Construction of a Flow-Simulating Method with Finite Volume Based on a Voronoi Diagram*

Nobuyuki TANIGUCHI**, Chuichi ARAKAWA***
and Toshio KOBAYASHI**

In flow simulations performed with the finite difference method or the finite volume method, it is a serious limitation that the calculating points must be ordered on the coordinates. Using the Voronoi diagram for the cell division of the finite volume method creates a new discretization form which permits an arbitrary distribution of points. This paper constructs a new method for the flow simulations by a Voronoi diagram and shows the calculation results of two-dimensional flows.

Key Words: Computational Fluid Dynamics, Numerical Analysis, Finite Volume Method, Voronoi Diagram, Unstructured Grid

1. Introduction

In simulations of complicated flow fields, there are two difficulties : adaptability to the wall boundary and computational time. The former is due to the restrictions for calculational grids ; for example, in the structured grid, the points must be along a global coordinate system. The latter is mainly related to the efficiency of the solution methods for a system of discretized equations, especially on array-processor or multiprocessor systems of supercomputers.

Flow-simulating techniques based on the finite difference method (FDM) or the finite volume method (FVM) are superior in computational time because efficient iterative methods and algorithms can be easily applied to solve their equation systems. In order to introduce mathematical models for turbulence or other physical phenomena, they have another advantage in that their equations can be expressed as simple forms. However, it is often difficult to generate the appropriate grids, even in the case of a flow field with simple geometry. Although the body-fitted coordinate system has been successfully adopted in some simulations, it causes another problem of grid generation.

Therefore, we introduced the Voronoi diagram to derive the discretized equations on unstructured grids which are independent on any global coordinates. A Voronoi diagram defines a system of territories by the scattered points. In the fields of computational geometry, many studies[1] involving the Voronoi diagram have been conducted. According to these studies, we extended the FVM to the unstructured grid on the condition that iterative techniques of matrix solutions are as effective as in the previous methods[2] on the global coordinate system, which were successfully developed on the supercomputers. With respect to the incompressible flow, an effective algorithm, SIMPLE, can be applied with little expansion.

This paper will outline the present simulation method and indicate some results in two-dimensional laminar flows.

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** Institute of Industrial Science, University of Tokyo, 7-22-1 Roppongi, Minato-ku, Tokyo 106, Japan
*** Faculty of Mechanical Engineering, University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113, Japan

2. Discretizing Method

2.1 Control cells

According to the FVM, a differential equation is integrated in each control cell which fills the whole calculation domain. The integrated form for a general value ϕ, such as a component of the velocity vector or a scalar value, is

$$\int_{vol}\frac{\partial\phi}{\partial t}dV=\int_S \boldsymbol{J}\cdot d\boldsymbol{n}+\int_{vol}BdV \qquad (1)$$

where $\int_{vol}dV$: volume integration, $\int_S d\boldsymbol{n}$: surface integration, $\boldsymbol{J}(=\boldsymbol{v}\phi+\Gamma\nabla\phi)$: total flux vector, B : source of ϕ, $\boldsymbol{v}$: velocity vector, Γ : diffusive coefficient and $\boldsymbol{n}$: surface vector.

In previous methods[2], calculation grids are defined along a global coordinate system and their control cells are figured with a rectangular mesh in the two-dimensional case. According to these methods, the volume integrations in the right-hand side of Eq.(1) and the second in the left, are estimated by a constant value in each control cell ; the surface integrations of the first term in the left are discretized on each boundary face of the cell by the values on the points across it. The coefficient between these points is calculated by a flux interpolation on the face, or in other words, a scheme. The general form of the discretized equation is

$$a^C\phi^C=\sum a^{NB}\phi^{NB}+b \qquad (2)$$

where a : coefficient of linearized equation, b : source term, and suffixes C and NB : central point and neighbor points along the coordinates. Because the points related in Eq.(1) are ordered along the global coordinates, some effective iterative methods are available to solve its system, while the grid generation is as difficult as in the FDM.

Another concept of cell generation enables the derivation of a new discretizing method from Eq.(1) because there is no restriction for the cell figures. It is considered that they must be under the following conditions in order to retain the above advantages of the previous FVM.

(1) The whole region is filled with cells which are never superimposed on each other.

(2) Each cell has one calculation point in it.

