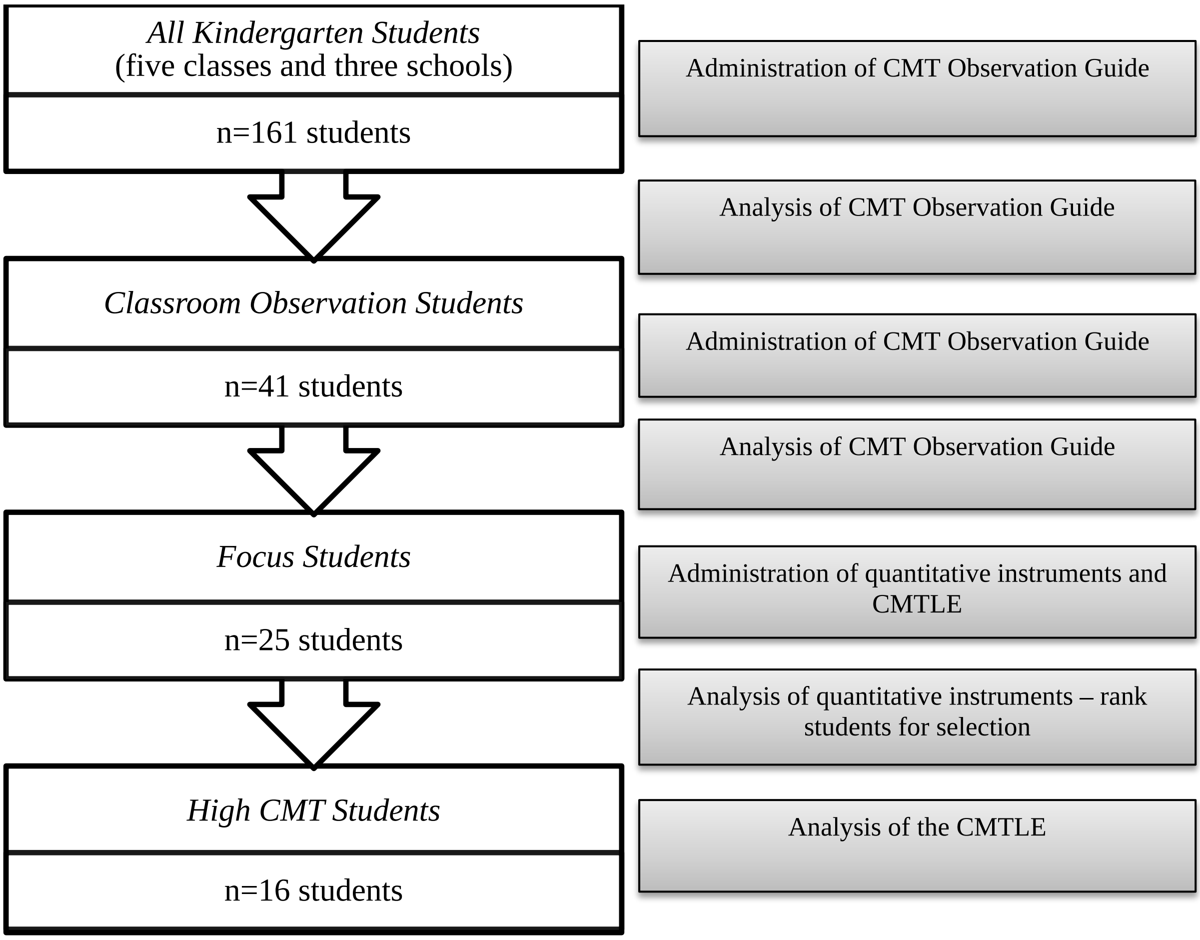


Figure 4.5. Summary of the Student Selection Process

4.5.1 Data Gathering Instruments

Five instruments were used to gather the data. These instruments were:

- One observation instrument:
    - (1): CMT Observation Guide (Section 4.6.1.1 and 4.6.1.2; Appendix A);

- Three quantitative instruments:
  - (2): Raven’s Progressive Matrices (Raven et al., 1998b);
  - (3): Slosson Intelligence Test (SIT-P) (Erford et al., 2008);
  - (4): Patterns and Structure Assessment (Mulligan et al., 2015)
- One Clinical interview:
  - (5) The Critical Mathematical Thinking Learning Experiences (CMTLE) (Table 4.7)

Each instrument is discussed in the sections that follow.

4.5.1.1 Development of the Qualitative Observation Instrument (CMT Observation Guide) – (Administered to All Kindergarten Students and Classroom Observation Students)

The CMT Observation Guide instrument consisted of 14 observation categories that were drawn from the literature (see Section 4.6.1). They included: mathematisation / grasping principles; estimating; evaluating; classifying; assuming / hypothesising; connections / noting relations among other relationships / transferring learning; argumentation/ offering opinions with reasons; number sense and mental computation; spatial and geometric reasoning; data and probability sense; making judgements with criteria / reasoning; inferring logically; self-correcting and questioning. The researcher listed each CMT category observation guide in a checklist format. One observation guide was used with each class that the researcher visited. The CMT Observation Guide, was used with *All Kindergarten Students* (Section 4.4.2.1) and the *Classroom Observation Students* (Section 4.4.2.2). Table 4.6 presents an example of a completed CMT Observation Guide for School C. All observation guides are provided in Appendix B.

Table 4.6
Example of a completed CMT Observation Guide for the Classroom Observation Students at School C (n=8)

CMT Observation Categories	Observation 1	Observation 2	Observation 3	Observation 4
Mathematisation / Grasping principles	S24 S21	S18	S19	S24 S19 S23
Estimating	S24 S20 S23 S25	S20 S21 S19		S24 S20
Evaluating				S24
Classifying			S24	
Assuming / Hypothesising	S18			
Connections / Noting relations among other relationships /transferring learning	S21 S19	S24 S23		
Argumentation/ Offering opinions with reasons				
Number sense and mental computation	S20 S21 S23	S18	S24 S23	S24 S21 S9 S25
Spatial and geometric reasoning				
Data and probability sense				
Making judgements with criteria / Reasoning		S21	S21	S21 S18
Inferring logically				S20
Self-correcting	S24 S18	S19		S20 S22
Questioning				S24

NB: S24 refers to Student 24.

4.5.2 Quantitative Instruments – (Administered to *Focus Students* (n=25))

Three quantitative instruments were used with the 25 *Focus Students*. These were Raven’s Progressive Matrices (Raven et al., 1998b), Slosson Intelligence Test (Erford et al., 2008), and the Patterns and Structure Assessment (Mulligan et al., 2015). These quantitative instruments are commonly used in education and clinical settings. Each instrument is described in the sections that follow.

It is noted in Raven’s Progressive Matrices manual that the measure should “form part of comprehensive evaluations which cover topics many of which cannot be adequately investigated with the range of tests currently available” (Raven, 1998a, section CPM53). Thus, the use of the Raven’s Progressive Matrices should not be a stand-alone measure of a student’s cognitive processes.

4.5.2.1 Quantitative Instrument – Raven's Progressive Matrices

The Raven's Progressive Matrices (Raven et al., 1998a) is a standardised measure that consists of three sections (including 12 questions in each section), resulting in a total of 36 questions. The questions require an individual to select a missing element from images provided (see Figures 4.6 and 4.7). The Raven's Progressive Matrices tests students' ability to complete a pattern without the use of language. The participant employs reasoning strategies for solving question items in each set. Figures 4.6 and 4.7 present two question items from the Raven's Progressive Matrices measure. For Figure 4.6 students are required to select 1 rectangle from the 5 options that completes the pattern. For Figure 4.7, students are required to select 1 shape from the 8 options that completes the pattern.

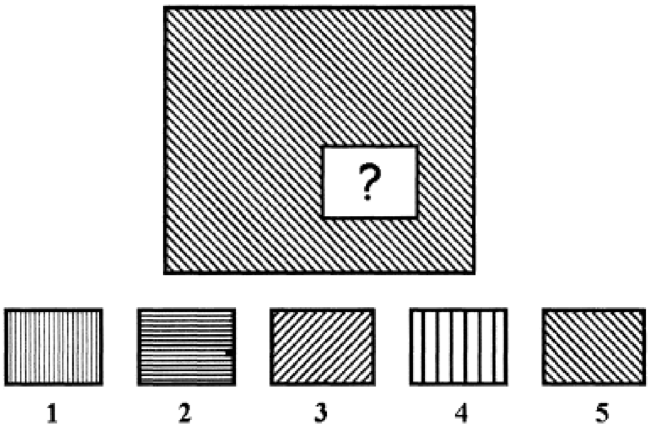


Figure 9.1 Pattern Completion Item Type.

Figure 4.6. Example 1A Question Item from the Raven's Progressive Matrices (Raven, 2003, p. 176)

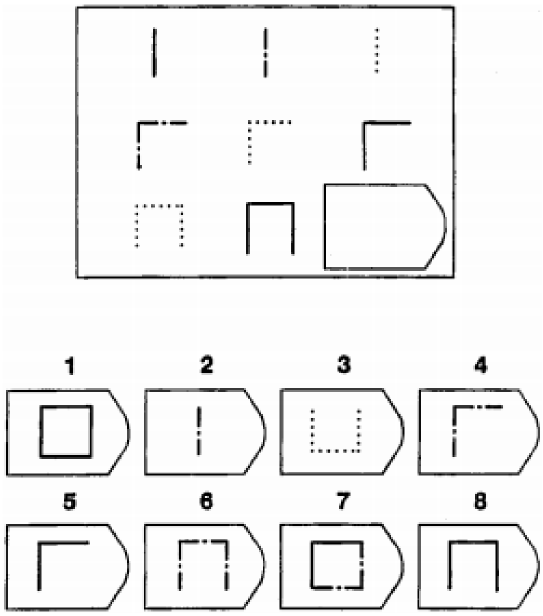


FIG. 1. Illustrative Progressive Matrices item. Respondents are asked to identify the piece required to complete the design from the options below. (The item shown here is not from the current range of tests.)

Figure 4.7. Example 1B Question Item from the Raven's Progressive Matrices (Raven, 2000, p. 2)

Data collection procedures for the Raven's Progressive Matrices are further discussed in Section 4.6.2.1.

4.5.2.2 Quantitative Instrument – Slosson Intelligence Test (SIT-P)

The Slosson Intelligence Test- Primary (Erford et al., 2008) was the second quantitative instrument administered individually to the *Focus Students* in the study. The test estimates participants' intellectual ability. The test consists of questions focusing on ascertaining six verbal cognitive domains:

- General information reflecting the learning of cultural knowledge (29 items)
- Comprehension reflecting one's knowledge of social behaviour (33 items)
- Quantitative reflecting the ability to do mental calculations (34 items)
- Similarities and differences reflecting one's ability in determining common attributes (30 items)
- Vocabulary reflecting the ability to use, understand and define words orally (33 items)
- Auditory memory reflecting one's ability to remember and repeat a random sequence of digits (28 items).

Examples from the test are not available for publication, hence cannot be provided in this present study. Data collection procedures for the Slosson Intelligence Test- Primary are further discussed in Section 4.6.2.2.

4.5.2.3 Quantitative Instrument - Patterns and Structure Assessment

The Patterns and Structure Assessment (PASA) (Mulligan et al., 2015) was the third quantitative instrument used in this study. The test focuses on students' awareness of "sequences, shape and alignment, equal spacing, structured counting and partitioning" (Mulligan & Mitchelmore, 2014, p. 4). The PASA provides data that align with students' general mathematical understandings (Mulligan & Mitchelmore, 2009). The PASA tasks include:

1. Partitioning lengths into thirds
2. Border patterns
3. Triangular arrays
4. Partitioning
5. Ten frames
6. Counting by threes: number track
7. Spatial pattern continuation
8. Square arrays
9. Structuring/using the hundred chart
10. Constructing analogue clock
11. Grid completion
12. Comparing triangles
13. Growing pattern continuation
14. Making a ruler
15. Constructing bar charts
16. Comparing capacities

(Mulligan & Mitchelmore, 2018, p. 20)

An example of a task from the PASA, focusing on border patterns, is displayed in Figure 4.8 below.

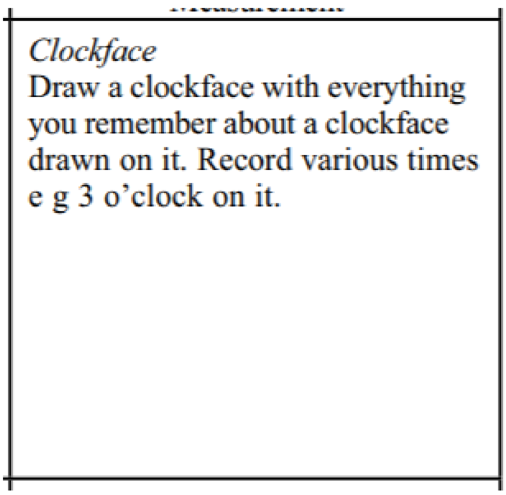

Clockface
Draw a clockface with everything you remember about a clockface drawn on it. Record various times e g 3 o'clock on it.

Figure 4.8. Example Question from the PASA (Mulligan et al., 2006, p. 378)

For this task students are required to draw components of an analogue clock. At the time of the interview the researcher noted the process a student takes. For example, the way a student spaces out the numbers 1 – 12; the inclusion and length of each clock hand. Data collection procedures for the PASA instrument are further discussed in Section 4.6.2.3.

4.5.3 Development of the Clinical Interview – Critical Mathematical Thinking Learning Experiences (CMTLE) (Administered to *Focus Group* students)

The Critical Mathematical Thinking Learning Experiences (CMTLE) was the final instrument to be delivered to the *16 Focus Students*. The construction of the eight learning experiences were underpinned by: (1) the CMT Conceptual Framework (see Section 3.4); (2) a synthesis of literature pertaining to learning experiences that enhance CMT (see Section 3.5.1); and, (3) the role of teacher questioning (see Section 3.5.3).

The CMTLE was designed as a clinical interview. The process of a clinical interview has roots in *Piaget's methode Clinique* (Hunting, 1997) that identifies children's cognitive ability within the social context of the learning. The method of clinical interviews was used by Piaget in 1952. Piaget included specific methods and strategies tailored to children so that he could observe their behaviour. In mathematics education, clinical research methods are common approaches to assessing young children's mathematical learning (Hunting, 1997; Ginsburg, 2005; Opper, 1977; Warren et al., 2012).

A clinical interview involves many methods and strategies. They include, intensively working one-on-one with an individual child, an ongoing conversation between the adult and the child and the inclusion of flexible questioning based on how the child is interacting (Ginsburg, 2005; Hunting & Doig, 1997; Posner & Gertzog, 1982). Clinical interview methods highlight language as a significant component. The dialogue between the participant and the researcher supported the researcher to clarify, ask questions and pose problems during the CMTLE interview. The design of questions posed to students is a key element of clinical interviews. According to Hunting and Doig (1997) the questioning should be open-ended in design, include freedom of choice in responses, maximise discussion opportunities and allow for the researcher and the student to reflect on the process.

Each learning experience (LE) was designed to include: (i) a pre-planned mathematics LE with an open-ended question posed to students; and, (ii) appropriate physical manipulatives (concrete material or a visual stimulus) to support the experience. Designing each LE to be an open-ended experience provided the opportunity for students to solve the problem in multiple ways. In addition to this, each LE covered a range of mathematical concepts.

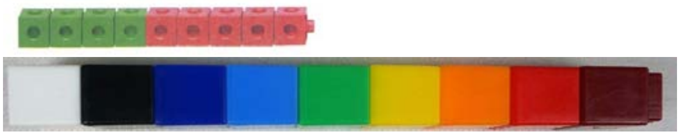
The purpose of the CMTLE was to identify the CMT young students bring to school. Therefore, as part of the process, prior to implementation of the CMTLE to young students, classroom teachers were interviewed to ensure these learning experiences (and relevant mathematical content) had not previously occurred in mathematics lessons.

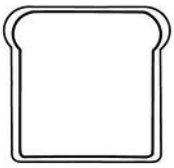
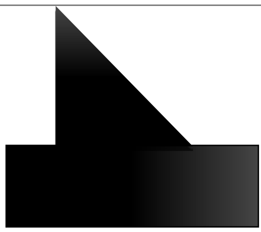
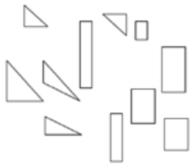
Table 4.7 presents the eight learning experiences including the physical manipulatives and examples of the teacher questions used when implementing the LE to the 25 *Focus Students*.

Table 4.7

A Brief Description of each Learning Experience

Learning Experience		Physical Manipulatives	Example teacher questions
LE1	**Framed photograph – finding the middle** This is a framed photograph of my son Joey. (*Hold up real framed photograph*.). I have a blank wall at home, and I would like to hang this photograph in the middle of that wall. Let's imagine this A3 piece of paper (hold up A3 paper) is the blank wall and this is a smaller picture frame (hold up small picture frame). Open-ended question: How can you find the exact place to hang up Joey's photograph?	 Photo frame (measures 20 cm x 15 cm)  Smaller laminated photo frame (measures 4 cm x 3 cm)	How can we check that's exactly the middle? What can we do to check? Is there a way that we can find exactly the middle of the entire wall?
LE2	**Mini bean bag – counting unseen objects** This is a mini bean bag (*show mini bean bag*). It is filled with many little beans like these (*show zip lock bag with some beans*). It's too tricky to count them one by one. Open-ended question: Can you think of another way to find out how many beans are in this mini bean bag?	 Bean bag filled with beans (used to hold a phone or device)  Beans (inside a zip lock bag)	I wonder if there's another way we can figure out how many are in there? But how else can we find Out how many are in there?
LE3	**Why is 3 + 3 the same as 4 + 2?** Open-ended question: Can you tell me why 3 + 3 is the same as 4 + 2? Why/why not?	3 + 3 4 + 4 4 + 2 6 + 2 Laminated cards	And how do you know that they are the same? Is there anything about the numbers that

	[If appropriate, change the numbers to 2-digit numbers. Ask students to provide two reasons why they are equal. Can you tell me another way you can work this out?]		you can think of?
LE4	**Towers – identifying which tower is taller** Here are two towers that I built earlier (*show readymade towers built with different sized blocks*). Open-ended question**:** Which tower do you think has more blocks? [If student answers incorrectly – teacher prompt: Let's count to see how many blocks make up each tower (*count to find that both have the same number of blocks*). How can you explain why this tower is taller than this tower?]	 2 Towers made from different interlocking cubes	How can you explain why this one is shorter than this one, then? Why is one taller than the other, if they've both got the same blocks?
LE5	**Teddy bears – real life number sentences** I had some bears in my pocket. Emily gave me some more. I counted and found I have 11 bears altogether. Open-ended question: How many did I start with and how many did Emily give me?	 11 teddy bear counters	How might I go about finding out how many I'll need?
LE6	**Cubby house – identifying number of tiles required** I have just finished building a cubby house for my children at home (*show picture of the cubby house*). I	 Image of a cubby house  Square tile (measures 5 cm x 5 cm)	So how do you know? How can I figure out how many tiles I need to put on

	would like to put these tiles down on the floor of the cubby house (*show square tile*). Open-ended question: How can I work out how many tiles I need?			the floor of the cubbyhouse?
LE7	**Sandwich – cutting and sharing equally** Open-ended question: How many different ways can you cut a sandwich in half? (*Provide several pieces of papers shaped as a sandwich*).	 Image of a slice of bread		How do you know that you get more or the same or less than me? Can you think of any other way to cut this bread so that it's shared equally?
LE8	**Shapes – replicating** Open-ended question: How many different ways, using the cut out shapes, can you re-create this shape? (*Provide students with the cut out shapes*).	 Image of the main shape	 Cut out shapes to replicate shape on the left	Does it look exactly the same?

See Section 4.7.3 for the reliability and validity for this instrument.

4.6 DATA GATHERING PROCEDURES

The mixed methods explanatory design (Creswell et al., 2003) included three phases of data gathering.

- Phase 1: Qualitative data gathering and analysis: CMT Observation Guide with *All Kindergarten Students* (n=161) and *Classroom Observation Students* (n=41)
- Phase 2: Quantitative data gathering and analysis: Raven's Progressive Matrices, Slosson Intelligence Test, Patterns and Structure Assessment, CMTLE with *Focus Students* (n=25)
- Phase 3: Qualitative data analysis: CMTLE with *High CMT Students* (n=16)

In addition to this, there were iterative phases of data gathering and analysis across the quantitative and qualitative data sets (see Figure 4.9). The purpose for this was to identify the *High CMT Students* whose data would be analysed to determine CMT capabilities. Figure 4.9 displays the iterative phases of the data gathering procedures of the study.

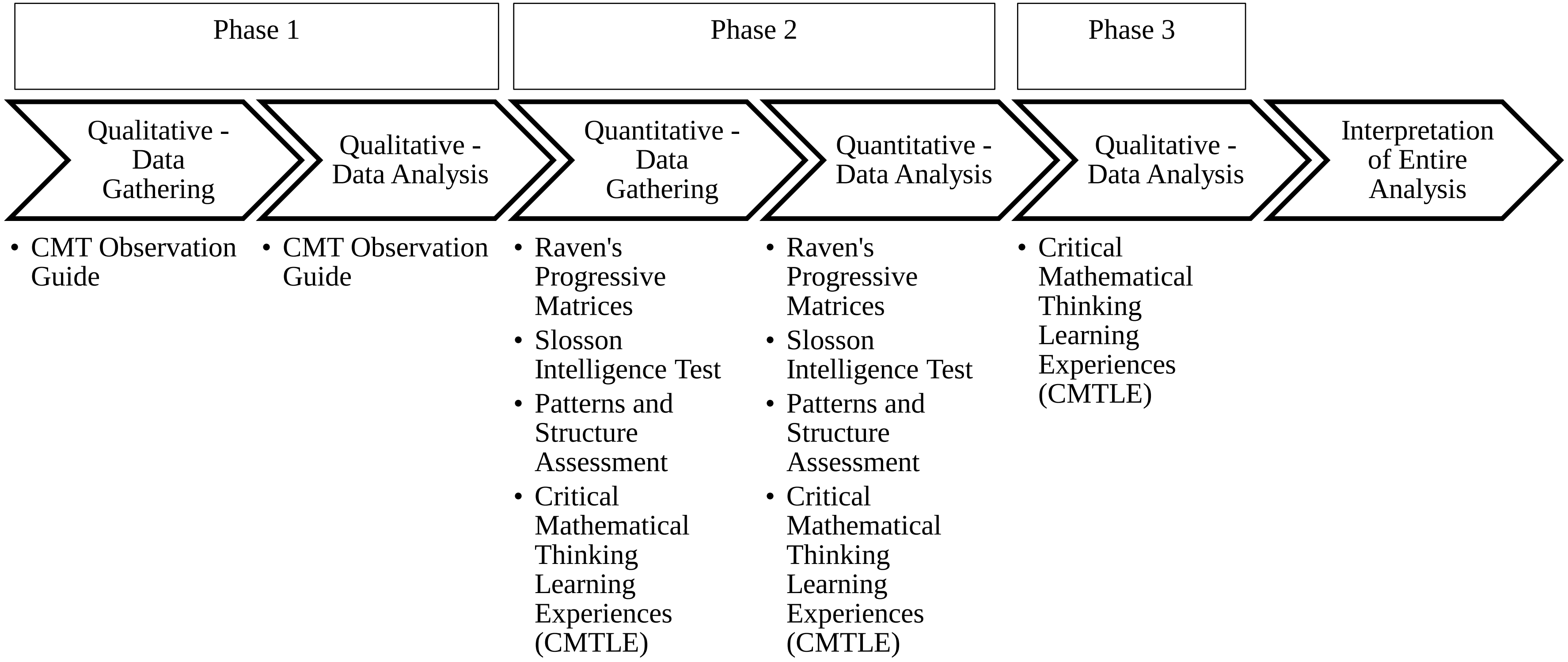


Figure 4.9. The Explanatory Design in this Study (adapted from Creswell et al., 2003)

Presented in Table 4.8 is the phase, purpose, participant and data gathering instruments used in this study to answer the research questions. The purpose of each data gathering instrument is presented to support the process of narrowing participants for in-depth analysis.

Table 4.8
Phase, Purpose, Participants, Data Gathering Instruments

Phase	Purpose	Participants	Data Gathering Instruments
1. Qualitative Data gathering	Identify CMT capabilities observed in all kindergarten classrooms Refine student selection to identify *Classroom Observation Students*	*All Kindergarten Students (n=161)*	CMT Observation Guide
	Refine student selection to identify *Focus Students*	*Classroom Observation Students (n=41)*	Re-administered CMT Observation Guide
2. Quantitative Data gathering	Determine if the quantitative measures provide different or complimentary data to that of the CMTLE	*Focus Students (n=25)*	Raven's Progressive Matrices Slosson Intelligence Test (SIT-P) Patterns and Structure Assessment
	Identify if young students can exhibit CMT Student selection for *High CMT Students* for in depth investigation	*Focus Students (n=25)*	The Critical Mathematical Thinking Learning Experiences (CMTLE)

NB: In Phase 3 no data were collected.

The next section describes the data gathering processes implemented in each phase.

4.6.1 Phase 1 - Qualitative: CMT Observation Guide with *All Kindergarten Students (n=161) and Classroom Observation Students (n=41)*

The CMT Observation Guide was administered in phase 1 of the data collection: first, with *All Kindergarten Students*, and then repeated with the

Classroom Observation Students. The phase 1 process occurred early in the school year. For School A and School C, this process occurred in April 2015. For School B, phase 1 occurred in May 2015. The Australian school year is organised into two semesters with a 6 week long break between December and January. Students begin the school year in the last week of January. The researcher began the data collection process in April 2015, 7 weeks after the Kindergarten students started their school year. Figure 4.10 displays the timeline for the data collection for each phase.

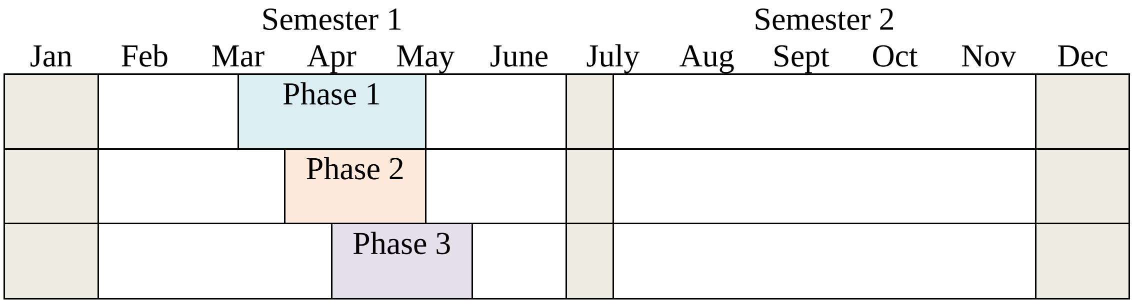


Figure 4.10. Timeline for the Data Collection Phases

4.6.1.1 Phase 1 - CMT Observations - All Kindergarten Students (n=161)

The researcher actively observed *All Kindergarten Students* (n=161) in the five kindergarten classes (during teacher led mathematics lessons) and used the CMT Observation Guide that emerged from the literature (Table 3.2 and Appendix A). Data collection included the researcher marking and noting individual students that exhibited CMT capabilities against the guide.

The observations occurred over four teacher-led mathematics lessons for each class (25 observations in total). The duration of each of the lessons was between 40 minutes to 1 hour. The lesson observations were not always conducted on consecutive days. This was mainly due to the school availability and the researcher's availability. The classroom visits by the researcher assisted in building rapport with the kindergarten students and the classroom teachers. The researcher did not teach during these visits. When the teacher-led sections of the lesson occurred, the researcher remained at the back of the classroom, recording the overview of each lesson, and identifying and recording students' interaction, engagement and responses. When students engaged in group work activities, the researcher roamed the room and spoke to groups or individual students. The researcher spoke briefly with the teacher after each mathematics lesson to discuss students that displayed CMT.

4.6.1.2 Phase 1 - CMT Observations – Classroom Observation Students (n=41)

The four whole class observations process resulted in the selection of 41 *Classroom Observation Students* (23 males and 18 females) across the five classes. The researcher attended one further classroom teacher-led mathematics lesson conducted at each school (five observations in total), which included the researcher exclusively observing the 41 *Classroom Observation Students* (School A = 18 students; School B = 15 students; School C = 8 students). The *Classroom Observation Students* engaged in the teacher-led mathematics lesson. During group work style activities, the researcher remained with these students and marked/noted individual student CMT capabilities (using the CMT Observation Guide).

A review of the CMT Observation Guide data for the *Classroom Observation Students*, including the five observations conducted with the whole and the additional observation conducted with the 41 *Classroom Observation students*, together with the classroom teacher, resulted in the identification of 25 students (*Focus Students*) who engaged in phase 2 of the data gathering process.

4.6.2 Phase 2 - Quantitative: Raven's Progressive Matrices, Slosson Intelligence Test , Patterns and Structure Assessment, Critical Mathematical Thinking Learning Experiences (CMTLE) with *Focus Students (n=25)*

The four quantitative instruments were administered individually and consecutively to each of the 25 *Focus Students*. The timing of the interviews was based on the structure of the school day, considering the start time, any scheduled breaks and the dismissal time. If there was not enough time to administer each instrument, the researcher waited until the students returned from a break to begin the next instrument. Students had short breaks between each test or when students requested them. The researcher utilised the break-out rooms that were adjacent to the Kindergarten classrooms to conduct the test. This was to provide a quiet space to work that was also close to the student's classroom teacher. The following presents the data gathering procedures for each test in the order they were administered.

The administration of the suite of three instruments took between 50 minutes to 65 minutes for each student. The researcher scheduled five full days per school to administer the instruments.

4.6.2.1 Phase 2 - Quantitative Instrument – Raven's Progressive Matrices with Focus Students (n=25)

The Raven's Progressive Matrices took 20 minutes to administer to each *Focus Student*. The researcher sat beside the student and the student pointed to the answer (out of 6 possible answers). The researcher read from a script and used the response sheet to record the student's response at the time of the interview. The Raven's Progressive Matrices comprises of 36 items, and students' responses to each item were scored as either correct (1) or incorrect (0) using an instrument specific answer sheet. Scoring procedures are discussed in Section 4.7.2.1.

4.6.2.2 Phase 2 - Quantitative Instrument – Slosson Intelligence Test – SIT-P with Focus Students (n=25)

Administration of the Slosson Intelligence Test took between 10 – 20 minutes for each student. As with the first instrument, the researcher sat beside the student. This instrument requires the researcher to ask questions to the student. The Slosson Intelligence Test has different starting points based on the participant's age and the calculation of the individual's basal item (question number at which the student answers 10 correct consecutive questions). The test is verbally administered, and the researcher reads each question to the participant and marks either correct (1) or incorrect (0) for each response on the instrument specific response sheet. The test ceases when the participant incorrectly answers 10 consecutive questions. The test provides results that identify:

- Chronological age + raw score = total standard score
- Mean age equivalent
- Percentile ranking

The analysis procedures for these data are discussed in Section 4.7.2.2.

4.6.2.3 Phase 2 - Quantitative Instrument – The Patterns and Structure Assessment with Focus Students (n=25)

The administration of the PASA took between 20 – 25 minutes per student. The researcher posed questions and asked the students to answer the question by providing a drawing, written representation or a representation with physical manipulatives. The researcher took photographs of their response and collected the work samples produced. During the interview, the researcher used the Patterns and Structure response booklet and the 1 – 5 categorisation guides to write notes beside each category based on the student's response.

4.6.2.4 Phase 2 – Clinical Interview – The Critical Mathematical Thinking Learning Experiences (CMTLE) with Focus Students (n=25)

The administration of the CMTLE took between 25 – 35 minutes for each student. The learning experiences were administered using a one-on-one clinical interview.

The CMTLE reflected the aspects associated with clinical interviews (see Section 4.5.3). The CMTLE was administered in a context that was familiar to the student. Included in the CMTLE were flexible open-ended questions during mathematics learning experiences. In addition, the researcher was conscious of avoiding leading the student in a particular direction, however, included ongoing dialogue to support the student to clarify responses.

All CMTLE interviews were video recorded to capture both the student responses and researcher's questioning. One video camera was set up on a tripod focusing on the student and researcher. These videos were downloaded at the conclusion of the student interviews and were later transcribed for data analysis. In addition to this, the researcher also took notes and collected student work samples produced during the CMTLE.

4.7 ANALYSIS OF DATA

The CMT Observation Guide, suite of quantitative instruments and the CMTLE required three phases of data analysis. First, the CMT Observation Guide analysis occurred at the conclusion of each lesson (phase one), in consultation with the classroom teachers. Next, the suite of quantitative instruments analysis (phase two), including a quantitative analysis of the results of the CMTLE took place in order to identify the *High CMT Students* for phase three of the study. Finally, the qualitative analysis of the CMTLE for the *High CMT Students* (phase three) included the researcher's designed approach discussed in Section 4.5.3.

4.7.1 Phase 1 - Qualitative Analysis: CMT Observation Guide with All Kindergarten Students (n=161) and Classroom Observation Students (n=41)

The process of summarising data into understandable information (DeWalt & DeWalt, 2011) was used during the CMT Observation Guide analysis for phase one. In this research, data were gathered for analysis from *All Kindergarten Students* (n=161) using the CMT Observation Guide (see Section 4.6.1). The researcher used thematic coding to add student names next to the CMT observation categories and found the total of the frequencies. This process was repeated after each classroom observation and in consultation with the classroom teacher. At the conclusion of the observations, all observational guides were collated for analysis. The researcher analysed the number of occurrences of CMT that a student demonstrated within a lesson and across lessons. The students identified formed the *Classroom Observation Students* (n=41) group.

The research repeated the CMT Observation Guide process with the *Classroom Observation Students* (n=41). This included, using the CMT Observation Guide, identifying CMT, using thematic coding and finding the total of the frequencies. At the conclusion of the observations, the researcher followed the same procedure (that is, all observational guides were collated for analysis by number of occurrences of CMT).

Following this, the information gathered and summarised was briefly discussed with the classroom teacher to identify *Focus Students (n=25)* for the quantitative data collection process.

4.7.2 Phase 2 - Quantitative Analysis: Raven's Progressive Matrices, Slosson Intelligence Test, Patterns and Structure Assessment, CMTLE with *Focus Students (n=25)*

The three quantitative instruments were analysed for each *Focus Student* (n=25) to develop datasets. The dataset data were organised in a spreadsheet and included instrument-based data to represent each student (Connolly, 2007).

4.7.2.1 Phase 2 - Analysis: Raven's Progressive Matrices with Focus Students (n=25)

Analysis of the Raven's Progressive Matrices commenced with finding the total of the individual's score out of 36 noted on each individual student answer sheet (by entering correct (1) or incorrect (0) in the answer sheet).

The total score and the student's age were then used to identify the percentile ranking that is aligned with the normative data in the Raven's Progressive Matrices Manual (Raven et al., 1998b) and then this score was aligned with a grading classification for intelligence as evident in Table 4.9 (Raven et al., 1998b). The Raven's Progressive Matrices classifies the respondents into five grades, that is, Grade I (intellectually superior), Grade II (above average), Grade III (intellectually average), Grade IV (below average), and Grade V (intellectually impaired) (Raven et al., 1998b, section CMP50).

Table 4.9
Raven's Progressive Matrices Percentile grading (Raven et al., 1998b)

	Grade I	Grade II	Grade III	Grade IV	Grade V
Percentile range	At or above 95th percentile	At or above 75th percentile	Between 25th – 75th percentile	At or below 25th percentile	At or below 5th percentile
Grading classification	Intellectually superior	Above average	Intellectual average	Below average	Intellectually impaired

The lower the grade classification, the better a student scored on the test. For example, if a student's percentile is 80 %, they are considered Grade II, at or above the 75th percentile and their grading classification is above average.

The Raven's Progressive Matrices raw score and the percentile range were entered in an Excel spreadsheet with all 25 *Focus Student* data. The spreadsheet was used to analyse the data by grouping students into the grades and conducting student comparisons between data sets and student performance (see Figure 4.11). The data analysis allowed the researcher to consider students that performed in the higher grades on this assessment. This included using additional instruments assisted to build a selection process of students for the qualitative component of this study.

4.7.2.2 Phase 2 - Analysis: Slosson Intelligence Test with Focus Students (n=25)

The data collected from the Slosson Intelligence Test were recorded on an instrument specific answer sheet. In order to make meaningful interpretations of the data, the total raw score, together with the participant's chronological age were used to find the percentile rank and the descriptor (range) of each rank (Erford et al., 2008). Table 4.10 displays the percentile ranks and ranges.

Table 4.10
Slosson Intelligence Test Percentile Rank and Grade Raven's Progressive Matrices Percentile grading (Erford et al., 2008)

Percentile Rank	Range
98 +	Very Superior
91 – 97	Superior
75 – 90	High Average
25 – 74	Average
9 – 24	Low Average
2 – 8	Borderline
.13 – 2	Mildly Deficient
.01 - .12	Moderately Deficient
<.01	Severely or Profoundly Deficient

The *Focus Student* Slosson Intelligence Test data were analysed to identify the number of students in each Slosson Intelligence Test range and percentile rank. The student performance data were also analysed by school and class to determine if student achievement varied within schools and classes.

The Slosson Intelligence Test inter range and the descriptor were later entered in a spreadsheet for all 25 *Focus Students*, together with their Raven's Progressive Matrices score. The Excel spreadsheet was used to analyse the data by grouping students into the range and conducting student comparisons between the groups of students.

4.7.2.3 Phase 2 - Analysis: Patterns and Structure Assessment (PASA) with Focus Students (n=25)

The data collected from the PASA were recorded on the PASA response booklet. The response booklet included the PASA 1 – 5 response categories that aligned with each of the 16 tasks. The response categories were assessed during each student's engagement of the PASA tasks. The researcher reviewed the student responses again, including student work samples, after administrating the PASA and aligned a 1 – 5 response. The guiding examples in the PASA Teacher Guide were used to verify the response category for each student's task. The 1 - 5 response categories are presented in Table 4.11.

Table 4.11
PASA 1 – 5 Response Categories (Elia et al., 2018, p.19)

Rank	Response Category	Brief Description	
5	Advanced structural	-	Accurate, efficient and generalised use of the underlying structure
4	Structural	-	Correct but limited use of the underlying structure
3	Partial structural	-	Shows most of the relevant features of the pattern, but the underlying structural organisation is inaccurate or incomplete
2	Emergent	-	Shows some relevant features of the pattern, but these are not organised in such a way as to reflect the underlying structure
1	Prestructural	-	If any response is given, it shows only limited and disconnected features of the pattern.

