On Completely Separable MAD Families

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Abstract

We survey some constructions of completely separable MAD families.

1 Introduction

An infinite family $\mathcal{A} \subseteq [\omega]^{\omega}$ is called *almost disjoint* (or *AD* for short)[1] if the intersection of any two of its elements is finite. An AD family is called MAD if it is maximal with this property. Almost disjoint families and MAD families are very important in set theory, topology and functional analysis (the reader may consult [18] and [17] for a survey on AD families and their applications). It is easy to prove that the Axiom of Choice implies the existence of MAD families; however, constructing MAD families with special combinatorial or topological properties is a very difficult task without appealing to an additional hypothesis beyond the usual axioms of set theory. The following is a quote of Hrušák and Simon (see [19]):

> Special MAD families are notoriously difficult to construct in ZFC alone. The reason being the lack of a device ensuring that a recursive construction of a MAD family would not prematurely terminate, an object that would serve a similar purpose as independent linked families do for the construction of special ultrafilters. The notion of a completely separable MAD family is a candidate for such a device an, moreover, is an interesting notion in its own right.

Before introducing the definition of a completely separable MAD family, we need the following notions:

Definition 1 *If $\mathcal{A}$ is an AD family, we define:*

1. *$\mathcal{I}(\mathcal{A})$ is the ideal generated by $\mathcal{A}$ (and all finite sets). In other words, $X \in \mathcal{I}(\mathcal{A})$ if and only if there are $A_0, ..., A_n \in \mathcal{A}$ such that $X \subseteq^* A_0 \cup ... \cup A_n$.*
2. *$\mathcal{I}(\mathcal{A})^+$ is the set of all subsets of ω that are not in $\mathcal{I}(\mathcal{A})$.*

[1] The definition of the undefined terms will be presented in the next section.

3. $\mathcal{I}(\mathcal{A})^{++}$ *is the set of all* $X \subseteq \omega$ *for which there is* $\mathcal{B} \in [\mathcal{A}]^{\omega}$ *such that if* $A \in \mathcal{B}$ *then* $X \cap A$ *is infinite.*

4. $\mathcal{A}^{\perp}$ *is the set of all* $X \subseteq \omega$ *such that* $|X \cap A| < \omega$ *for every* $A \in \mathcal{A}$.

5. *We say that* $\mathcal{A}$ *is* nowhere *MAD if for every* $X \in \mathcal{I}(\mathcal{A})^{+}$ *there is* $Y \in [X]^{\omega}$ *such that* $Y \in \mathcal{A}^{\perp}$.

6. *If* $X \in [\omega]^{\omega}$, *define* $\mathcal{A} \restriction X = \{A \cap X \mid A \in \mathcal{A} \wedge |A \cap X| = \omega\}$.

It is not difficult to prove that an AD family $\mathcal{A}$ is MAD if and only if $\mathcal{I}(\mathcal{A})^{+} = \mathcal{I}(\mathcal{A})^{++}$. Completely separable MAD families were introduced by Hechler in [16]. We recall the definition:

Definition 2 *Let* $\mathcal{A}$ *be a MAD family. We say that* $\mathcal{A}$ *is* completely separable *if for every* $X \in \mathcal{I}(\mathcal{A})^{+}$ *there is* $A \in \mathcal{A}$ *such that* $A \subseteq X$.

In other words, $\mathcal{A}$ is completely separable if every $X \subseteq \omega$ satisfies the following dichotomy: either X is almost covered by finitely many elements of $\mathcal{A}$, or X contains an element of $\mathcal{A}$. The following problem was posed by Erdős and Shelah in 1972 (see [10]):

Problem 3 (Erdős-Shelah) *Is there a completely separable MAD family?*

It is easy to prove that consistently the answer is affirmative. Most of the early work on Problem 3 was done by Balcar and Simon (see [1]). They proved that completely separable MAD families exist assuming one of the following axioms:

$$\mathfrak{a} = \mathfrak{c}$$
$$\mathfrak{b} = \mathfrak{d}$$
$$\mathfrak{d} \leq \mathfrak{a}$$
$$\mathfrak{s} = \omega_1$$

A major advance on Problem 3 was done by Shelah himself in [28]. He proved that there is a completely separable MAD family in the following cases:

$$\mathfrak{s} < \mathfrak{a}$$
$$\mathfrak{s} = \mathfrak{a} + \text{a certain “PCF hypothesis”}$$
$$\mathfrak{s} > \mathfrak{a} + \text{a certain “PCF hypothesis”}$$

In [22] Mildenberger, Raghavan and Steprāns (building from results in [24]) were able to eliminate extra hypothesis in the second case and provide an uniform proof for the first and second cases. In this way, it follows that $\mathfrak{s} \leq \mathfrak{a}$ implies that there is a completely separable MAD family. We will talk more about the “PCF hypothesis” mentioned above later on, but for now, let us just mention that such hypothesis holds in case the continuum is smaller than $\aleph_{\omega}$. In this way, we have the following:

Theorem 4 (Mildenberger, Raghavan, Shelah, Steprāns) *There is a completely separable* MAD *family under the following assumptions:*

1. $\mathfrak{s} \leq \mathfrak{a}$.
2. $\mathfrak{c} < \aleph_\omega$.

The purpose of the present survey is to provide a proof of the above Theorem. In [18] a uniform proof of the existence of a completely separable MAD family was provided in the case that $\mathfrak{c}$ is less than $\aleph_\omega$.

The technique initially developed by Shelah and further improved and extended in [24] and [22] is very powerful and has more applications beyond the construction of completely separable MAD families. We list some examples:

1. (Raghavan and Steprāns [24]) There is a weakly tight MAD family if $\mathfrak{s} \leq \mathfrak{b}$.
2. (G. [14]) There is a +-Ramsey MAD family.
3. (G.) There is no Katětov-top MAD family if $\mathfrak{s} \leq \mathfrak{b}$.
4. In his master's thesis, Miroslav Olšák constructed examples of m-tuples of topological spaces such that the product of the m-tuple is not Fréchet but all smaller subproducts are Fréchet.

Moreover, Raghavan has many more applications of this method, unfortunately, his results remain unpublished. The impression of the author is that although this technique for building MAD families is extremely powerful, it is not very well-known. The author chose this topic for the current survey as well as for his mini course at the RIMS Set Theory Workshop in the hope to get more people interested in it.

The current survey is heavily based on [24], [22] and [18]. Neither the theorems, nor the proofs found in here are due to the author. The author only claims ownership of the mistakes or inaccuracies found in this paper.

2 Cardinal Invariants of the Continuum

The *cardinal invariants of the continuum* play a fundamental role in the construction of completely separable MAD families. The following is a quote from Raghavan (see [23]):

> A cardinal invariant of the continuum marks the place where a given type of diagonalization argument that works for any countable ordinal first fails; a cardinal invariant can be associated with each type of diagonalization argument. Moreover, there is always a set of

size $\mathfrak{c}$ for which these diagonalization arguments fail, so that every cardinal invariant lies between ω_1 and $\mathfrak{c}$ (since the diagonalization always works for countable ordinals).

The reader may find a lot of information on cardinal invariants in [2], [29], [30] and [23]. In here, we will just recall the basic notions that will be needed in the paper.

Let $f, g \in \omega^\omega$, define $f \leq g$ if and only if $f(n) \leq g(n)$ for every $n \in \omega$ and $f \leq^* g$ if and only if $f(n) \leq g(n)$ holds for all $n \in \omega$ except finitely many. We say a family $\mathcal{B} \subseteq \omega^\omega$ is *unbounded* if $\mathcal{B}$ is unbounded with respect to $\leq^*$. We say that S *splits* X if $X \cap S$ and $X \setminus S$ are both infinite. A family $\mathcal{S} \subseteq [\omega]^\omega$ is a *splitting family* if for every $X \in [\omega]^\omega$ there is $S \in \mathcal{S}$ such that S splits X. If $A, B \in [\omega]^\omega$, by $A \subseteq^* B$ we mean that $A \setminus B$ is finite and we say that A is an *almost subset of* B. We will say that $A \in [\omega]^\omega$ is a *pseudointersection* of $\mathcal{H} \subseteq [\omega]^\omega$ if A is almost contained in every element of $\mathcal{H}$.