The present paper proposes applying a Voronoi diagram for the cell generation satisfying the above conditions. Concerning the n points (X^n) scattered in a calculational region, a Voronoi diagram defines their territories,

$$V(X^i)=\bigcap_{j=1}^{n}\{X|d(X^i,X)\leqq d(X^j,X)\}$$

where $d(X,X^i)$: distance between X and X^i, X : a point in the domain. In the two-dimensional case, its concept is illustrated in Fig. 1, where the bold lines indicate the Voronoi diagram for the mother points as small circles and define their cell boundaries. It is known in computational geometry that such a closed cell should be always and uniquely generated for each point without the boundary and also that it should be a convex polygon. For any grid, the Voronoi diagram can be drawn automatically from its characteristic that a segment is defined as the bisector of two neighboring points and their intersection is the center of a circumscribed circle for their mother points.

The method of discretizing Eq.(1) on the cell system by the Voronoi diagram will be summarized as follows.

2.2 Flux interpolation

According to the FVM, the first term in the right-hand side of Eq.(1) can be calculated if the total flux, $\boldsymbol{J}$, is approximated to be constant on a side of the polygon by Voronoi diagram. Then, it is noted that the arrangement of points across a face is the same in the cell system by the Voronoi diagram as in the previous method on the Cartesian coordinates where each cell face is a normal bisector of two neighboring points, for example, the face "a" and the points X^0 and X^1 in Fig. 1. Therefore, the flux through the face, $J(=\boldsymbol{J}\cdot\boldsymbol{n})$, can be interpolated by the schemes on the Cartesian coordinates[3]. In this research, we adopted a scheme based on the analytical solution of the one-dimensional equation, $(dJ/dx)=0$, that is,

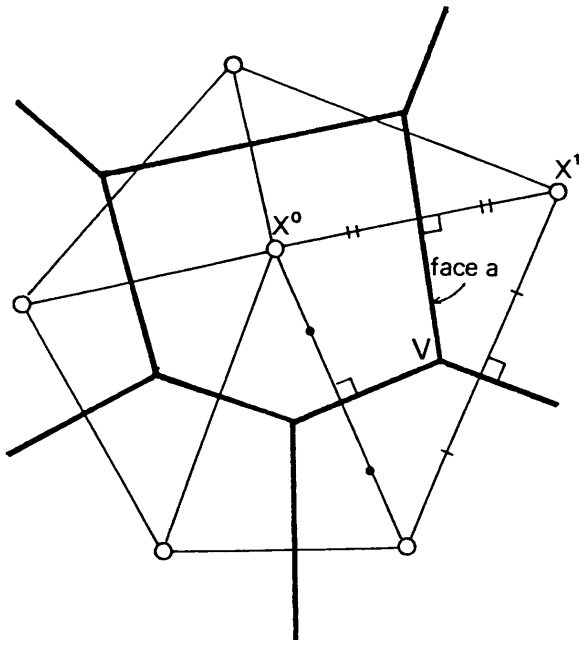

Fig. 1 Cell system by Voronol diagram

$$J=u\left\{\frac{\exp(Pe)}{1-\exp(Pe)}\phi^0+\frac{1}{1-\exp(Pe)}\phi^1\right\} \quad (3)$$

where Pe $(=uL/\Gamma)$: cell Pecret number, ϕ^0 and ϕ^1 : values at X^0 and X^1, L : distance between the two points, u : velocity component in their direction. Equation (3) is approximated as follows :

$$J=(\Gamma/L)F(Pe)(\phi^1-\phi^0)+u\phi^{UP}$$

$$\phi^{UP}=\begin{cases}\phi^0 & u\geq 0\\ \phi^1 & u<0\end{cases} \quad (4)$$

where $F(Pe)$, the function of Pecret number Pe, is expressed for each scheme as in Table 1. The power-law scheme is used in the present calculations. The scalar product, $\boldsymbol{J}\cdot\boldsymbol{n}$, can be easily calculated because the direction of the flux J in Eq.(3) is defined as normal to the cell face.

2.3 Source terms

A source term, the second in the right of Eq.(1), is dependent on the variable ϕ. In the flow simulations by turbulence models such as the k-ε model, it is expressed by the first-order differential coefficients of other values ; for example, a pressure gradient term appears for a component of velocity, or the product by the strain velocity tensors for a scalar value k or ε. A gradient vector, $\nabla\phi$, can be estimated in the following way. In this section, superscript and subscript symbols respectively indicate the position and the component of a vector, and the summing symbol refers to the position.