An instrument specific Excel proforma (worksheet) was then used to calculate the scale score, error and Awareness of Mathematical Pattern and Structure (AMPS) level. The process included the researcher entering the student's name, age and gender into an Excel worksheet developed by ACER and Mulligan et al. (2015). The researcher transferred the 1 – 5 response categories rank for each task into the worksheet. The worksheet automatically calculated the scale score, error and AMPS level for each student.

The AMPS scale scores were calculated using a pre-determined formula on the Excel worksheet. For example, when all 16 PASA 1 – 5 rank responses (see Table 4.12) are entered into the worksheet, an overall AMPS Rank is provided between 1 – 4. The AMPS were identified and described on a 4 level scale, with the lowest being 1 and the highest level 4. Table 4.12 provides an overview and description of the 4 level AMPS scales. In the particular table, the examples are drawn from different contexts, identification of common structures and an ability to look for patterns.

Table 4.12
PASA AMPS Scale Scores (Elia et al., 2018, p.21)

Rank	AMPS Scale Description	Example
1	Struggle to recognise simple patterns	Copy block patterns by matching the blocks one by one
2	Recognise simple patterns	Can identify unit of repeat and show some awareness of relations with other patterns
3	Aware of fundamental structures	Aware of underlying structure of alignment in grids and arrays
4	Aware of the generality of fundamental structures and can extend these	Extend a growing pattern

The PASA AMPS data were placed in a table in alignment with the AMPS scale description (see Section 5.3.2.3). The same data were used to identify student PASA AMPS performance according to school and class.

The PASA score and the PASA AMPS data were then entered into the Excel spreadsheet with all 25 *Focus Students*, together with the Slosson Intelligence Test and the Raven's Progressive Matrices data.

4.7.2.4 Phase 2 - Analysis: Quantitative Component of the Interview – The Critical Mathematical Thinking Learning Experiences (CMTLE) with Focus Students (n=25)

The data collection of the one-on-one clinical interview using the CMTLE included both a quantitative and qualitative component. The quantitative data gathering occurred in phase two and the qualitative data gathering occurred in phase three (see Section 4.7.3).

The data analysis process for the results CMTLE interview involved identifying elements of the data to extend, expand and provide themes and relating the themes with the CMT Conceptual Framework identified in the literature (see Section 3.3.3) (Wolcott, 1994). The data analysis used the transcripts of the video-recordings of each *Focus Student*, which occurred in phase two of the data collection process. All video recordings for the interviews were transcribed using de-identified codes for each student (for example, S2 = Student 2).

At the completion of the data collection process, the video recordings of each *Focus Student* were viewed and dissected. The field notes were gathered. The data collected were discussed with experienced mathematics educators to determine the analysis tool for use.

The CMTLE Analysis Tool was developed to analyse the results of the CMTLE interview data (component 2). The CMTLE scoring tool consisted of a six-point measure identifying student application of a strategy and CMT for each of the eight learning experiences. Table 4.13 provides an overview of the structure of the CMTLE scoring tool for each learning experience.

Table 4.13
CMTLE Scoring tool - 6-Point Measure

CMT Scoring Tool for each Learning Experience	
Strategy	CMT
5. Rigorous Method	5. High Levels
4. Thorough Method	4. Thorough Levels
3. Sound Method	3. Sound Levels
2. Emergent Method	2. Emergent Levels
1. Limited Method	1. Limited Levels
0. Inappropriate Method	0. No evidence

Descriptors for each numerical scale were developed for each learning experience. Table 4.14 provides the descriptor used to analyse learning experience one.

Table 4.14
CMT Scoring tool Measure and Descriptors for Learning Experience One

CMTLE Scoring Tool for Learning Experience 1			
Strategy	Strategy Descriptor	CMT	CMT Descriptor
Means of measurement or estimation		Estimating, evaluating, assumption, noting relationships, visual relationships, argumentation / reasoning	
5.Rigorous Method	High mathematical knowledge used to find middle without using a ruler or folding the paper (for example, diagonal lines to find the centre/middle)	5.High Levels	High levels of justification for placement and supports placement with mathematical language, knowledge and/or skill Uses understanding from other mathematical concepts to support placement
4.Thorough Method	Thorough mathematical knowledge to find the middle using informal measurement (for example, horizontal or vertical lines)	4.Thorough Levels	Provides justification for placement Mainly focuses on visual relationships
3.Sound Method	Sound mathematical knowledge used to find the middle using informal measurement (for example, hand/finger span)	3.Sound Levels	Provides justification of placement Mainly focuses on estimating
2.Emergent Method	Emergent mathematical knowledge including informal measurement however does not accurately find the middle	2.Emergent Levels	Provides limited justification of placement Mainly focuses on assumption
1.Limited Method	Limited mathematical knowledge used by placing the photograph close to middle with no supporting mathematical thinking	1.Limited Levels	Provides incorrect justification of placement
0.Inappropriate Method	Places the photograph inappropriately	0.No evidence	No evidence

The researcher re-viewed each focus student CMTLE interview and analysed student responses against the CMTLE 6-point measure and descriptors for each of the eight learning experiences. In order to validate the process, experienced mathematics educators viewed and analysed the *Focus Student* data using the CMTLE scoring tool. This analysis was cross referenced with the researcher's version. Adjustments were made.

The CMTLE scoring tool was used to provide a score out of 80 for each student (that is: 5 points for CMT Strategy + 5 points for CMT Descriptor = 10 points per learning experience (x8)). The results for the CMT score consisted of 2 dimensions for each task, namely, a score out of 5 for the strategy used and a score out of 5 for the level of CMT exhibited by the student. Thus, the test results consisted of 16 items (two for each learning experience), and each student was allocated a mark for each of these items.

The score out of 80 allowed for the *Focus Student* CMT data to be ranked from highest to lowest CMT score. The trend in the students' total score for the CMT were reviewed. In order to examine the trend in students' score across the eight learning experiences, the mean average student score for each learning experience was calculated (out of 10). A comparison of student data in schools and classes occurred. Analysis of how the students performed was reviewed in graphical form. The spread of students results were clustered. The average score of all *Focus Student* data against each of the eight learning experiences were analysed to provide the order of difficulty.

For internal consistency, that is, to ascertain how closely related a set of items were as a group, a Cronbach alpha was performed. The results of the Cronbach alpha are presented in Table 4.15.

Table 4.15
Results of the Cronbach Alpha for the CMT Scale (Reliability Statistics)

Cronbach's Alpha	Cronbach's Alpha Based on Standardised Items	N of Items
.910	.915	16

The Cronbach's alpha was 0.910 which is considered to be very high and significant $F(24) < 0.001$. Removing any items from the scale did not increase the instrument's reliability.

Hence this evidences that the CMT scale is uni-modal and can be considered the one measure of high levels of CMT. The internal reliability of the CMT scale was high. In addition, most items appeared worthy of retention, as the deletion of any of the items did not result in an increase in Cronbach alpha.

Validity of the CMT scale was established as all items have been examined for content validity and member checking by three experienced mathematics educators.

Thus, the resultant CMT scale was considered to a be a reliable and valid measure of young students' Critical Mathematical Thinking.

All of the data were analysed at the conclusion of the data analysis phase using the statistical Package SPSS. This analysis focussed on identifying trends within each quantitative instrument and between the quantitative instruments.

4.7.2.5 Phase 2 - Analysis: All Quantitative Instruments/Clinical Interview

The data sets obtained from each quantitative instrument/interview were entered into a spreadsheet. Prior to all data collection and analysis each child was allocated a student code (for example, S2). This was to ensure that all data was de-identified at each stage of the data collection. The data listed in the spreadsheet included the following student data: student number (or code), age at the time of the test, school and class. The Raven's Progressive Matrices data was listed: raw score out of 80 and the Raven's percentile ranking. The Slosson Intelligence data included: Slosson inter range and the descriptor. Data from the PASA included the PASA score and the PASA AMPS. The raw numerical score for the CMT was entered into the spreadsheet. Figure 4.11 provides a screenshot of student one's data entered into the Excel spreadsheet.

				RAVEN'S PROGRESSIVE MATRICES		SLOSSON INTELLIGENCE TEST		PATTERNS AND STRUCTURE ASSESSMENT	
Stud. No	Age at test	School	Class	RAW SCORE /80	RAVEN %	SlossonInter Range	Descriptor	PASA score	PASA AMPS
S1	5y8m	B	K1	58	50	91 - 97	Superior	113.9	3

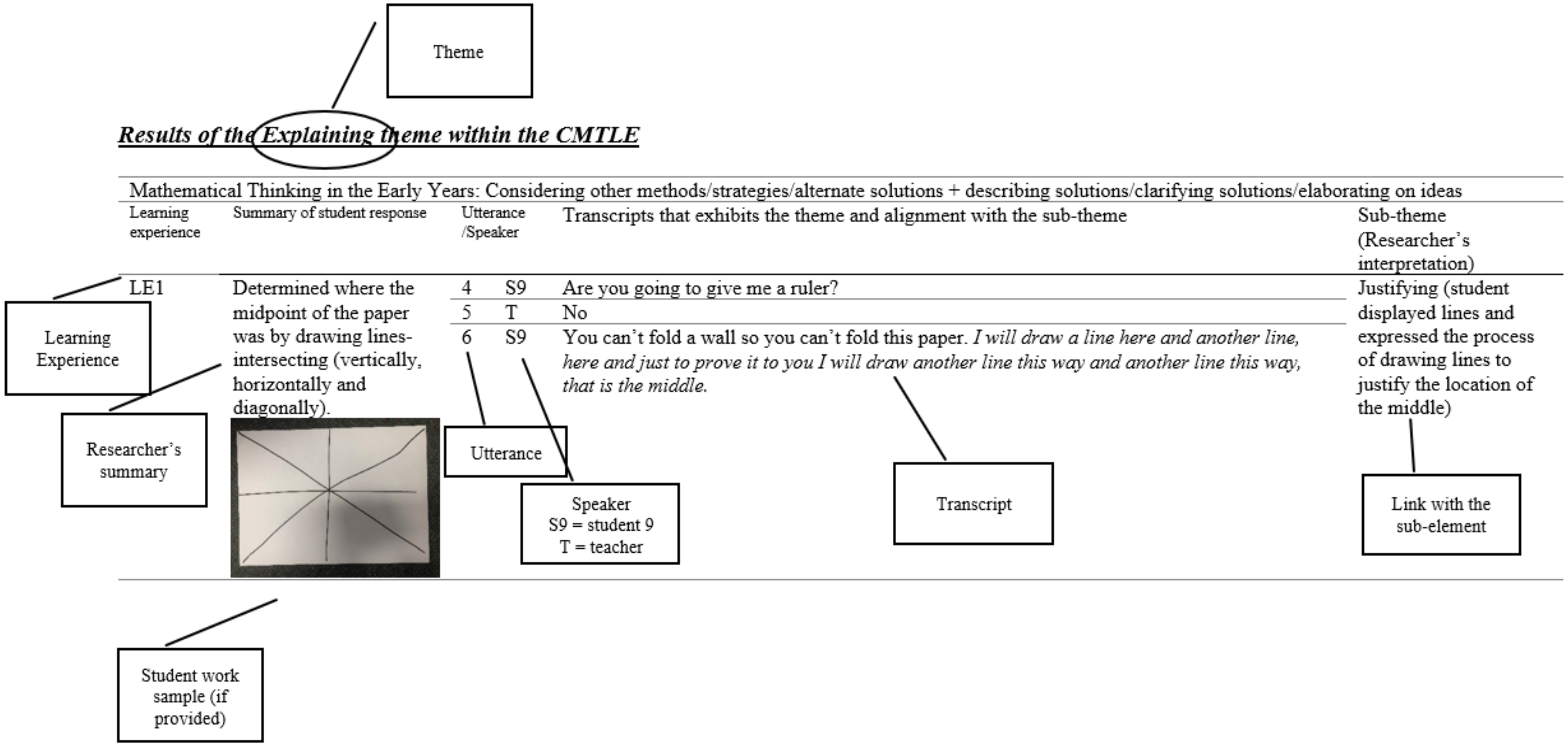
Figure 4.11. Student One's Data in the Excel Spreadsheet

The scores for three quantitative instruments and CMT were used to select the *High CMT Students (n=16).* In order to ascertain relationships between the CMT scale scores and teach of the three quantitative instruments scale scores (adjusting for the age of the student) partial correlations were performed. The level of significance of the correlation between the two scales is reported in Chapter 5.

4.7.3 Phase 3 – Qualitative Analysis: Critical Mathematical Thinking Learning Experiences (CMTLE) with High CMT Students (n=16)

At the conclusion of the quantitative analyses, the *High CMT Students (n=16)* student videos were transcribed. The researcher conducted iterative cycles of analysis on the interview transcripts. The first analysis cycle focused on students displaying CMT capabilities. The second focused on teacher questioning.

To analyse CMT capabilities, the researcher identified components within each *High CMT Student* transcript that aligned with the CMT Conceptual Framework (see Section 4.2.2.1). The research identified the student's and researcher's utterance in alignment with the transcript. The CMT Conceptual Framework Theme and sub-themes were identified. The researcher identified frequencies of themes and sub-themes and provided a summary of the student responses. If a work sample was available or produced by the student, the research included the work sample in the table and was a part of the analysis process. Figure 4.12 provides an example of the coding of one student's response in a learning experience within a transcript. All *High CMT Student* responses can be viewed in Appendix C.



Results of the Explaining theme within the CMTLE

Mathematical Thinking in the Early Years: Considering other methods/strategies/alternate solutions + describing solutions/clarifying solutions/elaborating on ideas					
Learning experience	Summary of student response	Utterance /Speaker		Transcripts that exhibits the theme and alignment with the sub-theme	Sub-theme (Researcher's interpretation)
LE1	Determined where the midpoint of the paper was by drawing lines- intersecting (vertically, horizontally and diagonally).	4	S9	Are you going to give me a ruler?	Justifying (student displayed lines and expressed the process of drawing lines to justify the location of the middle)
		5	T	No	
		6	S9	You can't fold a wall so you can't fold this paper. *I will draw a line here and another line, here and just to prove it to you I will draw another line this way and another line this way, that is the middle.*	

Figure 4.12. Example of Coding of Transcripts using the Themes and Sub-themes of the CMT Conceptual Framework

The teacher questions were extracted from the 16 student videos' transcriptions for each CMTLE. The researcher listed the questions asked into five groups. These groups aligned with five themes of the CMT Framework used to categorise the student data. They include: (1) the student interpreting the task - Learning Experience Questions (LEQUE); (2) the student analysing the task; (3) the student evaluating the task; (4) the student explaining the answer; and (5) the student showing or creating. First, the researcher grouped each question according to when it occurred within the clinical interview. For example, was the student interpreting, analysing, evaluating, explaining or creating. Second each question was further classified as a guiding, factual or probing question. Finally, each question was further coded deductively using the CMT Conceptual Framework (see Section 4.2.2.1). For example, the question was classified as clarifying if the teacher was endeavouring to clarify the student's response. An in-depth examination of the codes that emerged from the teacher questions were analysed. Figure 4.13 provides an example of the coding of teacher questioning for CMTLE 1. All teacher questioning results can be viewed in Appendix D.

Learning Experience

Results of teacher questioning within the CMTLE 1

Classification of Teacher Questions

Alignment with Critical Mathematical Thinking	Teacher Questions – CMTLE 1	LEQUE	TQ Factual	TQ Probing	TQ Guiding
			Maths related question	Clarifying Decoding Noting relationships Offering opinions with reasons	Assessing Understanding Self-regulating
Stage 1 – the student interpreting the task	What can I do to find the middle of the wall?				
	How could we find the middle of the wall?				
	How might we find the middle?				
	Where can I place this so that it's in the middle and how can I make sure?				
	How can I find the middle?				
	How can I find the middle?				
	How can I find the middle?				
	How can I find out where the middle of the wall and how do I know that that is the middle?				
	How can I find the middle?				
	How can I find how to cut this exactly in half? So, we've got two equal parts?				
	Can you think of a way to find the middle?				
	How do you think we can find the middle?				
Group 2 – the student analysing the task	How do I measure it?				
	How do I know it's the middle that way?				
	How do you know that's the middle?				
	So, tell me why this isn't the middle.				

Teacher questions posed

Stage of the learning experience

Codes based on CMT Conceptual Framework

Figure 4.13. Example of Coding of Teacher Questions for CMTLE 1

4.8 TRUSTWORTHINESS

4.8.1 Trustworthiness with the Quantitative Data

As indicated in Section 4.5.1, the quantitative data included test instruments that are well established for education and clinical settings. The first instrument is the Raven's Progressive Matrices (Raven et al., 1998b) which measures an individual's level of intellectual development. The administration of the Raven's Progressive Matrices follows a script, the use of a test booklet and an individual answer sheet. The second measure is the Slosson Intelligence Test (SIT-P) (Erford et al., 2008) which is a screening measure for developmental early childhood skills. The administration and scoring follow a structured process (see Section 4.5.2.2) to determine the establishment of the basal and the ceiling for each student participant. Construct validity is established in the Raven's Progressive Matrices and the Slosson Intelligence test quantitative data measures as both are commonly used in education situations.

The third quantitative measure is the Patterns and Structure Assessment (Mulligan et al., 2015). This instrument allows for the student score to be aligned with Awareness of Mathematical Patterns and Structure (AMPS) levels. The AMPS levels were validated and correlated by the researchers against another mathematics standardised measure, that is, the Progressive Achievements Assessment (PAT) Maths Test (Mulligan et al., 2015).

The reliability of the three quantitative measures was considered in this study. The administration of each test was conducted in the same test conditions. This included: individual student administration, teacher/researcher sitting next to the student, the use of the same administration instructions for all student participants, and the analysis of the data using the same measures. The tests were administered in the same order (as listed in Section 4.5.2) for each student participant. The duration differed for each student; however, each student completed all three quantitative measures within 50 – 65 minutes.

4.8.2 Trustworthiness with the Qualitative Data

The trustworthiness of the qualitative component in this study focuses on "credibility, dependability, conformability, transferability and authenticity" (Elo et al., 2014, p. 2).

Credibility refers to establishing if the research findings display plausible information that was interpreted from the participant's original data. Further, credibility refers to the correct data interpretation from the participant's original work (Korstjens & Moser, 2018; Lincoln & Guba, 1986). During the use of the CMT Observation Guide and CMTLE interview, credibility was maintained with the following actions: researcher engagement in the teacher-led mathematics lessons, use of the CMT Conceptual Framework to guide the observations (see Section 4.2.2.1) and the review of data using member checking (Lincoln & Guba, 1985).

Dependability refers to the constancy of data within different conditions (Elo et al., 2014). Confirmability considers processes of data interpretation to ensure the research path is transparent (Korstjens & Moser, 2018). Dependability and confirmability were achieved by having the CMT Observation Guide data and CMT scale scores data reviewed by another researcher (discipline expert/member checking) to ensure similar interpretations and conclusions were made (Lincoln & Guba, 1985).

Transferability describes the way in which the work can extend from one study and be applied to other situations (Merriam & Tisdell 2015). To ensure transferability, the same CMT Observation Guide and the CMTLE were used for each participant from the three schools and five kindergarten classrooms. This allowed for the researcher to determine the transferability of the findings (Creswell, 2013). The approach adopted in the research includes a detailed analysis of interviews and student work samples in alignment with the CMT Conceptual Framework.

Authenticity refers to accurate analysis of the data (Elo et al., 2014). There were multiple instances in the research process to ensure the authenticity principle was enacted. The CMT Observation Guide development and data gathering considered various stakeholder input. The stakeholders included: the researcher (experienced early years mathematics teacher), the supervisory team (expert discipline researchers) and the classroom teacher. The CMTLE interview development and data analysis included the researcher and the supervisory team contesting and critiquing the analysis process to ensure there was limited subjectivity.

The administration phase of the qualitative data instruments aligned with the CMT Conceptual Framework which emerged from the literature (see Section 4.2.2.1). Thus, the interpretation of the data included participant voice and work samples which allowed for clear alignment with themes and sub-themes identified in the conceptual framework. The researcher’s viewpoints did not interfere in the interpretation of the data. Further, in this study, the viewpoints are grounded in the literature sources that have supported the establishment of the CMT Conceptual Framework.

4.9 ETHICAL CONSIDERATIONS

The collection of data did not commence until ethical procedures were completed. In adhering to ethical guidelines set out by the Australian Catholic University (ACU) and in conjunction with the Australian National Statement on Ethical Conduct in Human Research (Silverman, 2009), clearance to conduct this research was required before data collection commenced. This process allowed for ethical research clearance to be obtained by Sydney Catholic Schools (SCS) (formally known as Catholic Education Office (CEO)) (Appendix E and F). Risks were assessed and the management of these risks took place to ensure participants and data adhere to the ethical guidelines. The ACU Information Letters (Appendix G and H), CEO Guidelines to Conduct Research (Appendix F) and the CEO Risk Assessment Form (Appendix F) provided a statement about the possible risks associated with this research.

The CEO suggested schools and principals that might be interested in engaging in this study. The researcher communicated with interested school principals in the first instance and invited the principal to engage in the study without pressure of coercion, with an opportunity to withdraw at any time. This process involved an initial letter to the Principal (Appendix G). The next step included gaining consent from the teacher participants. The teacher participants were requested to complete a Consent Form (Appendix H) after reading the Teacher Participant Information Letter

(Appendix H). Details about the project; associated risks; their involvement; time allocation; benefits; opportunity to withdraw; results; questions and complaints was made available to the teachers. The same project detail information was provided to the parents of the students involved in this research. The researcher's email contacts were made available to teachers and parents if any questions or issues were to arise.

The main participants in this research were children aged between 5 years and 1 month to 6 years and 8 months of age. A Parent Information Sheet describing the research was sent to each parent of a selected child (Appendix I). A Parent Consent Form was provided to the parent or carer of the child (Appendix I). Assent for the child participant was conducted by a yes/no question at the beginning of the delivery of the CMTLE (McNeilly et al., 2020; Oates, 2020). The researcher circled yes or no on the recording sheet. The data collected was concealed and each student was issued with a code to ensure anonymity (for example,S16 Student 16).

In this study, video recordings were used during the CMTLE data collection process with the children participants. The use of recording devices were clearly stated in the information documentation sent to all stakeholders. The data collection, both in its raw state and after analysis was stored as a hard copy and an electronic copy. The hard copy is stored in a locked filing system at the researcher's work office and the electronic copy on the researcher's password secured laptop computer. The researcher's work office is secured with a lock.

Issues of confidentiality and anonymity are significant in this research. The completion of a consent form was required by all persons involved in the research. The reasons for a consent form include:

- To strengthen confidentiality and anonymity in the process;
- As this research includes children as participants; parental consent and verbal consent from the child is required;
- As teachers will also be involved in the process; individual teacher consent is required.

Towards the end of the project, opportunity for discussion with all stakeholders was made available to ensure a two-way information exchange opportunity was provided (Alderson & Morrow, 2004).

4.10 CHAPTER REVIEW

The purpose of this chapter was to outline and provide justification of the research design adopted. The chapter commenced with an outline of the theoretical perspective and the research design. As this study was to investigate CMT in young students and teaching actions/questions that help these young students exhibit their CMT, the interpretative paradigm was used. An explanation of the case study methodology using a mixed methods explanatory approach followed. The participants, *All Kindergarten Students, Classroom Observation Students, Focus Students* and *High CMT Students* were explained and justified. The involvement of each participant was discussed in the chapter in alignment with the research methodology used in the study. The data gathering procedures for both the qualitative (CMT Observation Guide with *All Kindergarten Students* and *Classroom Observation Students*; CMTLE with *Focus Students)* and quantitative instruments (Raven's Progressive Matrices, Slosson Intelligence Test and Patterns and Structure Assessment with *Focus Students*) followed. The analysis of data was outlined for each qualitative and quantitative instrument implemented. Trustworthiness and ethical considerations were delineated. The following chapter reports on the collection of data using the mixed methods approach and the results and findings from the study.

The next chapter also presents the analysis of data. The analysis of the results will be presented within the following themes:

- Young students' ability to exhibit Critical Mathematical Thinking
- Critical Mathematical Thinking Capabilities in Young Students
- Exhibiting Critical Mathematical Thinking – The Role of the Learning Experience

Chapter 5: Results and Findings

5.1 CHAPTER OVERVIEW

Presented in this chapter are the results and findings of the data analysis in relation to the two research questions:

1. *What are the CMT capabilities young students exhibit as they begin formal schooling?*
2. *What types of teacher questions help these young students exhibit their CMT?*

The analysis of the results of the CMTLE and the teacher questions revealed several general findings. These findings are presented under the broad themes that guided the research (see Section 5.6) and data analysis, including:

- Young students' ability to exhibit Critical Mathematical Thinking
- Critical Mathematical Thinking Capabilities in Young Students
- Exhibiting Critical Mathematical Thinking – The Role of Teacher Questioning

Generally, the first two themes (young students' ability to exhibit CMT and the CMT capabilities in young students) align with the first research question. The final theme (exhibiting CMT) aligns with the second research question.

Broadly, the findings of the research indicated that there were defined CMT capabilities presented by young students. Additionally, there was no relationship between these young students' performance on the CMTLE and their performance on the three measures of Higher Order thinking, namely, the Slosson Intelligence Test and the Raven's Progressive Matrices. Thus, CMT appeared to be different from Higher Order thinking. However, there was a positive relationship between students' performance on the CMTLE and their performance on PASA, suggesting the ability to perceive structure and pattern in mathematical tasks seemed to be related to the ability to engage in CMT. Finally, teacher questioning, and particular probing questions, helped young students to exhibit their CMT.

In this chapter, each section begins with a brief overview for each phase of the data collection and analysis. The chapter concludes with a summary of findings. Figure 5.1 provides an overview of the chapter.

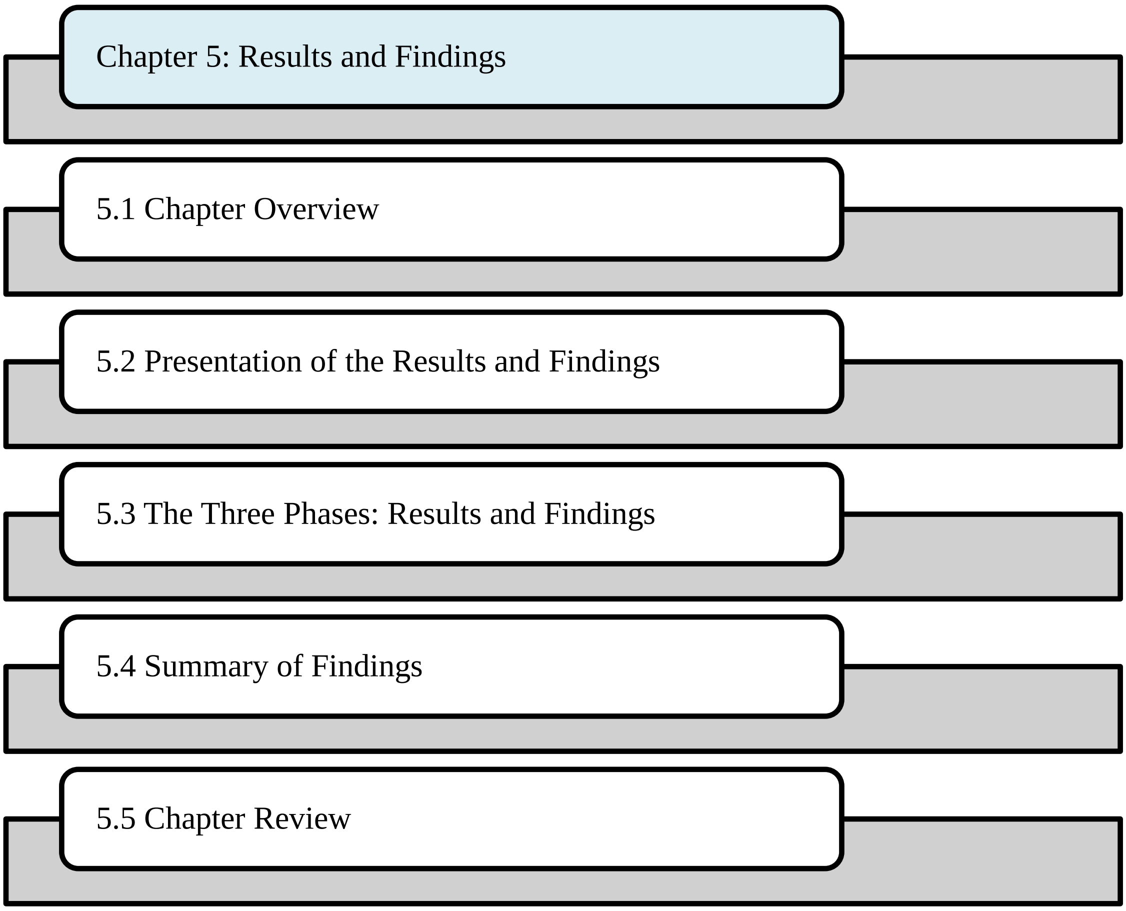


Figure 5.1. Overview of Chapter 5

5.2 PRESENTATION OF THE RESULTS AND FINDINGS

The presentation of the results and findings for this chapter occurs under the three phases identified in the research design chapter, namely:

- Phase 1: Qualitative data gathering and analysis: CMT Observation Guide with *All Kindergarten Students* (n=161) and *Classroom Observation Students* (n=41)
- Phase 2: Quantitative data gathering and analysis: Raven's Progressive Matrices, Slosson Intelligence Test, Patterns and Structure Assessment, and CMTLE with *Focus Students* (n=25)
- Phase 3: Qualitative data analysis: CMTLE with *High CMT Students* (n=16)

Five instruments were used to gather the data. These instruments include the CMT Observation Guide, three quantitative instruments (Raven’s Progressive Matrices, Slosson Intelligence Test (SIT-P), Patterns and Structure Assessment), and The Critical Mathematical Thinking Learning Experiences (CMTLE) interview. The five instruments were implemented with selected participants during the study. Table 5.1 provides an overview of the instruments, interview and the student participants.

Table 5.1
Overview of the Instruments, Interview and Participants

Phase	Data Gathering Instruments	Participants
1	Qualitative Instrument: o CMT Observation Guide	*All Kindergarten Students (n=161)*
1	Re-administered CMT Observation Guide	*Classroom Observation Students (n=41)*
2 3	Quantitative Instruments: o Raven’s Progressive Matrices o Slosson Intelligence Test o Patterns and Structure Assessment Clinical Interview: o The CMTLE	*Focus Students (n=25)*
3	Clinical Interview – in depth analysis: o The CMTLE	*High CMT Students (n=16)*

Each qualitative and quantitative instrument were described in Section 4.5.2.

The CMT Observation Guide included 14 observational categories derived from the literature (see Section 4.5.1.1). The CMT Observation Guide assisted in the selection of students for titled the *Classroom Observation Students* (from n = 161 to n = 41). From the 41 students, 25 students (*Focus Students*) were selected to participate in the one-on-one CMTLE interview. In addition to the CMTLE interview, three quantitative instruments were administered to the *Focus Students.*

From the data generated, sixteen students (*High CMT Students*) were identified for an in-depth data analysis. Figure 5.2 provides an overview of the kindergarten students who participated in this study.

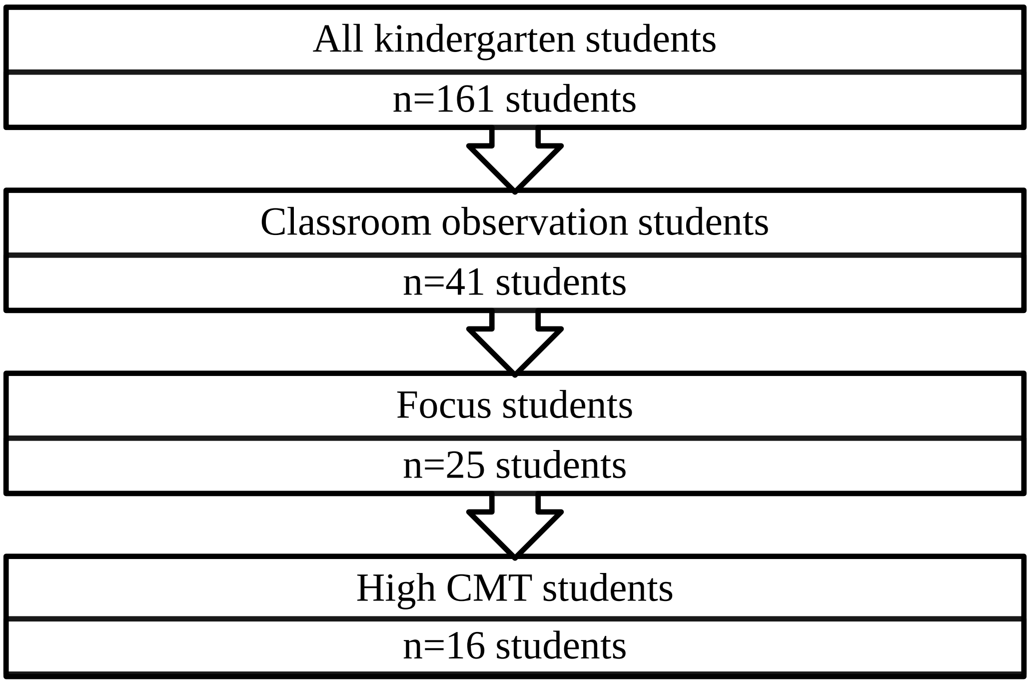


Figure 5.2. Overview of Kindergarten Students that Participated in the Study

5.3 THE THREE PHASES: RESULTS AND FINDINGS

The results and findings are presented in order of the phases delineated for a mixed methods explanatory design (Creswell et al., 2003). The phases are displayed in Table 5.1 in the preceding section.

5.3.1 Phase 1 – Qualitative Results and Findings: CMT Observation Guide with All Kindergarten Students (n=161) and Classroom Observation Students (n=41)

The CMT Observation guide was implemented with *All Kindergarten Students* (n=161 students) to identify CMT capabilities observed in all kindergarten classrooms, and to assist in selecting the *Classroom Observation Students* (n=41 students). The observations took place five times for each kindergarten classroom over a 2 week period per class. The student data were classified into the 14 observation categories presented in the CMT Observation Guide (see Section 4.5.1.1). Student initials were noted and tallied next to the CMT capabilities in the CMT Observation Guide. The selection of 41/161 students (23 males and 18 females) (students that displayed CMT capabilities several times) participated in the next phase of the data gathering process (*Classroom Observation Students*).

In order to select the *Focus Students* from the *Classroom Observation Students* (n = 41) students were observed for an additional lesson using the CMT Observation Guide. In all, the *Focus Students* were observed five times. An analysis of this data, in consultation with the classroom teacher, resulted in the identification of 25 students (*Focus Students*) who engaged in phase two of the data gathering process. The observation data for the selected *Focus Students* were entered into a spreadsheet. The average age of these 25 focus students was 6 years of age. These 25 focus students were selected on the basis of the number of times each displayed CMT capabilities (see Table 5.1).

Table 5.1 presents the analysis of the CMT Observation Guide in accordance with the school and class (for example, A K1 – School A Kindergarten classroom 1; B K1 – School B Kindergarten classroom 1) over the five observation visits for the 25 focus students. Included in the table are codes assigned for each student (for example, S4 refers to Student 4). If a student displayed the same CMT at a different observation, the occurrences were noted in Table 5.2 (for example, S4 displayed the CMT of questioning two times over the five visits).

Table 5.2

Observations of CMT Observation Categories for the 25 Focus Students arranged by School (3) and Class (5)

CMT Observation Categories	Observation 1 School/Classes					Observation 2 School/Classes					Observation 3 School/Classes					Observation 4 School/Classes				
	AK1	AK2	BK1	BK2	CK	AK1	AK2	BK1	BK2	CK	AK1	AK2	BK1	BK2	CK	AK1	AK2	BK1	BK2	CK
Mathematisation / Grasping principles	S13				S24 S21				S9	S18	S12				S19			S1	S3 S6	S24 S19 S23
Estimating	S12	S10 S17	S6 S1 S2 S5	S3	S24 S20 S23 S25	S12	S17		S9 S3	S20 S21 S19		S10	S5				S10	S2		S24 S20
Evaluating																				S24
Classifying															S24					
Assuming / Hypothesising					S18												S17			
Noting relationships /transferring learning			S6		S21 S19		S10			S24 S23										
Argumentation/ Offering opinions	S14						S11													
Number sense and mental computation	S13 S15	S16	S1		S20 S21 S23			S1	S6	S18	S12			S6	S24 S23	S12 S13				S24 S21 S9 S25
Spatial and geometric reasoning				S9																
Judgements / Reasoning									S9	S21				S9	S21					S21 S18
Inferring logically																				S20
Self-correcting		S10		S3 S8	S24 S18					S19										S20 S22
Questioning			S7	S4		S12								S3 S4						S24

An analysis of the CMT observation data revealed that all 25 *Focus Students* displayed at least one CMT observation category over the course of the five observations. The occurrences of CMT per student varied between 1 to 11 occurrences.

Presented in Table 5.3 is the frequency of observations over all the elements of the CMT Observation Guide over the five observations for the 25 *Focus Students* selected to engage in the interview and quantitative instruments.

Table 5.3
Results of the Observation Guide over 4 Visits for the 25 Focus Students

	CMT Observation Categories													
	Estimating	Number sense and mental computation	Mathematisation / Grasping principles	Making judgements with criteria / Reasoning	Self-correcting	Connections / Noting relationships /transferring learning	Questioning	Argumentation/ Offering opinions with reasons	Assuming / Hypothesising	Evaluating	Classifying	Spatial and geometric reasoning	Inferring logically	Data and probability sense
Number of students (n= 25)	15	13	11	3	8	6	5	2	2	1	1	1	1	0
Number of student occurrences observed	33	27	15	8	8	6	6	2	2	1	1	1	1	0
Student codes	S1 (x2) S2(x2) S3 (x2) S5 (x2) S6 S9 S10 (x2) S12 (x2) S17 (x2) S19 S20 (x3) S21 S23 S24 (x2) S25	S1 (x2) S6 (x2) S9 S12 (x2) S13 (x2) S15 S16 S18 S20 S21 (x2) S23 (x2) S24 (x2) S25	S1 S3 S6 S9 S12 S13 S18 S19 (x2) S21 S23 S24 (x2)	S9 (x2) S18 S21 (x3)	S3 S8 S10 S18 S19 S20 S22 S24	S6 S10 S19 S21 S23 S24	S3 S4 (x2) S7 S12 S24	S11 S14	S17 S18	S24	S24	S9	S20	

Over the four classroom visits, dimensions of the CMT Observation Guide most frequently observed were: estimation (33 times involving 15 students), followed by the dimension of number sense and mental computation (27 times involving 13 students), and the dimension of grasping principles (15 times involving 15 students). The frequency of occurrence of the remaining CMT dimensions ranged from 8 (involving 3 students) to 0 (involving 0 students).