Definition 5

1. *By* $\mathfrak{c}$ *we demote the size of the continuum.*
2. *The* bounding number $\mathfrak{b}$ *is the smallest size of an* $\leq^*$*-unbounded family of functions.*
3. *The* splitting number $\mathfrak{s}$ *is the smallest size of a splitting family.*
4. *The* almost disjointness number $\mathfrak{a}$ *is the smallest size of a MAD family.*

The following is a very useful notion when working with the bounding number:

Definition 6 *Let* $\mathcal{B} \subseteq \omega^\omega$. *We say that* $\mathcal{B}$ *is a* $\mathfrak{b}$-scale *if the following conditions hold:*

1. $(\mathcal{B}, \leq^*)$ *is well-ordered of order type* $\mathfrak{b}$.
2. $\mathcal{B}$ *is unbounded.*
3. *Every element of* $\mathcal{B}$ *is an increasing function.*

It is easy to see that there are $\mathfrak{b}$-scales. An important features of $\mathfrak{b}$-scales is that they are not only unbounded with respect to total functions, but with infinite partial functions as well:

Lemma 7 *Let* $\mathcal{B} \subseteq \omega^\omega$ *be a* $\mathfrak{b}$*-scale,* $A \in [\omega]^\omega$ *and* $g : A \longrightarrow \omega$. *There is* $f \in \mathcal{B}$ *such that* $f \restriction A \nleq^* g$ *(i.e. there are infinitely many* $n \in A$ *such that* $f(n) > g(n)$*).*

Proof. Take an increasing enumeration $A = \{a_n \mid n \in \omega\}$. For every $n \in \omega$, define an interval P_n as follows: $P_0 = [0, a_0]$ and $P_{n+1} = (a_n, a_{n+1}]$. We now define $\overline{g} : \omega \longrightarrow \omega$ where $\overline{g}(i) = g(a_{n+1})$ for all $i \in P_{n+1}$. Since $\mathcal{B}$ is an unbounded family, there is $f \in \mathcal{B}$ such that $\overline{g}$ does not dominate f. We claim that $f \restriction A \nleq^* g$. Since f is not dominated by $\overline{g}$, we know that there are infinitely many $i \in \omega$ such that $\overline{g}(i) < f(i)$. Now, if $i \in P_{n+1}$, then we get that $g(a_{n+1}) < f(i)$. Since $i \leq a_{n+1}$ and f is increasing, it follows that $g(a_{n+1}) < f(i) \leq f(a_{n+1})$. ■

It is not hard to prove that $\mathfrak{b} \leq \mathfrak{a}$. There is no other provable relation between $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{s}$. Each of the following statements are consistent:

1. $\mathfrak{s} = \mathfrak{b} = \mathfrak{a}$ (this holds under CH or MA).
2. (Dow [7]) $\mathfrak{s} < \mathfrak{b} = \mathfrak{a}$ (this holds in the Laver model).[2]
3. (Shelah [27]) $\mathfrak{s}, \mathfrak{b} < \mathfrak{a}$ (a model for this inequality is obtained by iterating along a template. See also [4]).
4. (Shelah [25], see also [26], [6] and [15]) $\mathfrak{b} = \mathfrak{a} < \mathfrak{s}$ (This can be done with a countable support iteration of proper forcings. It is also possible to do it with matrix iteration, see [5]).
5. (Shelah [25], see also [6] and [15]) $\mathfrak{b} < \mathfrak{s} = \mathfrak{a}$ (This can be done with a countable support iteration of proper forcings. It is also possible to achieve it using matrix iteration with the aid of a measurable cardinal, see [5] and [3]. See also [8]).
6. (Fischer and Mejia [12]) $\omega_1 < \mathfrak{s} < \mathfrak{b} < \mathfrak{a}$ (see also [21] and [11]).

It is worth pointing out that there are still open problems regarding this cardinal invariants. The following is a very interesting problem of Brendle and Raghavan:

Problem 8 (Brendle, Raghavan [6]) *Does* $\mathfrak{b} = \mathfrak{s} = \omega_1$ *imply* $\mathfrak{a} = \omega_1$*?*

3 Basic results on Completely Separable MAD families

In this section we will prove some basic facts of MAD and completely separable MAD families. Although not all of the results in this section will be used in the paper, we included them because we believe they are useful for getting insight on completely separable MAD families.

[2] The hard part of this result is that $\mathfrak{s} = \omega_1$ holds in the Laver model, which was proved by Alan Dow.

The following result is fundamental when working with AD families:

Proposition 9 *Let $\mathcal{A}$ be an AD family and $\{Y_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ a decreasing sequence. There is $X \in \mathcal{I}(\mathcal{A})^+$ that is almost contained in each Y_n.*

Proof. Let $\mathcal{Y} = \{Y_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})^+$ be a decreasing family. If there is $Z \in \mathcal{A}^\perp$ such that Z is a pseudointersection of $\mathcal{Y}$ we are done. We now assume $\mathcal{Y}$ does not have a pseudointersection in $\mathcal{A}^\perp$. We recursively find a family $\mathcal{B} = \{B_n \mid n \in \omega\} \subseteq \mathcal{I}(\mathcal{A})$ such that the following holds:

1. Each B_n is a pseudointersection of $\mathcal{Y}$ and $B_n \subseteq Y_n$.
2. There is $A_n \in \mathcal{A}$ such that $B_n \subseteq A_n$.
3. If $n \neq m$ then $A_n \neq A_m$.

Let $X = \bigcup_{n\in\omega} B_n$, it is easy to see that X is the set we were looking for. ■

We know that if $\mathcal{A}$ is a completely separable MAD family, then every element of $\mathcal{I}(\mathcal{A})^+$ contains an element of $\mathcal{A}$. In fact, it must contain a lot of them:

Lemma 10 *Let $\mathcal{A}$ be a completely separable MAD family. If $X \in \mathcal{I}(\mathcal{A})^+$, then the set $\{A \in \mathcal{A} \mid A \subseteq X\}$ has size $\mathfrak{c}$.*

Proof. Since $X \in \mathcal{I}(\mathcal{A})^{++}$, we know there is a family $\{A_n \mid n \in \omega\} \subseteq \mathcal{A}$ such that $A_n \cap X$ is infinite for every $n \in \omega$ and $A_n \neq A_m$ whenever $n \neq m$. We can now find an almost disjoint family $\mathcal{B} \subseteq [X]^\omega$ of size $\mathfrak{c}$ such that $B \cap A_n$ is infinite for every $n \in \omega$ and $B \in \mathcal{B}$. Since $\mathcal{A}$ is completely separable, for every $B \in \mathcal{B}$, there is $A_B \in \mathcal{A}$ such that $A_B \subseteq B$. Finally, note that if $B, C \in \mathcal{B}$ and $B \neq C$, then $A_B \neq A_C$ since B and C are almost disjoint. ■

The following notion is closely related to the completely separable MAD families.

Definition 11 *Let $\mathcal{A}$ be a MAD family. We say that $\mathcal{A}$* has true cardinality $\mathfrak{c}$ *if for every $X \in \mathcal{I}(\mathcal{A})^+$, the set $\{A \in \mathcal{I}(\mathcal{A}) \mid |A \cap X| = \omega\}$ has size $\mathfrak{c}$.*

It follows by Lemma 10 that every completely separable MAD family has true cardinality $\mathfrak{c}$. A MAD family of true cardinality $\mathfrak{c}$ may not be completely separable, nevertheless, we have the following result:

Lemma 12 *The following statements are equivalent:*

1. *There is a completely separable MAD family.*
2. *There is a MAD family of true cardinality $\mathfrak{c}$.*

Proof. By the remark above, it is enough to prove that we can construct a completely separable MAD family from one of true cardinality $\mathfrak{c}$. Let $\mathcal{A}$ a MAD family of true cardinality $\mathfrak{c}$. Take an enumeration $[\omega]^{\omega} = \{X_\alpha \mid \alpha \in \mathfrak{c}\}$ such that every infinite subset of ω appears cofinally many times. We will recursively build a family $\mathcal{B} = \left\{B_\alpha^i \mid i < 2 \wedge \alpha \in \mathfrak{c}\right\} \subseteq [\omega]^{\omega}$ such that for every $\alpha \in \mathfrak{c}$, the following holds:

1. $B_\alpha^0 \cap B_\alpha^1 = \emptyset$.
2. There is $A \in \mathcal{A}$ such that $A = B_\alpha^0 \cup B_\alpha^1$.
3. $\mathcal{B}$ is almost disjoint.
4. Let $\mathcal{B}_{<\alpha} = \left\{B_\xi^i \mid \xi < \alpha \wedge i < 2\right\}$. If $X_\alpha \in \mathcal{I}(\mathcal{B}_{<\alpha})^{++}$, then $B_\alpha^0 \subseteq X_\alpha$ or $B_\alpha^1 \subseteq X_\alpha$.