Firstly, concerning the points X^0 and X^{nb} across a cell face "nb", the difference between them expresses the projection component of the gradient vector, $\nabla\phi$ in this direction "nb", as follows :

$$(\nabla\phi^{nb})_{nb}=\frac{\phi^{nb}-\phi^0}{\|x^{nb}-x^0\|} \quad (5)$$

where $\|x^{nd}-x^0\|$ is the length of the two points. In addition, the difference from the value in the central point x^0

$$\delta\phi^{nb}=\frac{\phi^{nb}-\phi^0}{\|x^{nb}-x^0\|}-\nabla\phi\cdot\boldsymbol{e}^{nb} \quad (6)$$

can be derived on each face of a cell, where $\boldsymbol{e}^{nb}$ is the unit vector normal to the face "nb" which agrees with the direction "nb".

The weighted mean square of Eq.(6) for all faces of a cell,

Table 1 Functions the flux interpolation

· Central dif.	F (Pe) = 1 - 0.5* | Pe |
· Upwind	F (Pe) = 1
· Hybrid	F (Pe) = max (0 , 1- .5* | Pe |)
· Power Law	F (Pe) = max (0 , (1- .1* | Pe |)**5)

$$\Psi=\sum_{nb}(g^{nb}\delta\psi^{nb}\delta\psi^{nb}) \quad (7)$$

gives a definition of $\nabla\psi$, as it is minimized for a set of components. It is expressed by

$$\frac{\partial\Psi}{\partial(\nabla\psi)_a}=0 \quad (8)$$

Here, orthogonal coordinates should be adopted for the components ; generally, the Cartesian is available. The weighted coefficient, g^{nb} is defined by the angle of vision for the face segment in the present calculations.

Finally, the following system of linear equations is derived :

$$[A]\nabla\psi=\boldsymbol{r} \quad (9)$$

$$A_{ab}=\sum_{nb}\{g^{nb}(e^{nb})_a(e^{nb})_b\}$$

$$r_a=\sum_{nb}\{g^{nb}(e^{nb})_a(\nabla\psi^{nb})_a\}$$

where suffixes a and b : components of the orthogonal coordinates. As matrix [A] is assured to be regular on the cell system by the Voronoi diagram, the components of $\nabla\psi$ can be calculated.

According to the above techniques, Eq.(1) can be discretized in the linear form

$$a^C\phi^C=\sum a^{NB}\phi^{NB}+b \quad (10)$$

where suffixes C and NB are the central and neighboring points defined by the Voronoi diagram. It is noted that the number of neighboring points is dependent on the figure of each cell. Equation (10) is equivalent to Eq.(2) for the grid along the Cartesian coordinates because the cell system obtained by the Voronoi diagram coincides with that derived by the coordinates.

2.4 Simple method

Concerning the continuity equation, the SIMPLE method which is a most popular method in steady simulations, was modified by Rhie and Chow(4) for the nonstaggered grid system. It is also available for the present cell system.

Discretizing the continuity equation in the cells by the Voronoi diagram, we can derive the next form,

$$\Sigma(uS)^{nb}=0 \quad (11)$$

where u : velocity component normal to the surface, S : area of the surface, and nb : cell surface index. The velocity, u, in Eq.(11) is calculated by the averaged momentum equation with respect to two points x^0 and x^{nb} across the face "nb". That is,

$$u=\hat{u}-d(\nabla P)_u$$

$$\hat{u}=\left[\left\{\left(\sum a^{NB}u^{NB}+b^1\right)/a^C\right\}^1+\left\{\left(\sum a^{NB}u^{NB}+b^1\right)/a^C\right\}^0\right]/2 \quad (12)$$

$$d=[(V/\rho a^C)^1+(V/\rho a^C)^0]/2$$

where $(\ )_u$: component in the direction of u, a : coefficients in Eq.(10). b': the source term without pressure, V : volume of cell, ρ : density, and $\{\ \}^0$,$\{\ \}^1$

: the values at x^0, x^1. Following Rhie and Chow, the pressure term is rewritten by the two-point difference.

$$(\nabla P)_u=(P^1-P^0)/L,$$

where L is the distance from x^0 to x^1.