It is evident from the data collected using the CMT Observation Guide that this particular tool was appropriate for identifying and selecting the 25 *Focus Students* who participated in the one-on-one interview. In the next sections, for each of the four instruments the overall results for the 25 *Focus Students* are presented.

5.3.2 Phase 2 – Quantitative Results and Findings: Raven's Progressive Matrices, Slosson Intelligence Test, Patterns and Structure Assessment, CMTLE with *Focus Students (n=25)*

Three quantitative instruments were administered individually and consecutively to each of the 25 *Focus Students*. The first three instruments include Raven's Progressive Matrices, Slosson Intelligence Test (SIT-P), Patterns and Structure Assessment. The fourth set of quantitative data arose from the CMTLE clinical interview (see Section 4.7.2.4), the CMT scale scores. The purpose of the three quantitative instruments was to determine if the quantitative measures provide different or complimentary data to the CMT scale scores. Results and findings for each quantitative instrument are presented and discussed below. The CMT scale scores provided identification measures to select 16 *High CMT Students* for in depth investigation (see Section 4.7.2.4).

5.3.2.1 Phase 2 – Quantitative Results and Findings: Raven's Progressive Matrices with Focus Students (n=25)

The 25 *Focus Students* were assessed using the Raven's Progressive Matrices. Table 5.4 presents the results from the Raven's Progressive Matrices for the 25 students including frequencies for each Grade. Included in the table are codes assigned for each student (for example, S2 refers to student 2).

Table 5.4
Results of the Raven's Progressive Matrices Score for the 25 Focus Students

	Raven's Progressive Matrices Five Grades				
	Grade V	Grade IV	Grade III	Grade II	Grade I
	At or below 5^{th} percentile	At or below 25^{th} percentile	Between 25^{th} – 75^{th} percentile	At or above 75^{th} percentile	At or above 95^{th} percentile
Raven's Progressive Matrices Classification	Intellectually impaired	Below average	Intellectually average	Above average	Intellectually superior
Number of students (n= 25)	0	1 (4%)	3 (12%)	13 (52%)	8 (32%)
Students		S14	S1 S10 S18	S3 S4 S5 S6 S8 S11 S13 S15 S17 S19 S21 S22 S23	S2 S7 S9 S12 S16 S20 S24 S25

An analysis of the Raven's Progressive Matrices data revealed that 21 (84%) of the 25 *Focus Students* were considered *above average and intellectually superior*. The exception included one student who was classified as *below average* (4%) and three students who were classified as *intellectually average* (12%).

5.3.2.2 Phase 2 – Quantitative Results and Findings: Slosson Intelligence Test with Focus Students (n=25)

The twenty-five focus students were also assessed and aligned with the Slosson Intelligence Test percentile ranking and range. The students' results presented in Table 5.5 provide an overview of the results from the 25 *Focus Students*. Included in the table are codes assigned for each student (for example, S3 refers to student 3).

Table 5.5
Results of Slosson Intelligence Test – SIT-P for the 25 Focus Students

	Focus Students' Results				
Slosson Intelligence Test Range	Low Average or Below	Average	High Average	Superior	Very superior
Percentile Rank	9 –0	25 – 74	75 – 90	91 - 97	98 +
Number of students	0	2	7	10	6
Student codes		S14 S15	S8 S10 S11 S16 S19 S21 S23	S1 S6 S7 S9 S13 S17 S18 S22 S24 S25	S2 S3 S4 S5 S12 S20

The analysis of the Slosson Intelligence data revealed that 21 (84%) of the focus students were considered above *high average, superior* or *very superior*. Two students were classified as *average* (8%).

5.3.2.3 Phase 2 – Quantitative Results and Findings Patterns and Structure (PASA) (n=25)

The PASA provided specific mathematics data that focused on patterns and structure performance of the students. Twenty-five focus students were classified into the four PASA categories of Awareness of Mathematical Pattern and Structure (AMPS).

Table 5.6 presents the results of the Awareness of Mathematical Pattern and Structure (AMPS) level for the 25 *Focus Students*. Included in the table are codes assigned for each student (for example, S5 refers to student 5).

Table 5.6

Results of The Patterns and Structure Assessment (PASA) for the 25 Focus Students

	AMPS Levels and Scales			
	1	2	3	4
AMPS Scale Description	Struggle to recognise simple patterns	Recognise simple patterns	Aware of fundamental structures	Aware of the generality of fundamental structures and can extend these
Number of students	0	1	18	6
Student codes		S14	S1 S12 S2 S13 S4 S15 S5 S16 S6 S17 S7 S18 S8 S19 S9 S22 S10 S25	S3 S11 S20 S21 S23 S24

An analysis of the PASA data revealed that 24 of the 25 *Focus Students* (96%) performed within the 2 highest AMPS levels on the PASA and were classified as either AMPS level 4, *aware of the generality of fundamental structures* and can extend these or AMPS level 3, *aware of fundamental structures.* The exception included one student who was classified as AMPS level 1, *able to recognise simple patterns*.

5.3.2.4 Phase 2 – Quantitative Results and Findings: Critical Mathematical Thinking Learning Experiences (CMTLE) (n=25)

The CMTLE interview comprised eight learning tasks. Each *Focus Student* (n=25) was assigned a score out of 10 for each item. Thus, the highest score a student could score was 80. For each learning experience students were allocated a score out of 5 for the strategy they used and a score out for 5 for the level of CMT they exhibited as they explained their solution (see Section 4.7.2.4, Table 4.13 for a full description on how the CMTLE as scored). The results presented in Table 5.7 identify each individual student’s school and class, their age at the time of testing, and their CMT scale score. The analysed data are ranked highest student score to the lowest student score.

Table 5.7
Focus Students' CMT Scale Scores

Student	School/ Class	Age	CMT Scale Score /80	Rank on CMT
S24	C K	6y2m	74	1st
S20	C K	6y3m	72	2nd
S21	C K	6y0m	70	Equal 3rd
S9	B K2	6y1m	70	Equal 3rd
S18	C K	6y3m	70	Equal 3rd
S19	C K	6y4m	66	6th
S3	B K2	6y2m	64	Equal 7th
S23	C K	6y0m	64	Equal 7th
S12	A K1	6y1m	60	9th
S10	A K2	6y2m	59	Equal 10th
S6	B K1	5y11m	59	Equal 10th
S1	B K1	5y8m	58	Equal 12th
S25	C K	6y2m	58	Equal 12th
S13	A K1	5y9m	52	14th
S2	B K1	5y3m	51	Equal 15th
S17	A K2	6y1m	51	Equal 15th
S5	B K1	6y1m	42	17th
S16	A K2	5y3m	41	18th
S15	A K1	6y0m	40	19th
S11	A K2	5y9m	38	20th
S8	B K2	6y8m	34	21st
S22	C K	6y0m	32	22nd
S7	B K1	5y11m	30	23rd
S4	B K2	5y1m	29	24th
S14	AK1	6y3m	26	25th

Column two in Table 5.7 identifies the school and class each *Focus Student* attended. It is evident that the students are spread across the 3 schools and 5 classes. The trend in the students' CMT scales scores indicated that the scores tended to be clustered with the largest drop in scores occurring between S17 and S5. The top 16 students (S24 to S17) were initially chosen as the *High CMT students*. Figure 5.3 presents the graph relating to these trends.

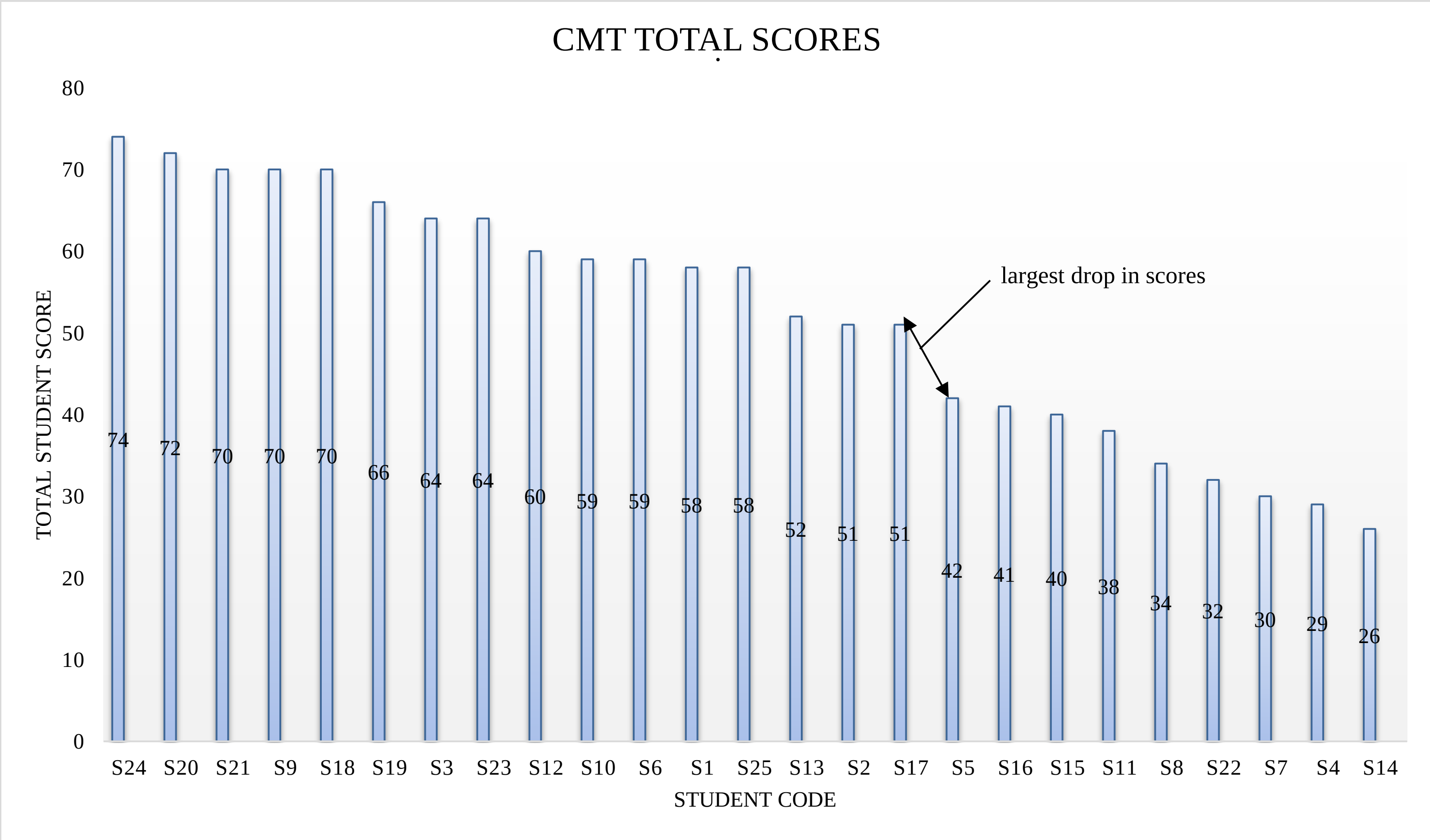
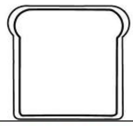
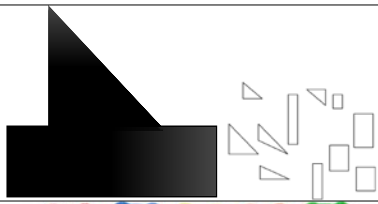


Figure 5.3. Trend in Students' Scores for the CMT

In order to examine the trend in students' score across the eight learning experiences, the mean average student score for each learning experience was calculated (out of 10). Table 5.8 presents each learning experience together with the average score students obtained for each learning experience. The table is organised from the highest to lowest average score for each learning experience.

Table 5.8
Average Focus Student's (n = 25) Score for each of the Eight Learning Experiences out of 10 Presented in Order of Difficulty

Brief Descriptions of each Learning Experience		Physical Manipulatives	Students' Average CMT Score /10
LE1	**Framed photograph – finding the middle** The problem posed was if you had a blank wall and a framed photograph to hang on the wall, how could you find the exact place to hang the photograph so it was in the middle of the wall.		7.4
LE3	**Why is 3 + 3 the same as 4 + 2?** Students were asked to provide two reasons why 3 + 3 is the same as 4 + 2.	3 + 3 4 + 2	7.32
LE6	**Cubby house – identifying number of tiles required** Students were shown a picture of a cubby house and given a small square tile. They were asked if they could work out how many tiles the needed for the floor of the cubby house		7.16
LE7	**Sandwich – cutting and sharing equally** Students were asked to show many different ways a sandwich can be cut in half.		7.08
LE4	**Towers – identifying which tower is taller** Students were asked to explain why one tower is taller than the other tower		6.72
LE8	**Shapes – replicating** Students were asked to find different ways to re-create this shape		6.56
LE5	**Teddy bears – real life number sentences** Students provided an answer to describe subtraction with unseen items		5.4
LE2	**Mini bean bag – counting unseen objects** Students were asked to provide a strategy to count the beans in the bean bag without touching them		4.76

NB: The full description of each Learning Experience is located in Table 4.7.

The average score out of 10 ranges from 7.4 (highest) to 4.76 (lowest). The two learning experiences that students had most difficulty with were LE 5 and LE 2.

Finding 1: Young students as they begin formal schooling were capable of engaging in critical mathematical thinking.

5.3.3 Phase 2 – Results and Findings: Relationship between the CMT Scale Scores and the Three Quantitative Instruments

To assess the relationship between students' scale scores for the CMT and each of the three quantitative scales (Raven's Progressive Matrices, Slosson Intelligence Test, and Patterns and Structure Assessment (PASA) AMPS), partial correlations adjusting for the age of the student were performed. The partial correlation is chosen because at this age, even small differences in age are related to improvements in CMT.

Table 5.9 reports the value of the partial correlation between the CMT scores and the three quantitative tests when the age of the student is controlled. The level of significance of the correlation between the two scales is also reported for each relationship. The significance is based on the use of a two-tailed test of significance, which is appropriate when testing for significance difference between two scales.

Table 5.9
Results of the Partial Correlation analysis between CMT and Raven's Progressive Matrices, Slosson Intelligence Test and PASA AMPS of the Focus Students (n=25). (Degrees of freedom = 22)

Scale	Correlation (*r*)	Significance (*p*)
CMT - Raven RAW	.349	.095
CMT - Slosson verb Raw	.339	.105
CMT - PASA AMPS	.518	.005**

*** $p < .05$ – significant correlations.*

The relationship between the CMT scale scores and the PASA AMPS scale scores was the only relationship that significantly correlated ($p < .05$). In addition, it appears that there was no statistically significant relationship between age appropriate intelligence tests, namely, Slosson Intelligence Test and the Raven's Progressive Matrices, and the CMT scale scores. This is further evidence that the CMTLE interview is testing something other than intelligence, suggesting that high levels of CMT are not assessed in the same ways as intelligence assessment measures. In addition, there appears to be a significant positive relationship between CMT and an underlying awareness of pattern and structure.

Finding 2: There was no statistically significant relationship between CMT scores and the scores for Slosson Intelligence test and the Raven's Progressive Matrices tests. Thus, it appears that high levels of Critical Mathematical Thinking are not assessed in the same ways as intelligence assessment measures.

Finding 3: There was a positive statistically significant relationship of student performance between the scores for CMT and PASA. Thus, it appears that there is a relationship between Critical Mathematical Thinking and the underlying awareness of pattern and structure.

5.3.4 Phase 2 – Analysis of Data to Identify the High CMT Students (n=16)

This section shares the findings pertaining to this selection process to identify *High CMT Students* to answer the first research question, namely: *What are the CMT capabilities young students exhibit as they begin formal schooling?*

Since one aim of the study was to identify CMT in young students, only the top performing students (*High CMT Students)* were chosen. The selection process was initially based on students' results for the CMTLE.

Analysis of the top 16 students (*High CMT Students*) in the CMTLE (see Figure 5.3) indicated that particular learning experiences provide a platform for students to exhibit CMT. Table 5.10 presents each learning experience together with the average score the top 16 students obtained for each learning experience. The table is organised from the highest to lowest average score for each learning experience.

Table 5.10
Average High CMT Student (n = 16) score for each of the Eight Learning Experiences out of 10 Presented in Order of Difficulty

Learning Experience and Description		High CMT Students Average Score /10
LE6	Cubby house – identifying number of tiles required	8.87
LE7	Sandwich – cutting and sharing equally	8.68
LE3	Why is 3 + 3 the same as 4 + 2?	8.43
LE1	Framed photograph – finding the middle	8.375
LE4	Towers – identifying which tower is taller	8
LE8	Shapes – replicating	8
LE5	Teddy bears – real life number sentences	6.56
LE2	Mini bean bag – counting unseen objects	5.75

Table 5.10 presents each task together with the average score students (n=16) obtained for each task. The table is organised from the highest to lowest average score for each task. The sequence of learning experiences from easiest to most difficult was LE6, LE7, LE3, LE1, LE4, LE8, LE5 and LE2.

A comparison of the data from Table 5.8 (*Focus Students* n=25) and Table 5.10 *(High CMT Students* n=16) displays a difference in the CMT average score and the order of difficulty. For example, LE6 average score for the focus students (n=25) was 7.16 and was 8.87 for the high CMT students (n=16). Overall, the high CMT student average score was between 0.975-1.71 marks higher than the *Focus Students'* score.

In addition, the order of difficulty changed between the two data sets. The final four (most difficult) learning experiences remain in the same order (that is, LE4, 8, 5 and 2), however, the order of the top four (easiest learning experiences) change.

Table 5.11
Comparison of the Top Average High CMT Students (n = 16) score and the Focus Students (n=25) for each of the Eight Learning Experiences

Order of Difficulty			
High CMT Students (n=16)		Focus Students (n=25)	
LE6	Cubby house – identifying number of tiles required	LE1	Framed photograph – finding the middle
LE7	Sandwich – cutting and sharing equally	LE3	Why is 3 + 3 the same as 4 + 2?
LE3	Why is 3 + 3 the same as 4 + 2?	LE6	Cubby house – identifying number of tiles required
LE1	Framed photograph – finding the middle	LE7	Sandwich – cutting and sharing equally
LE4	Towers – identifying which tower is taller	LE4	Towers – identifying which tower is taller
LE8	Shapes – replicating	LE8	Shapes – replicating
LE5	Teddy bears – real life number sentences	LE5	Teddy bears – real life number sentences
LE2	Mini bean bag – counting unseen objects	LE2	Mini bean bag – counting unseen objects

Table 5.11 displays the change in order of difficulty between the two data sets. The section in grey identifies the four learning experiences that did not change in order. LE 6, 7, 3 and 1 remain the easiest four learning experiences for both groups, however, the order changes.

The initial selection of *Focus Students* was based on their scores on the CMT scale. In line with the research questions, the highest performing students were selected for further analysis. As indicated in Figure 53 (Section 5.3.2.4) there was a marked drop in students' scores on the CMT scale between Student 17 and Student 5. This became the cut off point for the study. Thus, 16 students, high CMT students, selected were S24, S20, S21, S9, S18, S19, S3, S23, S12, S10, S6, S1, S25, S13, S2, and S17. These were the 16 students who scored higher than S5 on the CMTLE (see Figure 5.3).

Figure 5.4 presents the top 16 students (*High CMT students*) results for the three quantitative tests, namely the Slosson Intelligence Test, Raven's Progressive Matrices and the PASA test. Students' results for the Slosson and Intelligence Test and Raven's Progressive Matrices were initially presented in a 4x4 grid. The students with the highest scores for both tests (S2, S12 and S20) are situated in the top right-hand corner of the grid. The student with the lowest scores for both tests (S10) is situated in the bottom left hand corner of the spread of students' scores. With regard to the PASA test results only 5 students scored a PASA

AMP of 4 (S3, S20, S24, S21 and S23). These students appear in red. The remaining 11 students scored a PASA AMP of 3.

Slosson Intelligence Test	Grade V	Grade IV	Grade III	Grade II	Grade I
Very Superior				S3	S2 S12 S20
Superior			S1 S18	S13 S17 S6	S9 S24 S25
High Average			S10	S19 S21 S23	
Average					
Low Average					
	Raven's Progressive Matrices				

Note: Students in red scored a PASA AMP of 4. The rest scored a PASA AMP of 3

Figure 5.4. Top 16 High CMT Students Results for the Three Quantitative Tests.

As evidenced in the above table, all *High CMT Students* (n=16) also performed in the top categories on the Slosson Intelligence test, Raven's Progressive Matrices and the PASA.

5.3.5 Phase 3 - Qualitative Analysis: Critical Mathematical Thinking Learning Experiences (CMTLE) with *High CMT Students (n=16)*

The qualitative element of the CMTLE interview included a detailed analysis of student transcripts with consideration of the learning experience, the summary of student responses, the timing when the responses was evident in the transcript between the teacher and student, and the alignment with the sub-theme identified in CMTFYS that emerged from the literature (see Table 3.4 Section 3.3.3). The purpose of this process was to identify the capabilities of *High CMT Students.*

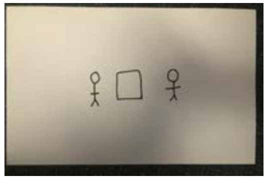
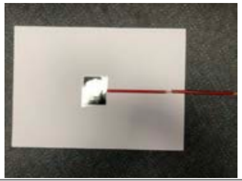
Tables 5.12 – 5.16 present a representative selection of the analysis of the *High CMT Student* data. The full analysis of the *High CMT Student* data is presented in Appendix C. The data were coded (see Section 4.7.3) and are presented under the following themes identified in the CMT Conceptual Framework (Table 3.4 Section 3.3.3), namely:

- Interpreting (Table 5.12);
- Analysing (Table 5.13);
- Evaluating (Table 5.14);
- Explaining (Table 5.15); and,
- Creating (Table 5.16).

The third column (Utterance/Speaker) records when the particular response occurred in the interview. For example, 3 indicates it is the third utterance in the interview and T indicates that the teacher said this utterance. In the next row, 4 indicates it is the fourth utterance in the interview and S13 indicates that Student 13 said this utterance. The tables are presented in order according to the phases of CMT, and the discussion of the findings pertaining to these tables is presented at the end of this sequence. The full analysis can be located in Appendix C.

Table 5.12

*Results of the **Interpreting** Theme within the CMTLE Interview*

Mathematical Thinking in the Early Years: Clarifying concepts and ideas = making sense of mathematics					
Learning experience	Summary of student response	Utterance/	Speaker	Transcripts that exhibits the theme and alignment with the sub-theme	Sub- theme (researchers' interpretation)
LE1	Student drew objects on either side of the photograph to find the middle. 	3	T	Well, how can we check that's exactly the middle? What can we do to check?	Clarifying (student clarified the notion of middle relation to the object's sides – it is somewhere between the sides)
		4	S13	*Because if you know these are the sides of it and there's no other one [student pointed to the other side of the paper] in there, that would actually be the middle.*	
		7	T	Maybe do you want to make any drawings to show me?	
		8	S13	Because if that person was next to the middle, like say I was next to the middle, and if you draw another person there next to my brother, all of.	
		10	S13	*Say if I was over here and my brother was over here and that was in the middle, that would count that that's the middle.*	Clarifying (student used a real-life context to clarify 'middle')
LE1	Student uses a lead pencil as an informal unit of measure 	3	S6	Measure. How about I measure it?	Estimating (student used a concrete object as an informal unit of measure to estimate the location of the middle)
		4	T	I'll give you this [teacher hands student a pencil] if you want to do anything with it. Do you think that's the middle? How did you find that out?	
		5	S6	W*e bring the pencil here, moving it over.*	
		12	S13	Well, you can actually feel inside your hand. *There could be a hundred or more*, so if you just pinch them and you can count them. And you can go all the way around and go up from here, you can go to the bottom, and pinch them a little bit so you can count them.	
LE2	Student considered the use of a machine to count the beans in the bean bag. The 2nd idea was to use a bowl, find out how many are in the bowl and then counting similar groups of that amount.	2	S3	Using a machine?	Relating (student related the use of counting machine to the process of addition)
		3	T	What sort of machine?	
		4	S3	A number machine, count how much there are in there.	
		5	T	Oh, I've never seen a number machine before. Can you explain that to me?	
		6	S3	*So, you have a machine, and then you can put it over here and it can count me how much beans there are in the bean bag. And then it can tell you on a thing, and then you know much are in there.*	

NB: Interpreting was evidenced 8 times across the high CMT students (n=16). Three examples are listed above. All examples are presented in Appendix C.

Table 5.13

*Results of the **Analysing** Theme within the CMTLE Interview*

Mathematical Thinking in the Early Years: Connecting understandings and procedures: noting relationships					
Learning experience	Summary of student response	Utterance/	Speaker	Transcripts that exhibits the theme and alignment with the sub- theme	Sub- theme (researchers' interpretation)
LE1	Student used his knowledge of making paper planes to find the middle.	10	S18	*I'm used to folding it this way because I make paper planes every day.*	Applying (student applied already known knowledge - concept of a half – to find the middle)
		11	T	Do you? Do your planes fly very far? They must be very good paper planes. All right, so now where do you think I can hang that photograph of Joey? Do you want to place it there and show me? Is there any other way that you can check that that is the middle? How?	
		12	S18	I could see *if I could fold this [the piece of A3 paper] half in.*	
LE5	Student began by asking the teacher if they used a particular strategy. Student went on to determine the answer and make connections with their mathematical understanding of odd/even and partitioning.	1	T	We played this game and we were putting our teddies [11 teddies] in our pockets. And I put some teddies in my pocket, and my friend put some teddies in her pocket.	
		2	S20	*Did you go one, one, two, three...?*	Querying (student queried how the teacher put the teddies in her pocket)
		3	T	What could we have done? And there were no more teddies left on the desk. So, there was some here and there was some in your pocket. Or in my friend's pocket. How many could it be... could be in my pocket?	
		4	S20	One, one, two, two.	
		5	T	So how many are in your pocket?	
		6	S20	One, two, three, four, five is in your pocket. One, two, three, four, five, six. It's an odd number, so one gets more than the other.	
		7	T	Very good. And then when we... when it was time to pack up, we found this many more teddies on the floor. I wonder whose pocket they could go into.	Noting relationships (student noted the relationship that if the total is an odd number then it's not the sum of two even numbers)
		8	S20	One... Oh, wait. Let me count. One, two, three, four, five, six seven. One, two, three, four, five, six, seven. Now it's even.	
		9	T	Now it's even. Okay.	
		10	S20	*And you have an odd number in that box.*	

NB: Analysing was evidenced 7 times across the high CMT students (n=16). Three examples are listed above. All examples are presented in Appendix C.

Table 5.14

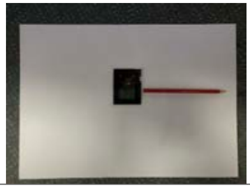
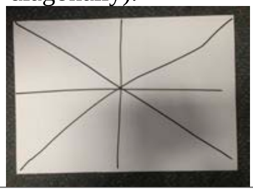
*Results of the **Evaluating** Theme within the CMTLE Interview*

Mathematical Thinking in the Early Years: Evaluating the strategies being used + reasoning and justifying strategies					
Learning experience	Summary of student response	Utterance/ Speaker		Transcripts that exhibits the theme and alignment with the sub- theme	Sub- theme (researchers' interpretation)
LE6	Student considered two different ways to determine the number of tiles required: placing the tiles in rows; using a measuring tape.	22	S23	You could *get a measuring tape and then pull it across like that, where the corners are.* Then you'll get five tiles, five tiles, five tiles. You keep getting five tiles and then you start laying them out in a row of five, like this. One, two, three, four, five.	Assessing claims and arguments (student assessed claims and arguments with regard to the area being repeated rows by measuring the width with a piece of wood and then using repeated pieces of wood to make the area)
		23	T	So, do you think that only five tiles can fit across here? Is that what you think?	
		24	S23	Well maybe you could use wood to do it.	
		25	T	Wood? Explain how you could use wood.	
		26	S23	So, pretend this is a whole piece of wood. And this is a whole piece of wood. It would go across the cubby house. And this is a piece of wood. And then that's a piece of wood. This is a piece of wood. And those last other ones could go, these last other ones, could go on top here. And one will go here. *And so, what you need to buy is one more tile.*	
LE1	Student self-corrected when drawing a ruler like impression to determine the middle. Student continued to provide another way of identifying the middle i.e. folding the paper. 	2	S24	Maybe, maybe you can count numbers that's here. So, you can go one, and then a finger space, and then two, and *then a finger space, and one, then a finger space*. So, you know which is the middle it's coming to. So, if you finish on six, then that would be the middle.	Offering opinions and reasons (student demonstrated, provided an opinion and reasoned by discussing and showing how to find the middle of the picture using the width of his finger)
		3	T	Do you want to have a go at showing me? It's a bit tricky now you're going the opposite way.	
		20	S24	So, I'm going to do 2, 2, 4. 1, 2, 4.	
		21	T	1, 2, 4?	
		22	S24	1, 2, 4 again. I'm *just going to make sure it's the same.*	
		23	T	Good checking.	
		3	T	Oh, okay. What about these two then, if that's a good way of thinking. Why is four plus four the same as six plus two?	
		4	S12	Because if you put it a bit more into this one, it would get to, and it would still make the add, the same.	
LE7	Student described equality by counting and drawing squares to determine the area of	2	S3	They *both have the same.* This has ten, one, two, three, four, five, six, seven, eight, nine, ten. One two, three, four, five, six, seven, eight, nine, ten. They're the same.	Making judgements with criteria (student provided a judgement and justified it with
		3	T	Well, how can you explain why this one is shorter than this one, then?	
		4	S3	*Because you're using little-er blocks. And these are bigger.*	

	each slice of toast	3	T	Why is three plus three the same as four plus two?	using the criteria of bigger and smaller)
		4	S10	Because four plus two is six.	
		5	T	So, we know that four and two is six. What do we know about three plus three?	
		6	S10	Three plus three can be 12 or six.	
		7	T	Tell me how it can be 12.	
		8	S10	I am not really sure.	
		9	T	Tell me how it can be six. Show me how three and three is six.	
		10	S10	*Because when you play three fingers here and three fingers here, it is the same and it makes six.*	
LE6	Student determined the process required to identify the number of tiles needed to tile a cubby house floor. Student described identifying the number of tiles required per rows, then placed the tile to determine how many rows will be required.	2	S6	Count the spaces there and count the spaces here.	Solving (student solved the problem by identifying the required number for 1 row of tiles required and then provided a process to purchase more tiles)
		3	T	Give me some more ideas there. What do you mean by count the spaces?	
		4	S6	Count them, and you know how many. Then you *get by this, the many that there and by the titles. Then see if there's enough. If there's not, just buy a few more and see if there's enough. Then put them together for the floor.*	
		5	T	How will I know how many I need to buy?	
		6	S21	I think you might know. I actually have a clever idea.	
		7	T	I like clever ideas.	
		8	S21	You could actually *build them all around the floor, and you could buy...* Well, you could put the first row and then buy more, and then put in another row, another row, and buy some more, and put another one, and buy some more.	

NB: Evaluating was evidenced 12 times across the high CMT students (n=16). Four examples are listed above. All examples are presented in Appendix C.

Table 5.15
*Results of the **Explaining** Theme within the CMTLE Interview*

Mathematical Thinking in the Early Years: Considering other methods/strategies/alternate solutions + describing solutions/clarifying solutions/elaborating on ideas					
Learning experience	Summary of student response	Utterance/	Speaker	Transcripts that exhibits the theme and alignment with the sub- theme	Sub- theme (researchers' interpretation)
LE2	Student considered counting in larger quantities to determine how many beans are in the bean bag.	2	S17	*The quickest way* you could do it is count by like fours or fives or six or sevens or...	Stating (student stated the quickest way to solve the task)
LE1	Student imagined a pencil was a measuring tape and acted out the actions required to measure to find the middle. 	13	S23	So, like this. *I'm pretending that this is a measuring tape, that one.*	Presenting (student presented a rule to find the middle)
		14	T	Right.	
		15	S23	So, *I'll put some numbers there. Which way is the five?*	
		16	T	So, I'm going to write it on the table.	
		17	S23	So, you said you wanted to put it in the middle. So, I'll *count the other way around.*	
LE1	Determined where the midpoint of the paper was by drawing lines-intersecting (vertically, horizontally and diagonally). 	4	S9	Are you going to give me a ruler?	Justifying (student displayed lines and expressed the process of drawing lines to justify the location of the middle)
		5	T	No	
		6	S9	You can't fold a wall so you can't fold this paper. *I will draw a line here and another line, here and just to prove it to you I will draw another line this way and another line this way, that is the middle.*	

NB: Explaining was evidenced 20 times across the high CMT students (n=16). Three examples are listed above. All examples are presented in Appendix C.

Table 5.16
*Results of the **Creating** Theme within the CMTLE Interview*

Mathematical Thinking in the Early Years: Tackling complex problems in a novel way					
Learning experience	Summary of student response	Utterance/	Speaker	Transcripts that exhibits the theme and alignment with the sub- theme	Sub- theme (researchers' interpretation)
LE3	Student thought of one idea which included changing the cards around, then realised that would not work and then discussed the commutative strategy	2	S23	Because four plus two is six, and three plus three is six. So, we can make a change with these cards if we just... We could make a change to add up different numbers. So, we put them together like this. Four plus two plus three and plus three. Maybe we can make new numbers if we add these two up together.	Self-regulating (student identified own misconceptions and self-regulated to find another solution)
		3	T	That's true. But let's think about just them as two separate cards for now.	
		4	S23	You know how you said they're both six?	
		5	T	You told me that.	
		6	S23	So maybe I can make a change... *Oh, that won't work.* I said... *I was almost going to say go backwards but it'll still be the same. Two plus four will still be the same.*	
		7	T	But why do you think that they're both equal six, Zara?	
		8	S23	Maybe it's because they're both similar? *Maybe they're both just similar?*	
		9	T	They're both similar?	
		10	S23	Or maybe they go in numbers. See, two, three and four.	
		11	T	Yeah. That's true. But is there something about these numbers that make them both add up to six? What do we know?	
		12	S23	They both have a plus.	
		13	T	They both have plus? Yes.	
		14	S23	But they both have different numbers on them.	
		15	T	Is there anything about the numbers that you can think of?	
		16	S23	Well I do know something about the numbers.	
		17	T	Excellent. What do you know?	
		18	S23	I just realized that each number that's there is the same small. So that's small. That's medium, and that's big. So, if we realize they're different, it goes in similar order. Small, medium and big. So that's why it goes in that order.	
		19	T	Okay. Well keep that thinking. What about now? Six plus two and four plus four. Why are they the same?	
		20	S23	Because they both... Well, see. It's like with the three. Four plus four... Each number that makes it... See. Five plus five is 10, and *I know another way to make 10. Three plus... Okay. Five... Okay. It's actually two, six plus four equals 10 as well.*	
LE1	Student counted from 1 – 6 from the edge of the	2	S24	Maybe, maybe you can *count numbers that's here. So, you can go one, and then a finger space, and then two, and then a finger space, and one, then a finger space. So, you know which is the*	Non – algorithmic decision making

page to find the middle. The student demonstrated by creating a ruler that went forwards from left to right and backwards from right to left.			*middle it's coming to. So, if you finish on six, then that would be the middle.*	(student used informal finger spacing (non-algorithmic) to find the middle.
	3	T	Do you want to have a go at showing me?	
	4	S24	Sure. Yeah. This one.	
	5	T	It's a bit tricky now you're going the opposite way.	
	6	S24	That was-	
	7	T	That's all right. I know what you're doing.	
	8	S24	I just don't know where.	
	9	T	Trying all your fingers now. Seeing which one's the best one.	
	10	S24	Yeah, I think this one is. Oh, I went the wrong way.	

NB: Creating was evidenced 6 times across the high CMT students (n=16). Two examples are listed above. All examples are presented in Appendix C**.**

Across the full analysis of the data (see Appendix C) pertaining to the alignment of 16 student transcripts there were 53 instances (of the 16 *High CMT Students*) that evidenced CMT (note: occurrences may appear more than once in a student's transcript). An analysis of the frequency of themes and sub-themes was reviewed based on the terms of difficulty of each learning experience (presented in Tables 5.10 and 5.12). Table 5.17 presents the theme, the sub-theme and the frequency for each learning experience. The data are presented in order of least difficult to most difficult Learning Experiences.

Table 5.17
Frequency of CMT Theme and Sub-themes across the High CMT Students (n=16) Transcripts in Order of Learning Experience Difficulty

Themes	Sub-themes	LE6	LE7	LE3	LE1	LE4	LE8	LE5	LE2	TOTAL for each sub-theme
Interpreting	Clarifying				4					4
	Estimating				2				1	3
									Interpreting Theme Total	**7**
Analysing	Applying				1					1
	Noting relationships			2				3	1	6
	Querying							1		1
									Analysing Theme Total	**8**
Evaluating	Assessing claims and arguments	1								1
	Offering opinions and reasons		1	2	1					4
	Making judgements with criteria		1							1
	Solving	4		1			1			6
									Evaluating Theme Total	**12**
Explaining	Stating			1	1	1			1	4
	Presenting				2					2
	Justifying	4	4		5	1				14
									Explaining Theme Total	**20**
Creating	Self regulating			1						1
	Non-algorithmic decision making	4			1					5
									Creating Theme Total	**6**
TOTAL for each Learning Experience		13	6	7	17	2	1	4	3	53

Students exhibited responses pertaining to all five elements of the Conceptual Framework for Critical Mathematical Thinking. The most frequent theme identified in the analysis of the transcripts was Explaining (20), followed by Evaluating (12). The least frequent theme identified in the analysis of the transcripts was Creating (6).

Table 5.18 presents a description of the sub-themes as exemplified by the High CMT students participating in this study. These descriptions are drawn from the analysis of the transcripts presented in Appendix C. The order of the sub-themes within each theme reflects the frequency in which they occurred in the data (see Table 5.21).

Table 5.18
Description of the Sub-themes Emerging from the Responses Pertaining to the High CMT Students (n=16)

Theme	Sub-themes	Description emerging from the data
Interpreting	Clarifying	Clarifying a concept
	Estimating	Estimating how many
Analysing	Noting relationship	Noting relationships between two different mathematical concepts
	Applying	Applying a known concept to a new context
	Querying	Querying the teacher's approach
Evaluating	Solving	Solves mathematical problem using problem solving strategies
	Offering opinions and reasons	Offering reasons for their claims and opinions
	Assessing claims & arguments	Assessing the claims made with regard to the relationship they noted
	Making judgements with criteria	Making judgements using criteria (for example, smaller and larger)
Explaining	Justifying	Justifying a solution by using a detailed explanation and argument
	Stating	Making statements with regard to the solution to the problem
	Presenting	Presenting a strategy for finding a solution with the inclusion of mathematical language
Creating	Non-algorithmic decision making	Using non-algorithmic decision making to apply mathematical ideas
	Self regulating	Reflecting on misconceptions and self-regulating to find another solutions

The descriptions that emerged from the data analysis provided a nuanced understanding of some of the sub-themes when young students are engaged in Critical Mathematic Thinking.