Assume we are at step α of the construction. If $X_\alpha \notin \mathcal{I}(\mathcal{B}_{<\alpha})^{++}$, we choose any $A \in \mathcal{A}$ that does not contain an element of $\mathcal{B}_{<\alpha}$ and we take any two disjoint $B_\alpha^0, B_\alpha^1 \in [A]^{\omega}$ such that $A = B_\alpha^0 \cup B_\alpha^1$. Now, assume that $X_\alpha \in \mathcal{I}(\mathcal{B}_{<\alpha})^{++}$. Note that this implies that $X_\alpha \in \mathcal{I}(\mathcal{A})^{+}$. Since $\mathcal{A}$ has true cardinality $\mathfrak{c}$, we can find $A \in \mathcal{A}$ that does not contain any element of $\mathcal{B}_{<\alpha}$ and $A \cap X$ is infinite. We find disjoint $B_\alpha^0, B_\alpha^1 \in [A]^{\omega}$ such that $A = B_\alpha^0 \cup B_\alpha^1$ and (at least) one of them is contained in X. This finishes the construction.

It is easy to see that $\mathcal{B}$ is a completely separable MAD family. ■

It is easy to see that completely separable MAD families may consistently exist:

Proposition 13 $\mathfrak{a} = \mathfrak{c}$ *implies that there is a completely separable MAD family.*

Proof. By Lemma 12 it is enough to prove that there is a MAD family of true cardinality $\mathfrak{c}$. In fact, we claim that every MAD family has true cardinality $\mathfrak{c}$. Let $\mathcal{A}$ be a MAD family and $X \in \mathcal{I}(\mathcal{A})^{+}$. Note that $\mathcal{A} \upharpoonright X$ is a MAD family on X (since $\mathcal{A}$ is maximal), so it must have size $\mathfrak{c}$. ■

It is worth noting that Problem 3 can be solved (without the need of an additional axiom beyond ZFC) if we do not demand maximality. In general, we say that an AD family $\mathcal{A}$ is *completely separable* if for every $X \in \mathcal{I}(\mathcal{A})^{++}$ there is $A \in \mathcal{A}$ such that $A \subseteq X$ (recall that if $\mathcal{A}$ is maximal, then $\mathcal{I}(\mathcal{A})^{+} = \mathcal{I}(\mathcal{A})^{++}$). The following is a remarkable theorem of Petr Simon:

Theorem 14 (Simon [13]) *There is a completely separable nowhere MAD family.*

Using a completely separable nowhere MAD family, Galvin constructed a Čech function (see [13]).

In this paper we will focus on two constructions of completely separable MAD families. To learn more about them as well as applications, the reader may consult [1], [19], [13] and [18] among others.

4 Splittings

In order to build a completely separable MAD family, we need to be able to "split the right sets at the right time". Probably this statement does not make any sense to the reader at the moment, but it as we continue with our quest to build a completely separable MAD family.

We say that $\mathcal{P} = \{P_n \mid n \in \omega\}$ is an *interval partition* if it is a partition of ω into consecutive intervals. Given interval partitions $\mathcal{P}$ and $\mathcal{Q}$ define $\mathcal{Q} \leq \mathcal{P}$ if for every $P_n \in \mathcal{P}$ there is $Q_m \in \mathcal{Q}$ such that $Q_m \subseteq P_n$ (in other words, every interval in $\mathcal{P}$ contains at least one interval of $\mathcal{Q}$) and $\mathcal{Q} \leq^* \mathcal{P}$ if for almost all $P_n \in \mathcal{P}$ there is $Q_m \in \mathcal{Q}$ such that $Q_m \subseteq P_n$ (i.e. almost every interval in $\mathcal{P}$ contains at least one interval of $\mathcal{Q}$). The proof of the following useful lemma can be found in [2].

Lemma 15 $\mathfrak{b}$ *is the smallest size of an unbounded family of partitions (using the order* $\leq^*$*).*

The following is a stronger notion than of a splitting family:

Definition 16

1. *Let* $S \in [\omega]^\omega$ *and* $\mathcal{P} = \{P_n \mid n \in \omega\}$ *be an interval partition. We say* S block-splits $\mathcal{P}$ *if both of the sets* $\{n \mid P_n \subseteq S\}$ *and* $\{n \mid P_n \cap S = \emptyset\}$ *are infinite.*

2. *A family* $\mathcal{S} \subseteq [\omega]^\omega$ *is called a* block-splitting family *if every interval partition is block-split by some element of* $\mathcal{S}$*.*

3. *By* $\mathfrak{bs}$ *we denote the smallest size of a block-splitting family.*

It is easy to see that every block-splitting family is splitting. The reader may complain that the notation $\mathfrak{bs}$ is inconvenient, since it could be confused with the (cardinal) product of $\mathfrak{b}$ and $\mathfrak{s}$. Fortunately, this is not an issue by following result of Kamburelis and Weglorz:

Proposition 17 ([20]) $\mathfrak{bs} = max\{\mathfrak{b}, \mathfrak{s}\}$.

Proof. Obviously $\mathfrak{s} \leq \mathfrak{bs}$ and now we will prove that $\mathfrak{b} \leq \mathfrak{bs}$. It is enough to show that no family of size less than $\mathfrak{b}$ is a block-splitting family. Let $\mu < \mathfrak{b}$ and $\mathcal{S} = \{S_\alpha \mid \alpha < \mu\}$ be a family of infinite subsets of ω. For every $\alpha < \mu$ define an interval partition $P_\alpha = \{P_n(\alpha) \mid n \in \omega\}$ such that each $P_n(\alpha)$ has non-empty intersection with both S_α and $\omega \setminus S_\alpha$. Since $\mu < \mathfrak{b}$ then there is an interval partition $\mathcal{R} = \{R_n \mid n \in \omega\}$ $\leq^*$-dominating each P_α i.e. almost all intervals of $\mathcal{R}$ contains one of P_α. It is easy to see that no element of $\mathcal{S}$ can block-split $\mathcal{R}$.

Now we will construct a block-splitting family of size $\max\{\mathfrak{b}, \mathfrak{s}\}$. First find an unbounded family of interval partitions $\mathcal{B} = \{P_\alpha \mid \alpha < \mathfrak{b}\}$ (where $P_\alpha = \{P_\alpha(n) \mid n \in \omega\}$) and a splitting family $\mathcal{S} = \{S_\beta \mid \beta < \mathfrak{s}\}$. Given $\alpha < \mathfrak{b}$ and $\beta < \mathfrak{s}$ define $D_{\alpha,\beta} = \bigcup_{n \in S_\beta} P_\alpha(n)$ we will prove that $\{D_{\alpha,\beta} \mid \alpha < \mathfrak{b}, \beta < \mathfrak{s}\}$ is a block-splitting family. Let $\mathcal{R} = \{R_n \mid n \in \omega\}$ be an interval partition. Since $\mathcal{B}$ is unbounded, there is $\alpha < \mathfrak{b}$ such that P_α is not dominated by $\mathcal{R}$. We can then find an infinite set $W = \{w_n \mid n \in \omega\}$ such that for every $n < \omega$ there is $k < \omega$ for which $R_k \subseteq P_\alpha(w_n)$ (this is possible since P_α is not dominated by $\mathcal{R}$). Since $\mathcal{S}$ is a splitting family, there is $\beta < \mathfrak{s}$ such that both $S_\beta \cap W$ and $(\omega \setminus S_\beta) \cap W$ are infinite. It is easy to see that $D_{\alpha,\beta}$ block-splits $\mathcal{R}$. ■

The following is an easy, yet fundamental result. It shows that with a block-splitting family we can split any positive element (with respect to an AD family) in two positive sets.