According to the SIMPLE method, the velocity and pressure divided into the assumed values, u^* and P^*, and the corrective values u' and P',

$$u=u^*+u', \quad P=P^*+P' \tag{13}$$

are substituted into Eqs. (11) and (12) when the corrective velocity terms are neglected with respect to the neighboring points.

Finally, the equation of corrective pressure is derived in the same form as Eq.(10), which has the same reference points. The calculating algorithm follows the previous method on the Cartesian coordinates.

4. Calculating Procedures

Adopting the above-mentioned techniques to the flow simulations, we completed the following procedures :

(1) Defining the grid points,
(2) Generating the cell system by Voronoi diagram,
(3) Calculating the geometrical information,
(4) Setting the reference index for matrix calculations,
(5) Solving flow by the SIMPLE method.

In order to increase the computational speed, the last procedure should be paid the most attention because it uses the most CPU time and can be highly accelerated by parallel processing. It is noted that, for the matrix solution of Eq.(10), point iterative methods are vaild without preconditioning. In the present study, the multicolored SOR (successive over-relaxation) technique is adopted, optimizing the calculating order of SOR for parallel processing. It is useful especially for the unstructured grid whose points are not previously ordered. As mentioned, the grid in the present method can be as unstructured as that in a finite element method (FEM). In the calculation program, the neighboring points are called through the list-vectors which are arranged in procedure (4) following the multicolor technique.

5. Calculation Results

Firstly, a steady and laminar incompressible flow in a lid-driven square cavity is simulated on a structured (type A) and an unstructured (type B) grid with the same number (1600) of points by the same program based on the present method. The grids are illustrated in Fig. 2, where the bold lines indicate the systems of cells. In type A, the present cell system coincides with that on the Cartesian coordinates and the discretized equations are also identical to the previous method. The results obtained by both grids have a satisfactory agreement, such as the streamlines in Fig. 3, where the Reynolds number is 1 000 based on the width of the cavity and the lid-driving velocity.

In the present method, the calculation points can be optionally added and purged. Figure 4 shows a sample grid by the addition of points to the Cartesian grid, where the symbols are the additional points and the bold lines represent the modified cell system. The result for 1 000 Reynolds numbers is shown in the next