Finding 4: There are defined CMT capabilities in Young Students and these align with aspects of the Conceptual Framework for Critical Mathematical Thinking that emerged from the literature.

5.3.6 Phase 3 - Qualitative Analysis: Teaching Questioning Results and findings: Critical Mathematical Thinking Learning Experiences (CMTLE) with *High CMT Students (n=16)*

In order to understand how teacher questions may support young students to exhibit CMT, the teacher questioning was reviewed within the transcripts of the *High CMT Students* (n=16) in this study. The teacher questioning data were initially grouped according to the particular students' theme they appeared in. For example, one group was the questions that occurred in the Evaluating theme of the student data (see Table 5.14 for example of the teacher questions that occurred during the Evaluating theme of the student data). The teacher questioning data were then classified according to three categories of questioning identified in the literature: *factual, probing and guiding* questions. Teacher questions that were considered to be factual questions were related to students' mathematical knowledge (for example, What is 2+5). The probing questions were those questions asked to assist these young students to; (a) move from lower to higher levels of thinking, and (b) encourage these young students to exhibit their thinking. Guiding questions were identified as those that supported the problem solving process. In addition to the factual, probing and guiding questions, the questions that were used in the initial setup for each learning experience were coded as Learning Experience Questions (LEQUE) (see Appendix D).

The analysis of the transcripts revealed that there were 437 questions posed to the 16 *High CMT Students* across the interviews. Table 5.19 presents the number of questions asked during each learning experience together with the number of questions in each category (LEQUE, Factual, Probing, Guiding).

Table 5.19
Frequency of Types of Questions asked for the High CMT Students (n=16) across the 8 Learning Experiences

Learning experience	LEQUE	Factual	Probing	Guiding	TOTAL
LE1	13	11	49	16	89
LE2	21	6	43	8	78
LE3	22	8	19	11	60
LE4	9	9	14	6	38
LE5	7	2	15	7	31
LE6	15	6	25	13	59
LE7	9	11	22	17	59
LE8	8	0	9	6	23
TOTAL	104	53	196	84	437

The number of questions asked varied across the learning experiences with most questions posed during LE1 (89 questions) and the least number of questions posed during LE8 (23 questions). Across learning experiences, the types of questions posed also varied. Most questions either occurred during the initial setup (104 questions) or were probing questions (196 questions).

In order to ascertain the trend in the questions being asked across the learning experiences, the frequencies of the three main questioning types (factual, probing, and guiding) were plotted against each learning experience. Figure 5.5 presents the trends in teacher question across the learning experiences.

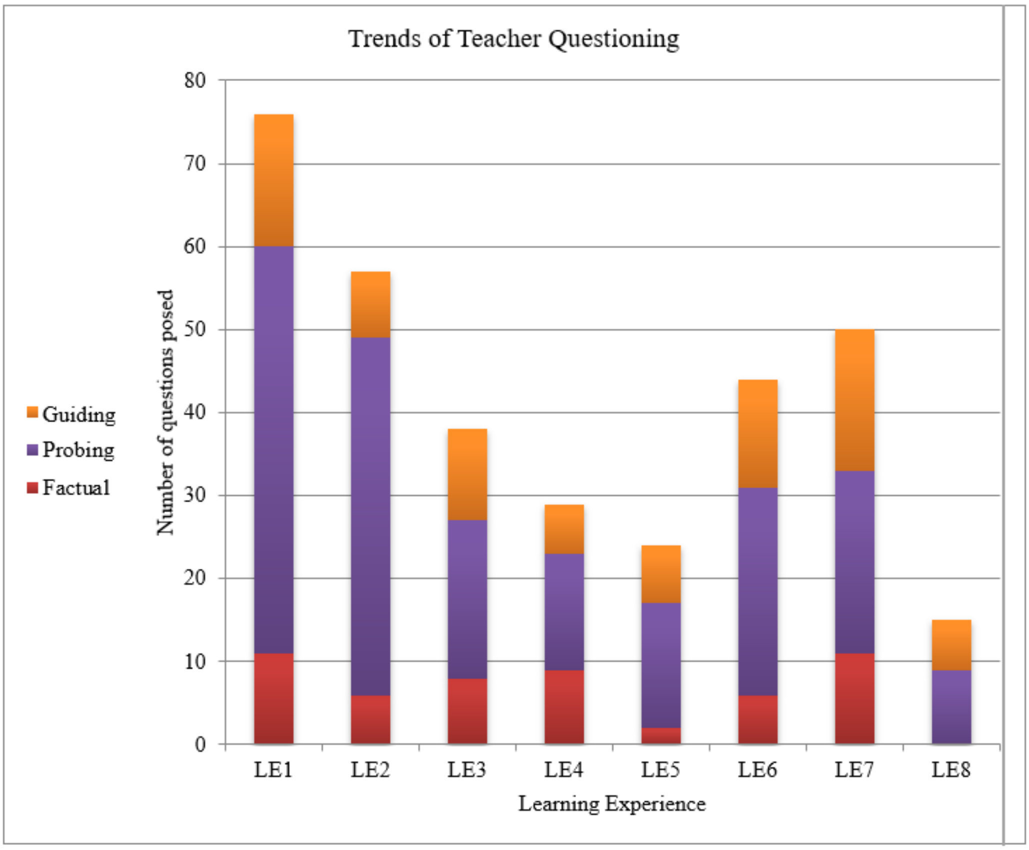


Figure 5.5. The Trend in the Types of Questions asked across the Task

For all eight learning experiences the most common type of teacher question posed were probing questions, followed by guiding questions (with the exception of LE4) and then factual questions.

To answer the research question, *What types of teacher questions help these young students exhibit their CMT?*, the teacher's probing and guiding questions were further analysed in alignment with the themes. Factual questions were least common, and often asked initially during the learning experiences. In addition, the factual questions predominantly pertained to ascertaining mathematical facts, and thus were not related to gauging CMT. Therefore, the analysis of factual questions was not considered to answer this research question. The following section presents these data.

5.3.6.1 Phase 3 – Qualitative: The Role of Probing Teacher Questions in the CMTLE

The analysis of the transcripts revealed that there were 196 probing questions posed to the 16 high CMT students. Table 5.20 presents the number of probing questions asked during each learning experience together with the number of questions that occurred in each student theme (for example, Interpreting, Analysing, Evaluating, Explaining and Creating).

Table 5.20
Frequency of Probing Questions asked for the High CMT Students (n=16) across the 8 Learning Experiences

		PROBING QUESTIONS				
LE	Interpreting	Analysing	Evaluating	Explaining	Creating	Total
LE1	0	3	10	26	9	49
LE2	0	5	11	24	3	43
LE3	0	1	2	13	3	19
LE4	0	1	2	11	0	14
LE5	0	1	2	12	1	15
LE6	0	4	4	13	4	25
LE7	0	3	4	12	4	22
LE8	0	1	4	4	0	9
Total	0	19	39	115	24	196

In total, 196 probing questions were asked (see Table 5.20). The most common probing student theme in which they occurred was explaining (115 questions), followed by evaluating (39 questions), then creating (24 questions), followed by analysing (19 questions). There were no questions asked in the student theme of Interpreting. The number of probing questions asked varied across the learning experiences with most questions posed during LE1 (49 questions) and the least number of questions posed during LE8 (9).

Finding 5A: *The role of the teacher questioning supported young students to exhibit their CMT.*

5.3.6.1.1 Results pertaining to the use of Probing Teacher Questions that help students Explain their CMT

Probing questions are commonly identified as questions that probe students to extend their mathematical thinking (Franke et al., 2009; Sahin & Kulm, 2008; Barnes, 1976; Martino & Maher 1999; Rigelman, 2007). Probing questions are those that support students from low to high levels of mathematical thinking and allow a student to explain their mathematical thinking. Thus, further analysis of the teacher probing questions that help students explain their CMT within each LE was conducted.

In total, 115 probing teacher questions that help students explain their CMT were identified (see Table 5.20). These teacher questions were classified as Probing (question classification), Explaining (student theme in which the question occurred), Teacher Questions (PETQs). The PETQs were reviewed and coded in accordance with the sub-themes of clarifying, noting relationships and offering opinions with reasons CMT. For example, if the role of the question was to help the teacher clarify students' explanations then the question was coded as clarifying. The role of researcher during this process was not to act as a co-learner, but to be in the role of a teacher who is trying to understand and exhibit student thinking. Although the sub-themes were originally used to identify and analyse students' CMT, the researcher adapted the sub-themes during analysis of the teacher questioning in order to further understand what teachers ask as students explain their thinking. These adaptions were as follows:

- *Clarifying* questions are those that were asked when the researcher wanted further clarification of their mathematical thinking. The researcher can rephrase or re-use the students' terms to gain further insight into their mathematical thinking.
- *Noting relationship* questions are those that the researcher asked to further probe students mathematical thinking. This included questions to gain further understanding of the relationships students were seeing across mathematical concepts.
- *Offering opinion* questions are those that the researcher asks to redirect thinking.

Tables 5.21 – 5.28 provide a list of the PETQs together with their coding against the codes clarifying, noting relationships and offering opinions with reasons. The tables are organised by learning experience.

Table 5.21

Frequency of Probing Teacher Questions posed to the High CMT Students (n=16) in LE1to Explain their CMT

LE1 Probing – Explaining	Clarifying	Noting relationships	Offering opinions with reasons
How did you find that out?	■		
What do you think? Is it in the middle?			■
So, where's the middle?	■		
So, does that make the middle? What about this way? How do you know it’s the middle?		■	
Is this the middle of the wall?	■		
Is this the middle down here?	■		
Can you think of any other way to show me how to find the middle without using your arms? Can you think of any other way?		■	
Well, how can we check that's exactly the middle? What can we do to check?		■	
Do you think it's the middle now?	■		
Can you think of any other way to find the middle?		■	
And what is it?	■		
No. Why not?			■
Okay, so what are you going to do next?	■		
Okay, so then where would you place the photograph?	■		
Is it? Is that the middle there? Is that the middle there?	■		
Can you think of any other way?			■
What are you going to do next?	■		
So, what do we need to do?	■		
Do you want to try it on the other side?	■		
A different way. Okay. What are you thinking this time?			■
All right, so now that we've got the dot, what are we going to do?	■		
Okay, so what? What if I do a dot there or a cross. How do we know it's the middle that way?			
All right, so what have you found?	■		
Is there a way that we can find exactly the middle of the entire wall?		■	
What do you think you've discovered now?			■
No other way? So where would we put Joey's photo if you think that's the middle? So, is that the middle? Are you able to check any other way?			■
Is there anything else you'd like to talk to me about with this picture?		■	
Sub-totals	14	6	6
TOTAL		26	

Table 5.22

Frequency of Probing Teacher Questions posed to the High CMT Students (n=16) in LE2 to Explain their CMT

LE2 Probing – Explaining	Clarifying	Noting relationships	Offering opinions with reasons
Hear how many?	■		
Any other way?		■	
What does it say?	■		
All right, can we think of another way?		■	
And what is it?	■		
How else can we find out if we don't count?		■	
Can you think of any other way that we can find out how many mini beans are in there?		■	
So now how many teddies are there altogether?	■		
More, really? I wonder if we can think of a way to find out how many are in there? How could we check?			■
I wonder if there's another way we can figure out how many are in there?		■	
You can guess. What else can you do?		■	
Any other way?		■	
Oh, what might the tag say? What might the tag say?	■		
Can you think of any other way?			■
Can you explain that to me?			■
Is it one of your inventions?	■		
Can you think of any other way?		■	
One hundred, you think there's a hundred?	■		
Is there another way? Can you think of any other way?		■	
What do you mean by that?	■		
So, can we do it another way?		■	
But how else can we find out how many how many are in there? Without unzipping it?		■	
But how can we find out how many are actually in there?		■	
So, any other ideas about how we can find out?		■	
Sub-totals	8	13	3
TOTAL		24	

Table 5.23

Frequency of Probing Teacher Questions posed to the High CMT Students (n=16) in LE3 to Explain their CMT

LE3 Probing – Explaining	Clarifying	Noting relationships	Offering opinions with reasons
What does it mean? What do you mean by opposite sides?	■		
Why is that?		■	
And how do you know that they're the same?	■		
What did you mean by that?	■		
Yeah? Tell me about that	■		
Why else are they the same?		■	
Yeah? Tell me more.	■		
And what about here?	■		
Yeah, and why do they equal the same?	■		
What do you mean by that?	■		
What makes them the same?		■	
But why do you think that they're both equal six?	■		
Is there anything about the numbers that you can think of?		■	
Sub-totals	9	4	0
TOTAL		13	

Table 5.24

Frequency of Probing Teacher Questions posed to the High CMT Students (n=16) in LE4 to Explain their CMT

LE4 Probing – Explaining	Clarifying	Noting relationships	Offering opinions with reasons
So, you're telling me that this one has more blocks, does it?			■
Is there a way that you can check?		■	
So, which one has more blocks?	■		
Why were you counting to five and then starting at one again?	■		
What if, whoa, what if I added two more blocks here? Which one would have more blocks?	■		
Well, how can you explain why this one is shorter than this one, then?			■
Okay, so what if I added three more blocks here? Which one would have more?	■		
Which one would have more?	■		
Why is one taller than the other, if they've both got the same blocks?	■		
What if I added two more here and one more here? Which one would have more?	■		
If we added two more blocks here, this one would still have more?	■		
Sub-totals	8	1	2
TOTAL		11	

Table 5.25
Frequency of Probing Teacher Questions posed to the High CMT Students (n=16) in LE5 to Explain their CMT

LE5 Probing – Explaining	Clarifying	Noting relationships	Offering opinions with reasons
What do you mean by count the spaces?	■		
How might I go about finding out how many I'll need?		■	
How many did my friend put in her pocket?	■		
Now, how many are in my pocket?	■		
How many are in my friend's pocket?	■		
Now how many teddies do I have?	■		
Where did you get six from?	■		
How can you check?		■	
But what about the rest?	■		
What could we have done?		■	
And how many in my other pocket?	■		
What do you mean by count the spaces?	■		
	9	3	0
TOTAL		12	

Table 5.26
Frequency of Probing Teacher Questions posed to the High CMT Students (n=16) in LE6 to Explain their CMT

LE6 Probing – Explaining	Clarifying	Noting relationships	Offering opinions with reasons
Can you explain what that would look like?			■
Can you think of any other way?		■	
So how do you know? How can I figure out how many tiles I need to put on the floor of the cubbyhouse?		■	
So how will I measure?	■		
So, I've measured and this is, let's say two meters long, what do I do next?	■		
So how would we lay these tiles down there?	■		
And how would you know how many more?	■		
Yeah, is there any other way that we can share it equally?		■	
Why not?		■	
How could I figure out how many tiles I need to buy to cover the area of the floor?	■		
Okay, so we're going to use one tile to help us, is what you're thinking?		■	
So now you've given me another design, but still my question is how many might I need to tile this front area? How can I work that out?		■	
I don't know how many rows I need. I don't know. How can I find out how?		■	
	5	7	1
TOTAL		13	

Table 5.27
Frequency of Probing Teacher Questions posed to the High CMT Students (n=16) in LE7 to Explain their CMT

LE7 Probing – Explaining	Clarifying	Noting relationships	Offering opinions with reasons
No. Why not?	■		
Are they exactly the same?	■		
And then how can you check?		■	
Can you think of any other way to cut this bread so that it's shared equally?		■	
What could you do?		■	
Oh, are you trying to get from that second little dash?	■		
And how do I know that they are exactly the same?		■	
How do you know that you get more or the same or less than me? Is there a way we can check?		■	
Is there a way to know if this piece is bigger than this piece?		■	
Why is the bottom smaller than the sides?		■	
Which two would you get?	■		
No. Why not?	■		
	5	7	0
TOTAL		12	

Table 5.28
Frequency of Probing Teacher Questions posed to the High CMT Students (n=16) in LE8 to Explain their CMT

LE8 Probing – Explaining	Clarifying	Noting relationships	Offering opinions with reasons
You're happy with that? Does it look exactly the same?	■		
So, you've drawn that exact shape, haven't you?	■		
What about now? I'm going to just slide these ones away.	■		
Let's not mix them up. What about this shape with these yellow pieces?			
Don't know? You want to try and make it as close as you can?	■		
	4	0	0
TOTAL		4	

Table 5.29 provides a summary of the frequency of types of PETQs were used across the eight learning experiences. The three codes clarifying, noting relationships and offering opinions with reasons are listed in column 1. The frequency of each is stated under each learning experience.

Table 5.29
Summary of Probing Explaining Teacher Questions (PETQs) posed to the High CMT Students (n=16)to Explain their CMT in all Learning Experiences

Sub-themes	Learning Experiences								
	LE1	LE2	LE3	LE4	LE5	LE6	LE7	LE8	TOTAL
Clarifying	14	8	9	8	9	5	5	4	62
Noting relationships	6	14	4	1	3	7	7	0	42
Offering opinions with reasons	6	2	0	2	0	1	0	0	11
TOTAL	26	24	13	11	12	13	12	4	115

In summary, a total of 115 PETQ s were asked of the 16 *High CMT Students* across the eight learning experiences. There was no identifiable pattern in how many types of PETQs were used. The learning experiences that are the easiest (LE1) and most difficult (LE2) had the highest number of PETQs. Most of the PETQs (61 out of 115 questions) were used for the teacher to clarify her own understanding of student thinking. The teacher probing student thinking pertaining to ascertaining the relationship students were noting between mathematical concepts was the next most frequent questions asked (42 out of 115).

Finding 5B: When using questioning to help exhibit CMT in young students, it appeared that clarifying probing questions (teachers endeavouring to clarify their own understanding of student thinking) followed by noting relationships probing questions (teaching endeavouring to understand the particular relationships students are noting) were the most frequent type of questions asked.

5.3.6.2 Phase 3 – Qualitative: The Role of Guiding Teacher Questions in the CMTLE

The analysis of the transcripts revealed that there were 84 guiding questions posed to the 16 *High CMT Students*. Table 5.30 presents the number of guiding questions asked during each student theme in which they occurred (Interpreting, Analysing, Evaluating, Explaining and Creating).

Table 5.30
Frequency of Guiding Questions asked for the 16 High CMT Students across the 8 Learning Experiences

GUIDING QUESTIONS						
LE	Interpreting	Analysing	Evaluating	Explaining	Creating	Total
LE1	0	0	3	4	9	16
LE2	0	0	3	5	0	8
LE3	0	3	2	5	1	11
LE4	0	0	3	1	2	6
LE5	0	2	0	2	3	7
LE6	0	1	0	6	6	13
LE7	0	1	0	7	9	17
LE8	1	0	0	2	3	6
Total	1	7	11	32	33	84

In total, 84 guiding questions were asked (see Table 5.30). The most common student theme in which the guiding questions occurred was creating (33 questions), followed by explaining (32 questions), then evaluating (11 questions), followed by analysing (7 questions), and then interpreting (1 question). The number of guiding questions asked varied across the learning experiences with most questions posed during LE7 (17 questions) and the least number of questions posed during LE8 (6). As identified in Section 5.3.5, the theme creating (6) (within the CMTFYS) was less evidenced than the theme of explaining (20) as young students exhibited their CMT. Thus, the in-depth examination of the types of guiding questions presented in the next section focuses on the data pertaining to the theme of students explaining.

5.3.6.2.1 Results pertaining to the use of Guiding Teacher Questions that help students Explain their CMT

In total, 32 explaining questions were identified. These teacher questions are identified as Guiding (question classification), Explaining (student theme in which they occurred), Teacher Questions (GETQs). The GETQs were reviewed and coded in accordance with the sub-themes of assessing, understanding and self-regulating CMT (Section 3.3). The adaptions were as follows:

- *Assessing* questions are those that the researcher asked during the interview to identify if the student had a firm foundation of the concept and the learning experience as well as an ability to communicate their ideas.
- *Understanding* questions were asked by the researcher to verify the students' understanding.

- *Self-regulating* questions were asked by the researcher to assist students; to monitor and evaluate their thinking during the problem solving.

Tables 5.31 – 5.38 provide a list of the GETQs and their coding against the terms assessing, understanding and self-regulating. The tables are organised by learning experience. The GETQs are listed and coded in alignment with assessing, understanding and self-regulating.

Table 5.31
Frequency of Guiding Teacher Questions posed to the High CMT Students (n=16) in LE1 to Explain their CMT

LE1 Guiding – Explaining	Assessing	Understanding	Self-regulating
What if I want to put it in the centre, right in the middle of everything? How can I find out where that is?	■		
What about this way? So, is this the middle up here?		■	
Okay, so is that in the middle now?	■		
Okay, so you think that's the middle?			■
Sub-totals	2	1	1
TOTAL		4	

Table 5.32
Frequency of Guiding Teacher Questions posed to the High CMT Students (n=16) in LE2 to Explain their CMT

LE2 Guiding – Explaining	Assessing	Understanding	Self-regulating
What if we don't count? Is there another way we can find out how many beans are in there?			■
I wonder if you can think of a way to find out how many beans are in there without counting?			■
How can we find out? What could you do?	■		
Yeah? Any other way that might not include counting?		■	
And then what happens if they're in equal groups?		■	
Sub-totals	1	2	2
TOTAL		5	

Table 5.33

Frequency of Guiding Teacher Questions posed to the High CMT Students (n=16) in LE3 to Explain their CMT

LE3 Guiding – Explaining	Assessing	Understanding	Self-regulating
As I said, you're very good at explaining, aren't you?	■		
Is there something that makes them the same?		■	
You told me that you know that three and three is six. What do you know about four and two?			■
And why would they both add up to eight?		■	
Is there anything about the numbers that you can think of?	■		
Sub-totals	2	2	1
TOTAL		5	

Table 5.34

Frequency of Guiding Teacher Questions posed to the High CMT Students (n=16) in LE4 to Explain their CMT

LE4 Guiding – Explaining	Assessing	Understanding	Self-regulating
So why is one taller than the other than?		■	
Sub-totals	0	1	0
TOTAL		1	

Table 5.35

Frequency of Guiding Teacher Questions posed to the High CMT Students (n=16) in LE5 to Explain their CMT

LE5 Guiding – Explaining	Assessing	Understanding	Self-regulating
10 plus 10 equals 11?		■	
What if I found I had more bears? What happens now?		■	
Sub-totals	0	2	0
TOTAL		2	

Table 5.36

Frequency of Guiding Teacher Questions posed to the High CMT Students (n=16) in LE6 to Explain their CMT

LE6 Guiding – Explaining	Assessing	Understanding	Self-regulating
How can I count the floor and then count the tiles? Explain that to me		■	
Okay, so how many do you think would fit?		■	
I don't know how many either, but how can I find out? What could I do?		■	
Tiles? You think I need 101 tiles? I really like your thinking around using a measuring tape that shows me that you... Have you ever used a measuring tape before?		■	
Is there a way we can find out how many I need to buy?		■	
Can you think of any other way we can find out without drawing it? How many I need?			■
Sub-totals	0	5	1
TOTAL		6	

Table 5.37
Frequency of Guiding Teacher Questions posed to the High CMT Students (n=16) in LE7 to Explain their CMT

LE7 Guiding – Explaining	Assessing	Understanding	Self-regulating
Is this piece the same size as this piece?	■		
Okay. So now, which pieces will you get?		■	
How can you know, how can you check that this side is exactly the same as this side?	■		
And so then how many pieces will I get and how many would you get if we were sharing it?		■	
How can I check that this side is exactly the same as this side? Can you think about what we might do?	■		
Is there a way you can check?	■		
Would it be fair if I got these two pieces and you got those two pieces?	■		
Sub-totals	5	2	0
TOTAL		7	

Table 5.38
Frequency of Guiding Teacher Questions posed to the High CMT Students (n=16) in LE8 to Explain their CMT

LE8 Guiding – Explaining	Assessing	Understanding	Self-regulating
How did you go? Have you created it?	■		
What do you need to add to make it look like this?		■	
Sub-totals	1	1	0
TOTAL		2	

Table 5.39 provides a summary of the frequency of types of GETQs used across the eight Learning Experiences. The three terms assessing, understanding and self-regulating are listed in column 1. The frequency of each GETQ is stated under each learning experience.

Table 5.39
Summary of Guiding Explaining Teacher Questions (GETQs)asked for the High CMT Students (n=16) to Explain their CMT in all Learning Experiences

Sub-themes	Learning Experiences								
	LE1	LE2	LE3	LE4	LE5	LE6	LE7	LE8	TOTAL
Assessing	2	1	2	0	0	0	5	1	11
Understanding	1	2	2	1	2	5	2	1	16
Self-regulating	1	2	1	0	0	1	0	0	5
TOTAL	4	5	5	1	2	6	7	2	32

In summary, there were a total of 32 GETQs asked of the 16 *High CMT Students* across the eight learning experiences. There were no identifiable patterns in how many types of GETQs were used. Learning experiences seven and six had the highest number of GETQs. Learning experience seven was the 4th easiest task in the suite of questions and learning experience seven was the 2nd easiest task.

Finding 5C: When using questioning to help exhibit CMT in young students, it appeared that understanding guiding questions (questions aimed at verifying students' understanding) followed by assessing questions (questions aimed at assessing students having a firm foundation of the concept and the learning experience as well as an ability to communicate their ideas) were the most frequent type of questions asked.

5.4 SUMMARY OF FINDINGS

The analysis of the results of the CMTLE and the teacher questions used revealed several general findings. These findings are presented under the broad themes that guided this research. These are as follows:

- Young Students' Critical Mathematical Thinking
- Critical Mathematical Thinking Capabilities in Young Students
- Exhibiting Critical Mathematical Thinking – The role of Teacher Questioning

5.4.1 Young Students' Critical Mathematical Thinking

Finding 1: Young students as they begin formal schooling were capable of engaging in critical mathematical thinking.

Nineteen of the twenty-five students scored 50% or more on the CMT scale, with nine of these students scoring 75% or more .

The average score on the 10 CMT scale items for the twenty-five students ranged from 4.85 to 7.6 out of a score of 10 for each item.

The student participant groups in this study were refined from *All Kindergarten Students* (n=161) to *Classroom Observation Students* (n=41) to *Focus Students* (n=25) and further refined to *High CMT Students* (n=16).

Sixteen students were selected as the top performers on the CMT scale. The analysis of the data identified that sixteen students in study displayed the capability of engaging in CMT.

Finding 2: There was no statistically significant relationship between student performance on the CMT scale and on the Slosson Intelligence Test and the Raven's Progressive Matrices scales.

Analysis of the CMT scale score and comparison of the Slosson Intelligence Test and the Raven's Progressive Matrices found that performance on the CMT scale does not relate to performance on age appropriate intelligence tests. Furthermore, the analysis has demonstrated that it appears the CMT scale is measuring something other than intelligence. Additionally, the finding validates that high levels of CMT is not assessed in the same ways as intelligence.

Finding 3: There is a significant positive relationship between student performance on the CMT and PASA AMPS scale.

There is a positive relationship between students exhibiting CMT and their awareness of pattern and structure in mathematics (see Table 5.6). The CMT scale and the PASA AMPS scale showed significant correlation. The testing of significance of a two tailed test identified similarities with both testing measures. The PASA AMPS is a mathematics assessment specific to patterns and structural awareness. In addition, there appears to be a significant positive relationship between CMT and an underlying awareness of pattern and structure (see Table 5.6).

5.4.2 Critical Mathematical Thinking Capabilities in Young Students

Finding 4: There were defined CMT capabilities in Young Students.

Analysis of the CMTLE interview data against the CMT themes and sub-themes has identified varied evidence of students using CMT in the learning experiences presented to them. All 25 *Focus Students* chosen for the one-on-one interviews exhibited the ability to engage in CMT (See Table 5.7).

The *High CMT Students* (n=16) chosen for an in-depth analysis of their interviews exhibited high levels of CMT with scores ranging from 51 (64%) to 74 (93%) out of a total of 80 (see Table 5.7).

In summary, there was no relationship between intelligence measures and CMT, suggesting that (a) there is a difference between these two constructs, or (b) CMT is not assessed in the same way as intelligence measures (see Table 5.9).

Across the 16 *High CMT Students* there was evidence of CMT pertaining to all five themes identified in the literature (see Section 3.3), namely, Interpreting, Analysing, Evaluating, Explaining and Creating (see Tables 5.12 – 5.15). The most common theme evidenced in the data was Explaining followed by Evaluating, Interpreting, and Analysing. The least common theme evidenced was Creating.

Within each theme, various sub-themes identified in the literature (see Section 3.3) were identified in the data. The number of sub-themes for each theme varied from 2 to 4. For each of the sub-themes identified there was a range of ways that the young students expressed this thinking. For example, for the sub-theme *Clarifying* students used language to describe a concept, related the concept to a real-world context, and physically modelled the concept using materials (see Table 5.12). In addition, the data analysis identified that while all five themes were present in the data not all sub-themes were evidenced in the young students in the study.

5.4.3 Exhibiting Critical Mathematical Thinking – The Role of Teacher Questioning

Finding 5: The role of the teacher questioning supported young students to exhibit their CMT.

Probing questions that support students to exhibit their mathematical thinking were found to be the most common used teacher questions in the study. When PETQs were posed to the students in this study, CMT themes and sub-themes were exhibited in student responses (see Section 5.4.3). The following questions were most frequently asked and supported young students to exhibit CMT:

- Clarifying probing questions (teachers endeavouring to clarify their own understanding of students thinking);
- Noting relationships probing questions (teaching endeavouring to understand the particular relationships students are noting);
- Guiding questions (questions aimed at verifying students' understanding);

- Assessing questions (questions aimed at assessing students having a firm foundation of the concept and the learning experience as well as an ability to communicate their ideas).

5.5 CHAPTER REVIEW

In this chapter, the results and findings of the analysis of data were presented in relation to the two research questions:

1. *What are the CMT capabilities young students exhibit as they begin formal schooling?*
2. *What types of teacher questions help these young students exhibit their CMT?*

The analysis of the results of the CMTLE and the teacher questions were presented under the following broad themes:

- Young students' ability to exhibit Critical Mathematical Thinking
- Critical Mathematical Thinking Capabilities in Young Students
- Exhibiting Critical Mathematical Thinking – The Role of the Learning Experience

In Chapter 6, these findings are discussed in the light of the research literature and context of this study.

Chapter 6: Discussion

6.1 CHAPTER OVERVIEW

In this chapter, the findings of the study are presented and discussed in terms of the literature pertaining to Mathematical Thinking, Critical Thinking and Teacher Questioning. The aims of this study were to investigate (a) Critical Mathematical Thinking (CMT) in young students, and (b) teaching actions that help these young students exhibit their CMT. The literature was presented within four overarching areas, namely, mathematical thinking, critical thinking, critical mathematical thinking and teacher actions that enhance CMT. As a result of the literature review, two research questions emerged. These were:

1. *What are the CMT capabilities young students exhibit as they begin formal schooling?*
2. *What types of teacher questions help these young students exhibit their CMT?*

The structure of this chapter is guided by three broad themes that emerged from the findings of the research and which aligned with the literature reviewed (see Section 5.4). Thus, the next three sections in this chapter are:

- Section 6.2: Young Students' Critical Mathematical Thinking
- Section 6.3: Critical Mathematical Thinking Capabilities in Young Students
- Section 6.4: Exhibiting Critical Mathematical Thinking – The role of Teacher Questioning

Within each section the contributions that this study makes to future research and education in general are delineated. The discussion begins with a focus on Sections 6.2 and 6.3, under the broad heading of CMT in Young Students. Within these, the CMT capabilities of young students together with young students' ability to exhibit CMT are discussed in light of the development of the CMT Conceptual Framework, critical thinking literature, and mathematical thinking literature. Section 6.4 discusses the types of teacher questions that assist young students exhibit their

CMT. This section concludes with exemplars and indicators of CMT in classroom practice. Figure 6.1 provides an overview of the chapter.

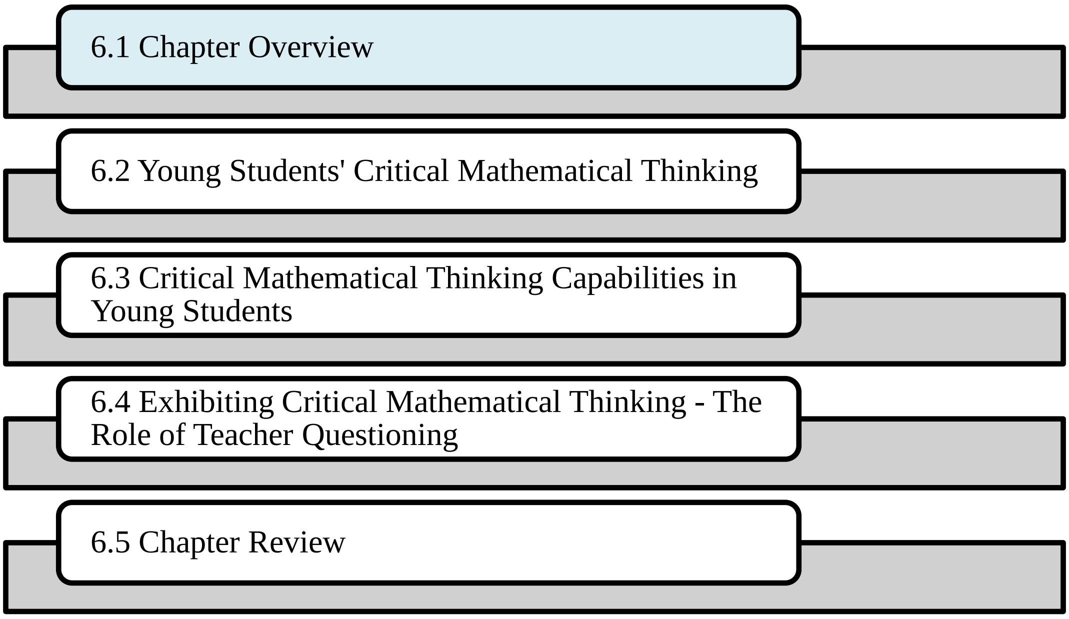


Figure 6.1. Overview of Chapter 6

6.2 YOUNG STUDENTS' CRITICAL MATHEMATICAL THINKING

Young students are capable of engaging in CMT as they enter school (Finding 1). This research finding adds to past studies which evidenced that many young students are mathematically competent prior to entering formal schooling (Bobis, 1996; Clarke et al., 2006; Clements & Sarama, 2011; Hunting et al., 2009; MacDonald & Carmichael, 2018; Papic & Mulligan, 2005; Perry et al., 2015; Scammacca et al., 2020) and mathematical thinking begins at a young age (Bobis et al 2005; Doig & Ompok, 2010; Samara & Clements, 2009). In particular, the new research findings from this present study contribute to the range of mathematical thinking capabilities of young students in prior to school and beginning school settings as identified in the literature review (see Table 3.2). The findings also provide clear insights into how young students can think critically in mathematics as they enter formal school settings. These additional insights are further discussed in Section 6.3.

In this study, there was no significant relationship between students' performance on the CMT scale and measures of mathematical reasoning (Finding 2).

Thus, CMT and mathematical reasoning appear to be different constructs for young students. Past studies have indicated that mathematical reasoning draws on a range of reasoning processes including analogical reasoning (English & Watters, 2004; White, 1998), and spatial quantitative reasoning (Carpenter et al., 1990). Additionally, some scholars have indicated that there are links between these reasoning processes and young students' cognitive ability (for example, Raven, 2000). However, the findings of this research suggest that links between these reasoning processes and CMT are weak (see Section 5.3.3). It is conjectured that this occurred for two reasons. First, CMT is different from reasoning processes and cognitive ability. Second, the measure used to ascertain high levels of CMT (that is, the CMTLE) differs from the ways that intelligence is commonly measured. Both these conjectures would benefit from further investigation.

However, there was a significant positive relationship between students' performance on CMT scale and their performance on PASA AMPS scale (a measure of pattern and structure awareness) (Finding 3). Thus, there appears to be a positive relationship between students' ability to ascertain mathematical patterns (the predictable regularity within mathematical relationships) (Mulligan et al., 2006; Orton & Orton, 1999; Warren & Cooper, 2005), the way this regularity is organised (Mulligan et al., 2009), and CMT. The findings from past research acknowledge the links between an understanding of pattern and structure and; (a) the transition from a concrete mathematical thinking approach to a more abstract approach (Mulligan et al., 2004) and, (b) mathematical achievement in later years (Papic et al., 2009; Papic et al., 2011). This new research contributes to the literature by suggesting that there is also a link between and an awareness of pattern and structure and CMT. Thus, as students enter school some already have a high awareness of pattern and structure and these students can also engage in high levels of CMT. This finding together with a more in-depth understanding of this link has implications for both research and teaching in the early years.

6.3 CRITICAL MATHEMATICAL THINKING CAPABILITIES IN YOUNG STUDENTS

The CMT Framework that emerged from the analysis of the literature pertaining to mathematical thinking and critical thinking served to aid the investigation of the research problem and interpretation of the research findings (see Section 4.2.2.1). The findings of this study served to refine this framework so that it was applicable to a particular cohort of students: students in the early years of formal schooling Analysis of the qualitative and quantitative data affirmed several themes and sub-themes presented in the CMT Conceptual Framework and also identified and defined new elements not currently presented in the mathematical thinking literature (see Table 5.18). Thus, a further contribution of this study to future research is twofold. The first is the conceptualisation, and refinement of the CMT Conceptual Framework to form the CMT Conceptual Framework for Young Students (CMTFYS), and the second is the of development of a valid and reliable CMT scale that can be used to measure CMT in young student. These contributions are significant for the following reasons:

- To date, literature pertaining to the mathematical thinking of young students as they begin formal schooling is very limited (see Section 3.2.3).
- There is a lacuna in the literature that examines high levels of mathematical thinking of these young students (see Section 3.2.3).
- The results being to address the lack of clarity with regard to what constitutes CMT in early years context (see Section 3.3).
- The results suggest that critical thinking within a subject specific domain (for example, mathematics) requires drawing on constructs of thinking from both domains (critical thinking domain and the subject specific domain), with both contributing to our understanding of domain specific critical thinking.
- The study also begins to define the terms and sub-themes of the Critical Mathematical Thinking Framework for Young Students (CMTFYS), terms that are often unclear, disjointed or not defined within the current literature (Cengiz et al., 2011; Facione, 1990; Fraivillig et al., 1999; Jacobs et al., 2010; Lai, 2011; Lipman, 1987; Miri et al., 2007; Resnick, 1987; Siswono, 2011; Stein et al., 1996).
- The study evidences the importance of the types of questions teachers

ask in assisting young students to exhibit their CMT.

- Presently there is no psychometric scale that purports to measure CMT in young students.

This section comprises two sub sections: Refining the CMT Conceptual Framework to form the CMTFYS; and defining the themes and sub-themes of CMTYS.