Proposition 18 ([24]) *Let $\mathcal{S}$ be a block-splitting family, $\mathcal{A}$ an AD family and $X \in \mathcal{I}(\mathcal{A})^+$. There is $S \in \mathcal{S}$ such that $X \cap S$, $X \setminus S \in \mathcal{I}(\mathcal{A})^+$.*

Proof. We first assume that there is $Y \in [X]^\omega$ such that $Y \in \mathcal{A}^\perp$. This is the easy case, we just take any $S \in \mathcal{S}$ that splits X (recall that $\mathcal{S}$ is a splitting family).

We now assume that X does not contain any infinite subset in $\mathcal{A}^\perp$. Since $X \in \mathcal{I}(\mathcal{A})^+$, it follows that $X \in \mathcal{I}(\mathcal{A})^{++}$. This means that there is $\{A_n \mid n \in \omega\} \subseteq \mathcal{A}$ such that $X \cap A_n$ is infinite for every $n \in \omega$. Now, define an interval partition $\mathcal{P} = \{P_n \mid n \in \omega\}$ such that if $i \leq n$ then $P_n \cap (A_i \cap X) \neq \emptyset$. Since $\mathcal{S}$ is a block-splitting family, there is $S \in \mathcal{S}$ that block-splits $\mathcal{P}$. It follows that $X \cap S$, $X \cap (\omega \setminus S) \in \mathcal{I}(\mathcal{A})^+$. ■

It is worth pointing out that for the previous proposition, the full strength of a block-splitting family was not needed. It is enough to assume that $\mathcal{S}$ is a "(ω, ω)-splitting family", which we define now:

Definition 19 *Let $S \in [\omega]^\omega$ and $\overline{X} = \{X_n \mid n \in \omega\} \subseteq [\omega]^\omega$.*

1. *We say that S (ω, ω)-splits $\overline{X}$ if both the sets $\{n \mid |X_n \cap S| = \omega\}$ and $\{n \mid |X_n \cap (\omega \setminus S)| = \omega\}$ are infinite.*

2. *We say that $\mathcal{S} \subseteq [\omega]^{\omega}$ is an* (ω, ω)-splitting family *if every countable collection of infinite subsets of ω is (ω, ω)-split by some element of S.*

It is easy to see that a block-splitting family is (ω, ω)-splitting. Furthermore, Proposition 18 is true for (ω, ω)-splitting families. In [22], it was proved that there is an (ω, ω)-splitting family of size $\mathfrak{s}$. Hence, there is always a splitting family of size $\mathfrak{s}$ satisfying the conclusion of the above proposition[3]. For the purpose of this survey, there is no advantage in using (ω, ω)-splitting instead of block-splitting, so we stick with the later notion. However, there could be a difference in future applications.

The following is well-known:

Lemma 20 *$\mathfrak{s}$ has uncountable cofinality.*

Proof. We argue by contradiction. Let $\mathcal{S}$ be a splitting family of size $\mathfrak{s}$. We can then find $\{\mathcal{S}_n \mid n \in \omega\}$ such that $\mathcal{S} = \bigcup \mathcal{S}_n$ and each $\mathcal{S}_n$ has size less than $\mathfrak{s}$ (so they are "nowhere splitting"). We can then recursively find a decreasing sequence $\mathcal{P} = \{A_n \mid n \in \omega\}$ such that no element of $\mathcal{S}_n$ splits A_n. Let B be a pseudointersection of $\mathcal{P}$. It is easy to see than no element of $\mathcal{S}$ splits B, which is a contradiction. ■

It was a long open problem if $\mathfrak{s}$ was a regular cardinal. This problem was finally solved by Dow and Shelah in [9] where they proved that $\mathfrak{s}$ may be singular.

It is easy to see that $\mathfrak{b}$ is a regular cardinal, so we get the following:

Corollary 21 *$\mathfrak{bs}$ has uncountable cofinality.*

Naturally, when we think of a cardinal invariant that allows to split some sets, the splitting number is the one that first comes into mind. However, there are some kinds of sets that can be splitted using $\mathfrak{b}$. This is the content of the following lemma. It will not be needed until the case of $\mathfrak{a} < \mathfrak{s}$.

Lemma 22 *Let $\mathcal{B} = \{B_n \mid n \in \omega\} \subseteq [\omega]^{\omega}$ be a $\subseteq$-decreasing sequence. There is a family $\mathcal{L}(\mathcal{B}) = \{L_\alpha \mid \alpha < \mathfrak{b}\} \subseteq B_0$ such that if $\mathcal{A}$ is any AD family and $X \in [\omega]^{\omega}$ that satisfy the following property:*

$*\,(\mathcal{B}, \mathcal{A}, X)$ *There are infinitely many $n \in \omega$ for which there is $A_n \in \mathcal{A}$ such that $A \subseteq^* B_n \setminus B_{n+1}$ and $X \cap A_n \neq \emptyset$.*

Then, there is $\alpha < \mathfrak{b}$ such that $L_\alpha \cap X,\ X \setminus L_\alpha \in \mathcal{I}(\mathcal{A})^+$.

[3]This is a very interesting proof. Since we do not need it, we will not review it here, but we recommend the reader to study it at some point.

Proof. Define $C_n = B_n \setminus B_{n+1}$ and let $H = \{n \mid |C_n| = \omega\}$. If H is finite, there is nothing to do, so we assume H is infinite. Let $\mathcal{B} = \{f_\alpha \mid \alpha < \mathfrak{b}\}$ be a $\mathfrak{b}$-scale. For every $\alpha < \mathfrak{b}$, define $L_\alpha = \bigcup\limits_{n\in\omega} (f_\alpha(n) \cap C_n)$. We claim that $\mathcal{L}(\mathcal{B}) = \{L_\alpha \mid \alpha < \mathfrak{b}\}$ has the desired properties. Let $\mathcal{A}$ be an AD family and $X \subseteq \omega$ such that $*(\mathcal{B}, \mathcal{A}, X)$ holds. Let K be the infinite subset of all $n \in \omega$ for which there is $A_n \subseteq^* B_n \setminus B_{n+1}$ such that $X \cap A_n \neq \emptyset$. We may assume that $X = \bigcup\limits_{n\in K} (X \cap A_n)$.

First consider the case where there is $Y \in [X]^\omega$ such that $Y \in \mathcal{A}^\perp$. Note that $Y \cap A_n$ is finite for every $n \in K$. Let $W = \{n \mid Y \cap A_n \neq \emptyset\}$ and define $g : W \longrightarrow \omega$ such that $Y \cap A_n \subseteq g(n)$ for every $n \in \omega$. Now, find $\alpha < \mathfrak{b}$ such that $f_\alpha \restriction W$ is not dominated by g (see Lemma 7). In this way, $L_\alpha \cap Y$ is infinite. It follows that $L_\alpha \cap X$, $X \setminus L_\alpha \in \mathcal{I}(\mathcal{A})^+$ and we are done.

We now assume that $[X]^\omega \cap \mathcal{A}^\perp$ is empty. In this way, we may find $\{D_n \mid n \in \omega\}$ and $\{E_n \mid n \in \omega\}$ with the following properties:

1. $\{D_n \mid n \in \omega\} \subseteq \mathcal{A}$.
2. $D_n \neq A_m$ for every n, m.
3. E_n is an infinite subset of $X \cap D_n$.

As in the previous case, for every $n \in \omega$ we can find $\alpha_n < \mathfrak{b}$ such that $L_{\alpha_n} \cap E_n$ is infinite for every $n \in \omega$. Let $\alpha = \bigcup\limits_{n\in\omega} \alpha_n$ and note that $\alpha < \mathfrak{b}$ since $\mathfrak{b}$ is regular. It follows that $L_\alpha \cap X$, $X \setminus L_\alpha \in \mathcal{I}(\mathcal{A})^+$ and we are done. ■

The reader should compare Lemma 22 and Proposition 18. In some way, Lemma 22 seems like a more restricted version of Proposition 18. In the case of $\mathfrak{s} \leq \mathfrak{a}$, we will use Proposition 18, but in case of $\mathfrak{a} < \mathfrak{s}$ that is not possible, so we need to settle with the more complicated Lemma 22.