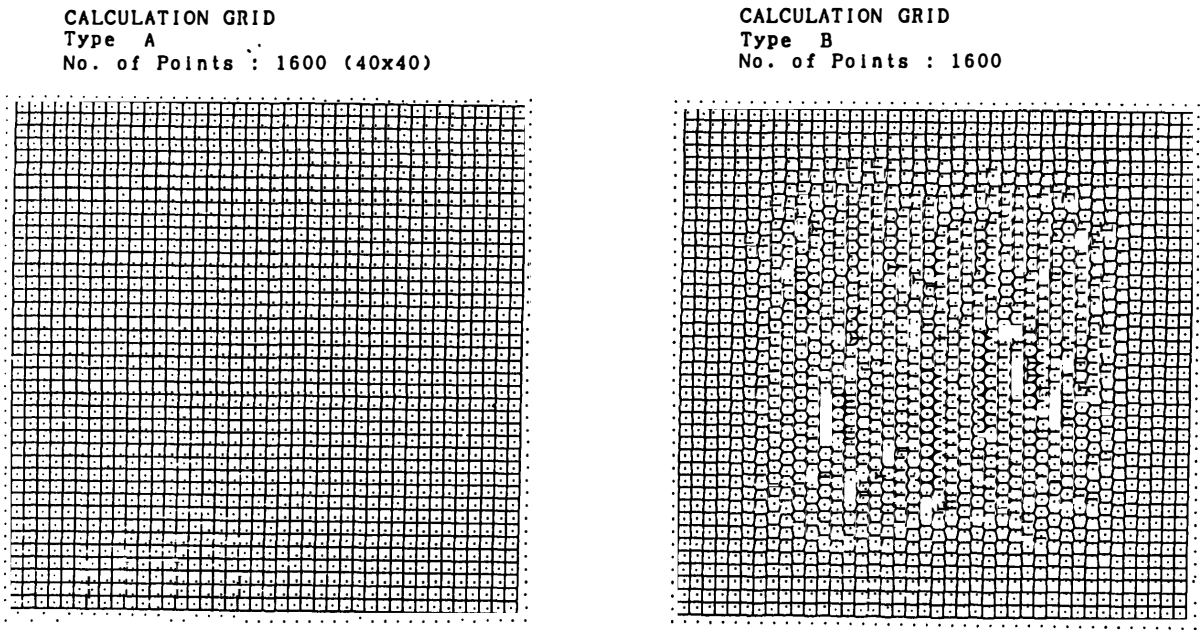

Fig. 2 Two types of grids on a square cavity

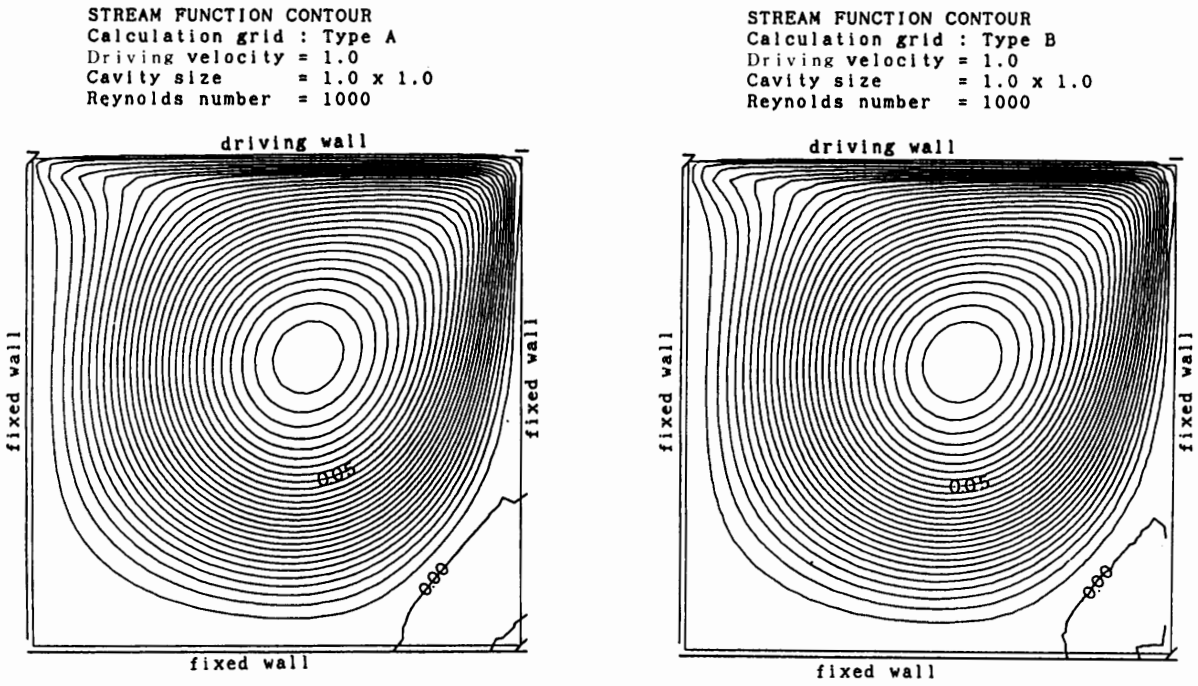

Fig. 3 Computational streamlines of the square cavity flow

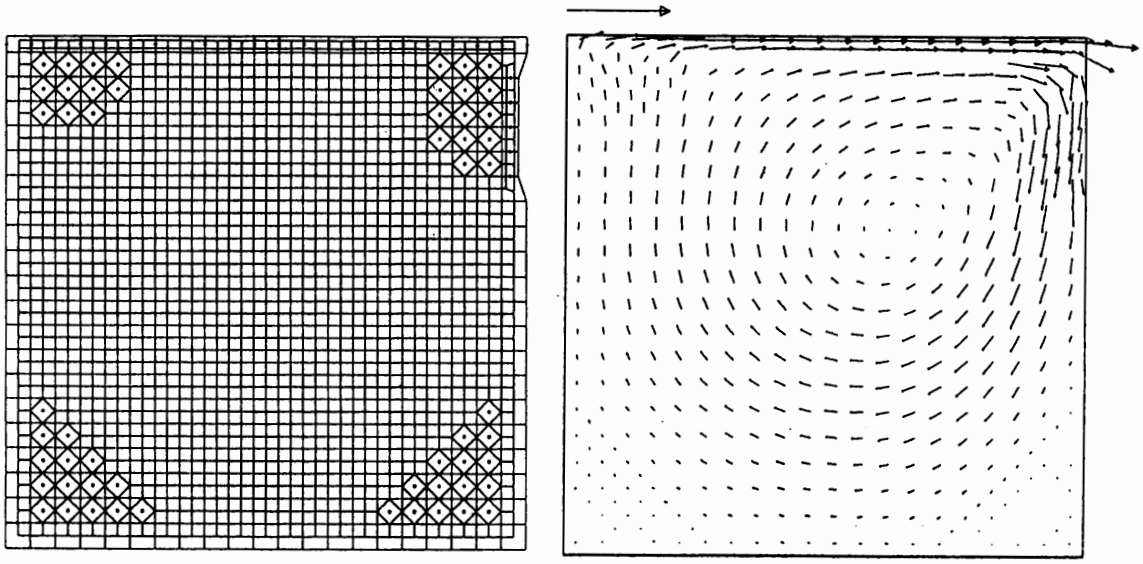

Fig. 4 A cell system of the grid with adding points and corputed velocity

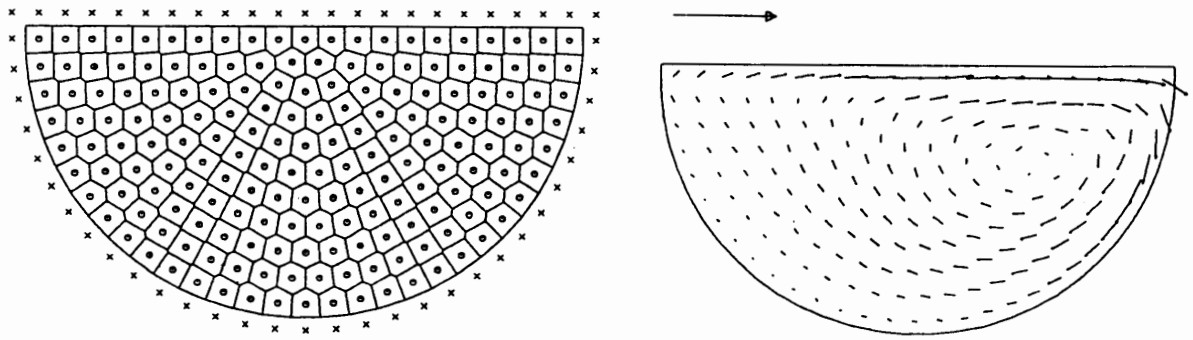

Fig. 5 A cell system of semicircular cavity and a computed velocity

Table 2 Computational time for the procedures

	M680	on benchmark	S820
Dividing cells	6.4 sec		
Calculating inform. of fig.	2.6 sec		
Setting ref. index	0.32sec		
Solving flow	113 sec	628	4.2 sec
Total	122 sec	678	
FVM on Cartesian		65	

figure.

Figure 5 shows a grid on the semicircular cavity and the predicted velocities by the straight driving lid. It is such a typical case with the grid singularity problem that it is impossible to structure the smooth grid on any global coordinates without singularity points.

Finally, the computational time of each procedure is arranged in Table 2, in the first case with grid type A.

The convergence criterion for the iterative procedure of SIMPLE is given as the condition that the nondimensional residuals of the discretized equation become less than 10^{-3}. The calculations are performed by the general-purpose computer (M 680 H by Hitachi) or the supercomputer (S 820/80), whose performances of scalar computation are almost the same.

It is clear that the iterative procedure uses much more CPU time than the former procedures and was accelerated sufficiently by the parallel computations. Compared with the previous method, the present one has about one-tenth performance with respect to the computational speed based on the benchmark problem[5]. The referential data are given by the FVM based on the Cartesian coordinates and the SIMPLE method with the staggered grid system, which is the most efficient method for the steady flow simulations. The performance of the present method can be satisfied from two points of views in that an arbitrary unstructured grid can be adopted and that high acceleration is expected for the parallel computation.

6. Conclusions

By adopting the concept of the Voronoi diagram, a new method on an arbitrary grid system is constructed for the incompressible steady flow simulations. This method effectively simulates two-dimensional laminar flow in lid-driven cavities. Compared with the previous method on FVM, its high performance is confirmed for computational time and grid dependency.

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