6.3.1 Refining the Critical Mathematical Thinking (CMT) Conceptual Framework

A CMT Conceptual Framework emerged from the literature review process (see Section 3.3.2). The CMT Conceptual Framework was constructed through the analysis of literature in the areas of mathematical thinking and critical thinking. A detailed literature review resulted in the identification of five themes and 32 sub-themes that defined the framework (see Section 3.3.2). In-depth data analysis found that while all five themes were present in the qualitative data not all sub-themes were evidenced in the young students in the study. Figure 6.2 presents a summary of the five themes and 14 sub-themes that emerged from the data analysis (see Section 5.3.5). The sub-themes not evidenced in the data analysis are presented in grey and within a box in Figure 6.2.

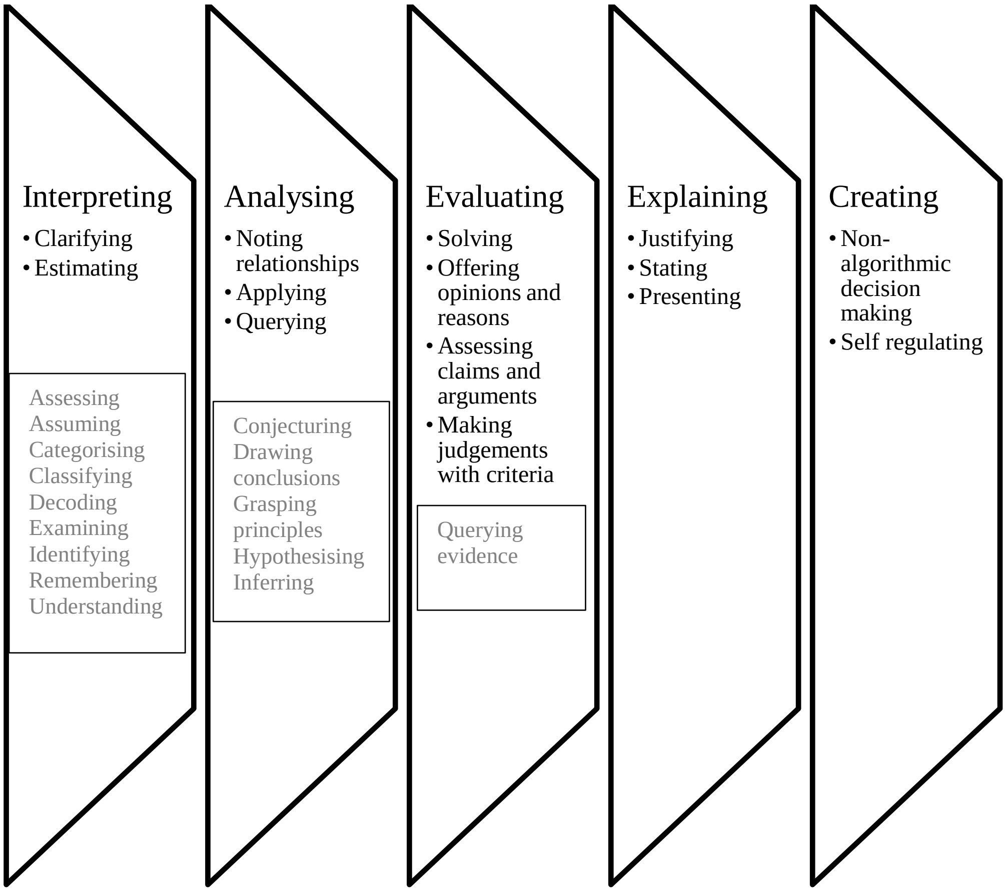


Figure 6.2. CMT Conceptual Framework

Results from this study refined the identified themes and sub-themes in the literature and provided a succinct version of a framework that begins to structure the CMT found in young students (between 5 years and 1 month to 6 years and 8 months of age) who participated in this study. There were two key findings in relation to the refinement of the CMT conceptual framework.

First, the initial identified five themes were evident in the student responses, however some were more evident than others (see Section 5.3.5). It appeared for these young students that ***Explaining*** was the most prominent critical thinking skill displayed in their responses. This was followed by ***Evaluating.*** There was limited evidence of ***Creating, Interpreting and Analysing*** in the data. The trend is further discussed at the conclusion of this section.

Second, the findings from the study identified fewer sub-themes than originally presented in the CMT Conceptual Framework (see Figure 6.2). For example, the theme in which most sub-themes were not evidenced was the theme ***Interpreting*** and thus will be used as an illustrative example. The identified sub-themes drew on several researchers' work in relation to mathematical and critical thinking (for example, Facione, 1990; Lipman, 1987; Siswono, 2011). For this particular theme, the two sub-themes that were most identified in the *High CMT Students* (n=16) were clarifying and estimating. These two CMT sub-themes occurred during two learning experiences (LE1 and LE2) (see Table 5.20).

As only two of the eleven sub-themes were evidenced in the theme ***Interpreting***, it could, therefore, be hypothesised that this occurred for one or more of the following reasons:

(a) The other sub-themes were difficult to identify in student responses from the suite of eight CMTLE presented to these young students.

(b) There is overlap between the sub-themes identified in the critical thinking literature (for example, aspects of categorising and examining form dimensions of noting relationships).

(c) Some of the sub-themes are predominantly observed in older students.

Thus, in general, the definition of the sub-themes and the overlap between and across sub-themes identified in the literature requires further investigation.

Figure 6.3 presents the final version of the CMTFYS with the themes and sub-themes colour coded according to the body of literature they emerged from. In addition, sub-themes within each theme are ordered from the most frequent to the least frequent. Colour coding aligns the literature sources that were used to develop the CMTFYS. The colour coding presents the themes and sub-themes as identified solely in the critical thinking literature in yellow, sub-themes identified solely in the mathematical thinking literature highlighted in green and sub-themes presented in both the critical thinking and mathematical thinking literature are highlighted in blue.

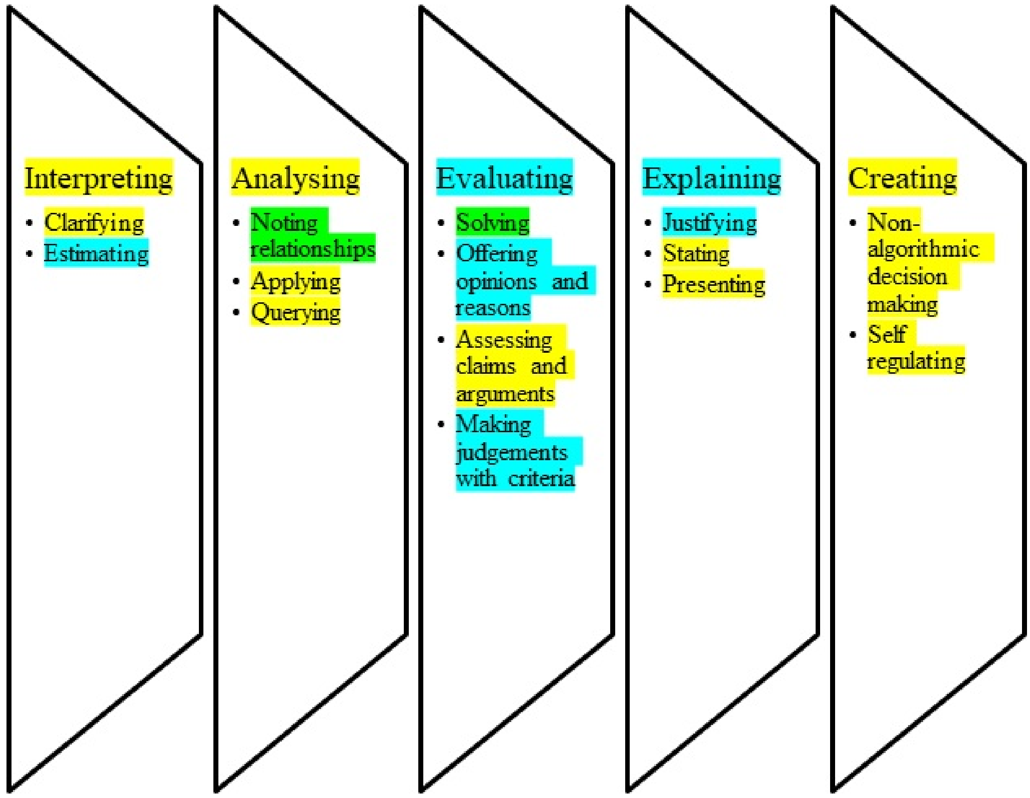


Figure 6.3. CMT Conceptual Framework for Young Students (CMTFYS)

As evidenced in Figure 6.3 both sets of literature, the critical thinking literature and the mathematical thinking literature are represented in the CMTFYS. Of importance is the literature that pertains solely to each (yellow = critical thinking and green = mathematical thinking). Thus, the unique contribution that CMFYS makes to the research literature lies in the blending of both sets of literature (critical thinking and mathematical thinking) to form the notion of CMT, domain specific thinking with critical thinking. As discussed in Section 3.3, there are differing views as to whether critical thinking is subject specific or standalone. A body of literature supports domain specific knowledge as a requirement for critical thinking (Bailin, 2002; Willingham, 2007), whereas Facione (1990) and Khun (1999) believe critical thinking is across all content or subject areas. The CMFYS supports and extends on the literature that critical thinking is subject specific.

6.3.2 Defining the Five Themes and Sub-themes of the CMTFYS

Within the literature, the meaning of the terms utilised to describe critical thinking (Section 3.3) and mathematical thinking (Section 3.2) tend not to be clearly defined and often appear to entail some overlap. For example, for mathematical thinking, differing descriptions include: Burton (1984) - content and techniques are required; Schoenfeld (1992) - sense making is a component; Clements and Sarama (2014) – there is hierarchy in relation to mathematical thinking; Suzuki (1998) - problem solving includes mathematical thinking; Wood et al. (2006) - mental activities is required from the individual. Similarly, with critical thinking, authors have no agreed upon definition. For example, Heard et al. (2020) provide an overarching reference that critical thinking includes skills, qualities and competencies that are across philosophy, psychology and education. However, other authors include other descriptions: Sternberg (1986) - mental processes; Lipman (1987) - decision making; Lai (2011) - inductive and deductive reasoning; Siswono (2010) - problem solving; Paul and Elder (2020) - construction of own thinking.

The next sections begin to readdress this lack of clarity by defining the themes and accompanying sub-themes of the CMTFYS. To help clarify the themes and sub-themes of the CMTYFYS exemplars have been drawn from the student data and are presented at the end of each section.

The five themes that emerged from the results of this study aligned with the five themes that were identified in the literature review. The themes, ***Interpreting***, ***Analysing***, and ***Creating*** emerged from the critical thinking literature while ***Evaluating*** and ***Explaining*** emerged from both critical thinking and mathematical thinking literature (see Figure 6.3).

6.3.2.1 The Theme of Explaining and Corresponding Sub-Themes

Explaining: Explaining was defined in the CMTFYS as students providing and justifying insights into their thinking, making direct statements about their thinking, and presenting strategies for finding solutions. Defining ***Explaining*** within CMTFYS encompassed drawing on aspects of mathematical thinking and critical thinking. ***Explaining*** is linked to (a) enriching classroom discourse which results in extending all students' mathematical thinking (Cengiz et al., 2011; Franke & Kazemi, 2001; Wood et al., 2006), and (b) providing insights into how students are engaging in mathematical learning experiences and what they have learnt (Cengiz et al., 2011; Franke & Kazemi, 2001; Hufferd-Ackles et al., 2004; Hunter & Anthony, 2011; McDonough & Clarke 2003; Way 2008; Wood et al., 2006). ***Explaining*** is also closely aligned with the cognitive processes of student thinking. For example, Facione (1990) suggested that asking students to articulate their thinking helps them further clarify their understanding.

An examination of the three sub-themes (*Justifying, Stating* and *Presenting*), helped to further define ***Explaining*** within CMTFYS.

Explaining **-** *Justifying****:*** The sub-theme of *Justifying* in this study involved young students using a detailed explanation and argument as they shared their solution (see Table 6.1). Justifying is a term frequently found in both the critical and mathematical thinking literature. It appears that the term justifying is not easily defined, and in most cases is associated with the term reasoning (Diezmann et al., 2001; Facione, 1990; Hunter & Anthony, 2011; Papic et al., 2009; Papic et al., 2011; Vale et al., 2017). However, the results from past research have contributed to the identification of common features pertaining to *Justifying* (see Section 3.2.2.2). These include defending solutions to problems (Wood et al., 2003), using facts and properties to test conjectures, (Diezmann et al., 2001), presenting logical arguments (Melhuish et al., 2020), and verifying predictions (Alised, 2003). The results from this present study aligned with these common features.

Explaining **–** *Stating* and ***Explaining*** - *Presenting:* In this study *Stating* involved young students making a direct statement when explaining a solution to a problem (Fennema et al., 1998) (see Table 6.1) and, *Presenting* entailed presenting a strategy for finding a solution with the inclusions of mathematical language (see Table 6.1). Both of these sub-themes emerged from the critical thinking literature. Within the critical thinking literature Facione (1990) combines *Stating* with the terms *Justifying* and *Presenting* rather than directly defining the terms themselves (see Section 3.2.2.2). This study contributes to the literature by beginning to define these two terms for young students.

Thus, it is inferred that the sub-themes for ***Explaining*** (*Justifying, Stating and Presenting)* add depth to our understanding of ***Explaining*** in young students. Table 6.1 includes the themes sub-themes and description as presented in Table 5.18.

The final column presents exemplars of each theme and sub-theme drawn from the data (see Appendix C).

Table 6.1
Description and Examples of Student Evidence of Explaining within the CMT Conceptual Framework

Themes	Sub-theme	Description	Examples
Explaining	Justifying	Justifying a solution by using a detailed explanation and argument	• Expressing the process of drawing lines to justify the location of the middle • Providing an argument for why a particular strategy will not work • Arguing and demonstrating that folding a piece of paper in half 'points to the middle' • Using the notion of fairness to justify the concept of equality
	Stating	Making statement with regard to the solution to the problem	• Stating the most effective approach to solving the problem • Stating that expression must be the same when their total is the same • Stating the towers have the same number of block – the difference is the size of the blocks not the number
	Presenting	Presenting a strategy for finding a solution with the	• Presenting a range of strategies to find the middle using mathematical language • Presenting a rule to find the middle

6.3.2.2 The Theme of Evaluating and Corresponding Sub-Themes

Evaluating: Evaluating was defined in CMTFYS as the process that occurred when students were assessing their thinking and offered opinions about their thinking as they solved mathematical problems. This definition aligns with both the mathematical thinking literature and the critical thinking literature. Within the mathematical thinking literature, Williams (2000) considered *Evaluating* occurring as students consider mathematical ideas from multiple perspectives. Moreover, *Evaluating* is recognised as a demonstrable process as students demonstrate their mathematical thinking (for example, Ball, 1993; Maričić & Špijunović, 2015; Steinberg et al., 2004; Wood et al., 2006).

Within the critical thinking literature, Lai (2011) suggests that *Evaluating* is a part of a skill set that also includes: "analyzing arguments, making inferences using inductive and deductive reasoning… and making decisions or solving problems" (p. 2). Lipman (1987) also identifies that *Evaluating* something requires one to think critically.

An examination of the three sub-themes (*Solving*, *Offering opinions with reasons*, and *Assessing claims*), helped to further define Explaining within CMTFYS.

Evaluating* – *Solving: The results from this present study identified young students' *Solving* a mathematical problem as young students using well known or invented strategies to reach solutions (see Table 6.2**)**. Three differing perspectives with regard to the sub-theme *Solve* emerged from the mathematics education literature. These are (a) students inventing their own strategies to solve problems (Fennema et al., 1998), (b) students using known problem-solving strategies to solve problems (Cheeseman, 2008; Franke & Carey, 1997), and (c) young students using manipulatives to solve problems (Fennema et al., 1998). The Critical thinking literature also acknowledged the sub-theme *Solving*, however, it is not clearly defined and appears with the notion of problem-solving (Kim, 2011; Lai, 2011, Lewis & Smith, 1993; Siswono, 2010). Thus, the definition emerging from this present study aligns with perspectives (a), (b) and (c) presented in mathematical thinking.

***Evaluating* – *Offering opinions with reasons*:** Within the critical thinking literature while *Offering opinions with reasons* is acknowledged as one of the hall marks of 'good' thinking (Lipman, 1987) (see Section 3.3.1), the term itself is not clearly defined. The CMTFYS findings suggest that young students are *offering opinions and reasons* when they provide valid reasons for their claims and opinions (see Table 6.2). Additionally, young students also often support these reasons with the use of physical gestures (see Table 6.2).

***Evaluating* - *Assessing Claims*:** *Assessing Claims*, according to both Ellerton (2018) and Facione (1990) is an evaluative process that requires students to demonstrate inferential critical thinking skills (see Section 3.3.1). The results from this present study suggest that while young students may not clearly demonstrate inferential critical thinking, they certainly can assess their claims and provide arguments for relationships they have noted (see Table 6.2).

***Evaluating** - Assessing Claims**:** Assessing Claims,* according to both Ellerton (2018) and Facione (1990) is an evaluative process that requires students to demonstrate inferential critical thinking skills (see Section 3.3). The results from this present study suggest that while young students may not clearly demonstrate inferential critical thinking, they certainly can assess their claims and provide arguments for relationships they have noted (see Table 6.2).

Evaluating –** Making judgements with criteria**: In this study, young students made 'well informed' *Judgements* based on their understandings of other concepts or ideas (see Table 6.2). This definition of *Making Judgements* aligns with the critical thinking literature where *Making Judgements* was identified as well informed decision (for example, with specific criteria), that an individual exhibits when displaying critical thinking in a range of learning areas (Facione, 1990; Lipman, 1987).

Thus, it is suggested that the sub-themes for ***Evaluating*** (*Solving, Offering opinions and reasons, Assessing claims and Making judgements with criteria)* add depth to our understanding of ***Evaluating*****.** As young students are evaluating their CMT as they solve problems they offer opinions, assess their claims and make informed judgements. Table 6.2 includes the themes, sub-themes and description for ***Evaluating*** together with examples drawn from the data (see Appendix C).

Table 6.2

Description and Examples of Student Evidence of Evaluating within the CMT Conceptual Framework

Theme	Sub-theme	Description	Examples
Evaluating	Solving	Solves mathematical problems using problem solving strategies	• Providing and evaluating a solution to the problem
	Offering opinions and reasons	Offering reasons for their claims and opinions	• Providing a range of valid reasons as to why their thinking is correct • Using physical gestures to support their reasoning
	Assessing claims	Assessing the claims made with regard to the relationships they noted	• Providing an argument as to while a claim about a relational between two different mathematical concepts holds.
	Making judgements with criteria	Making judgements based on understanding of smaller and larger	• Making a judgement using a criterion (for example, this block is longer because of the size of the blocks not because of the number of blocks)

The examples provided in Table 6.2 are a result of the identification of common themes presented in student responses.

The next section focuses on defining the three themes ***Creating***, ***Interpreting*** and ***Analysing***. Within this present study, the frequency of occurrence of these three themes was substantively less than for the two preceding themes, ***Explaining*** and ***Evaluating***. Hence, the evidence for the definition of these terms within the CMTFYS is not as strong. The discussion relating to why this trend occurred in the CMT in the early years of schooling is at the end of this section, Section 6.3.2.6.

6.3.2.3 The Theme of Creating and Corresponding Sub-Themes

Creating: Creating was defined in the CMTYS as occurring when young students were engaging in *Non-algorithmic* ways of solving problems and *Self-regulating* their thinking. The initial definition of *Creating* was drawn from the critical thinking literature (Anderson, 1999; Lewis and Smith, 1993; Wilson, 2016), as the term creating is not commonly linked with mathematical thinking. Within this critical thinking literature ***Creating*** is often considered the most difficult mental function (Anderson, 1999; Wilson, 2016). ***Creating*** is exhibited when students are: (a) joining elements to form a whole understanding (Wilson 2016), (b) reorganising elements to identify a new structure (Wilson, 2016), (c) engaging in non-algorithmic decision-making (Siswono, 2010), and (d) self-regulating one's thinking (Facione, 1990). It was the latter two that appeared in the data from this study, and hence they formed the definition of ***Creating*** for CMTFYS.

The two sub-themes pertaining to ***Creating*** (the least demonstrate Sub-theme) in the CMTFYS are *Non-algorithmic decision making and Self-regulating.*

Creating – *Non-algorithmic decision making*: The results from this present study support the notion that young students are inclined to use *Non-algorithmic decision making* to find the solution to a problem that does not have an instant answer (see Table 6.3). *Non-algorithmic decision making* is considered a complex or higher order thinking skill (Miri et al., 2007; Resnick, 1987; Siswono, 2010).

***Creating* –** *Self-regulating:* Within the CMTFYS, young students exhibited the ability to self-regulate when they identified their own misconceptions and then regulated their thinking to find a new solution (see Table 56.3). *Self-regulating* is (a) a cognitive skill that an ideal critical thinker possesses when self-evaluating their own inferences (Facione, 1990), and/or an action that can occur when one is engaging in higher order thinking (Resnick, 1987) (see Section 3.3).

Thus, the proposition presented in this section is that ***Creating*** and the sub-themes of *Non-algorithmic decision making* and *Self-regulating* enhance a young student's ability to present CMT. Table 6.3 comprises the themes and sub-themes together with descriptions and examples for ***Creating*** derived from the data (see Appendix C).

Table 6.3
Description and Examples of Student Evidence of Creating within the CMT Conceptual Framework

Theme	Sub-theme	Description	Examples
Creating	Non – algorithmic decision making	Uses non-algorithmic decision making to apply mathematical ideas	• Uses other problem-solving strategies, that are non-algorithmic, to find the solution to a problem
	Self-regulating	Reflecting on misconceptions and self-regulating to find another solution	• Identifying own misconceptions and self-regulating to find another solution

The examples provided in Table 6.3 are a compilation of the results and findings of this present study.

6.3.2.4 The Theme of Interpreting and Corresponding Sub-Themes

Interpreting: Interpreting was defined in the CMTFYS as when young students were clarifying meaning or using estimating to help them to make sense of an idea. This definition aligns with Facione (1990) who states that an individual is interpreting when they are using skills, such as, categorising to clarify meaning.

The results identified two sub-themes that emerged from the ***Interpreting*** theme. These were *Clarifying* and *Estimating*.

Interpreting- *Clarifying****:*** In this present study *Clarifying* supported young students as they ***Interpreted*** learning experiences. For example, they used mathematical language or physically modelled an idea in order to 'clarify concepts and ideas – making sense of mathematics' (Fennema et al., 1996; Fennema et al., 1998; Wood et al., 2006) and 'describe solutions or elaborate on ideas' (Clements & Sarama, 2014; Empson, 2011; English & Watters, 2004; Fraivillig et al., 1999; Mulligan et al., 2012; Mulligan et al., 2004) (see Section 3.3.1). Evidence of *Clarifying* in young students involved using a variety of strategies to make sense of a particular concept (see Table 6.4).

Interpreting- *Estimating****:*** Extending on what was identified in the literature, that is that *Estimating* is a critical thinking and reasoning skill (Lipman, 1987), young students in this present study estimated by using informal units and other known techniques and strategies to help them understand an idea (see Table 6.4).

Additionally, even though Clarifying and Estimating are often presented as discrete constructs in the literature, the results pertaining to the use of *Clarifying* in this study evidenced they never occurred in isolation. When coupled with *Estimating*, *Clarifying* was a powerful component of the CMT theme of ***Interpreting***. Table 6.4 identifies the themes, sub-themes and description for ***Interpreting*** together with examples drawn from the data (see Appendix C).

Table 6.4
Description and Examples of Student Evidence of Interpreting within the CMT Conceptual Framework

Theme	Sub-theme	Description	Examples
Interpreting	Clarifying	Clarifying a concept (middle)	• Uses language to describe the concept • Relates the concept to a real-world context • Models the concept (physically, using materials)
	Estimating	Estimating how many	• Uses an informal unit to help estimate (which is longer) • Uses counting techniques to help estimate (how many)

The examples for young students ***Interpreting*** and each sub-theme for this theme, can be used by teachers to identify the CMT presented by young students (see Table 6.4).

6.3.2.5 The Theme of Analysing and Corresponding Sub-Themes

Analysing**: **Analysing was defined in the CMTFYS as when students noted relationships between concepts, applied a concept to a new context or queried others' reasoning and ideas. Within the critical thinking literature *Analysing* as identified by Facione (1990) is when an individual examines ideas, identifies arguments and analyses arguments, a definition that lacks the notion of applying concepts to new contexts.

The results identified three sub-themes that emerged from the ***Analysing*** theme, namely *Noting Relationships, Applying*, and *Querying.*

Analysing – *Noting Relationships:* The results in this present study found that *Noting Relationships* occurred when a young student related a mathematical concept with another concept or method. For example, they related the concept of 'same amount' to the concept of 'even distribution' (see Table 6.5). Commonly within the literature, the notion *Noting Relationships* linked (a) with either relating to real-life applications (see Section 3.3.1) or (b) relating concepts, questions and methods (Fraivillig et al., 1999; House & Coxford, 1995; Stein et al., 1996). The CMTFYS definition aligns with the second notion, relating concepts.

Analysing – *Applying:* In this present study *Applying* occurred when a known concept was applied to a new context. It is noted in the literature, that in order to

Apply a concept, it first must be understood (Wilson, 2016) (see Section 3.3). This was evidenced in CMTFYS when a student used the concept of half to find the middle of an object.

***Analysing* – *Querying*:** Evidence of the presence of this sub-theme was weak in the data. Thus, the definition of *Querying* for CMTFYS is limited. One example of *Querying* was evidenced when a young student queried the teacher's approach to gauging how many unseen items were originally stated in the learning experience. The critical thinking literature review identified *Querying* as a way to inference and draw conclusions (Facione, 1990) (see Section 3.3.1).

The results pertaining to ***Analysing*** provide further insight into how a young student can display this CMT (see Table 6.5). The findings support that ***Analysis*** on its own is not enough, and that the sub-themes provide greater insights into young students' ability to analyse (see Appendix C).

Table 6.5
Description and Examples of Student Evidence of Analysing within the CMT Conceptual Framework

Theme	Sub-theme	Description	Examples
Analysing	Noting relationships	Noting relationships between two different mathematical concepts	• Noting the relationship between addition and equivalence • Noting the relationship between total and the sum of odd and even numbers • Noting the relationship between same amount and even distribution
	Applying	Applying a known concept to a new context	• Using the concept of a half to find the middle
	Querying	Querying the teacher's approach	• Querying how many items are unseen

The theme ***Analysing*** is best understood through the sub-themes of *Noting Relationships, Applying*, and *Querying.* Young students' responses were categorised to formulate a description and examples of what the theme and sub-themes could be presented by young students.

6.3.2.6 Themes and Sub-themes of the CMTFYS - Concluding comments

The findings pertaining to the CMTFYS indicate that themes and sub-themes can be identified and defined for young students (Finding 4). When reviewing each sub-theme, however, it is evident that some learning experiences provide a greater opportunity for young students to display components of the CMTFYS. Therefore, the CMTFYS is a framework with aspects of possible ways CMT is exhibited in young students. Although critical thinking and mathematical thinking literature has supported the formation of the original CMT Framework (see Section 3.3), the literature reviewed was derived from students in all year groups. The newly refined CMTFYS (see Section 6.3.1) contributes to the body of existing critical thinking and mathematical thinking literature and presents a formal framework with definitions of themes and exemplars of these themes that young students can exhibit.

The themes *Explaining* and *Evaluating* were evidenced most frequently in this study. This could be for two reasons. First, research focusing on exhibiting young students mathematical thinking resulted in an emphasis on encouraging students to explain and evaluate their thinking. Second, as evidenced in both the mathematical thinking literature and the critical thinking literature, explaining and evaluating one's thinking are key components of thinking, from both learning and clarifying learning perspective (Cengiz et al., 2011; Facione, 1990; Kim, 2011; Lai, 2011; Lipman, 1987; Papandreou & Tsiouli, 2020; Wood et al., 2006).

By contrast, within this present study, the frequency of occurrence of the three themes *Creating*, *Interpreting* and *Analysing* were substantively less than for the two preceding themes. As these three themes emerged from the critical thinking literature, it could be suggested that their absence within the mathematical thinking literature could pertain to the perceived nature of mathematics within the educational community. That is a rigid approach to teaching young students (Stanley, 2008), limited teacher content knowledge (Cohrssen et al., 2013; Lee & Ginsburg, 2009; MacDonald, 2020) and teacher beliefs about what young students can do in mathematics (Stipek et al., 2001). Even though the themes are valued notions in mathematics education, they are not presently included in the mathematical education curriculum (ACARA, 2009, 2018; DEEWR, 2009).

6.4 CRITICAL MATHEMATICAL THINKING – THE ROLE OF TEACHER QUESTIONING

The role of teacher questioning is found to be significant in promoting students' mathematical thinking (Fraivillig et al., 1999; Franke et al., 2009; Jacobs et al., 2010; Martino & Maher, 1999; McCullough & Findley, 1983; Rigelman, 2007; Sahin & Kulm, 2008; Sukmadewi, 2014; Wood & McNeal, 2003). There are many different ways and terms related to teacher questioning. Teacher questioning is deemed important, however, there is little evidence that clearly links teacher question to mathematical thinking. Teacher questioning was found to play a significant role in this study (Finding 5). The findings in this present study indicate that particular types of questions presented to young students assist student talk in mathematics learning experiences. This notion is well supported in mathematics education literature. For example, studies that encourage student talk supported students to discuss their mathematical ideas (Diezmann et al., 2002; Hunter & Anthony, 2011; Papic et al., 2009). Cengiz and colleagues (2011) also support this notion. They identified that the most common method that supported mathematical thinking with upper primary aged students included asking students to evaluate their own or other students' ideas or solutions. Similarly, Papic et al. (2009) identified that questions that require a young learner to 'explain' encouraged student talk in the early years. Findings from this present study closely align with Papic's finding. As indicated in previous research, student talk is important as it supports students to discuss their thoughts regarding CMT.

Three particular types of questions were utilised in this present study, namely factual, probing and guiding. The selection of these three types of questions was guided by the literature review process (see Section 3.5.3). Literature focusing on ways to exhibit high levels of mathematical thinking in students identified three main ways in which teachers frame their questioning. These were ***Factual*** (Di Teodoro et al., 2011; King, 1989; Sahin & Kulm, 2008; Shahrill, 2013; Vacc, 1993), ***Probing*** (Barnes, 1976; Franke et al., 2009; Martino & Maher 1999; Rigelman, 2007; Sahin & Kulm, 2008) and ***Guiding*** (Funahashi & Hino, 2014; Kojo et al., 2018; Mata-Pereira & da Ponte, 2017; Ortenzi, 2002; Sahin & Kulm, 2008; Sukmadewi, 2014). The teacher questioning data in this present study were coded according to these three main types of teacher questions.

The findings in the present study identified that the most frequent use of questions occurred when students were ***Explaining*** their thinking**,** and these questions were found to probe students and extend young students' mathematical thinking (Barnes, 1976; Franke et al., 2009; Martino & Maher 1999; Rigelman, 2007; Sahin & Kulm, 2008). Although mathematics education literature identifies students' ***Explaining*** as an outcome of teacher questioning (Diezmann et al., 2002; Hunter & Anthony, 2011; Papic et al., 2009) the direction of the present study allowed for further identification and analysis of the types of questions used when students were ***Explaining*** their thinking.

The teacher questions were then coded into the following three categories: ***Factual, Probing and Guiding*.** Factual (lower order) (Shahrill, 2013), probing (extending understanding) (Franke et al., 2009; Rigelman, 2007; Sahin & Kulm, 2008) and guiding questions (direct students) (Funahashi & Hino, 2014; Kojo et al., 2018; Mata-Pereira & da Ponte, 2017) are teacher questions that are found to support students during mathematics learning experiences. Although these questions are found to be beneficial during mathematics learning experiences, what is lacking are specific teacher questions that are appropriate for young learners and can elicit CMT.

The coding process identified that categorising teacher questioning into ***Factual, Probing and Guiding*** provided some findings, however, further scrutiny of the questions was required. At that point of the analysis of data the CMTFYS was reviewed and refined as a framework. To address the research question, *What types of teacher questions help these young students exhibit their CMT?*, the teacher's probing and guiding questions were further analysed in alignment with the themes of the CMTFYS (see Section 5.4.3). This process extends on current research where categorisation was specific to factual, probing and guiding questions.

Factual questions were least common and often asked initially during the learning experiences, therefore, not discussed. This aligns with the current research that factual questions are lower order in nature and provide very little understanding of a student's thinking (Sahin & Kulm, 2008; Shahrill, 2013). Thus, one contribution of the findings of this study into past research is the further refinement of the types of questions that can be used to elicit critical mathematical thinking in young students, namely, Probing and Guiding.

6.4.1 The Role of *Probing* Questioning

The sub-themes within the CMTFYS were used as a lens to further understand the types ***probing*** questions the teacher used as students exhibit their thinking. Although previous studies have identified probing questions as a way to support students to move from lower level to higher levels of thinking (Barnes, 1976; Franke et al., 2009; Martino and Maher 1999; Rigelman, 2007; Sahin & Kulm, 2008), the alignment of teacher probing questions with the CMTFYS provided further insight into specific questions that can elicit young students' CMT.

It was found that the researcher moved across three themes and one sub-theme from each theme when using teacher probing questioning. The three themes and sub-themes were as follows:

- ***Interpreting*** *-Clarifying,*
- ***Analysing*** - *Noting Relationships*
- ***Evaluating*** - *Offering Opinions*

It also should be noted that as the researcher moved across these three themes as she asked her questions. In order to review the teacher questions, the researcher developed statements, guided by the literature, to describe the three sub-themes (*Clarifying, Noting Relationships and Offering Opinions)* in light of teacher questioning. This process extends on the use of CMTFYS as a support framework for teachers of young students in mathematics. This additional classification provides examples for teachers to use with their students during mathematics learning experiences.

6.4.1.1 Teacher Interpreting – Clarifying Questioning to Exhibit CMT

Clarifying questions were asked when the researcher wanted further clarification of students' mathematical thinking. The researcher rephrased or re-used the students' terms to gain further insight into their mathematical thinking (see Section 5.4.3, Appendix D). Teacher questions were categorised to provide a description and examples of ***Interpreting***- *Clarifying* teacher questions (see Table 6.6). Assumptions could be made that using the lens of *Clarifying,* teacher questioning supports young students to exhibit CMT.

6.4.1.2 Teacher Analysing – Noting Relationships Questioning to Exhibit CMT

Noting Relationships questions were used to further probe students' mathematical thinking. This included questions to gain further understanding of the relationships students were seeing across mathematical concepts (see Section 5.4.3, Appendix D). The findings from this study support teacher questions aligned with ***Analysing*** *– Noting Relationships* as a way to support young students to formulate a description and examples of their CMT (see Table 6.6).

6.4.1.3 Teacher Evaluating – Offering Opinions Questioning to Exhibit CMT

Offering Opinions teacher questions were used by the researcher to redirect young students' thinking (see Section 5.4.3, Appendix D). Teacher questions were categorised with ***Evaluating*** *– Offering Opinions* to provide teachers with a description and examples (see Table 6.6). Within this component of teacher questioning, findings supported the inclusion of ***Evaluating*** *– Offering Opinions* teacher questions as a platform to support young students to exhibit their CMT.

Table 6.6 provides an overview of the components of the CMTFYS that can be used to delineate the types of teacher questions that support young students to explain their CMT. The questions were classified according to the CMTFYS. These were, ***Interpreting*** *– Clarifying;* ***Analysing*** *– Noting Relationship; and* ***Evaluating*** *– Offering Opinion.* The second column provides a description of the teacher questions used to support students to explain their CMT. The final column in Table 6.6 lists some examples of the types of teacher questions asked for that particular theme.

Table 6.6
CMTFYS: Types of Teacher Questions to Support Students to Explain their CMT

Alignment with the CMTFYS	Teacher Questions to Support Students to Explain their CMT	Example Teacher Questions
Interpreting *Clarifying*	*Clarifying* teacher questions were asked when the researcher wanted further clarification of students' mathematical thinking. The researcher rephrased or re-used the students' terms to gain further insight into their CMT.	How did you find that out? What do you mean by that? How do you know they're the same?
Analysing *Noting Relationships*	*Noting relationship* teacher questions were used to further probe students' CMT. This included questions to gain further understanding of the relationships students were seeing across mathematical concepts.	What about this way? How do you know it is the middle? How else can we find out if we do not count? Is there another way? Can you think of another way?
Evaluating *Offering Opinions*	*Offering opinion* teacher questions are those that the researcher asked to redirect student thinking.	What do think? Is it in the middle? Can you explain that to me? Why else are they the same? Can you explain what that would look like?

6.4.2 The Role of *Guiding* Questioning

The findings in the present study identified that the second most frequent use of questions occurred when students were asked ***Guiding*** teacher questions to support their CMT. ***Guiding*** questions direct students to derive concepts or procedures to solve problems (Funahashi & Hino, 2014; Kojo et al., 2018; Mata-Pereira & da Ponte, 2017; Ortenzi, 2002; Sahin & Kulm, 2008; Sukmadewi, 2014). These types of questions were found to promote students' mathematical thinking (Sahin & Kulm, 2008), interpret their own and others thinking, reason, and evaluate their thinking (Mata-Pereira & da Ponte, 2017). The researcher presented ***Guiding*** questions to the young students in this study and the findings indicate that further classification of the ***Guiding*** questions posted was required.

Identified sub-themes of the CMTFYS were used as a platform to further understand the types of teacher questions used. The researcher moved across two themes and sub-themes within those themes when posing teacher questioning. Therefore, the themes and sub-themes were as follows:

- ***Interpreting*** *-Assessing,*
- ***Interpreting*** - *Understanding*
- ***Creating*** *– Self-regulating*

In order to review the teacher questions, the researcher developed statements, guided by the literature, to describe the three sub-themes (*Assessing, Understanding and Self-regulating)* in light of teacher questioning. This process extends on the use of CMTFYS as a support framework for teachers, with example questions, for young students during mathematics learning experiences.

6.4.2.1 Teacher Interpreting – Assessing Questioning to Exhibit CMT

Assessing teacher questions are those that the researcher asked during the interview to identify if the student had a firm foundation of the concept and the learning experience as well as an ability to communicate their ideas (see Section 5.3.5, Appendix D). Teacher questions were categorised with ***Interpreting*** *– Assessing Questions* to interpret the data and offer teachers a description and examples (See Table 6.6). Within this component of teacher questioning, findings from this study advocated for the inclusion of ***Interpreting*** *– Assessing Questions* teacher questions as a way teachers support young students to exhibit their CMT.