5 A completely separable MAD family from $\mathfrak{s} \leq \mathfrak{a}$

We are now ready to build a completely separable MAD family assuming that the splitting number is at most the almost disjointness number. As mentioned in the introduction, Shelah proved that there is a completely separable MAD family if $\mathfrak{s} < \mathfrak{a}$. He also proved that there is such family assuming $\mathfrak{s} = \mathfrak{a}$ plus an extra hypothesis ([28]). Later, Mildenberger, Raghavan and Steprāns eliminated the need for the extra hypothesis in the case of $\mathfrak{s} = \mathfrak{a}$ and provided a uniform proof for both cases (see [22] and [24]). The proof we present here is the one from [22].

For convenience, given $X \subseteq \omega$ we denote $X^0 = X$ and $X^1 = \omega \setminus X$. We can now prove the following:

Theorem 23 (Mildenberger, Raghavan, Shelah, Steprāns) *If* $\mathfrak{s} \leq \mathfrak{a}$, *then there is a completely separable MAD family.*

Proof. Assume that $\mathfrak{s} \leq \mathfrak{a}$ (which obviously implies that $\mathfrak{bs} \leq \mathfrak{a}$). Fix $\mathcal{S} = \{S_\alpha \mid \alpha < \mathfrak{bs}\}$ a block-splitting family. By Proposition 18, if $\mathcal{A}$ is any AD family and $X \in \mathcal{I}(\mathcal{A})^+$, then there are $\alpha < \mathfrak{bs}$ and $\tau_X^{\mathcal{A}} \in 2^\alpha$ such that:

1. If $\beta < \alpha$ then $X \cap S_\beta^{1-\tau_X^{\mathcal{A}}(\beta)} \in \mathcal{I}(\mathcal{A})$ (hence $X \cap S_\beta^{\tau_X^{\mathcal{A}}(\beta)} \in \mathcal{I}(\mathcal{A})^+$).
2. $X \cap S_\alpha, X \setminus S_\alpha \in \mathcal{I}(\mathcal{A})^+$.

Clearly $\tau_X^{\mathcal{A}} \in 2^{<\mathfrak{s}}$ is unique and if $Y \in [X]^\omega \cap \mathcal{I}(\mathcal{A})^+$ then $\tau_Y^{\mathcal{A}}$ extends $\tau_X^{\mathcal{A}}$. Let $[\omega]^\omega = \{X_\alpha \mid \alpha < \mathfrak{c}\}$. We will recursively construct $\mathcal{A} = \{A_\alpha \mid \alpha < \mathfrak{c}\}$ and $\{\sigma_\alpha \mid \alpha < \mathfrak{c}\} \subseteq 2^{<\mathfrak{bs}}$ such that for every $\alpha < \mathfrak{c}$ the following holds (where $\mathcal{A}_{<\alpha} = \{A_\xi \mid \xi < \alpha\}$):

1. $\mathcal{A}_{<\alpha}$ is an AD family.
2. $\sigma_\alpha \in 2^{<\mathfrak{bs}}$.
3. If $\beta < \alpha$, then $\sigma_\alpha \nsubseteq \sigma_\beta$.
4. If $X_\alpha \in \mathcal{I}(\mathcal{A}_{<\alpha})^+$ then $A_\alpha \subseteq X_\alpha$.
5. If $\xi < dom(\sigma_\alpha)$ then $A_\alpha \subseteq^* S_\xi^{\sigma_\alpha(\xi)}$.

It is clear that if we manage to satisfy this requirements, then we will have build a completely separable MAD family. Assume we already have $\mathcal{A}_{<\delta} = \{A_\xi \mid \xi < \delta\}$. Let $X = X_\delta$ if $X_\delta \in \mathcal{I}(\mathcal{A}_{<\delta})^+$ and in the other case, let $X = \omega$ (or any other element of $\mathcal{I}(\mathcal{A}_{<\delta})^+$). We recursively find $\{X_s \mid s \in 2^{<\omega}\} \subseteq \mathcal{I}(\mathcal{A}_{<\delta})^+$,$\{\eta_s \mid s \in 2^{<\omega}\} \subseteq 2^{<\mathfrak{bs}}$ and $\{\alpha_s \mid s \in 2^{<\omega}\}$ as follows:

1. $X_\emptyset = X$.
2. $\eta_s = \tau_{X_s}^{\mathcal{A}_\delta}$ and $\alpha_s = dom(\eta_s)$.
3. $X_{s^\frown 0} = X_s \cap S_{\alpha_s}$ and $X_{s^\frown 1} = X_s \cap (\omega \setminus S_{\alpha_s})$.

Note that if $t \subseteq s$ then $X_s \subseteq X_t$ and $\eta_t \subseteq \eta_s$. On the other hand, if s is incompatible with t, then η_s and η_t are incompatible. For every $f \in 2^\omega$ let $\eta_f = \bigcup_{n\in\omega} \eta_{f\restriction n}$. Since $\mathfrak{bs}$ has uncountable cofinality, each η_f is an element of $2^{<\mathfrak{bs}}$. Furthermore, if $f \neq g$, then η_f and η_g are incompatible nodes of $2^{<\mathfrak{bs}}$. Since δ is smaller than $\mathfrak{c}$, me can find $f \in 2^\omega$ such that there is no $\alpha < \delta$ such that σ_α extends η_f. Now, $\{X_{f\restriction n} \mid n \in \omega\}$ is a decreasing sequence of elements in $\mathcal{I}(\mathcal{A}_{<\delta})^+$, so by Proposition 9 we know there is $Y \in \mathcal{I}(\mathcal{A}_{<\delta})^+$ such that $Y \subseteq^* X_{f\restriction n}$ for every $n \in \omega$.

Letting $\beta = dom(\eta_f)$, we claim that if $\xi < \beta$, then $Y \cap S_\xi^{1-\eta_f(\xi)} \in \mathcal{I}(\mathcal{A}_{<\delta})$. To prove this, let n be the first natural number such that $\xi < dom(\eta_{f\restriction n})$. By construction, we know that $X_{f\restriction n} \cap S_\xi^{1-\eta_f(\xi)} \in \mathcal{I}(\mathcal{A}_{<\delta})$ and since $Y \subseteq^* X_{f\restriction n}$ the claim follows.

For every $\xi < \beta$, find $F_\xi \in [\mathcal{A}_{<\delta}]^{<\omega}$ such that $Y \cap S_\xi^{1-\eta_f(\xi)} \subseteq^* \bigcup F_\xi$ and let $W = \{A_\alpha \mid \sigma_\alpha \subseteq \eta_f\}$. Denote $\mathcal{D} = W \cup \bigcup\limits_{\xi<\beta} F_\xi$ and note that $\mathcal{D}$ has size less than $\mathfrak{bs}$, hence it has size less than $\mathfrak{a}$. In this way, $Y \restriction \mathcal{D}$ is not a MAD family, so there is $A_\delta \in [Y]^\omega$ that is almost disjoint with every element of $\mathcal{D}$. Define $\sigma_\delta = \eta_f$. We claim that A_δ is almost disjoint with $\mathcal{A}_{<\delta}$.

Let $\alpha < \delta$, we need to argue that $A_\alpha \cap A_\delta$ is finite. In case that $A_\alpha \in W$ we already know it, so assume $A_\alpha \notin W$. Letting $\xi = \triangle(\sigma_\delta, \sigma_\alpha)$ we know that $A_\alpha \subseteq^* S_\xi^{1-\sigma_\delta(\xi)}$ so $A_\alpha \cap A_\delta \subseteq^* \bigcup F_\xi$ but since $F_\xi \subseteq \mathcal{D}$ we conclude that A_δ is almost disjoint with $\bigcup F_\xi$ and then $A_\alpha \cap A_\delta$ must be finite. ■

It is worth noting that in the proof above, each $\mathcal{A}_{<\delta} = \{A_\xi \mid \xi < \delta\}$ is nowhere MAD. In [14] the previous ideas were used to construct a +-Ramsey MAD family under $\mathfrak{s} \leq \mathfrak{a}$.

It is worth pointing out that if we used an (ω, ω)-splitting family instead of a block-splitting family, we could have worked with $2^{<\mathfrak{s}}$ instead of $2^{<\mathfrak{bs}}$. This makes no difference in the present paper, but we point it out because it could be important in future applications.

6 The extra hypothesis

In the introduction, we just mentioned that the case $\mathfrak{s} > \mathfrak{a}$ (and $\mathfrak{a} = \mathfrak{s}$ in the original proof of Shelah) required a certain "PCF hypothesis" and provided no more details. We will explain what this mysterious assumption is in this chapter. By a PCF hypothesis, one usually means a hypothesis on $\mathsf{cof}([\kappa]^\omega)$[4] for some cardinal κ. This hypothesis usually hold in case $\kappa < \aleph_\omega$. Personally, the author prefers to think in this hypothesis like some sort of "diamond" or "guessing principle" instead of a hypothesis of $[\kappa]^\omega$. The content of this section and the following one is based on [18].

We start with the following well-known fact:

Lemma 24 *$cof([\omega_n]^\omega) = \omega_n$ for every $n \geq 1$.*

[4] $\mathsf{cof}([\kappa]^\omega)$ is the smallest size of a family $\mathcal{C} \subseteq [\kappa]^\omega$ such that for every $A \in [\kappa]^\omega$ there is $C \in \mathcal{C}$ such that $A \subseteq C$. Note that if is regular, then $\kappa \leq \mathsf{cof}([\kappa]^\omega)$.

Proof. We already know that $\omega_n \leq \mathsf{cof}([\omega_n]^\omega)$. It remains to prove that for every $n \geq 1$ there is a cofinal family of size ω_n. We proceed by induction on n. We start with $n = 1$. In this case, it is clear that $\{\alpha \mid \omega \leq \alpha < \omega_1\} \subseteq [\omega_1]^\omega$ is a cofinal family.

Assume the lemma is true for n, we will prove it is also true for $n+1$. Let $A = \{\alpha \mid \omega_n \leq \alpha < \omega_{n+1}\}$. By the inductive hypothesis, for every $\alpha \in A$ there is a cofinal family $\mathcal{C}_\alpha \subseteq [\alpha]^\omega$ of size ω_n. Let $\mathcal{C} = \bigcup_{\alpha \in A} \mathcal{C}_\alpha$. It is clear that $|\mathcal{C}| = \omega_{n+1}$ and since ω_{n+1} has uncountable cofinality, $\mathcal{C}$ is a cofinal family. ■

If W is a set of ordinals, by $\mathsf{OT}(W)$ we denote the order type of W. We now introduce the following principle:

Definition 25 *Let κ be a cardinal such that $\mathfrak{b} \leq \kappa$. The principle $\mathsf{P}(\mathfrak{b}, \kappa)$ is the following statement: There is a family $\{U_\alpha \mid \omega \leq \alpha < \kappa\}$ with the following properties:*

1. *$U_\alpha \subseteq \alpha$ and $\mathsf{OT}(U_\alpha) = \omega$.*
2. *For every $X \subseteq \kappa$, if $\mathsf{OT}(X) = \mathfrak{b}$, then there is $\alpha < \bigcup X$ such that $|U_\alpha \cap X| = \omega$.*

We want to convince the reader that $\mathsf{P}(\mathfrak{b}, \kappa)$ is a very mild assumption. We start with the following lemma:

Lemma 26 *Let W be a countable set of ordinals. There is a family $\mathcal{H}(W) \subseteq [W]^\omega$ with the following properties:*

1. *$|\mathcal{H}(W)| = \mathfrak{b}$.*
2. *Every element of $\mathcal{H}(W)$ has order type ω.*
3. *For every $X \in [W]^\omega$ there is $H \in \mathcal{H}(W)$ such that $|X \cap H| = \omega$.*

Proof. We prove the lemma by induction on the order type of W, which we call α. The cases $\alpha = \omega$ or α a successor are trivial. Assume that $\alpha > \omega$ is a limit ordinal. We first consider the case that there is $\beta < \alpha$ such that $\alpha = \beta + \omega$. Let W_0 be the first β elements of W and $W_1 = W \setminus W_1$. Clearly $\mathcal{H}(W) = \mathcal{H}(W_0) \cup \mathcal{H}(W_1)$ has the desired properties.

We now assume $\alpha > \omega$ is a limit but it is not of the form $\beta + \omega$ for any $\beta < \alpha$. In here, we can find $\{W_m \mid m \in \omega\}$ such that for every $n \in \omega$, the following conditions hold:

1. $W = \bigcup_{m \in \omega} W_m$.
2. If $\xi \in W_n$ and $\eta \in W_{n+1}$, then $\xi < \eta$.

3. W_n is infinite and has order type less than α.

For every $n \in \omega$, take an enumeration $W_n = \{\beta_n(i) \mid i \in \omega\}$. Let $\mathcal{B} = \{f_\xi \mid \xi < \mathfrak{b}\} \subseteq \omega^\omega$ be a $\mathfrak{b}$-scale. For every $\xi < \mathfrak{b}$, define $A_\xi = \{\beta_n(i) \mid i \leq f_\xi(n)\}$. Using Lemma 7, it is easy to see that $\mathcal{H}(W) = \{A_\xi \mid \xi < \mathfrak{b}\} \cup \bigcup_{n\in\omega} \mathcal{H}(W_n)$ has the desired properties. ■

We conclude the following:

Corollary 27 *Let κ be a cardinal such that $\mathfrak{b} \leq \kappa$ and $\mathsf{cof}([\kappa]^\omega) = \kappa$. There is a family $\mathcal{S}(\kappa) \subseteq [\kappa]^\omega$ with the following properties:*

1. *$|\mathcal{S}(\kappa)| = \kappa$.*
2. *Every element of $\mathcal{S}(\kappa)$ has order type ω.*
3. *For every $X \in [\kappa]^\omega$ there is $H \in \mathcal{S}(\kappa)$ such that $|X \cap H| = \omega$.*

We can naturally extend the definition of $\mathsf{P}(\mathfrak{b}, \kappa)$ to any ordinal:

Definition 28 *Let γ be an ordinal such that $\mathfrak{b} \leq \gamma$. The principle $\mathsf{P}(\mathfrak{b}, \gamma)$ is the following statement: There is a family $\{U_\alpha \mid \omega \leq \alpha < \gamma\}$ with the following properties:*

1. *$U_\alpha \subseteq \alpha$ and $\mathsf{OT}(U_\alpha) = \omega$.*
2. *For every $X \subseteq \gamma$, if $\mathsf{OT}(X) = \mathfrak{b}$, then there is $\alpha < \bigcup X$ such that $|U_\alpha \cap X| = \omega$.*

If this is the case, we will say that $\{U_\alpha \mid \omega \leq \alpha < \gamma\}$ is a $\mathsf{P}(\mathfrak{b}, \gamma)$-sequence. We can now prove the following:

Proposition 29 *If γ is an ordinal such that $\mathfrak{b} \leq \gamma < \aleph_\omega$, then $\mathsf{P}(\mathfrak{b}, \gamma)$ is true.*

Proof. We proceed by induction over γ. The base case $\gamma = \mathfrak{b}$ holds by Lemma 24 and Corollary 27. The successor case is trivial. Moreover, if γ is not an indecomposable ordinal[5], then the result easily follows from the inductive hypothesis. We now assume that γ is an indecomposable ordinal and let $\beta = \mathsf{cof}(\gamma)$.

Fix $C = \{\gamma_\xi \mid \xi < \beta\}$ be a club on γ with the following properties:

1. $\gamma_0 = 0$ and $\gamma_1 = \mathfrak{b}$.

[5] An indecomposable ordinal is an ordinal closed under (ordinal) addition.

2. $2\gamma_\xi < \gamma_{\xi+1}$ for every $\xi < \beta$.