6.4.2.2 Teacher Interpreting – Understanding Questioning to Exhibit CMT

Understanding teacher questions were asked by the researcher to verify the students' understanding (see Section 5.3.5, Appendix D). ***Interpreting*** *– Understanding* teacher questions were categorised in this way to provide teachers with a description of this type of questioning and examples that could be used (see Table 6.6). Findings from this present study support the inclusion of ***Interpreting*** *– Understanding* teacher questions as a process used for students to exhibit their CMT.

6.4.2.3 Teacher Creating – Self-Regulating Questioning to Exhibit CMT

Self-regulating questions were asked by the researcher to assist students to monitor and evaluate their thinking during the problem solving (see Section 5.3.5, Appendix D). ***Creating*** *– Self-regulating* teacher questions provide teachers with a description of this type of questioning and examples that could be posed by the teacher (see Table 6.6). ***Creating*** *– Self-regulating* teacher questions are now considered as a way teachers can support students to exhibit their CMT.

Table 6.7 provides an overview of the components of the CMTFYS that can be used to delineate the types of teacher questions to support and guide students' CMT. The questions were classified according to the CMTFYS. These were, ***Interpreting** – Assessing Questions; **Interpreting** – Understanding; and **Creating** – Self-regulating.* The second column provides a description of the teacher questions used to support students and guide students' CMT. The final column in Table 6.7 lists some examples of the types of teacher questions asked for that particular theme.

Table 6.7
CMTFYS: Types of Teacher Questions to Support and Guide Students' CMT

Alignment with the CMTFYS	Teacher Questions to Support and Guide Students' CMT	Example Teacher Questions
Interpreting *Assessing*	*Assessing* teacher questions are those that the researcher asked during the interview if the student had a firm foundation of the concept and the learning experience as well as an ability to communicate their CMT.	What if I want to put it in the centre, right in the middle of everything? How can I find out where that is?
Interpreting *Understanding*	*Understanding* teacher questions were asked by the researcher to verify the students' understanding.	What about this way? Is this the middle up here? How can I count the tiles? Explain that to me.
Creating *Self-regulating*	*Self-regulating* teacher questions were asked by the researcher to assist students to monitor and evaluate their CMT during the problem solving.	You think that is the middle? Can you think of any other way we can find out without drawing it?

The findings pertaining to teacher questioning practice indicate that the researcher moved across all themes of the CMTFYS. However, distinct themes and sub-themes, as presented in column one of Tables 6.6 and 6.7, were identified as the significant ways in which the researcher posed teacher questioning.

6.5 CHAPTER REVIEW

Within this chapter, the findings that emerged from this present study were explored, reviewed and discussed in alignment to the literature and conceptual framework. An understanding of CMT for young students was reviewed and supported the refinement of the CMT conceptual framework now presented as the critical mathematical thinking for young students (CMTFYS). This new conceptual framework presents a scaffold with aspects of possible ways CMT is exhibited in young students. The defined themes and sub-themes within the CMTFYS are exemplars of ways young students can exhibit CMT. The discussion relating to teacher questioning emphasised the importance of supporting a young student to exhibit their CMT. The CMTFYS was found to have components that can support a teacher to provide questioning to exhibit CMT in young students. Themes and sub-themes of the CMTFYS were discussed as ways to use teacher questioning.

Chapter 7 addresses the research questions, presents the limitations of the study, identifies recommendations and further research considerations.

Chapter 7: Conclusion and Recommendations

7.1 CHAPTER OVERVIEW

This closing chapter reviews the findings of the study in alignment with the research questions. In addition, it presents a Conceptual Framework (CMTFYS) applicable to teachers of young students. Conclusions are provided, implications for teaching are discussed, and suggested areas of further research are outlined. Figure 7.1 presents an overview for the chapter.

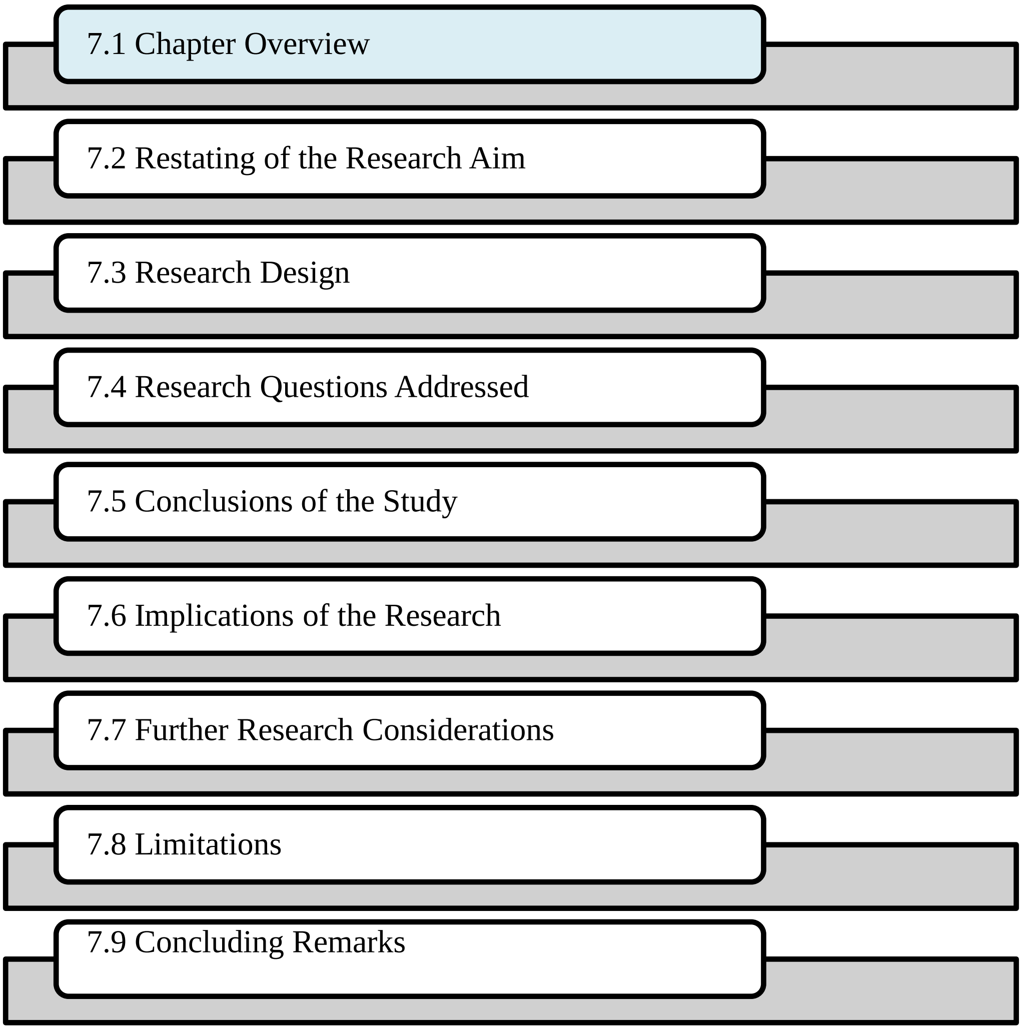


Figure 7.1. Overview of Chapter 7

7.2 RESTATING OF THE RESEARCH AIM

The aims of the study were to investigate (a) Critical Mathematical Thinking (CMT) in young students, and (b) teaching actions/questions that help these young students exhibit their CMT. To support the first part of the study, *CMT in young students,* processes were taken to develop a CMT Conceptual Framework. These processes involved synthesising a wide range of literature pertaining to mathematical thinking and critical thinking. The purpose of the literature review was to firstly determine the status of mathematical thinking and critical thinking in mathematics teaching and learning, and secondly to identify and synthesise key themes that emerged from the literature. To understand critical thinking, a review of literature sources and educational policies occurred. It was found that there was a lack of literature addressing the term critical mathematical thinking (CMT), and in particular CMT in young students. Therefore, an opportunity was provided to combine both critical thinking and mathematical thinking to develop a literature based conceptual framework for CMT. The data analysis resulted in the refinement of the CMT Conceptual Framework to form CMT Framework for Young Students (CMTFYS). The formation of the CMTFYS presents new knowledge in recent literature with regard to how young students display CMT.

In line with the second aim of the study, *teaching actions/questions that help these young students exhibit their CMT*, sub-themes within the CMTFYS were used as a lens to analyse the types of teacher questions asked in the interviews. Three themes and one sub-theme from each theme of the CMTFYS were explored to identify an array of types of teacher questions asked. The teacher questioning findings build on current research with regard to how teaching actions, specifically questioning, help exhibit CMT.

7.3 RESEARCH DESIGN

The study contributes to the research related to young students' CMT. In order to investigate CMT and ways to identify users of CMT, the following research questions guided the design of the study:

1. *What are the CMT capabilities young students exhibit as they begin formal schooling?*

2. *What types of teacher questions help these young students exhibit their CMT?*

Because the study explored CMT in young students, and teaching actions/questions that help these young students exhibit their CMT, an interpretive paradigm was adopted for the research. A constructivism epistemology was adopted to inform the instructional strategies used in this study. Thus, the interpretative paradigm together with the constructivism epistemology allowed the researcher to interpret or gain in-depth insights through the use of multiple forms of data. Further, an explanatory case study methodology was implemented for the study, which included the collection and analysis of both quantitative and qualitative data to answer the research questions.

The research was conducted across five kindergarten classrooms from three schools located in New South Wales, Australia. The 161 student participants were all in their first six months of their first year of formal schooling. The average age for these students was six.

To explore young students' CMT and what teaching actions/questions that help these young students exhibit CMT, a range of data gathering instruments and interviews were used. These were:

- One observation instrument:
  - CMT Observation Guide;
- Three quantitative instruments:
  - Raven's Progressive Matrices (Raven et al., 1998b);
  - Slosson Intelligence Test (SIT-P) (Erford et al., 2008);
  - Patterns and Structure Assessment (Mulligan et al., 2015)
- One Clinical interview:
  - The Critical Mathematical Thinking Learning Experiences (CMTLE)

Students were selected to participate in specific stages of the study. A narrowing of the initial student sample of 161 students occurred as the study progressed. The reason of this purposeful selection process was to identify students that were considered to exhibit high levels of CMT. An analysis of the quantitative data assisted this process. The four groups of student participants were selected across the three schools and were identified as*:*

- *All Kindergarten Students* (n=161);
- *Classroom Observation Students* (n=41);
- *Focus Students* (n=25) and
- *High CMT Students* (n=16).

The researcher played a pivotal role during the qualitative component of the study. The teaching actions, in particular questioning, of the researcher encouraged students to elaborate on their responses while refraining from steering students towards a desired answer.

7.4 RESEARCH QUESTIONS ADDRESSED

In order to address the aim of this research study, two questions were developed as a result of the synthesis of the literature. The main findings that emerged from this study are addressed in relation to the two research questions.

7.4.1 Research Question One

What are the CMT capabilities young students exhibit as they begin formal schooling?

The significant contributions to the literature and research field is the defining and conceptualisation of critical mathematical thinking for young students, and the development of a reliable and valid CMT scale to measure critical mathematical thinking in young students. The CMT Framework is unique and novel as it combines both mathematical thinking and critical thinking to form the new construct of critical mathematical thinking for young students.

As evidenced in this study, young students are capable of engaging in CMT as they enter school (see Section 5.3.5).

The CMT capabilities are described in the form of the Critical Mathematical Thinking For Young Students (CMTFYS) Framework which is a refined version of the CMT Conceptual Framework presented in the literature review (see Figure 7.2).

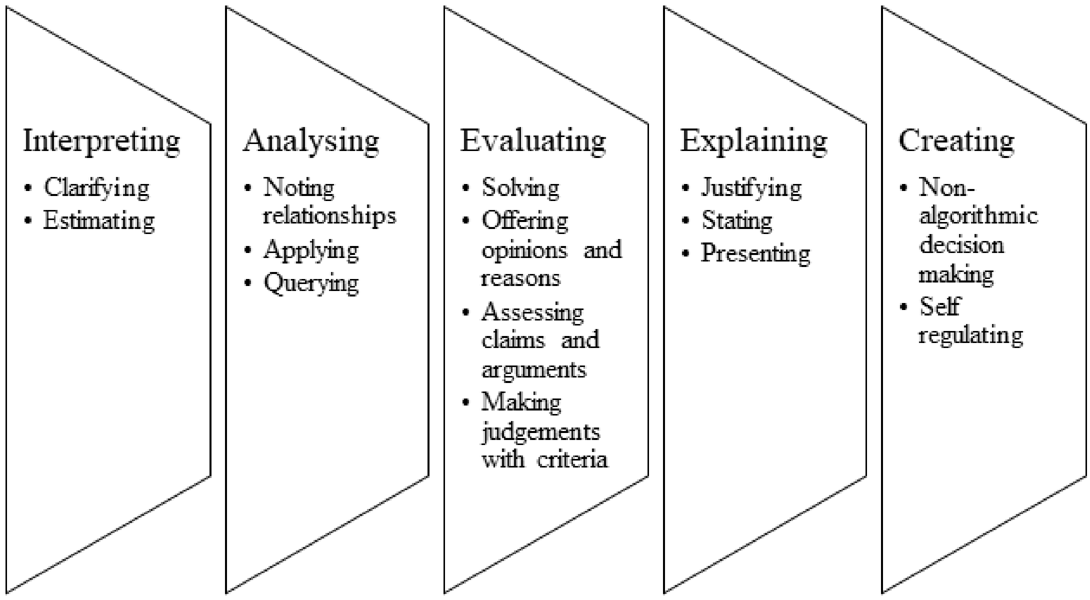


Figure 7.2. Critical Mathematical Thinking For Young Students (CMTFYS) Framework

7.4.2 Research Question Two

What types of teacher questions help these young students exhibit their CMT?

As indicated by the analysis of the data, particular types of teacher questions asked by the researcher support young students to exhibit their CMT. The researcher posed questions that moved across themes and sub-themes of the CMTFYS to assist students to exhibit their thinking. Thus, the CMTFYS provided a framework to identify the different types of questions which are highly effective in supporting young students to exhibit their CMT. The findings in the present study identified that the most frequent use of questions occurred when students were ***Explaining*** their thinking, and these questions were found to probe students and guide young students' mathematical thinking.

The types of probing questions teachers can ask to support students to explain their critical mathematical thinking include:

- ***Interpreting*** *– Clarifying*
- ***Analysing*** *– Noting Relationship*
- ***Evaluating*** *– Offering Opinions*

The types of guiding questions teachers can ask to support students to explain their critical mathematical thinking include:

- ***Interpreting*** *– Assessing Questions*
- ***Interpreting*** *– Understanding*
- ***Creating –*** *Self-regulating*

There are specific types of teacher questions, in alignment with the CMTYFS that assist young students to exhibit CMT (see Tables 5.20 and 5.39).

7.5 CONCLUSIONS OF THE STUDY

Overall, the findings from this study propose four conclusions in line with the two research questions. The first three conclusions relate to young students' CMT. The final conclusion is specific to teacher questioning that can exhibit CMT in young students.

7.5.1 Conclusion One

YOUNG STUDENTS' CRITICAL MATHEMATICAL THINKING – There are defined CMT capabilities in Young Students.

The findings of this study suggest that a literature and data informed conceptual framework – CMTFYS is required to support teachers in identifying CMT in young students. The analysis of data and identification of findings highlight that the development and use of a Conceptual Framework (CMTFYS) supports the definition of CMT capabilities in Young Students. The definition in the CMTFYS is presented as five themes and 14 sub-themes.

7.5.2 Conclusion Two

YOUNG STUDENTS' CRITICAL MATHEMATICAL THINKING – Intelligence Tests do not identify CMT in Young Students.

Findings from this study identify that intelligence tests, the Slosson Intelligence Test and the Raven's Progressive Matrices, do not measure CMT in young students. The CMTLE interview is identifying something other than intelligence. Findings indicate that CMT is different from reasoning processes and cognitive ability. Although intelligence tests do provide information for teachers, high levels of CMT in young students is not assessed in the same ways as intelligence.

7.5.3 Conclusion Three

YOUNG STUDENTS' CRITICAL MATHEMATICAL THINKING – The Patterns and Structure Assessment (PASA) has a relationship with the CMTLE.

The findings of this study suggested that there is a positive relationship between students exhibiting CMT and their awareness of pattern and structure in mathematics. Partial correlation data between the CMTLE scale and the PASA AMPS scale showed significant correlation (see Section 4.7.2). It is hypothesised that as the PASA is a mathematics assessment specific to patterns and structural awareness, and the CMTLE is also a mathematics based tool, there is a significant positive relationship between CMT and an underlying awareness of pattern and structure.

7.5.4 Conclusion Four

TEACHER QUESTIONING – The role of teacher questioning supported young students to exhibit CMT.

Findings indicate that teacher questioning was found to play a significant role in this study. There are particular types of questions that can be presented to young students to assist student talk in mathematics learning experiences. Teachers can support young students to explain their CMT by posing questions that align with themes and sub-themes in the CMTFYS.

7.6 IMPLICATIONS OF THE RESEARCH

Based on the conclusions of this study, implications of the research are categorised into recommendations for (a) curriculum, (b) teaching and learning, and (c) research.

7.6.1 Recommendations for Curriculum

The Australian Curriculum: Mathematics needs to include mathematics proficiencies that are specific to young students.

A recommendation of this study is that the *Australian Curriculum: Mathematics* for students in their first year of formal schooling includes the themes and sub-themes of the CMTFYS. This could be in the form of the mathematics proficiencies and supported through the critical thinking general capability. Specifically, CMT could be included as elaborations which show how young students engage in critical thinking in subject specific areas. Clarification of what specific mathematics proficiencies are for young students will assist teachers in identifying CMT during mathematics learning experiences. Furthermore, as Critical Thinking is an overarching theme in the Australian Curriculum and Mathematical Thinking is a well researched area in mathematics education, the inclusion of CMT in the curriculum will provide a better understanding within the specific subject domain.

7.6.2 Recommendations for Teaching and Learning

Mathematics teaching and learning approaches for young students need to be considered to allow for CMT to occur.

Young students need opportunities to engage in learning experiences that are designed to allow for CMT to occur. Therefore, teachers need to be willing to consider learning experiences that are open-ended in design and incorporate a range of concrete materials. These tasks also need to be underpinned by mathematical structures and representations. Furthermore, teachers need to prepare and have a teacher questioning repertoire that aligns with the CMTFYS that can promote students to exhibit CMT. Thus, it is imperative that the teaching and learning planning cycle includes teacher questioning as a core component in preparing students' learning.

Teachers need to be equipped to identify CMT in Young Students using the CMTFYS.

The results from this study can enhance teachers' knowledge about CMT. Therefore, it is recommended that teachers engage in professional learning to understand CMT. In addition, teachers would benefit from engaging in learning related to open-ended learning experience design. Furthermore, teachers require professional learning in identifying student responses and types of teacher questioning in alignment with the themes and sub-themes of the CMTFYS.

7.6.3 Recommendations for Research in Mathematics Education

Qualitative research needs to focus on young students' ways to exhibit CMT and ways teachers can support young student to exhibit CMT.

It is imperative that further research in CMT occurs to provide research informed practices for curriculum writers and teachers of young students. The findings of the study highlight the importance of the CMTFYS as a platform for teachers to support young students to exhibit CMT. It is by fully understanding CMT and the themes and sub-themes of the CMTFYS that teaching and learning practices will be modified to support young students in their mathematics learning.

7.7 FURTHER RESEARCH CONSIDERATIONS

There are several avenues worth pursuing through further research related to young students' CMT and what teaching actions/questions help these young students to exhibit their CMT.

First, the qualitative analysis component of the study included a small number of participants, that is, 16 students. To develop a more detailed picture of CMT capabilities, a larger study would be beneficial. Furthermore, this approach would provide further insight into the CMTFYS and its use with young students. Further data collection with a larger sample can strengthen the themes and sub-themes in the current CMTFYS.

Second, the design of the mathematics learning experiences was not a researched component of this study. It would be valuable to conduct further studies to investigate mathematics learning experience task design that can support young students to exhibit CMT. To begin with an examination of the types of learning experiences currently in use would provide a starting point. The data collected could be used to explore what modifications can be considered for young students to exhibit CMT.

Third, further studies into the correlation between the CMT and PASA are required. Findings from this study identified a correlation, however, further investigation may determine what elements within each instrument are related. Examination of the design of the questions, the mathematics content examined, teacher scaffolding, the marking of each instruments and the ways students respond to both the CMT and PASA would be beneficial to investigate and gain insight into why both instruments showed correlation. In addition, the CMT scale itself requires further investigation and refinement, with a focus on its transferability to other contexts and larger samples of students.

Finally, the findings of this study suggest that teacher questioning supports young students to exhibit CMT. It would be beneficial to engage in further studies to investigate the current questioning approaches used by teachers. Examination of what types of questions teachers are currently using would provide further evidence of how particular questions exhibit young students' CMT. Furthermore, this approach could provide a platform for areas of mathematical education professional learning for early years teachers.

7.8 LIMITATIONS

Limitations of this study are discussed in terms of the design of the study.

Participants

The study focused on a small sample of young students from similar schools in New South Wales. Hence, the study was bound by context. The researcher acknowledges that disparities are possible based on different school environments. In order to overcome this limitation, it is required to repeat the same study in various school and early learning settings to provide transferability of results.

The Researcher

A limitation identified from this study is that the researcher takes on the role of both data collector and the data analyst. To overcome pre-conceived ideas that can influence the research findings, the researcher embedded member checking. This was conducted by experienced mathematics education personnel during the data collection and analysis process.

Small Sample Size

Seeing that the study included a small sample size, the generalisations drawn from the data are limited. However, the data provided rich descriptive information about young students' CMT and teacher methods to exhibit CMT.

Timing of the Research

The study was bounded by time. The timeframe was purposively identified to occur early in the school year to ensure there was no teaching input that could skew the data. It would be beneficial to ensure a repeat of the study was to occur in advance of too much school-based teaching input.

Not all findings were encompassed

Finally, it is acknowledged that it is not possible to incorporate all findings related to young students' CMT and ways to exhibit CMT within the limits of this present study. It is anticipated that further studies and examinations into CMT will redress this limitation.

7.9 CONCLUDING REMARKS

Conclusions drawn from this study confirm that young students' CMT can be evidenced, and teacher questioning can support young students to exhibit CMT. New insights are gained into what CMT looks like in young students, and the important role the teacher has to support students to exhibit CMT.

First, findings presented from this study offer a unique contribution with the development of the CMTFYS. Furthermore, the CMTFYS is specific to young students and can be considered for use in both prior to school early learning settings and formal primary school classes.

Second, the data from the young students used for the refinement of the CMTFYS demonstrated sophisticated CMT. More instances of identifying such thinking from young students can shift the ways teachers plan for mathematics learning.

Finally, a proposed model to support teacher questioning to exhibit young students' CMT can influence approaches teachers use. The conclusions derived from the findings of the study recognise that teachers have a pivotal role in supporting young students to exhibit CMT. However, this is dependent on the teacher's ability to know the direction of the learning experience and the timing of the types of questions.

As a mathematics education teacher and lecturer in the early years and primary school, it has been a pleasure to work with young students in Kindergarten classes to define CMT and to refine a conceptual framework for CMT. The research and process I have undertaken to complete this thesis affirm that there are young students who can display CMT, and that further research and dissemination of results is required to profile this level of thinking evident in young students.

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Appendices

Appendix A – CMT Observation Guide Template

CMT Observation Categories	Observation 1	Observation 2	Observation 3	Observation 4
Mathematisation / Grasping principles				
Estimating				
Evaluating				
Classifying				
Assuming / Hypothesising				
Connections / Noting relations among other relationships /transferring learning				
Argumentation/ Offering opinions with reasons				
Number sense and mental computation				
Spatial and geometric reasoning				
Data and probability sense				
Making judgements with criteria / Reasoning				
Inferring logically				
Self-correcting				
Questioning				

Appendix B – Completed CMT Observation Guides for each School

CMT Observation Guide for the Classroom Observation Students at School A (n=18)

CMT Observation Categories	Observation 1	Observation 2	Observation 3	Observation 4
Mathematisation / Grasping principles	S13 *		S12	
Estimating	S12 S10 S17	S12 S17	S10 * * * *	S10 * *
Evaluating				
Classifying				
Assuming / Hypothesising				S17
Connections / Noting relations among other relationships /transferring learning		S10		
Argumentation/ Offering opinions with reasons	S14	S11 * * *		
Number sense and mental computation	S13 S15 S16			
Spatial and geometric reasoning				
Data and probability sense				
Making judgements with criteria / Reasoning				
Inferring logically				
Self-correcting	S10			
Questioning				

Note: Additional students not a part of the 25 Focus Students are identified as an asterix (*) in the above table

CMT Observation Guide for the Classroom Observation Students at School B (n=15)

CMT Observation Categories	Observation 1	Observation 2	Observation 3	Observation 4
Mathematisation / Grasping principles		S9		
Estimating	S6 S1 S2 S5 S3 *	S9 S3 *	S5	
Evaluating				
Classifying				
Assuming / Hypothesising				
Connections / Noting relations among other relationships /transferring learning	S6			
Argumentation/ Offering opinions with reasons			*	
Number sense and mental computation	S1	S1 S6	S6 *	
Spatial and geometric reasoning	S9			
Data and probability sense				
Making judgements with criteria / Reasoning	*	*	S9	
Inferring logically		S9		
Self-correcting	S3 S8			
Questioning	S4 S7		S3 S4	

Note: Additional students not a part of the 25 Focus Students are identified as an asterix (*) in the above table

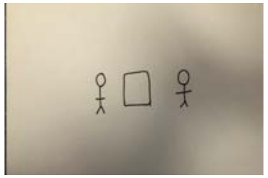
CMT Observation Guide for the Classroom Observation Students at School C (n=8)

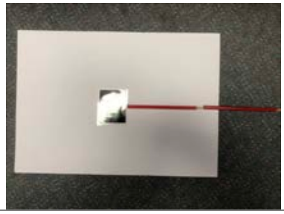
CMT Observation Categories	Observation 1	Observation 2	Observation 3	Observation 4
Mathematisation / Grasping principles	S24 S21	S18	S19	S24 S19 S23
Estimating	S24 S20 S23 S25	S20 S21 S19		S24 S20
Evaluating				S24
Classifying			S24	
Assuming / Hypothesising	S18			
Connections / Noting relations among other relationships /transferring learning	S21 S19	S24 S23		
Argumentation/ Offering opinions with reasons				
Number sense and mental computation	S20 S21 S23	S18	S24 S23	S24 S21 S9 S25
Spatial and geometric reasoning				
Data and probability sense				
Making judgements with criteria / Reasoning		S21	S21	S21 S18
Inferring logically				S20
Self-correcting	S24 S18	S19		S20 S22

Note: Additional students not a part of the 25 Focus Students are identified as an asterix (*) in the above table

Appendix C – Full Analysis of the High CMT Student Data

Results of the Interpreting theme using Themes and Sub-themes of the CMT Conceptual Framework

Mathematical Thinking in the Early Years: Clarifying concepts and ideas = making sense of mathematics					
Learning experience	Summary of student response	Utterance/ Speaker		Transcripts that exhibits the theme and alignment with the sub-theme	Sub-theme (researchers' interpretation)
LE1	Student drew objects on either of the photograph to find the middle. 	3	T	Well, how can we check that's exactly the middle? What can we do to check?	Clarifying (student clarified the notion of middle relation to the object's sides – its somewhere between the sides)
		4	S13	*Because if you know these are the sides of it and there's no other one [student pointed to the other side of the paper] in there, that would actually be the middle.*	
		7	T	Maybe do you want to make any drawings to show me?	
		8	S13	Because if that person was next to the middle, like say I was next to the middle, and if you draw another person there next to my brother, all of.	
		10	S13	*Say if I was over here and my brother was over here and that was in the middle, that would count that that's the middle.*	Clarifying (student used a real-life context to clarify 'middle')
LE1	Student used arm spans and pointed to spaces around the image to show that the spaces are the same.	4	S2	*So, you can see if it is the same.*	
		5	S2	Okay. So okay, you've made some changes there. What about if the photo was here?	
		6	T	Is that the middle?	Clarifying (student used own arm span to clarify 'middle')
		7	S2	No.	
		8	T	Why not?	
		9	S2	Because *it has to be like this [student used arms to demonstrate].*	
LE1	Student uses arms and hands as an informal unit of measure	3	T	Here's a pencil, if you want to.	Clarifying (student used own arm span to clarify 'middle')
		4	S10	So, you can try put in here and *you can think that's the middle and you can think that's the middle and maybe.*	
		5	T	How do I know if I'm right?	
		6	S10	*Because you can place your arms here [student demonstrated with arms and hands] and you can know that you are right because that's the middle.*	
		7	T	Okay, so is that in the middle now?	Estimating (student used a concrete object
		8	S10	Yeah. It's in the middle now.	

		9	T	Can you think of any other way to show me how to find the middle without using your arms? Can you think of any other way?	to estimate the location of the middle)
		10	S10	*You can use the pencil and you can put the picture here and we can see it's in the middle.*	
		11	T	Do you think it's the middle now?	
		12	S10	Yes.	
LE1	Student uses a lead pencil as an informal unit of measure 	3	S6	Measure. How about I measure it?	Estimating (student used a concrete object as an informal unit of measure to estimate the location of the middle)
		4	T	I'll give you this [teacher hands student a pencil] if you want to do anything with it. Do you think that's the middle? How did you find that out?	
		5	S6	W*e bring the pencil here, moving it over.*	
LE2	Student discussed counting a handful of beans and then counting by that amount to find an estimate of the total number of unseen beans.	1	T	Oh, you can count by twos. Is there any other way to find out? Can you think of any other way?	Estimating (student estimated the possible amount of beans in the bean bag)
		12	S13	Well, you can actually feel inside your hand. *There could be a hundred or more*, so if you just pinch them and you can count them. And you can go all the way around and go up from here, you can go to the bottom, and pinch them a little bit so you can count them.	
		3	T	What sort of machine?	
		4	S3	A number machine, count how much there are in there.	
		5	T	Oh, I've never seen a number machine before. Can you explain that to me?	
		6	S3	*So, you have a machine, and then you can put it over here and it can count me how much beans there are in the bean bag. And then it can tell you on a thing, and then you know much are in there.*	

Results of the Analysing theme using Themes and Sub-themes of the CMT Conceptual Framework

Mathematical Thinking in the Early Years: Connecting understandings and procedures: noting relationships					
Learning experience	Summary of student response	Utterance/	Speaker	Transcripts that exhibits the theme and alignment with the sub-theme	Sub-theme (researchers' interpretation)
LE1	Student used his knowledge of making paper planes to find the middle.	10	S18	*I'm used to folding it this way because I make paper planes every day.*	Applying (student applied already known knowledge - concept of a half – to find the middle)
		11	T	Do you? Do your planes fly very far? They must be very good paper planes. All right, so now where do you think I can hang that photograph of Joey? Do you want to place it there and show me? Is there any other way that you can check that that is the middle? How?	
		12	S18	I could see *if I could fold this [the piece of A3 paper] half in.*	
LE3	Student started by saying they both equal 6 and then went on to discuss why.	2	S13	Because they both equal six.	Noting relationships (student beginning to note the relationship that 3+3 is the same as 4+2)
		3	T	Why is that?	
		4	S13	Because, two and four *is close to three*. Like if you count your hands like four plus two equals six and yo*u can count your hands three plus three equals six.*	
LE3	Student solved and provided reasoning to explain why some expressions in addition are equivalent.	4	S2	Because it's just addition. So, we have t*o take one more from this and then it'll be four plus two.*	Noting relationships (student noted the relationship that 3+3 is the same as 4+2)
		5	T	Oh, I like that. That's a very good way of thinking. I'm going to give you another set of number sentences. The first one is 4 + 4 four, and 6 + 2. Now tell me why they're the same.	
		6	S2	This is four plus four. *If we take one from here and then two from here, then six here and two there.*	
LE5	Student talked through the strategies and identified how many teddies there would be altogether.	2	S23	Let me guess. *Well, it looks like there's the same amount*, so I think there's five.	Noting relationships (student noted the relationship between 'same amount' and 'even distribution')
		3	T	But there were 11 teddies altogether.	
		4	S23	Okay. So, let's see. I can figure out that. So, if I take five away, and I took the other five... If I took the other five away, then there's one teddy left.	
LE5	Student was able to solve	2	S25	Six in one [pocket].	Noting relationships

	the problem by sharing an addition fact.	3	T	And then how many in the other?	(student noted the relationship between known addition facts and the problem presented to them)
		4	S25	Five in the other.	
		5	S25	Good. Quick recall. How did you know that there were six and five?	
		6	S25	*Six and five equals 11.*	
LE5	Student began by asking the teacher if they used a particular strategy. Student went on to determine the answer and make connections with their mathematical understanding of odd/even and partitioning.	1	T	We played this game and we were puttingour teddies [11 teddies] in our pockets. And I put some teddies in my pocket, and my friend put some teddies in her pocket.	
		2	S20	*Did you go one, one, two, three...?*	Querying (student queried how the teacher put the teddies in her pocket)
		3	T	What could we have done? And there were no more teddies left on the desk. So, there was some here and there was some in your pocket. Or in my friend's pocket. How many could it be... could be in my pocket?	
		4	S20	One, one, two, two.	
		5	T	So how many are in your pocket?	
		6	S20	One, two, three, four, five is in your pocket. One, two, three, four, five, six. It's an odd number, so one gets more than the other.	
		7	T	Very good. And then when we... when it was time to pack up, we found this many more teddies on the floor. I wonder whose pocket they could go into.	Noting relationships (student noted the relationship that if the total is an odd number then it’s not the sum of two even numbers)
		8	S20	One... Oh, wait. Let me count. One, two, three, four, five, six seven. One, two, three, four, five, six, seven. Now it's even.	
		9	T	Now it's even. Okay.	
		10	S20	*And you have an odd number in that box.*	

Results of the Evaluating theme using Themes and Sub-themes of the CMT Conceptual Framework

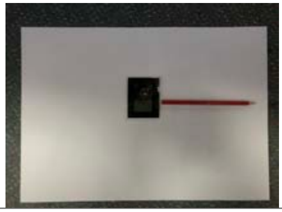
Mathematical Thinking in the Early Years: Evaluating the strategies being used + reasoning and justifying strategies					
Learning experience	Summary of student response	Utterance/ Speaker		Transcripts that exhibits the theme and alignment with the sub-theme	Sub-theme (researchers' interpretation)
LE6	Student considered two different ways to determine the number of tiles required: placing the tiles in rows; using a measuring tape.	22	S23	You could *get a measuring tape and then pull it across like that, where the corners are*. Then you'll get five tiles, five tiles, five tiles. You keep getting five tiles and then you start laying them out in a row of five, like this. One, two, three, four, five.	Assessing claims and arguments (student assessed claims and arguments with regard to the area being repeated rows by measuring the width with a piece of wood and then using repeated pieces of wood to make the area)
		23	T	So, do you think that only five tiles can fit across here? Is that what you think?	
		24	S23	Well maybe you could use wood to do it.	
		25	T	Wood? Explain how you could use wood.	
		26	S23	So, pretend this is a whole piece of wood. And this is a whole piece of wood. It would go across the cubby house. And this is a piece of wood. And then that's a piece of wood. This is a piece of wood. And those last other ones could go, these last other ones, could go on top here. And one will go here. *And so, what you need to buy is one more tile.*	
LE1	Student self-corrected when drawing a ruler like impression to determine the middle. Student continued to provide another way of identifying the middle i.e. folding the paper. 	2	S24	Maybe, maybe you can count numbers that's here. So, you can go one, and then a finger space, and then two, and *then a finger space, and one, then a finger space*. So, you know which is the middle it's coming to. So, if you finish on six, then that would be the middle.	Offering opinions and reasons (student demonstrated, provided an opinion and reasoned by discussing and showing how to find the middle of the picture using the width of his finger)
		3	T	Do you want to have a go at showing me? It's a bit tricky now you're going the opposite way.	
		20	S24	So, I'm going to do 2, 2, 4. 1, 2, 4.	
		21	T	1, 2, 4?	
		22	S24	1, 2, 4 again. I'm *just going to make sure it's the same.*	
		23	T	Good checking.	
LE3	Student solved and	2	S24	Because they're just using different numbers but this one *is 3 so then it's 3 and then*	Offering opinions and

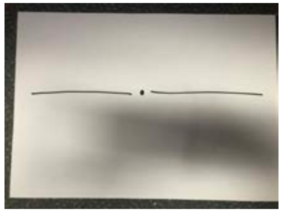
	provided reasoning to explain why some expressions in addition are equivalent			*you add 1 from this one to that so it's 4 and then 2 is left over so they have the same amount and they both add up 6.*	reasons (student provided three different reasons for her solution)
		3	T	That is a fabulous response. What about these two number sentences? What can you tell me?	
		6	S24	*The 6, is there, and then you go, that's 4, 5, 6, and then there's two left over because 4, and then you take away 2 and it equals 3*. So, then it's just the same.	
		7	T	And how do you know that they're the same?	
		8	S24	Because, it's the same reason as the other one, because, this one just has 4 so you just add 2 so it's 6. And like, that's it so it could've been 6 there, and 2 from there and then you could take, then there would be 2 left because of 4. Like, so if you want to make 4, so 2 plus 2 equals 4, d*o then you just take one away and it's the other way around, 2 minus 2 equals...*	
LE7	Student identified that cutting a piece of toast in half does not provide exact halves. Student considered ways to ensure each person receives the equal amount of toast.	2	S24	First you *have to make sure the toast is not leaning* that way and has a bit more on that side and it's not leaning that way and it has a bit more on that side.	Offering opinions and reasons (student offered opinions on how to cut a shape in half to ensure that the two pieces are exactly a half)
		3	T	Let's imagine it is this toast, which looks like it's pretty even.	
		4	S24	Yeah.	
		5	T	Okay, so how might we share it? You can draw on that if you like.	
		6	S24	So, we do, so... so then, I have to get this, then, kind of *sweep it that way and make it into tall. Make it that way*, oh. So, this way.	
LE3	Student solved and provided reasoning to explain why some expressions in addition are equivalent.	2	S12	Because, wait... Because you *added one more into here, and then there's two*.	Offering opinions and reasons (student reasoned and explained that the two amounts are the same by using the notion of equivalence)
		3	T	Oh, okay. What about these two then, if that's a good way of thinking. Why is four plus four the same as six plus two?	
		4	S12	Because if you put it a bit more into this one, it would get to, and it would still make the add, the same.	
LE7	Student described equality by counting and drawing squares to determine the area of each slice of toast	2	S3	They *both have the same.* This has ten, one, two, three, four, five, six, seven, eight, nine, ten. One two, three, four, five, six, seven, eight, nine, ten. They're the same.	Making judgements with criteria (student provided a judgement and justified it with using the criteria of bigger and smaller)
		3	T	Well, how can you explain why this one is shorter than this one, then?	
		4	S3	*Because you're using little-er blocks. And these are bigger.*	
LE3	Student provided	2	S10	Three plus three.	Solving (student

	reasoning to explain why some expressions in addition are equivalent	3	T	Why is three plus three the same as four plus two?	solved the problem by drawing on an understanding of addition and number facts)
		4	S10	Because four plus two is six.	
		5	T	So, we know that four and two is six. What do we know about three plus three?	
		6	S10	Three plus three can be 12 or six.	
		7	T	Tell me how it can be 12.	
		8	S10	I am not really sure.	
		9	T	Tell me how it can be six. Show me how three and three is six.	
		10	S10	*Because when you play three fingers here and three fingers here, it is the same and it makes six.*	
LE6	Described problem solving methods, with real-life applications, to determine how many tiles are needed in a cubby house.	8	S9	I'm counting. We must count how many bricks. You must *take it down and then you must get one that you want to put on the floor and get some tiles and place them on there*. But you'd better break the house down and then you better get the tiles, you better get the wooden thing and then put on the floor and get some tiles and put it in top. Then you'd figure out how many.	Solving (student provided a response that involved a real-life solution).
LE6	Student determined the process required to identify the number of tiles needed to tile a cubby house floor. Student described identifying the number of tiles required per rows, then placed the tile to determine how many rows will be required.	2	S6	Count the spaces there and count the spaces here.	Solving (student solved the problem by identifying the required number for 1 row of tiles required and then provided a process to purchase more tiles)
		3	T	Give me some more ideas there. What do you mean by count the spaces?	
		4	S6	Count them, and you know how many. Then you *get by this, the many that there and by the titles. Then see if there's enough. If there's not, just buy a few more and see if there's enough. Then put them together for the floor.*	
LE6	Student provided many ways to determine the number of tiles required to tile the cubby floor: placing the tiles in a square shape; drawing the amount required; repeating the process to	4	S20	Well, you can *get one tile and then do a square around it, and then keep doing that for all of the places where you need to and then count them after that.*	Solving (student solved the problem by repeatedly using one tile to determine the number of tiles required)
		5	T	Oh, that's a good idea. So, we take this tile, we place it on the floor, and we draw around it.	
		6	S20	Yeah. And then you move on to the next place.	
		7	T	So where might you put it next? Right.	
		8	S20	And then you draw it.	
		9	T	Yeah.	
		10	S20	Next, draw it. Next, draw it.	