Define $L_\xi = [\gamma_\xi, \gamma_{\xi+1})$ for every $\xi < \beta$. Let E_0 be the class of even ordinals and E_1 the class of odd ordinals. Now, by the inductive hypothesis, we can find $\{U_\alpha(\gamma_{\xi+1}) \mid \omega \leq \alpha < \gamma_{\xi+1} \wedge \alpha \in E_1\}$ that is a $\mathsf{P}(\mathfrak{b}, \gamma_{\xi+1})$-sequence. We now find $\{U_\alpha(\gamma) \mid \omega \leq \alpha < \gamma\}$ with the following properties:

1. If $\alpha \in E_1$ and $\alpha \in L_\xi$, then $U_\alpha(\gamma) = U_\alpha(\gamma_{\xi+1})$.
2. $\{U_\alpha(\gamma) \mid \gamma_\xi \leq \alpha < \gamma_{\xi+1} \wedge \alpha \in E_0\}$ has the following properties:
   (a) Each $U_\alpha(\gamma)$ has order type ω.
   (b) For every $Y \in [\gamma_\xi]^\omega$, there is $\alpha \in E_0$ with $\gamma_\xi \leq \alpha < \gamma_{\xi+1}$ such that $Y \cap U_\alpha(\gamma)$ is infinite.

The second point is possible since $2\gamma_\xi < \gamma_{\xi+1}$ as well as Lemma 24 and Corollary 27. We claim that $\{U_\alpha(\gamma) \mid \omega \leq \alpha < \gamma\}$ is the family we are looking for.

Let $X \subseteq \gamma$ of order type $\mathfrak{b}$. We first consider the case where there is $\xi < \beta$ such that $|L_\xi \cap X| = \mathfrak{b}$. In here, since $\{U_\alpha(\gamma_{\xi+1}) \mid \omega \leq \alpha < \gamma_{\xi+1} \wedge \alpha \in E_1\}$ is a $\mathsf{P}(\mathfrak{b}, \gamma_{\xi+1})$-sequence, we know that there is $\alpha \in E_1$ such that $\alpha < \bigcup(L_\xi \cap X) \leq \bigcup X$ and $U_\alpha(\gamma_{\xi+1}) = U_\alpha(\gamma)$ has infinite intersection with X.

We now assume that $|L_\xi \cap X| < \mathfrak{b}$ for all $\xi < \beta$. For every $n \in \omega$, let ξ_n be the n ordinal such that $L_{\xi_n} \cap X \neq \emptyset$. Define $\xi = \bigcup\limits_{n\in\omega} \xi_n$. Note that $\xi < \gamma$ and also $\delta_{\xi+1} < \bigcup X$. It follows that there is $\alpha \in E_0$ with $\gamma_\xi \leq \alpha < \gamma_{\xi+1}$ such that $Y \cap U_\alpha(\gamma)$ is infinite. ■

In particular, it follows that if $\mathfrak{b} \leq \mathfrak{s} < \aleph_\omega$, then $\mathsf{P}(\mathfrak{b}, \mathfrak{s})$ holds. This is the hypothesis we need in order to construct a completely separable MAD family.

7 A Completely Separable MAD family from $\mathfrak{a} < \mathfrak{s}$ $(+\ \varepsilon)$

We will now construct a completely separable MAD family from $\mathfrak{a} < \mathfrak{s}$ (and an extra hypothesis). The proof is very similar to the one of Theorem 23, but more complicated. We recommend the reader to only proceed if she\he has mastered Theorem 23. As with the previous section, the content of this section is based on [18].[6]

[6] In [18] there is an uniform proof (without case subdivisions) that there is a completely separable MAD family if $\mathfrak{c} < \aleph_\omega$.

Theorem 30 (Shelah) *If* $\mathfrak{a} < \mathfrak{s}$ *and* $\mathsf{P}(\mathfrak{b}, \mathfrak{s})$ *holds, then there is a completely separable* MAD *family.*

Proof. Assume $\mathfrak{a} < \mathfrak{s}$ and $\mathsf{P}(\mathfrak{b}, \mathfrak{s})$ holds. Fix a family $\{U_\alpha \mid \omega \leq \alpha < \mathfrak{s}\}$ with the following properties:

1. $U_\alpha \subseteq \alpha$ and $\mathsf{OT}(U_\alpha) = \omega$.
2. For every $X \subseteq \mathfrak{s}$, if $\mathsf{OT}(X) = \mathfrak{b}$, then there is $\alpha < \bigcup X$ such that $|U_\alpha \cap X| = \omega$.

For each $\alpha < \mathfrak{s}$, enumerate $U_\alpha = \{u_\alpha(n) \mid n \in \omega\}$ in an increasing way. Now, choose a partition $\{P_\alpha \mid \alpha \in \mathfrak{s}\}$ of $\mathfrak{s}$ with the following properties:

1. $|P_0| = \mathfrak{s}$ and $\omega \subseteq P_0$.
2. If $\alpha \neq 0$, the following holds:
   (a) $|P_\alpha| = \mathfrak{b}$.
   (b) $\alpha \leq \mathsf{min}(P_\alpha) < \mathsf{max}(P_\alpha) < \alpha + \mathfrak{b}$.

Since $\mathfrak{bs}$ is the maximum of $\mathfrak{b}$ and $\mathfrak{s}$ and $\mathfrak{b} \leq \mathfrak{a} < \mathfrak{s}$, we know that there is a block-splitting family of size $\mathfrak{s}$. Fix $\{S_\alpha \mid \alpha \in P_0\}$ a block-splitting family and take an enumeration $[\omega]^\omega = \{X_\alpha \mid \alpha \in \mathfrak{c}\}$.

Our goal is to recursively define $\{(A_\alpha, \sigma_\alpha, C_\alpha) \mid \alpha \in \mathfrak{c}\}$ such that for every $\alpha < \mathfrak{c}$, the following properties hold:

1. $\mathcal{A}_{<\alpha} = \{A_\xi \mid \xi < \alpha\} \subseteq [\omega]^\omega$ is an AD family.
2. $\sigma_\alpha \in 2^{<\mathfrak{s}}$.
3. If $\beta < \alpha$, then $\sigma_\alpha \nsubseteq \sigma_\beta$.
4. If $X_\alpha \in \mathcal{I}(\mathcal{A}_{<\alpha})^+$, then $A_\alpha \subseteq X_\alpha$.
5. $C_\alpha : 2^{<\mathfrak{s}} \longrightarrow \wp(\omega)$ and $A_\alpha \subseteq^* C_\alpha(\sigma_\alpha \restriction \xi)^{\sigma_\alpha(\xi)}$ for every $\xi < dom(\sigma_\alpha)$.

It is clear that if we manage to do this, then we will have achieved in constructing a completely separable MAD family. The requirements look very similar to the ones in the proof of Theorem 23, but there is an important difference: The presence of the function C_α. Note if we had $C_\alpha(\sigma) = S_{dom(\sigma)}$, then we will have exactly the same requirements as in Theorem 23. However, in this case, the function C_α (in general) will not be constant by levels and in usually $C_\alpha \neq C_\beta$ whenever $\alpha \neq \beta$.

The function C_α can be computed from $\mathcal{A}_{<\alpha}$ and $\{\sigma_\xi \mid \xi < \alpha\}$ in the following way: Let $\tau \in 2^{<\mathfrak{s}}$ and let $\xi < \mathfrak{s}$ such that $\tau \in 2^\xi$. The definition of $C_\alpha(\tau)$ is by cases:

Case 31 $\xi \in P_0$.

In here, we simply define $C_\alpha(\tau) = S_\xi$ (so in this case, C_α is constant by levels, but in the other case it may not be like that).

Case 32 $\xi \in P_\delta$ *with* $\delta \neq 0$ *(recall that* $\delta \leq \xi < \delta + \mathfrak{b}$*).*

In here, for every $n \in \omega$ we define:

$$B_n^\alpha(\tau) = \bigcap_{i \leq n} \left(C_\alpha(\tau \upharpoonright u_\delta(i))^{\tau(u_\delta(i))} \setminus A_{\tau \upharpoonright u_\delta(i-1)} \right)$$

Where $A_{\tau \upharpoonright u_\delta(i-1)}$ is defined as follows:

1. If there is $\beta < \alpha$ such that $\tau \upharpoonright u_\delta(i-1) = \sigma_\beta$, then $A_{\tau \upharpoonright u_\delta(i-1)} = A_\beta$ (note that at most one β can satisfy the requirement).
2. In the other case, let $A_{\tau \upharpoonright u_\delta(i-1)} = \emptyset$.