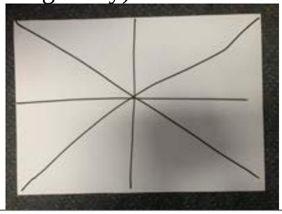
	check for accuracy.	11	T	Do I need to draw it all over the floor, or is there a quicker way?	
		12	S20	So, you need to do it all over the floor.	
		13	T	All over the floor?	
		14	S20	Yeah.	
LE6	Student considered the number of tiles required for row one, then continued to discuss how you would need to purchase each row as required. 	4	S21	You put some up. I think there might be four across. They might be three across. Yeah, three across, not four. *Four across, and those two, I have two each, so two of everything. At your home it has to be more than twice. It has to be more. It has to be more than two. We have to fill more up.*	Solving (student solved the problem using an understanding of area as an array (area consisting of repeated rows)
		5	T	How will I know how many I need to buy?	
		6	S21	I think you might know. I actually have a clever idea.	
		7	T	I like clever ideas.	
		8	S21	You could actually *build them all around the floor, and you could buy*... Well, you could put the first row and then buy more, and then put in another row, another row, and buy some more, and put another one, and buy some more.	
LE8	Student reviewed original response and considered ways to modify the shape, made the modification and confirmed that it was the same.	12	T	So, this is a shape that's made up of lots of little shapes. Can you try and make this shape here, with these shapes? We'll move that to the side.	Solving (student reviewed original response, made changes and solved the task correctly)
		13	S8	Move this. Move this.	
		14	T	You're happy with that? Does it look exactly the same?	
		15	S8	*I think we need it a bit bigger [student selected another piece that was larger and appropriate]. That's the same.*	

Results of the Explaining theme using Themes and Sub-themes of the CMT Conceptual Framework

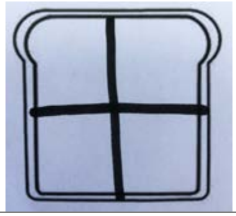
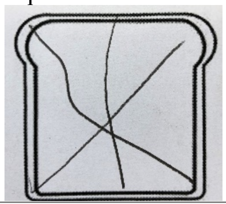
Mathematical Thinking in the Early Years: Considering other methods/strategies/alternate solutions + describing solutions/clarifying solutions/elaborating on ideas					
Learning experience	Summary of student response	Utterance/ Speaker		Transcripts that exhibits the theme and alignment with the sub-theme	Sub-theme (researchers' interpretation)
LE2	Student considered counting in larger quantities to determine how many beans are in the bean bag.	2	S17	*The quickest way* you could do it is count by like fours or fives or six or sevens or...	Stating (student stated the quickest way to solve the task)
LE3	Student stated why 3+3 is the same as 2+4	2	S17	*Because they both equal six*	Stating (student stated that expressions must be the same if their total is the same)
LE4	Student stated that the tower was taller as blocks from tower 2 were larger than those from tower 1 but the number of blocks in each was the same	5	T	Ah, they are the same. So, can you explain why this tower is taller and this tower is shorter?	Stating (student stated that both towers have the same number of blocks – the difference between the towers is their length not the number of blocks)
		6	S10	Because *all of the blocks here are bigger. All of the blocks here are smaller*.	
LE1	Student imagined a pencil was a measuring tape and acted out the actions required to measure to find the middle. 	13	S23	So, like this. *I'm pretending that this is a measuring tape, that one.*	Presenting (student presented a rule to find the middle)
		14	T	Right.	
		15	S23	So, *I'll put some numbers there. Which way is the five?*	
		16	T	So, I'm going to write it on the table.	
		17	S23	So, you said you wanted to put it in the middle. So, I'll *count the other way around.*	

LE1	Student considered many ways to determine if the photograph was in the middle i.e. using a ruler, using blocks, using teddy counter. Student used the counters and blocks to determine if the photograph was in the middle.	3	T	But how do we know that that's the middle?	Presenting (student presented strategies to find the middle, the use of technical mathematical language was evident in the explanation)
		4	S20	Well, I could draw lines.	
		5	T	All right, go ahead and draw some lines.	
		6	S20	*And I'm going to see if they're all the same length*, but the one... These two need to be the same length and these two need to be the same length.	
		7	T	And how might we check if they're the same length?	
		8	S20	With a ruler.	
		9	T	I don't have a ruler.	
		10	S20	I know where the rulers are.	
		11	T	Oh, you know where they are. Let's imagine we can't use a ruler. Is there any other way that you can think of to check? You did say we could measure if this line is the same as this line and this line is the same as this line, but we don't have a ruler. Can you think of any other way?	
		12	S20	Do we have little blocks or something?	
		13	T	Do we have little blocks? I have teddies and I have these blocks. Do think that will help? Let's see what you're planning.	
		14	S20	So, I can see if they have the same amount of blocks.	
		15	T	Okay. Let's see what else we've got. We'll put that aside and we'll use these ones.	
		16	S20	Yeah. And *we'll see if they all have the... they both have the same amount of blocks.*	Stating (student stated that having same number of blocks on each side assists in finding the middle)
LE1	Student displayed ways to draw lines to make the middle. 	2	S1	You can put a dot in the middle.	Justifying (student provided a solution and justified the solution by stating a way to check for its accuracy)
		3	T	How can I find that middle to put that dot?	
		4	S1	I don't know. You can go like this for the middle and go to the line like that [student used hand spans to locate the middle].	
		5	T	Can you say that again?	
		6	S1	To the middle and make a line like that [student drew a vertical line].	
		7	T	Can you draw what you're thinking please?	
		8	S1	*Draw a line to the middle.*	
		9	T	All right, so do you think that there is the middle? How can we check?	
		10	S1	*With a ruler.*	

LE1	Students provided four ways to identify the middle. 1) informally using hand spans, 2) using technical mathematical language, 3) using a ruler, 4) providing a reason that supported his strategies	2	S23	Because there's *half here, and half here, and half here, and there's half here.*	
		3	T	Is there any other way that you can think of, to check?	
		4	S23	That there on the sides. There, that's the long way of them. *And that is the length.*	
		5	T	The length, I heard that very special word length. How can we check it?	
		6	S23	By *getting rulers and counting* how much there are?	
		16	S23	So, you know how you said you wanted to put a photo of him in the middle, well this is the way I discovered. If you put it *with a measuring tape on the even number*, you know it's the middle because the odd number that doesn't add up, it would totally be in the middle. Because see, it's the same length as the other, so that's the middle. So, if you use the... So, you could draw a piece of paper, then you'll know where the middle is.	Justifying (student justified her response with a statement involving the use of a ruler)
		17	T	So, it's the middle that way. What about the middle that way?	
		18	S23	Well, the middle that way is like... I think it's the same. One, two, three.	
		19	T	You could use the same strategy? Okay. All right.	
		20	S23	And I also know another strategy to do it as well. So, I'll do another line here.	
		21	T	Yeah.	
		22	S23	But not just that line. I found another way to add it up. So, you know how you said that what you want to do is put this in the middle? Wel*l by adding up, you could draw a line like that, two lines, above one line, and keep...* And then make another one to show where your pictures standing, and then put the other line here. So, three lines, and then what you do is, if you want to find out if you needed to do it like that, here's how you do it. So, you can do those lines. And then you put the photo on. I might do a little thing to show sticky tape.	Stating (student stated another way to identify the middle)
		23	T	Oh, okay.	
		24	S23	And then what you do is you hang it up like that. Then you'll know which is the middle.	
		25	T	So, are you thinking that this is the wall now?	
		26	S23	Yes.	
		27	T	So not the big paper. Just that.	
		28	S23	Yes.	
		29	T	So, you wouldn't look at lines and you'd draw the lines to find the middle.	
		30	S23	Yes.	Justifying (student justified how to find the middle with a detailed explanation)

LE1	Determined where the midpoint of the paper was by drawing lines-intersecting (vertically, horizontally and diagonally). 	4	S9	Are you going to give me a ruler?	Justifying (student displayed lines and expressed the process of drawing lines to justify the location of the middle)
		5	T	No	
		6	S9	You can't fold a wall so you can't fold this paper. *I will draw a line here and another line, here and just to prove it to you I will draw another line this way and another line this way, that is the middle.*	
LE1	Student justified what is required to determine if the middle is or is not accurate.	2	S19	By *counting and then if it's not the same amount, then it's not the middle.*	Justifying (student provided a justification should the strategy not work)
LE1	Student used his knowledge of making paper planes to find the middle.	10	S18	I'm used to folding it this way because I make paper planes every day.	Justifying (student justified that it is the middle by explaining that both sides of the folded paper are pointing to the 'same spot'.)
		11	T	Do you? Do your planes fly very far? They must be very good paper planes. All right, so now where do you think I can hang that photograph of Joey? Do you want to place it there and show me? Is there any other way that you can check that that is the middle? How?	
		12	S18	I could see *if I could fold this [the piece of A3 paper] half in.*	
		13	T	Have a go, yeah.	
		14	S18	And then fold this all in.	
		15	T	What do you think you've discovered now?	
		16	S18	They're *both pointing in the same spot.*	
		17	T	*Yeah, they both matched the same spot, haven't they? But does that help you find the middle?*	
		18	S18	Mm-hmm (affirmative).	
		19	T	It does? How?	
		20	S18	Because then you just open this up and you can see.	
LE4	Student physically broke apart the tower to show why one tower was taller	9	S23	Well, they both have the same amount of blocks. So, what I'm thinking right now is, you know how these blocks are more thicker and taller? So, *if I break one off, you'll see the difference. S*o, you see that sometimes things can be taller or smaller. *If I put these together it*	Justifying (student justified that the difference in the towers

	than the other.			*makes a long tower and you see, if I break all of these off, it's small*. I'll show you an example with this red or green one.	was not the number of blocks but the length of the blocks in each, and the different lengths were dependent on the size of the blocks)
		10	T	Are you all right? Can I help you?	
		11	S23	Can you break one of these blocks off?	
		12	T	Just one maybe from this end. There we go. And we will put the rest together again.	
		13	S23	So, you see, this block is smaller than this block. So, what I'm thinking is that these blocks are the same as them. But this one is not the... But this one is thin, and this one is not that thick. So, what I'm thinking right now is that t*hey have the same amount but this one is just smaller than these ones*. So, maybe that's why they have the same amount. But together there are that... This one isn't that tall because that one is thinner.	
LE6	Student described using one tile, drawing around it to determine how many floor tiles are required altogether.	2	S19	I think so. So, you can count so you can measure the tile you can, *you can buy one tile and then measure it and then draw around it* and then do the same on the others and then count the squares and then put the tiles in the squares.	Justifying (student justified strategy to determine the number of tiles required)
		3	T	So, you're saying to take a tile and draw around it, trace the tile and then keep tracing to see how many tiles we need?	
		4	S19	Yeah.	
LE6	Student described using one tile, drawing around it to determine how many floor tiles are required altogether.	10	S17	Oh, so I can *place this on the floor and then I'll trace it.*	Justifying (student justified strategy to determine the number of tiles required)
		11	T	Yeah.	
		12	S17	And then I'll *keep on doing it.*	
LE6	Student considered using one tile, place it on the floor and see how many more are required.	2	S18	If it was from there to there, you'd put tiles just on the ground.	Justifying (student justified strategy to determine the number of tiles required)
		3	T	Yeah but I want to know how many to buy. I've only got this many and I know I'm going to need a lot more, but how many? How could I find out how many I need to buy?	
		4	S18	*Put them on the ground and then count all the empty spaces.*	
LE6	Student described using one tile, drawing around it to determine how many floor tiles are required altogether	2	S1	I think to measure it. To measure it.	Justifying (student justified what to do should you not have enough tiles)
		3	T	So how do I know how many tiles to buy? So, it fits on the floor in there.	
		4	S1	*You can measure and put the square. You can draw the squares.*	
		5	T	Can you show me what you mean? All right, I get what you mean now. Can you think of any other way we can find out without drawing it? How many I need?	
		6	S1	You can put tiles from the floor.	
		7	T	Yeah.	
		8	S1	And if you're missing one, you can put one more. You can put one more. I think.	
LE7	Student was justifying	2	S21	One bit for you, one bit for me.	

	how the sections of the toast are equal. He proceeded to cut the toast into four, which allowed for each person to have one small and one larger piece each. 	3	T	And how do I know that they are exactly the same?	Justifying (student used fairness to justify equality)
		4	S21	Because they're both the same.	
		5	T	How do I know?	
		6	S21	Because they're both the same, but when you cut it up, it got smaller. It got smaller pieces, so that's how I know.	
		7	T	Is there a way you can check?	
		8	S21	If you want to more pieces, and squares, two for you, and two for me.	
		9	T	Would it be fair if I got these two pieces and you got those two pieces?	
		10	S21	No, *because these two pieces are bigger than those two pieces.*	
LE7	Drew intersecting lines to show how to find a way to cut the toast in equal halves. 	12	S9	So, but imagine they weren't circles anymore, imagine they were lines. So, one if went there, one went there, and one went there, then you take it all away, but it was with bread-	Justifying (student drew lines then provided a justification by stated to remove two lines in order to have equal halves)
		13	T	Could you draw those lines that you're talking about?	
		14	S9	On here?	
		15	T	Yeah.	
		16	S9	Then and *imagine you took those two away*. *And then you cut it in half like that.*	
LE7	Student considered many ways to ensure that each person received the exact same amount of toast.	8	S20	So, we see if it's... There. We can see if we can get another piece of bread and slice it again.	Justifying (student justified and made modifications to ensure equality)
		9	T	Okay.	
		10	S20	Now let's see. We'll do mine with the beads first. (silence) Now this one's two. That one's two. Your one is... *There's still more space on yours.*	
		11	T	But it's pretty close. So, I get... I know how you're thinking now. So, you're going to use these to help find the exact middle.	
		12	S20	Yes.	
LE7	Student cut the toast into four pieces and determined which two	3	T	Can you show me a way?	Justifying (student justified response by determining which 2
		4	S23	So, you... Say if you have a rubber, that's like this one or something, you put it like that, and then where the rubber goes it goes in the middle bit, so like that.	

	pieced each person received to ensure they have the same amount. 	5	T	And so how can we check if that's half? How can we check that this half is exactly the same size as this half?	pieces each person received to ensure they had the same amount)
		6	S23	By doing this. If you wanted to have two pieces of toast if you were so hungry, maybe you could also do this. Two for me and two for you.	
		7	T	Which two would you get?	
		8	S23	*I'll get those two and you'll get those two. Let's see how...* One, two, three, four, five, six. One, two, three, four, five, six. One, two, three, four, five, six. One, two, three, four, five, six. So, they both have six.	

Results of the Creating theme using Themes and Sub-themes of the CMT Conceptual Framework

Mathematical Thinking in the Early Years: Tackling complex problems in a novel way					
Learning experience	Summary of student response	Utterance/	Speaker	Transcripts that exhibits the theme and alignment with the sub-theme	Sub-theme (researchers' interpretation)
LE3	Student thought of one idea which included changing the cards around, then realised that would not work and then discussed the commutative strategy	2	S23	Because four plus two is six, and three plus three is six. So, we can make a change with these cards if we just... We could make a change to add up different numbers. So, we put them together like this. Four plus two plus three and plus three. Maybe we can make new numbers if we add these two up together.	Self-regulating (student identified own misconceptions and self-regulated to find another solution)
		3	T	That's true. But let's think about just them as two separate cards for now.	
		4	S23	You know how you said they're both six?	
		5	T	You told me that.	
		6	S23	So maybe I can make a change... *Oh, that won't work.* I said... *I was almost going to say go backwards but it'll still be the same. Two plus four will still be the same.*	
		7	T	But why do you think that they're both equal six, Zara?	
		8	S23	Maybe it's because they're both similar? *Maybe they're both just similar?*	
		9	T	They're both similar?	
		10	S23	Or maybe they go in numbers. See, two, three and four.	
		11	T	Yeah. That's true. But is there something about these numbers that make them both add up to six? What do we know?	
		12	S23	They both have a plus.	
		13	T	They both have plus? Yes.	
		14	S23	But they both have different numbers on them.	
		15	T	Is there anything about the numbers that you can think of?	
		16	S23	Well I do know something about the numbers.	
		17	T	Excellent. What do you know?	
		18	S23	I just realized that each number that's there is the same small. So that's small. That's medium, and that's big. So, if we realize they're different, it goes in similar order. Small, medium and big. So that's why it goes in that order.	
		19	T	Okay. Well keep that thinking. What about now? Six plus two and four plus four. Why are they the same?	
		20	S23	Because they both... Well, see. It's like with the three. Four plus four... Each number that makes it... See. Five plus five is 10, and *I know another way to make 10. Three plus... Okay. Five... Okay. It's*	

				actually two, six plus four equals 10 as well.	
LE1	Student counted from 1 – 6 from the edge of the page to find the middle. The student demonstrated by creating a ruler that went forwards from left to right and backwards from right to left.	2	S24	Maybe, maybe you can *count numbers that's here. So, you can go one, and then a finger space, and then two, and then a finger space, and one, then a finger space. So, you know which is the middle it's coming to. So, if you finish on six, then that would be the middle.*	Non – algorithmic decision making (student used informal finger spacing (non-algorithmic) to find the middle.
		3	T	Do you want to have a go at showing me?	
		4	S24	Sure. Yeah. This one.	
		5	T	It's a bit tricky now you're going the opposite way.	
		6	S24	That was-	
		7	T	That's all right. I know what you're doing.	
		8	S24	I just don't know where.	
		9	T	Trying all your fingers now. Seeing which one's the best one.	
		10	S24	Yeah, I think this one is. Oh, I went the wrong way.	
LE6	Student described using one tile, drawing around it to determine how many floor tiles are required altogether.	2	S19	I think so. So, you can count so you can measure the tile you can, *you can buy one tile and then measure it and then draw around it* and then do the same on the others and then count the squares and then put the tiles in the squares.	Non-algorithmic decision making (student used one tile (non-algorithmic) to determine the number of tiles required)
		3	T	So, you're saying to take a tile and draw around it, trace the tile and then keep tracing to see how many tiles we need?	
		4	S19	Yeah.	
LE6	Student described using one tile, drawing around it to determine how many floor tiles are required altogether.	10	S17	Oh, so I can *place this on the floor and then I'll trace it.*	Non-algorithmic decision making (student used one tile (non-algorithmic) to determine the number of tiles required)
		11	T	Yeah.	
		12	S17	And then I'll *keep on doing it.*	
LE6	Student considered using one tile, place it on the floor and see how many more are required.	2	S18	If it was from there to there, you'd put tiles just on the ground.	Non-algorithmic decision making (student used a non-algorithmic strategy to determine how many tiles are needed)
		3	T	Yeah but I want to know how many to buy. I've only got this many and I know I'm going to need a lot more, but how many? How could I find out how many I need to buy?	
		4	S18	*Put them on the ground and then count all the empty spaces.*	
LE6	Student described using one tile, drawing around it to determine how many floor tiles are	2	S1	I think to measure it. To measure it.	Non-algorithmic decision making (student drew tiles (squares) (non-
		3	T	So how do I know how many tiles to buy? So, it fits on the floor in there.	
		4	S1	*You can measure and put the square. You can draw the squares.*	
		5	T	Can you show me what you mean? All right, I get what you mean now. Can you think of any other way	

required altogether			we can find out without drawing it? How many I need?	algorithmic) to determine the number of tiles required)
	6	S1	You can put tiles from the floor.	
	7	T	Yeah.	
	8	S1	And if you're missing one, you can put one more. You can put one more. I think.	

Appendix D – Coded Teacher Questioning Learning Experiences

Results of teacher questioning within the CMTLE 1

Alignment with Critical Mathematical Thinking	**Teacher Questions – CMTLE 1**	**LEQUE**	**TQ Factual**	**TQ Probing**	**TQ Guiding**
			Maths related question	**Clarifying Decoding Noting relationships Offering opinions with reasons**	**Assessing Understanding Self-regulating**
Phase 1 – the student interpreting the task	What can I do to find the middle of the wall?				
	How could we find the middle of the wall?				
	How might we find the middle?				
	Where can I place this so that it's in the middle and how can I make sure?				
	How can I find the middle?				
	How can I find the middle?				
	How can I find the middle?				
	How can I find out where the middle of the wall and how do I know that that is the middle?				
	How can I find the middle?				
	How can I find how to cut this exactly in half? So, we've got two equal parts?				
	Can you think of a way to find the middle?				
	How do you think we can find the middle?				
Phase 2 – the student analysing the task	How do I measure it?				
	How do I know it's the middle that way?				
	How do you know that's the middle?				
	So, tell me why this isn't the middle.				
Phase 3 – the student evaluates the task	Do you think that's the middle?				
	How do you know they're the same?				
	How do I know if I'm right?				
	How can we check?				
	Is there any other way that you can think of, to check?				
	How can we check?				

Alignment with Critical Mathematical Thinking	Teacher Questions – CMTLE 1	LEQUE	TQ Factual	TQ Probing	TQ Guiding
			Maths related question	Clarifying Decoding Noting relationships Offering opinions with reasons	Assessing Understanding Self-regulating
	Are they two equal parts? How can you check?				
	So that helped you find the exact middle?				
	So, do you think that's the middle?				
	Can you think of anything from before?				
	So, can you show me what that means to you? I'm happy for you to draw on the paper if you want to. What would you do?				
	Does that look like the middle of the wall? The center of the wall?				
	Yeah, they both matched the same spot, haven't they? But does that help you find the middle?				
	It does? How?				
	so, do you think that there is the middle? How can we check?				
	So, what do we need to do?				
	I haven't got a ruler. Any other way that you can think of?				
Phase 4 – student explains the answer	How did you find that out?				
	What do you think? Is it in the middle?				
	So, where's the middle?				
	So, does that make the middle? What about this way? How do you know it's the middle?				
	Is this the middle of the wall?				
	Is this the middle down here?				
	What if I want to put it in the centre, right in the middle of everything? How can I find out where that is?				
	What about this way? So, is this the middle up here?				
	Okay, so is that in the middle now?				
	Can you think of any other way to show me how to find the middle without using your arms? Can you think of any other way?				
	Well, how can we check that's exactly the middle? What can we do to check?				
	Do you think it's the middle now?				
	Can you think of any other way to find the middle?				

Alignment with Critical Mathematical Thinking	Teacher Questions – CMTLE 1	LEQUE	TQ Factual	TQ Probing	TQ Guiding
			Maths related question	Clarifying Decoding Noting relationships Offering opinions with reasons	Assessing Understanding Self-regulating
	And what is it?				
	No. Why not?				
	So that's the middle there. But how do we know it's the middle? That way?				
	Okay, so what are you going to do next?				
	Okay, so then where would you place the photograph?				
	Is it? Is that the middle there? Is that the middle there?				
	Let’s try a new word, what about the center?				
	So, where's the center?				
	How do you know that's the center?				
	So, can you show me where'd you put the photo of Joey? Is that in the middle of the wall? How can you tell me that's the middle?				
	Can you think of any other way?				
	What are you going to do next?				
	So, what do we need to do?				
	Do you want to try it on the other side?				
	A different way. Okay. What are you thinking this time?				
	All right, so now that we've got the dot, what are we going to do?				
	Okay, so what? What if I do a dot there or a cross. How do we know it's the middle that way?				
	All right, so what have you found?				
	Is there a way that we can find exactly the middle of the entire wall?				
	What do you think you've discovered now?				
	No other way? So where would we put Joey's photo if you think that's the middle? So, is that the middle? Are you able to check any other way?				
	Okay, so you think that's the middle?				
	So, it's the middle that way. What about the middle that way?				
	Is there anything else you'd like to talk to me about with this picture?				

Alignment with Critical Mathematical Thinking	Teacher Questions – CMTLE 1	LEQUE	TQ Factual	TQ Probing	TQ Guiding
			Maths related question	Clarifying Decoding Noting relationships Offering opinions with reasons	Assessing Understanding Self-regulating
Phase 5 – student displays/shows/creates	Is there a way you can check that that's the middle?				
	Is there anything else you need to do?				
	Do you want to show me?				
	Maybe do you want to make any drawings to show me?				
	So okay, you've made some changes there. What about if the photo was here? Is that the middle?				
	Do you want to have a go at showing me?				
	Did you want to show me?				
	Maybe there? So, is it the middle of the wall?				
	Can you draw a line?				
	Could you draw those lines that you're talking about?				
	You want to have a go and you can use a pencil if you like. Or what do you think?				
	Do you want to have a go?				
	How can we check that it's the same amount of paper on each side? Is there a way that we can check?				
	How can you show me how to do that?				
	Or do you want to try another strategy?				
	And how might we check if they're the same length?				
	It does? Okay, there's no other way you check? No?				
	Maybe you can think of another strategy. Is there another way you can find the middle?				
	So, you could draw a piece of paper, then you'll know where the middle is.				

Results of teacher questioning within the CMTLE 2

Alignment with Critical Mathematical Thinking	Teacher Questions – CMTLE 2	LEQUE	TQ Factual	TQ Probing	TQ Guiding
			Maths related question	Clarifying Decoding Noting relationships Offering opinions with reasons	Assessing Understandin g Self-regulating
Phase 1 – the student interpreting the task	Have you ever sat on a bean bag?				
	Have you ever sat on a bean bag?				
	Have you ever sat on a bean bag?				
	Did all of these things come out of it?				
	I'm wondering, how can we find out how many are in there?				
	Do you know those things that are really squishy, like those little balls?				
	A bean bag. Do you have one at home?				
	Do you know what they are called?				
	How can I try and figure out how many beans are in this bean bag?				
	Is there any other way that we can find out how many mini beans are in this beanbag?				
	What do you do with it?				
	Do you know what a bean bag is?				
	Have you ever seen anything like this before?				
	So how can we find out how many beans might be in there?				
	How can we try and find out how many beans are in this bean bag?				
	How can we find out how many?				
	What way can we find out how many beans are in this bean bag?				
	Can you think of a way that we can find out how many beans are in that bean bag?				
Phase 2 – the student analysing the task	Can you think of another way?				
	Can you think of any other way?				
	Is there any other way that you can think of?				
	Is there any other way that you can think of to find out how many beans are in this bean bag?				
	Can you think of any other way to find out how many beans are in this bean bag?				
	Can you think of another way we can find out how many beans are in there?				

Alignment with Critical Mathematical Thinking	Teacher Questions – CMTLE 2	LEQUE	TQ Factual	TQ Probing	TQ Guiding
			Maths related question	Clarifying Decoding Noting relationships Offering opinions with reasons	Assessing Understandin g Self-regulating
	What might we do?				
	Well I want you to think about how I can find out how many of these are in here? How can I find it? How can I figure it out?				
Phase 3 – the student evaluates the task	How do you know that there's 101?				
	So, you know that, that's 10 and then are you going to do next?				
	Is that the best way?				
	What sort of machine?				
	How long do you think that's going to take?				
	Do you need to go that way?				
	All right, so you've estimated 250. How can we check?				
	So how would you think?				
	And what does that mean to you				
	Can you think of another way we can find out how many are in there? Can you think of anything?				
	Can you think of another way that we can find out how many beans are in this bean bag?				
	Can you think of another way that's not counting them by ones? What are you thinking?				
	How would you put them into equal groups?				
	How do we know how many more there are?				
	Would that take a long time?				
Phase 4 – student explains the answer	Hear how many?				
	Any other way?				
	What does it say?				
	All right, can we think of another way?				
	And what is it?				
	How else can we find out if we don't count?				
	What if we don't count? Is there another way we can find out how many beans are in there?				
	Can you think of any other way that we can find out how many mini beans are in there?				

Alignment with Critical Mathematical Thinking	Teacher Questions – CMTLE 2	LEQUE	TQ Factual	TQ Probing	TQ Guiding
			Maths related question	Clarifying Decoding Noting relationships Offering opinions with reasons	Assessing Understanding Self-regulating
	So now how many teddies are there altogether?				
	I wonder if we can think of a way to find out how many are in there? How could we check?				
	And do you think there'll be far too many to count one by one?				
	I wonder if there's another way we can figure out how many are in there?				
	I wonder if you can think of a way to find out how many beans are in there without counting?				
	You can guess. What else can you do?				
	Can you think of another way to find out how many beans are in this bean bag?				
	How can we find out? What could you do?				
	Any other way?				
	Oh, what might the tag say? What might the tag say?				
	Can you think of any other way?				
	Can you explain that to me?				
	Is it one of your inventions?				
	Can you think of any other way?				
	One hundred, you think there's a hundred?				
	Can you think of another way that we can find out how many beans are in there?				
	Is there another way? Can you think of any other way?				
	What do you mean by that?				
	So, can we do it another way?				
	Is there any other way you can think of other than counting by ones, twos, and fives...				
	But how else can we find out how many how many are in there? Without unzipping it?				
	But how can we find out how many are actually in there?				
	Yeah? Any other way that might not include counting?				
	And then what happens if they're in equal groups?				
	So, you count by tens then?				

Alignment with Critical Mathematical Thinking	Teacher Questions – CMTLE 2	LEQUE	TQ Factual	TQ Probing	TQ Guiding
			Maths related question	Clarifying Decoding Noting relationships Offering opinions with reasons	Assessing Understandin g Self-regulating
	So, any other ideas about how we can find out?				
Phase 5 – student displays/shows/creates	Tell me what you know about this?				
	What would we do with all of these?				
	Magnifying glass? Would a magnifying glass be able to show you?				

Results of teacher questioning within the CMTLE 3

Alignment with Critical Mathematical Thinking	**Teacher Questions – CMTLE 3**	**LEQUE**	**TQ Factual**	**TQ Probing**	**TQ Guiding**
			Maths related question	**Clarifying Decoding Noting relationships Offering opinions with reasons**	**Assessing Understandi ng Self-regulating**
Phase 1 – the student interpreting the task	Can you tell me why three plus three is the same as four plus two?				
	Do you know what that says?				
	What is three plus three?				
	Why is three plus three the same as four plus two?				
	Why is three plus three the same as four plus two?				
	Why is three plus three the same as four plus two?				
	Why is four plus four the same as six plus two?				
	Why is three plus three the same as four plus two?				
	Why is three plus three the same as four plus two?				
	Why is three plus three the same as four plus two?				
	Why is three plus three the same as four plus two?				
	What do you know about three plus three?				
	What about these two number sentences? Why is six plus two the same as four plus four?				
	What is that? Can you read it for me?				
	What's three plus three?				
	Why is three plus three the same as four plus two?				
	Why is 4 plus 2 the same as 3 plus 3?				
	Question three, what do you know about this?				
	What does that say?				
	Why is six plus two the same as four plus four?				
	What about these two number sentences? Six plus two, and four plus four.				
	Why is this the same as this?				
Phase 2 – the student analysing the task	Yeah, and is there any other way you can think?				
	What about now?				
	Why do they both equal eight?				

Alignment with Critical Mathematical Thinking	Teacher Questions – CMTLE 3	LEQUE	TQ Factual	TQ Probing	TQ Guiding
			Maths related question	Clarifying Decoding Noting relationships Offering opinions with reasons	Assessing Understandi ng Self-regulating
	What do we know about three plus three?				
	And what about this one? What do we know about four plus two?				
	What about these two number sentences? What can you tell me?				
	So, if that's one more, what's that?				
Phase 3 – the student evaluates the task	Why is that so?				
	Because you don't know?				
Phase 4 – student explains the answer	As I said, you're very good at explaining, aren't you?				
	What does it mean? What do you mean by opposite sides?				
	Is there something that makes them the same?				
	You told me that you know that three and three is six. What do you know about four and two?				
	And how is it possible that they both equal eight?				
	Why is that?				
	And how do you know that they're the same?				
	What did you mean by that?				
	Now, tell me why these two are the same?				
	Yeah? Tell me about that				
	Why else are they the same?				
	What about the numbers? What do you know about the numbers?				
	Yeah? Tell me more.				
	And what about here?				
	And why would they both add up to eight?				
	Yeah, and why do they equal the same?				
	What do you mean by that?				
	What makes them the same?				
	Is there something that happens with the numbers?				
	They do both make six, but why? Why do they both equal six?				
	But why do you think that they're both equal six				

Alignment with Critical Mathematical Thinking	**Teacher Questions – CMTLE 3**	**LEQUE**	**TQ Factual**	**TQ Probing**	**TQ Guiding**
			Maths related question	**Clarifying Decoding Noting relationships Offering opinions with reasons**	**Assessing Understanding Self-regulating**
	Is there anything about the numbers that you can think of?				
Phase 5 – student displays/shows/creates	How can you check?				
	Is there any other way that you can check?				
	Can you think of any other numbers, number sentences, this is a number sentence, that you can think of where the answers are the same?				
	Can you think of any other ways you can show me your fingers that equals six?				
	What about with your fingers and four plus two how would you do that with your fingers?				
	Can you try counting that again?				

Results of teacher questioning within the CMTLE 4

Alignment with Critical Mathematical Thinking	**Teacher Questions – CMTLE 4**	**LEQUE**	**TQ Factual**	**TQ Probing**	**TQ Guiding**
			Maths related question	**Clarifying Decoding Noting relationships Offering opinions with reasons**	**Assessing Understanding Self-regulating**
Phase 1 – the student interpreting the task	Which one has more blocks?				
	Which one has more blocks?				
	Which one has more blocks?				
	Which tower do you think has more blocks?				
	Which tower has more blocks?				
	Which one do you think is taller?				
	Which one do you think has more blocks?				
	Which one has more blocks?				
	Which tower's got more blocks?				
	which tower has more blocks?				
Phase 2 – the student analysing the task	What if I added one more block here, and two more blocks here, which one?				
	If I added two more blocks to this tower, which one would have more blocks?				
	Which one's taller?				
	So, which one? What do you mean none of them? How do you, how do you know that none of them have more than the other				
	What about if I added three more here, which would have more blocks?				
Phase 3 – the student evaluates the task	How can you check?				
	How can you check?				
	How do you know?				
	How can you check which one's got more blocks?				
	Let's see if I put three more blocks here, which one would have more?				
	If I put one more block with this one here and two more blocks here, which tower would have more blocks?				
	If I was to add three more here and three more here, which one would have more blocks?				

Alignment with Critical Mathematical Thinking	Teacher Questions – CMTLE 4	LEQUE	TQ Factual	TQ Probing	TQ Guiding
			Maths related question	Clarifying Decoding Noting relationships Offering opinions with reasons	Assessing Understanding Self-regulating
Phase 4 – student explains the answer	So, can you explain why this tower is taller and this tower is shorter?				
	So, you're telling me that this one has more blocks, does it?				
	Is there a way that you can check?				
	So, which one has more blocks?				
	How many did you just count?				
	Why were you counting to five and then starting at one again?				
	What if, whoa, what if I added two more blocks here? Which one would have more blocks?				
	Well, how can you explain why this one is shorter than this one, then?				
	Okay, so what if I added three more blocks here? Which one would have more?				
	Which one would have more?				
	So why is one taller than the other than?				
	Why is one taller than the other, if they've both got the same blocks?				
	What if I added two more here and one more here? Which one would have more?				
	If we added two more blocks here, this one would still have more?				
Phase 5 – student displays/shows/creates	How can you check?				
	Can you check to see which one's got more blocks?				

Results of teacher questioning within the CMTLE 5

Alignment with Critical Mathematical Thinking	**Teacher Questions – CMTLE 5**	**LEQUE**	**TQ Factual**	**TQ Probing**	**TQ Guiding**
			Maths related question	**Clarifying Decoding Noting relationsh ips Offering opinions with reasons**	**Assessing Understa nding Self- regulating**
Phase 1 – the student interpreting the task	How many teddy bears do you think Sarah put in my pocket?				
	Do you have teddies in your classroom?				
	How many do you think I started with? If I had 11 teddies altogether, remember I said I left some there and then when I went home there was some more teddies and 11 where they all together.				
	How many do you think Emily put in my pocket?				
Phase 2 – the student analysing the task	And how many did I put in?				
	How many did I say they were altogether				
	How many do you think Emily put in my pocket?				
	So, if Emily put six in, how many did I put in?				
Phase 3 – the student evaluates the task	And how many did I put in my pocket?				
	So how many might I put in one pocket?				
	How many do you think Emily put in my pocket?				
Phase 4 – student explains the answer	What do you mean by count the spaces?				
	How might I go about finding out how many I'll need?				
	How many did my friend put in her pocket?				
	Now, how many are in my pocket?				
	How many are in my friend's pocket?				
	10 plus 10 equals 11?				
	Now how many teddies do I have?				
	what if I found I had more bears? What happens now?				
	And then how many in the other?				
	How did you know that there were six and five?				
	And so, if there were three here, how many are in my other pocket				

Alignment with Critical Mathematical Thinking	Teacher Questions – CMTLE 5	LEQUE	TQ Factual	TQ Probing	TQ Guiding
			Maths related question	Clarifying Decoding Noting relationships Offering opinions with reasons	Assessing Understanding Self-regulating
	Where did you get six from?				
	How can you check?				
	But what about the rest?				
	What could we have done?				
	And how many in my other pocket?				
Phase 5 – student displays/shows/creates	Can you show me what you might do?				
	Then how will you know how many of these to buy?				
	How can thinking like this help you do something somewhere else in the real world?				
	Okay, can you think of where I'd use something like that in real life? Where would we use that thinking about what we know about?				