It follows that $B^\alpha(\tau) = \langle B_n^\alpha(\tau) \rangle_{n \in \omega}$ is a decreasing sequence of subsets of ω. Fix a family $\mathcal{L}(\mathcal{B}^\alpha(\tau))$ of size $\mathfrak{b}$ as in Lemma 22. Take an enumeration $\mathcal{L}(\mathcal{B}^\alpha(\tau)) = \{L_\alpha^\tau(\nu) \mid \nu \in P_\delta\}$ (recall that P_δ has size $\mathfrak{b}$).

Finally, define $C_\alpha(\tau) = L_\alpha^\tau(\xi)$ (recall that ξ is the height of τ).

A key property of the functions $\langle C_\alpha \rangle_{\alpha < \mathfrak{c}}$ is that they are "coherent" in the following sense:

$$\text{If } \gamma < \alpha \text{, then} \quad \begin{array}{l} C_\gamma(\sigma_\gamma \upharpoonright \xi) = C_\alpha(\sigma_\gamma \upharpoonright \xi) \\ \text{for every } \xi < dom(\sigma_\gamma) \end{array}$$

This is the reason why we demanded that if $\beta < \alpha$, then $\sigma_\alpha \nsubseteq \sigma_\beta$. In this way, $C_\gamma(\sigma_\gamma \upharpoonright \xi)$ and $C_\alpha(\sigma_\gamma \upharpoonright \xi)$ are computed in the same way. In particular, we get the following:

$$\text{If } \gamma < \alpha \text{, then} \quad \begin{array}{l} A_\gamma \subseteq^* C_\alpha(\sigma_\gamma \upharpoonright \xi)^{\sigma_\gamma(\xi)} \\ \text{for every } \xi < dom(\sigma_\gamma) \end{array}$$

Before defining A_α and σ_α we need to point out the following: For every $X \in \mathcal{I}(\mathcal{A}_{<\alpha})^+$, there are $\xi < \mathfrak{s}$ and $\tau_X \in 2^\xi$ such that:

1. If $\beta < \xi$ then $X \cap C_\alpha(\tau_X \upharpoonright \beta)^{1-\tau_X(\beta)} \in \mathcal{I}(\mathcal{A}_{<\alpha})$ (so $X \cap C_\alpha(\tau_X \upharpoonright \beta)^{\tau_X(\beta)} \in \mathcal{I}(\mathcal{A}_{<\alpha})^+$).
2. $X \cap C_\alpha(\tau_X \upharpoonright \beta), X \setminus C_\alpha(\tau_X \upharpoonright \beta) \in \mathcal{I}(\mathcal{A}_{<\alpha})^+$.

This is because $\{S_\nu \mid \nu \in P_0\}$ is a block-splitting family (and by Proposition 18). Clearly $\tau_X \in 2^{<\mathfrak{s}}$ is unique and if $Y \in [X]^\omega \cap \mathcal{I}(\mathcal{A}_{<\alpha})^+$ then τ_Y extends τ_X.

We are now in position to define A_α and σ_α. Let $X = X_\alpha$ if $X_\alpha \in \mathcal{I}(\mathcal{A}_{<\alpha})^+$ and if not, let $X = \omega$ (or any other element of $\mathcal{I}(\mathcal{A}_{<\alpha})^+$). Using the same argument as in Theorem 23, we may find $\langle X_n \rangle_{n\in\omega}$, Y, σ and γ such that the following holds:

1. $X_0 = X$.
2. $\langle X_n \rangle_{n\in\omega}$ is a decreasing sequence of elements in $\mathcal{I}(\mathcal{A}_{<\alpha})^+$.
3. $\sigma = \bigcup_{n\in\omega} \tau_{X_n}$.
4. $\gamma = dom(\sigma)$.
5. $X_{n+1} = X_n \cap C_\alpha(\sigma \restriction \gamma_n)^{\sigma(\gamma_n)}$ (where $\gamma_n = dom(\tau_{X_n})$).
6. If $\beta < \alpha$, then σ_β does not extend σ.
7. $Y \in \mathcal{I}(\mathcal{A}_{<\alpha})^+$ and is a pseudointersection of $\langle X_n \rangle_{n\in\omega}$ contained in X_0.
8. If $\beta < \gamma$, then $Y \cap C_\alpha(\sigma \restriction \gamma)^{1-\sigma(\beta)} \in \mathcal{I}(\mathcal{A}_{<\alpha})$.

Now, our goal is to find a subset of Y that is almost disjoint with $\mathcal{A}_{<\alpha}$. With this objective in mind, we define W as the set of all $\xi < \gamma$ such that there is $\beta < \alpha$ with the following properties:

1. $|A_\beta \cap Y| = \omega$.
2. Either $\sigma_\beta = \sigma \restriction \xi$ or $\xi = \triangle(\sigma_\beta, \sigma)$.

Now we will prove the following:

Claim 33 $|W| < \mathfrak{b}$.

Assume this is not the case. Let W_0 be the first $\mathfrak{b}$-elements of W. By the principle $\mathsf{P}(\mathfrak{b}, \mathfrak{s})$, we know that there is $\delta < \cup W_0$ such that $U_\delta \cap W_0$ is infinite. Note that $\delta < \delta + \mathfrak{b} \leq \gamma$. Let ξ be the first element of P_δ (so $\delta \leq \xi < \delta + \mathfrak{b} \leq \gamma$). Since $U_\delta \cap W_0$ is infinite, it follows that $*(B^\alpha(\sigma \restriction \xi), \mathcal{A}_{<\alpha}, Y)$ holds (see Lemma 22). In this way, there is $\nu \in P_\delta$ (so $\delta \leq \nu < \delta + \mathfrak{b} \leq \gamma$) for which:

$$Y \cap L_\alpha^{\sigma\restriction\xi}(\nu), Y \setminus L_\alpha^{\sigma\restriction\xi}(\nu) \in \mathcal{I}(\mathcal{A}_{<\alpha})^+$$

By definition, we know that $C_\alpha(\sigma \restriction \nu) = L_\alpha^{\sigma\restriction\xi}(\nu)$, hence:

$$Y \cap C_\alpha(\sigma \restriction \nu), Y \setminus C_\alpha(\sigma \restriction \nu) \in \mathcal{I}(\mathcal{A}_{<\alpha})^+$$

But this is a contradiction! since $\nu < \gamma$, so it must be the case that $Y \cap C_\alpha(\sigma \restriction \nu)^{1-\sigma(\nu)} \in \mathcal{I}(\mathcal{A}_{<\alpha})$. This finishes the proof that W has size less than $\mathfrak{b}$.

For every $\xi \in W$, define $Z(\xi) \in [\mathcal{A}_{<\alpha}]^{<\omega}$ as follows:

1. $Z(\xi) = \{A_\beta\}$ if $|A_\beta \cap Y| = \omega$ and $\sigma_\beta = \sigma \restriction \xi$, or
2. $Y \cap C_\alpha(\sigma \restriction \xi)^{1-\sigma(\xi)} \subseteq^* \bigcup Z(\xi)$.

Let $\mathcal{D} = \bigcup_{\xi \in W} Z(\xi)$. It follows that $\mathcal{D}$ is an AD family of size less than $\mathfrak{b}$ (so it is less than $\mathfrak{a}$). In this way, $\mathcal{D} \restriction Y$ is not maximal, so we can find $A_\alpha \in [Y]^\omega \cap \mathcal{D}^\perp$. Define $\sigma_\alpha = \sigma$. We need to prove that A_α is almost disjoint with $\mathcal{A}_{<\alpha}$, but this follows by the same argument as the one of Theorem 23. ■

As discussed earlier, we get the following:

Corollary 34 *If $\mathfrak{c} < \aleph_\omega$, then there is a completely separable MAD family (in fact, $\mathfrak{s} < \aleph_\omega$ is enough).*

If there was a model where there are no completely separable MAD families, then $\mathfrak{a} < \mathfrak{s}$ and the negation of $\mathsf{P}(\mathfrak{b}, \mathfrak{s})$ must hold in that model. The author does not know if this is consistent.

Acknowledgement 35 *The author would like to thank Michael Hrušák for all the discussions of MAD families throughout all these years.*

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