Results of teacher questioning within the CMTLE 6

Alignment with Critical Mathematical Thinking	Teacher Questions – CMTLE 6	LEQUE	TQ Factual	TQ Probing	TQ Guiding
			Maths related question	**Clarifying Decoding Noting relationships Offering opinions with reasons**	**Assessing Understanding Self-regulating**
Phase 1 – the student interpreting the task	Can I stand in a cubby house?				
	Do you have a cubby house at home?				
	Would my head hit the roof?				
	How might I try and find out how many tiles I need?				
	Do you know what a cubby house is?				
	How can I find out how many tiles I need, to place on the floor of this cubby house?				
	How can I find out how many tiles I need to cover the floor of that cubby house?				
	Have you ever played in a cubby house? It's not as big as a real house, is it?				
	Have you ever been into a cubby house?				
	How can I find out how many I need?				
	How do I know how many tiles I need to buy?				
	How can we find out how many tiles I need to buy to fit on the floor of the cubby house?				
	How could I find out how many tiles I need to buy to put on the floor there?				
	How can if find out how many tiles I need to buy.				
	Is there a way that we can work out how many tiles I need to buy to fill this floor?				
Phase 2 – the student analysing the task	How can I find it out?				
	So how many tiles do you think would fit in that cubby house?				
	What might I do?				
	I've only got this many and I know I'm going to need a lot more, but how many? How could I find out how many I need to buy?				
	Does it? What if I buy too many tiles?				
	So how can I find out how many tiles I need to buy? How can I find out how many tiles I need to buy?				

Alignment with Critical Mathematical Thinking	Teacher Questions – CMTLE 6	LEQUE	TQ Factual Maths related question	TQ Probing Clarifying Decoding Noting relationships Offering opinions with reasons	TQ Guiding Assessing Understanding Self-regulating
	How can I work out how many tiles to buy for that space?				
Phase 3 – the student evaluates the task	How do I know? How can I find out? Is there something I can do to find out how many tiles could fit on that floor?				
	How would I find out how many of these tiles I need to buy so that I can cover the floor? What should I do?				
	So, you think that one tile is all I need to fill the entire big real life cubby house? That's just a picture remember?				
	But how many would I need to put down all on this section here of that cubby house?				
	How will I know how many I need to buy?				
Phase 4 – student explains the answer	Can you explain what that would look like?				
	Can you think of any other way?				
	So how do you know? How can I figure out how many tiles I need to put on the floor of the cubbyhouse?				
	How can I count the floor and then count the tiles? Explain that to me				
	Okay, so how many do you think would fit?				
	So how will I measure?				
	So, I've measured and this is, let's say two meters long, what do I do next?				
	So how would we lay these tiles down there?				
	And how would you know how many more?				
	Yeah, is there any other way that we can share it equally?				
	Why not?				
	How could I figure out how many tiles I need to buy to cover the area of the floor?				
	I don't know how many either, but how can I find out? What could I do?				
	Okay, so we're going to use one tile to help us, is what you're thinking?				
	How would I measure the floor?				
	How long do you think that section would be if I got a measuring tape out?				

Alignment with Critical Mathematical Thinking	Teacher Questions – CMTLE 6	LEQUE	TQ Factual	TQ Probing	TQ Guiding
			Maths related question	Clarifying Decoding Noting relationships Offering opinions with reasons	Assessing Understanding Self-regulating
	Tiles? You think I need 101 tiles? I really like your thinking around using a measuring tape that shows me that you... Have you ever used a measuring tape before?				
	Is there a way we can find out how many I need to buy?				
	Can you think of any other way we can find out without drawing it? How many I need?				
	So now you've given me another design, but still my question is how many might I need to tile this front area? How can I work that out?				
	I don't know how many rows I need. I don't know. How can I find out how?				
Phase 5 – student displays/shows/creates	And so, then what would I do next?				
	Can you show me how to mark it?				
	How can I share it equally with you? There's a pencil there, can you show me?				
	So, what are you doing?				
	How would we count them? Show me.				
	So where might you put it next?				
	Do I need to draw it all over the floor, or is there a quicker way?				
	So, you're saying to take a tile and draw around it, trace the tile and then keep tracing to see how many tiles we need?				
	So, do you think that only five tiles can fit across here? Is that what you think?				
	How many tiles do you think fit across here?				
	How many do you think fit down here?				

Results of teacher questioning within the CMTLE 7

Alignment with Critical Mathematical Thinking	**Teacher Questions – CMTLE 7**	**LEQUE**	**TQ Factual**	**TQ Probing**	**TQ Guiding**
			Maths related question	**Clarifying Decoding Noting relationships Offering opinions with reasons**	**Assessing Understanding Self-regulating**
Phase 1 – the student interpreting the task	What did you have for morning tea?				
	I've really felt like a piece of toast with maybe some honey on it. What would you put on your toast?				
	How might we cut it so that both of us have equal parts?				
	Do you like to eat toast?				
	Do you put jam on it?				
	How might you cut this piece of bread so that you and I get the exact same amount?				
Phase 2 – the student analysing the task	Tell me, which pieces would you have? Those two? Which would I have?				
	Do you know what that is?				
	Okay, so how might we share it?				
	Now, how can we check that you and I get the exact same amount? What can we do?				
	How can I share it equally with you?				
Phase 3 – the student evaluates the task	How do I know, so that it's fair, and you don't eat more toast than I do? How do I know that this side is exactly the same as that side?				
	Can you think of any other ways we can cut this bread?				
	Can you think of any other way to cut it?				
	Can you think of a way to share this toast so that you and I get the exact same amount?				
	So, who gets more?				
	Is there a way that we can equally share it so that you and I get the same amount of toast?				
	Well, that's not fair, is it?				
Phase 4 – student explains the answer	Is this piece the same size as this piece?				
	No. Why not?				
	Which pieces will you have, and which will I have? Does that make it more friendly and equal?				
	Okay. If we drew any other lines, can we still share it equally?				

	Okay. So now, which pieces will you get?				
	Are they exactly the same?				
	How can you know, how can you check that this side is exactly the same as this side?				
	And then how can you check?				
	Can you think of any other way to cut this bread so that it's shared equally?				
	And so then how many pieces will I get and how many would you get if we were sharing it?				
	How can I check that this side is exactly the same as this side? Can you think about what we might do?				
	What could you do?				
	Oh, are you trying to get from that second little dash?				
	And how do I know that they are exactly the same?				
	Is there a way you can check?				
	Would it be fair if I got these two pieces and you got those two pieces?				
	How do you know that you get more or the same or less than me? Is there a way we can check?				
	Is there a way to know if this piece is bigger than this piece?				
	Why is the bottom smaller than the sides?				
	Can you think of any other ways that we can cut this bread so that we still have the same amount each?				
	So, is that half the same as that half?				
	Which two would you get?				
Phase 5 – student displays/shows/creates	Okay, so if I want to share it equally with you, can you draw a line to show me how I could do that?				
	Oh, can you draw any other lines?				
	And then which pieces will you get?				
	Can you show me how we might do that with a pencil?				
	Now is this section here exactly the same size as this section here?				
	Do you want to get rid of a line?				
	How do we know that this side is the same as this side? Have a look at them. Do you, if that's yours and that's mine. Who gets more toast?				
	So, it's not even, is it? Is there another way that you can cut the toast so that both of us get the same amount?				
	So now you've got this line. How do we know that this amount is the same as this amount?				
	So, if I give you another slice of bread, can you have another go of trying to find out how to do it exactly right?				

	Can you show me with the pencil where you might cut it in half?				
	Use your pencil to show me. Draw it. I want you to really look at this now. Is this side exactly the same? Has it got the exact same space as this side?				
	Can you show me a way that we can share this bread?				
	How can you prove that this side is exactly the same as this side?				
	Okay, can you show me how?				
	Let's imagine this one is folded this way and this one is folded this way, like you said. How can you, if you had this and I had this, would we have the exact same amount? Why not?				
	How can I do that?				
	Can you show me a way?				
	And so how can we check if that's half? How can we check that this half is exactly the same size as this half?				

Results of teacher questioning within the CMTLE 8

Alignment with Critical Mathematical Thinking	Teacher Questions – CMTLE 8	LEQUE	TQ Factual	TQ Probing	TQ Guiding
			Maths related question	Clarifying Decoding Noting relationships Offering opinions with reasons	Assessing Understanding Self-regulating
Phase 1 – the student interpreting the task	Can you copy that shape using lots of little grey shapes?				
	Can you try and make this shape here, with these shapes?				
	Can you use these shapes to make this one?				
	Can you use these grey shapes to demonstrate this or to show me a shape that's the same as this shape?				
	Can you try and make this shape with the little shapes over here?				
	Can you try and make this black shape?				
	Can you try and use these grey shapes as many as you like to make the same shape, please?				
Phase 2 – the student analysing the task	Can I just move this a little bit closer to this one? Are they exactly the same?				
	Do you think that works?				
Phase 3 – the student evaluates the task	Are they exactly the same?				
	Why don't you use more than one shape for that?				
	Is it the same?				
	Yeah, you're happy with that?				
Phase 4 – student explains the answer	How did you go? Have you created it?				
	You're happy with that? Does it look exactly the same?				
	So, you've drawn that exact shape, haven't you?				
	What do you need to add to make it look like this?				
	What about now? I'm going to just slide these ones away. Let's not mix them up. What about this shape with these yellow pieces?				
	Don't know? You want to try and make it as close as you can?				
	Can you think of what you might need to do to it to make it look like that?				
	What about this shape?				

Alignment with Critical Mathematical Thinking	Teacher Questions – CMTLE 8	LEQUE	TQ Factual	TQ Probing	TQ Guiding
			Maths related question	Clarifying Decoding Noting relationships Offering opinions with reasons	Assessing Understanding Self-regulating
Phase 5 – student displays/shows/creates	All right, you want to have a go?				
	Maybe you can add something to it to make it longer?				

Appendix E - Australian Catholic University Ethics Approval Letter

AUSTRALIAN CATHOLIC UNIVERSITY

Human Research Ethics Committee
<u>Committee Approval Form</u>

Principal Investigator/Supervisor: Dr Roger Vallance
Co-Investigators: N/A
Student Researcher: : Ms Chrissy Monteleone

Ethics approval has been granted for the following project: The performance characteristics of early education children in mainstream classrooms with respect to critical mathematical thinking
for the period: 07/10/2014-31/05/2015
Human Research Ethics Committee (HREC) Register Number: 2014 212N

<u>Special Condition/s of Approval</u>
Prior to commencement of your research, the following permissions are required to be submitted to the ACU HREC:
Sydney Catholic Education Office

The following <u>standard</u> conditions as stipulated in the *National Statement on Ethical Conduct in Research Involving Humans* (2007) apply:

(i) that Principal Investigators / Supervisors provide, on the form supplied by the Human Research Ethics Committee, annual reports on matters such as:
- security of records
- compliance with approved consent procedures and documentation
- compliance with special conditions, and

(ii) that researchers report to the HREC immediately any matter that might affect the ethical acceptability of the protocol, such as:
- proposed changes to the protocol
- unforeseen circumstances or events
- adverse effects on participants

The HREC will conduct an audit each year of all projects deemed to be of more than low risk. There will also be random audits of a sample of projects considered to be of negligible risk and low risk on all campuses each year.

Within one month of the conclusion of the project, researchers are required to complete a *Final Report Form* and submit it to the local Research Services Officer.

If the project continues for more than one year, researchers are required to complete an *Annual Progress Report Form* and submit it to the local Research Services Officer within one month of the anniversary date of the ethics approval.

Signed: ...... [redacted] ...... Date: .... 08/10/2014.....
(Research Services Officer, McAuley Campus)

U:\Ethics\APPLICATIONS\Ethics Applications 2014\2014 212N Vallance\2014 212N Approval Formbris 2014.doc

Appendix F - Catholic Education Office Sydney Approval Letter

Catholic Education Office Sydney
www.ceosyd.catholic.edu.au

27 October 2014

Ref: Research Application 924

Ms Chrissy Monteleone
Australian Catholic University
Locked Bag 2002
STRATHFIELD NSW 2135

Dear Chrissy,

RE: RESEARCH APPLICATION REF: 924 – LETTER OF APPROVAL

Thank you for the submission of your application to conduct research in Archdiocesan Catholic Schools under the jurisdiction of the Catholic Education Office (CEO) Sydney. Approval is given by CEO Sydney to conduct this study. **This approval is granted subject to full compliance with NSW Child Protection and Commonwealth Privacy Act legislation.** It is the prerogative of any Principal or staff member whom you might approach to decline your invitation to be involved in this study or to withdraw from involvement at any time. Any study involving the participation of students will require written, informed consent by parents/guardians.

Permission is given for you to approach the Principal of the school nominated, listed below, requesting participants for your study: ***"The performance characteristics of early education children mainstream classrooms with respect to critical mathematical thinking".***

Our Lady of Fatima Catholic Primary School, Kingsgrove *Ms Anne Colreavy*

COMMONWEALTH PRIVACY ACT

The privacy of the school and that of any school personnel or students involved in your study must, of course, be preserved at all times and comply with requirements under the Commonwealth Privacy Amendment (Private Sector) Act 2000. In complying with this legislation, the CEO Sydney has decided that individual research participants should not be identified in the report.

38 Renwick Street, PO Box 217, Leichhardt NSW 2040 • Phone (02) 9569 6111 • Fax (02) 9550 0052
Eastern Region 33 Banks Avenue, Daceyville NSW 2032 • Phone (02) 8344 3000 • Fax (02) 8344 3097
Inner Western Region 3 Keating Street, Lidcombe NSW 2141 (Locked Bag 83, 1825) • Phone (02) 9643 3600 • Fax (02) 9643 3609
Southern Region 300 The River Road, Revesby Heights NSW 2212 • Phone (02) 9772 7000 • Fax (02) 9772 7009

FURTHER REQUIREMENTS

It is a condition of approval that when your research has been completed you will forward a **summary report of the findings and/or recommendations** to this office as soon as results are to hand.

All correspondence relating to this Research should note 'Ref: Research Application 924'.

Please contact me at this office if there is any further information you require. I wish you well in this undertaking and look forward to learning about your findings.

Yours sincerely,

[redacted]

Dr Michael Bezzina
Director of Teaching and Learning
Email: research.centre@syd.catholic.edu.au

Appendix G - Information Letter to the Principal

AUSTRALIAN CATHOLIC UNIVERSITY

Strathfield Campus (Mount St Mary)
25A Barker Road Strathfield NSW 2135
Locked Bag 2002 Strathfield NSW 2135 Australia
Telephone: **9701 4455**
Facsimile: **9701 4034**
Email: **roger.vallance@acu.edu.au**

www.acu.edu.au

INFORMATION LETTER TO PRINCIPAL

TITLE OF PROJECT: **The performance characteristics of early education children in mainstream classrooms with respect to critical mathematical thinking**

PRINCIPAL INVESTIGATOR: **Chrissy Monteleone**

Dear (insert Principal name),

I would like to invite you and your school to participate in a research project to investigate the performance characteristics of early education children in mainstream classrooms with respect to critical mathematical thinking. In 2012 I began the Doctor of Education degree at the Australian Catholic University, Strathfield. The coursework associated with this degree and the guidance from my supervisors has allowed me to build the literature and theoretical base for this research. In 2014, I took part in a confirmation of candidature process associated with this research with success. The ethics clearances from ACU and CEO Sydney have been met.

This research will include ten CEO Sydney Primary Schools. The model, explained below, will be replicated in each school. The research will address the following questions:

- What do teachers know about critical mathematical thinking processes?
- How do teachers identify children that display critical mathematical thinking?
- What do children, identified as being critical mathematical thinkers, display during tasks that measure high levels of mathematical thinking?
- How do children, identified as having critical mathematical thinking, perform in standardised tasks that measure high levels of mathematical thinking?

The participants of this research are expected to include: the Kindergarten Classroom Teacher and Kindergarten Children. Up to ten Kindergarten Children from each Kindergarten class will be involved.

Data collection is expected to be through the following methods:

TEACHER	Semi-structured interview: Pre-prepared semi-structured interview questions asking teachers about n current methods used by the teacher in respect to identification of critical mathematical thinkers.	Data collected Video Recording
CHILD	Observation: Researcher observes all kindergarten children during their mathematics lessons (up to 5 visits)	Observations made at the end using the unstructured observation protocol (Appendix 3).
	Focus Groups: Researcher observes selected kindergarten children during their mathematics lessons in their classroom (up to 5 visits)	Unstructured observation checklist used (Appendix 3)
	Semi-structured interview: Pre-prepared piloted stimulus questions in a one-on-one interview (Appendix 10)	Data collected on possible critical mathematical thinking skills Video Recording
	Standardised tests: Three standardised test, Slosson Intelligence Test, Raven's Progressive Matrices, Patterns and Structure Assessment, will be administered to the children	Data will be aligned with the qualitative summative checklist (Appendix 1(

It is anticipated that this project will commence in the March 2015 and will continue through 2015. You may cease your involvement in this project at any time, wholly at your discretion and without explanation.

The outcomes of this evaluation project are expected to include a report on outcomes relevant to your school, which at your discretion may be shared with the school community, and one or more staff and parent briefings of research findings. The research is expected to provide data for several academic publications that will describe identification methods of kindergarten children with high critical mathematical thinking capabilities. The research outcomes of this study may be summarised and appear in publications or may be provided to other researchers in a form that does not identify the participants in any way. Furthermore, this research will help staff reflect upon and better understand critical mathematical thinking.

The participation of all people will be fully voluntary. Data will be carefully safeguarded to ensure that the identity of all participants: your teachers, students and parents. We anticipate there will be no risk to the participants, and no pressure will be put on any participant at any stage. Participants may withdraw at any time without giving a reason.

Any questions related to the project can be directed to myself, Chrissy Monteleone, Australian Catholic University, 25A Barker Rd, Strathfield 2135 or by phone on (02) 9701 4203.

This study has been approved by the Human Research Ethics Committee at Australian Catholic University. In the event that you have any complaint or concern about the way you have been treated during the study, or if you have any query that the Investigator has not been able to satisfy, you may write to the Chair of the Human Research Ethics Committee

Chair, HREC
C/o Research Services
Australian Catholic University
Locked Bag 2002,
STRATHFIELD NSW 2135
Tel: 02 9701 4093 Fax: 02 9701 4350

Any complaint or concern will be treated in confidence and fully investigated. The participant and yourself will be informed of the outcome.

If you are willing that your school participate in this research about the performance characteristics of early education children in mainstream classrooms with respect to critical mathematical thinking, could I ask that you read the accompanying Consent Form. This form does not commit your staff to volunteer for this research, but does indicate your approval that staff be invited to voluntarily participate in this research. Please sign one copy of the Consent Form for School Principal and return this signed copy to me, and keep one copy for your records.

Yours sincerely

Chrissy Monteleone

AUSTRALIAN CATHOLIC UNIVERSITY

Strathfield Campus (Mount St Mary)
25A Barker Road Strathfield NSW 2135
Locked Bag 2002 Strathfield NSW 2135 Australia
Telephone: **9701 4455**
Facsimile: **9701 4203**
Email: **chrissy.monteleone@acu.edu.au**

www.acu.edu.au

Consent Form for School Principal (Principal's copy)

The performance characteristics of early education children in mainstream classrooms with respect to critical mathematical thinking

Name of Principal Investigator: Chrissy Monteleone

I , ..., (Principal's name)
have read and understood the information provided in the Letter to the Principal. Any questions I have asked have been answered to my satisfaction. I agree that my staff and kindergarten children may, if they wish, participate in all or part of the following:

- Teacher interviews,
- Child observation, interview and assessment.

I realise that I may withdraw any and all participants from this evaluation process at any time.
I agree that research data collected for the study may be published or may be provided to other researchers in a form that does not identify any member of the school community in any personal way.

Name of principal: ....

Signature: .... **Date:** ....

Signature of principal investigator: Date: ....

Appendix H – Participation letter to the Teacher

AUSTRALIAN CATHOLIC UNIVERSITY

Strathfield Campus (Mount St Mary)
25A Barker Road Strathfield NSW 2135
Locked Bag 2002 Strathfield NSW 2135 Australia
Telephone: **9701 4203**
Email: **chrissy.monteleone@acu.edu.au**

www.acu.edu.au

TEACHER PARTICIPATION LETTER

PROJECT TITLE: Mathematical Thinking in Young Children
PRINCIPAL INVESTIGATOR: Dr Roger Vallance
HIGHER DEGREE RESEARCH STUDENT: Chrissy Monteleone
STUDENT'S DEGREE: Doctor of Education

Dear Kindergarten Classroom Teacher at ___________________________________,

You are invited to participate in the research project described below:

What is the project about?

The research project investigates ways in which children that display high critical mathematical thinking in the early years can be identified. The study will involve both qualitative and quantitative measures including the use of piloted stimulus questions and standardised assessments. The aim of this research is to find a measure that can be used by mainstream classroom teachers to identify children with high critical mathematical thinking capability.

Who is undertaking the project?

This project is being conducted by Chrissy Monteleone and will form the basis for the degree of Doctor of Education at Australian Catholic University under the supervision of Dr Roger Vallance and Professor Paul White.

Are there any risks associated with participating in this project?

This project involves teachers and children. The children will be aged between 4 ½ - 6 years of age (kindergarten). There is a low level of risk associated with the research. Consent will be provided by the teacher and caregiver of the child. The child will be verbally asked if they wish to participate in this research as the first question in the semi-structured interview. The risks may include: distress or anxiety, embarrassment or loss of self-esteem if unable to answer the questions. As the researcher is an experienced teacher in the early years, with skills in working with young children, the research will be conducted in a non-threatening and calm environment. The classroom observations will allow the researcher to become familiar with children before the individual research commences.

What will I be asked to do?

The teacher:

- Provide opportunity for the researcher to visit classroom during mathematics lessons (5 visits)
- Commit to one semi-structured one-on-one interview, asking questions related to their understanding of critical mathematical thinking and their techniques in identifying these capabilities in children, that will involve digital audio recording

The child:

- Focus group observation in their classroom during mathematics lesson
- Semi-structured one-on-one interview, asking up to eight mathematics stimulus questions, that will involve video recording

- Structured one-on-one interview, administering three standardised tests (Slosson Intelligence Test, Raven's Progressive Matrices, Patterns and Structure Assessment , that will involve video recording

The location:

- Kindergarten Classroom
- Quiet room at the school

N.B. For your information, children will be withdrawn from your classroom for interviews

How much time will the project take?

The teacher:

- 5 mathematics lessons (negotiated with the teacher)
- 1 semi-structured one-on-one interview– up to 1 hour

The child:

- 5 mathematics lessons (observation and focus group)
- 1 semi-structured one-on-one interview – up to 1 hour
- 1 structured one-on-one interview – up to 1 hour

What are the benefits of the research project?

The teacher:

- A model for method of identification of children with high mathematical critical thinking capability

The child:

- Exposure to challenging mathematical problem solving tasks.
- Potentially an increased interest in mathematics

Can I withdraw from the study?

Participation in this study is completely voluntary. You are not under any obligation to participate. If you agree to participate, you can withdraw from the study at any time without adverse consequences. Children can withdraw at any time.

Will anyone else know the results of the project?

The study will be published as a thesis and scholarly journals. All participants will be non-identified. The data collection, both in its raw state and after analysis will be stored as a hard copy and an electronic copy. The hard copy will be stored in a locked filling system at the researcher's work office and the electronic copy on the researcher's password secured laptop computer. The researcher's work office is secured with a lock. Journal publications or conference presentations will emerge from the data collected and from the findings of analysis of the data collected from teachers and children.

Will I be able to find out the results of the project?

A detailed letter with aggregated research findings, so that individual schools, teachers and children are not identified, will be presented to Dr Dan White, Executive Director of CEO Sydney, respective Regional Consultants and school Principals and Leadership Teams. Opportunity to discuss findings will be made available to the classroom teachers and parents through a visit from the researcher to the school.

Who do I contact if I have questions about the project?

Student Researcher: Chrissy Monteleone Email: chrissy.monteleone@acu.edu.au Phone: 02 9701 4203	Principal Supervisor: Dr Roger Vallance Email: roger.vallance@acu.edu.au Phone: 02 9701 4455

What if I have a complaint or any concerns?

The study has been reviewed by the Human Research Ethics Committee at Australian Catholic University (approval number 2014 212N). If you have any complaints or concerns about the conduct of the project, you may write to the Chair of Research Ethics Manager:

Research Ethics Manager (ResEthics.Manager@acu.edu.au)
Office of the Deputy Vice-Chancellor (Research)
Australian Catholic University
North Sydney Campus

PO Box 968
North Sydney NSW 2059.

Any complaint or concern will be treated in confidence and fully investigated. You will be informed of the outcome.

I want to participate! How do I sign up?

Participants accepting to take part in this research are requested to complete the consent form and provide a signed copy to the researcher.

Yours sincerely,

[redacted] [redacted]

Chrissy Monteleone Dr Roger Vallance

Strathfield Campus (Mount St Mary)
25A Barker Road Strathfield NSW 2135
Locked Bag 2002 Strathfield NSW 2135 Australia
Telephone: **9701 4203**
Email: **chrissy.monteleone@acu.edu.au**

www.acu.edu.au

TEACHER CONSENT FORM

Copy for Researcher / Copy for Participant to Keep

TITLE OF PROJECT: Mathematical Thinking in Young Children

Student Researcher: Chrissy Monteleone Email: chrissy.monteleone@acu.edu.au Phone: 02 9701 4203	Principal Supervisor: Dr Roger Vallance Email: roger.vallance@acu.edu.au Phone: 02 9701 4455

I .... *(the Kindergarten Classroom Teacher)* have read and understood the information provided in the Letter to Participants. Any questions I have asked have been answered to my satisfaction. I agree to participate in this research, as outlined below, realising that I can withdraw my consent at any time. I agree that research data collected from this study may be published or may be provided to other researchers in a form that does not identify me in any way.

Teacher Participation:

- Kindergarten Classroom Teacher semi-structured one-on-one interview including audio recording (1 visit) – up to 1 hour
- Agree to have my class observed during mathematics lessons in Kindergarten (5 visits) including one focus group observation (selected Kindergarten children during mathematics lesson) – up to 1 hour per lesson

NAME OF TEACHER:
SIGNATURE: DATE:

NAME OF PRINCIPAL SUPERVISOR: Dr Roger Vallance
SIGNATURE: [redacted] DATE:

NAME OF STUDENT RESEARCHER: Chrissy Monteleone
SIGNATURE: [redacted] DATE:

Appendix I – Information Letter to the Parent/Carer

AUSTRALIAN CATHOLIC UNIVERSITY

Strathfield Campus (Mount St Mary)
25A Barker Road Strathfield NSW 2135
Locked Bag 2002 Strathfield NSW 2135 Australia
Telephone: **9701 4203**
Email: **chrissy.monteleone@acu.edu.au**

www.acu.edu.au

CHILD PARTICIPATION LETTER

PROJECT TITLE: Mathematical Thinking in Young Children
PRINCIPAL INVESTIGATOR: Dr Roger Vallance
HIGHER DEGREE RESEARCH STUDENT: Chrissy Monteleone
STUDENT'S DEGREE: Doctor of Education

Dear Parent of ________________________at ________________________,
You are invited to read the proposed research and consider agreeing to allow your child to participate, if your child would like to participate, in this research.

What is the project about?
The research project investigates ways in which children that display high critical mathematical thinking in the early years can be identified. The study will involve both qualitative and quantitative measures including the use of piloted stimulus questions and standardised assessments. The aim of this research is to find a measure that can be used by mainstream classroom teachers to identify children with high critical mathematical thinking capability.

Who is undertaking the project?
This project is being conducted by Chrissy Monteleone and will form the basis for the degree of Doctor of Education at Australian Catholic University under the supervision of Dr Roger Vallance and Professor Paul White.

Are there any risks associated with participating in this project?
This project involves teachers and children. The children will be aged between 4 ½ - 6 years of age (kindergarten). There is a low level of risk associated with the research. Consent will be provided by the teacher and caregiver of the child. The child will be verbally asked if they wish to participate in this research as the first question in the semi-structured interview. The risks may include: distress or anxiety, embarrassment or loss of self-esteem if unable to answer the questions. As the researcher is an experienced teacher in the early years, with skills in working with young children, the research will be conducted in a non-threatening and calm environment. The classroom observations will allow the researcher to become familiar with children, and the children with the researcher, before the individual research commences. At the time of the one-on-one interview with the child, the researcher will explain that the questions will cover mathematics questions and that the information will help understand the way in which children think during mathematics, the researcher will then ask the individual child if they understand the research and they assent to being part of this research project. If a child does not wish to participate, the interview will conclude with no further questions asked of that individual child and no data collected from the child will be used.

What will my child be asked to do? (an overview of school participation is included as a quick glance table)
Data collected relevant to your child will be:

- Observation during mathematics lessons (5 visits)
- Focus group informal interview in their classroom during mathematics lesson
- Semi-structured one-on-one interview, asking up to eight mathematics stimulus questions, that will involve video recording
- Structured one-on-one interview, administering three standardised tests (Slosson Intelligence Test, Raven's Progressive Matrices, Patterns and Structure Assessment) , that will involve video recording

The location:

- Kindergarten Classroom
- Quiet room at the school

How much time will the project take?

Your child

- 5 mathematics lessons (observation and focus group)
- 1 semi-structured one-on-one interview – up to 1 hour
- 1 structured one-on-one interview – up to 1 hour

What are the benefits of the research project?

The child:

- Exposure to challenging mathematical problem solving tasks

Can I withdraw from the study?

Participation in this study is completely voluntary. You are not under any obligation to participate. If you agree to participate, you can withdraw from the study at any time without adverse consequences. Children can withdraw at any time.

Will anyone else know the results of the project?

The study will be published as a thesis and scholarly journals. All participants will be non-identified. The data collection, both in its raw state and after analysis will be stored as a hard copy and an electronic copy. The hard copy will be stored in a locked filling system at the researcher's work office and the electronic copy on the researcher's password secured laptop computer. The researcher's work office is secured with a lock. Journal publications or conference presentations will emerge from the data collected and from the findings of analysis of the data collected from teachers and children.

Will I be able to find out the results of the project?

A detailed letter with research findings will be presented to Dr Dan White, Executive Director of CEO Sydney, respective Regional Consultants and school Principals and Leadership Teams. Opportunity to discuss findings will be made available to the classroom teachers and parents through a visit from the researcher to the school.

Who do I contact if I have questions about the project?

Contact person: Chrissy Monteleone Email: chrissy.monteleone@acu.edu.au Phone: 02 9701 4203	Supervisor: Roger Vallance Email: roger.vallance@acu.edu.au Phone: 02 9701 4455

What if I have a complaint or any concerns?

The study has been reviewed by the Human Research Ethics Committee at Australian Catholic University (approval number 2014 212N If you have any complaints or concerns about the conduct of the project, you may write to the Chair of the Research Ethics Manager:

Research Ethics Manager
(ResEthics.Manager@acu.edu.au)
Officer of the Deputy Vice – Chancellor (Research)
Australian Catholic University
North Sydney Campus
PO Box 967
North Sydney NSW 2059

Any complaint or concern will be treated in confidence and fully investigated. You will be informed of the outcome.

I want to participate! How do I sign up?
Participants accepting to take part in this research are requested to complete the consent form and provide a signed copy to the researcher.

Yours sincerely,

[redacted] [redacted]

Chrissy Monteleone Dr Roger Vallance

Strathfield Campus (Mount St Mary)
25A Barker Road Strathfield NSW 2135
Locked Bag 2002 Strathfield NSW 2135 Australia
Telephone: **9701 4203**
Email: **chrissy.monteleone@acu.edu.au**

www.acu.edu.au

CHILD CONSENT FORM

Copy for Researcher / Copy for Participant to Keep

TITLE OF PROJECT: Mathematical Thinking in Young Children

Student Researcher: Chrissy Monteleone Email: chrissy.monteleone@acu.edu.au Phone: 02 9701 4203	Principal Supervisor: Dr Roger Vallance Email: roger.vallance@acu.edu.au Phone: 02 9701 4455

I .... *(the Parent/ Carer)* have read (*or, where appropriate, have had read to me)* and understood the information provided in the Letter to Participants. Any questions I have asked have been answered to my satisfaction. I agree to participate in this research, as outlined below, realising that I can withdraw my consent at any time. I agree that research data collected from this study may be published or may be provided to other researchers in a form that does not identify me in any way.

I agree my child participates in the following:

- Observation of mathematics lessons in Kindergarten (5 visits) including one focus group observation (selected Kindergarten children during mathematics lesson)
- One-on-one semi structured interviews with selected Kindergarten children
- On-on-one administration of standardised tests with selected Kindergarten children

I have spoken with my child about being interviewed and they are happy about it
Y / N (please circle).
I have spoken with my child about being filmed during the interview and they are happy about it
Y / N (please circle).
I know that before the research commences, my child will be asked whether they are happy to filmed Y / N (please circle).

NAME OF TEACHER:
SIGNATURE: DATE:

NAME OF PRINCIPAL SUPERVISOR: Dr Roger Vallance
SIGNATURE: [redacted] DATE:

NAME OF STUDENT RESEARCHER: Chrissy Monteleone
SIGNATURE: [redacted] DATE:

Appendix J – Participation letter to the Teacher

AUSTRALIAN CATHOLIC UNIVERSITY

Strathfield Campus (Mount St Mary)
25A Barker Road Strathfield NSW 2135
Locked Bag 2002 Strathfield NSW 2135 Australia
Telephone: **9701 4203**
Email: **chrissy.monteleone@acu.edu.au**

www.acu.edu.au

TEACHER PARTICIPATION LETTER

PROJECT TITLE: Mathematical Thinking in Young Children
PRINCIPAL INVESTIGATOR: Dr Roger Vallance
HIGHER DEGREE RESEARCH STUDENT: Chrissy Monteleone
STUDENT'S DEGREE: Doctor of Education

Dear Kindergarten Classroom Teacher at ___________________________________,

You are invited to participate in the research project described below:

What is the project about?

The research project investigates ways in which children that display high critical mathematical thinking in the early years can be identified. The study will involve both qualitative and quantitative measures including the use of piloted stimulus questions and standardised assessments. The aim of this research is to find a measure that can be used by mainstream classroom teachers to identify children with high critical mathematical thinking capability.

Who is undertaking the project?

This project is being conducted by Chrissy Monteleone and will form the basis for the degree of Doctor of Education at Australian Catholic University under the supervision of Dr Roger Vallance and Professor Paul White.

Are there any risks associated with participating in this project?

This project involves teachers and children. The children will be aged between 4 ½ - 6 years of age (kindergarten). There is a low level of risk associated with the research. Consent will be provided by the teacher and caregiver of the child. The child will be verbally asked if they wish to participate in this research as the first question in the semi-structured interview. The risks may include: distress or anxiety, embarrassment or loss of self-esteem if unable to answer the questions. As the researcher is an experienced teacher in the early years, with skills in working with young children, the research will be conducted in a non-threatening and calm environment. The classroom observations will allow the researcher to become familiar with children before the individual research commences.

What will I be asked to do?

The teacher:

- Provide opportunity for the researcher to visit classroom during mathematics lessons (5 visits)
- Commit to one semi-structured one-on-one interview, asking questions related to their understanding of critical mathematical thinking and their techniques in identifying these capabilities in children, that will involve digital audio recording

The child:

- Focus group observation in their classroom during mathematics lesson
- Semi-structured one-on-one interview, asking up to eight mathematics stimulus questions, that will involve video recording

- Structured one-on-one interview, administering three standardised tests (Slosson Intelligence Test, Raven's Progressive Matrices, Patterns and Structure Assessment , that will involve video recording

The location:

- Kindergarten Classroom
- Quiet room at the school

N.B. For your information, children will be withdrawn from your classroom for interviews

How much time will the project take?

The teacher:

- 5 mathematics lessons (negotiated with the teacher)
- 1 semi-structured one-on-one interview– up to 1 hour

The child:

- 5 mathematics lessons (observation and focus group)
- 1 semi-structured one-on-one interview – up to 1 hour
- 1 structured one-on-one interview – up to 1 hour

What are the benefits of the research project?

The teacher:

- A model for method of identification of children with high mathematical critical thinking capability

The child:

- Exposure to challenging mathematical problem solving tasks.
- Potentially an increased interest in mathematics

Can I withdraw from the study?

Participation in this study is completely voluntary. You are not under any obligation to participate. If you agree to participate, you can withdraw from the study at any time without adverse consequences. Children can withdraw at any time.

Will anyone else know the results of the project?

The study will be published as a thesis and scholarly journals. All participants will be non-identified. The data collection, both in its raw state and after analysis will be stored as a hard copy and an electronic copy. The hard copy will be stored in a locked filling system at the researcher's work office and the electronic copy on the researcher's password secured laptop computer. The researcher's work office is secured with a lock. Journal publications or conference presentations will emerge from the data collected and from the findings of analysis of the data collected from teachers and children.

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Student Researcher: Chrissy Monteleone Email: chrissy.monteleone@acu.edu.au Phone: 02 9701 4203	Principal Supervisor: Dr Roger Vallance Email: roger.vallance@acu.edu.au Phone: 02 9701 4455

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Office of the Deputy Vice-Chancellor (Research)
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Yours sincerely,

[redacted] [redacted]

Chrissy Monteleone Dr Roger Vallance

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TEACHER CONSENT FORM

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Student Researcher: Chrissy Monteleone Email: chrissy.monteleone@acu.edu.au Phone: 02 9701 4203	Principal Supervisor: Dr Roger Vallance Email: roger.vallance@acu.edu.au Phone: 02 9701 4455

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Teacher Participation:

- Kindergarten Classroom Teacher semi-structured one-on-one interview including audio recording (1 visit) – up to 1 hour
- Agree to have my class observed during mathematics lessons in Kindergarten (5 visits) including one focus group observation (selected Kindergarten children during mathematics lesson) – up to 1 hour per lesson

NAME OF TEACHER:
SIGNATURE: DATE:

NAME OF PRINCIPAL SUPERVISOR: Dr Roger Vallance
SIGNATURE: [redacted] DATE:

NAME OF STUDENT RESEARCHER: Chrissy Monteleone
SIGNATURE: [redacted] DATE: