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AUTHOR(S):

WATANABE, Sennosuke; FUKUDA, Akiko; SEGAWA,
Etsuo; SATO, Iwao

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Limit theorem of the max-plus walk

By

Sennosuke WATANABE*, Akiko FUKUDA,**
Etsuo SEGAWA*** and Iwao SATO†

Abstract

The max-plus algebra is a semiring on $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with addition \oplus and multiplication \otimes defined by $\oplus = \max$ and $\otimes = +$, respectively. It is known that eigenvalues of max-plus matrices are equivalent to the maximal average weight of the corresponding directed graph. In [9], authors introduced the max-plus walk which is a walk model on one dimensional lattice on \mathbb{Z} over max-plus algebra, and discussed its properties such as the conserved quantities and the steady state. In this paper, we will discuss the limit measure of the max-plus walk.

§ 1. Introduction

Studies on a spatial discretization of the Schrödinger equation is known as the discrete-Schrödinger equation. Quantum walk can be regarded as a temporal discretization of a discrete-Schrödinger equation [6, 7]. This is a reason that quantum walk is recently known as a quantum simulator as envisioned by Feynman [2]. On the other hand, there is a further discretization; so called the ultradiscretization, which is introduced in [8]. Ultradiscretization is a technique which transform a difference equation

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*The University of Fukuchiyama, Kyoto 620-0886, Japan.

e-mail: sewatana@fukuchiyama.ac.jp

**Shibaura Institute of Technology, Saitama 337-8570, Japan.

e-mail: afukuda@shibaura-it.ac.jp

***Yokohama National University, Yokohama 240-8501, Japan.

e-mail: e.segawa17@gmail.com

†National Institute of Technology, Oyama college, Tochigi 323-0806, Japan.

e-mail: isato@oyama-ct.ac.jp

into piecewise linear equation and appears in the context of integrable systems. It is based on the following formula.

$$\lim_{\epsilon \rightarrow +0} \epsilon \log(e^{A/\epsilon} + e^{B/\epsilon}) = \max\{A, B\}.$$

Since ultradiscrete equations can be written by using operations “max” and “+”, they can be considered over max-plus algebra. In max-plus algebra, the sum of two elements is their maximum and the product of two elements is their sum. This algebraic structure is known as idempotent semiring. In [9], a new walk model on one-dimensional lattice over max-plus algebra is introduced. Such a walk is called the max-plus walk.

In the usual quantum walk (resp. random walk) on one dimensional lattice, the limit theorem can be characterized as follows [4]: (1) the scaling order is linearly proportional (resp. proportional to square root) to time steps and (2) the limit distribution is described by the Konno distribution (resp. the Gaussian distribution). In [9], we find the conserved quantities which are independent of the time step, such as the ℓ^2 -norm conservation in the unitary quantum walk, and a useful necessary and sufficient condition of the setting of the max-plus walk for the conservation. The conserved quantities are given by the summation of eigenvalues of the state decision matrices over all the lattice points. In this paper, we define a measure on the lattice points under the condition. We also obtain a limit theorem on this measure in the long-time limit. This corresponds to the original unitary quantum walk.

This paper is organized as follows. In sections 2 and 3, we give a short review on the max-plus algebra and max-plus walk, respectively. In section 4, under a certain conserved condition, we obtain the limit theorem corresponding to the usual quantum walk. Finally, in section 5, we give concluding remarks.

§ 2. Max-plus algebra

Max-plus algebra is defined in a set $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with two binary operations

$$a \oplus b = \max\{a, b\}$$

and

$$a \otimes b = a + b$$

for $a, b \in \mathbb{R}_{\max}$. Addition \oplus is commutative, associative and has the identity element $\varepsilon := -\infty$. Multiplication \otimes is also commutative, associative and has the identity element $e := 0$. \otimes is distributive with respect to \oplus . There exists an inverse element with respect to \otimes for any elements in $\mathbb{R}_{\max} \setminus \{\varepsilon\}$. Note that there is not exist inverse elements with respect to \oplus . \oplus is idempotent, namely, $a \oplus a = a$ for $a \in \mathbb{R}_{\max}$.

As in the conventional algebra, we extend the operations \oplus and \otimes to matrices. Let $\mathbb{R}_{\max}^{m \times n}$ be the set of $m \times n$ matrices whose entries are in \mathbb{R}_{\max} and $[X]_{ij}$ be the (i, j) entry of the matrix X . For $A = (a_{ij}), B = (b_{ij}) \in \mathbb{R}_{\max}^{m \times n}$, their sum $A \oplus B \in \mathbb{R}_{\max}^{m \times n}$ is defined by

$$[A \oplus B]_{ij} = a_{ij} \oplus b_{ij}.$$

For $A = (a_{ij}) \in \mathbb{R}_{\max}^{m \times k}$ and $B = (b_{ij}) \in \mathbb{R}_{\max}^{k \times n}$, their product $A \otimes B \in \mathbb{R}_{\max}^{m \times n}$ is defined by

$$[A \otimes B]_{ij} = \bigoplus_{l=1}^k a_{il} \otimes b_{lj}.$$

We denote $A^{\otimes k}$ as the k -th power of A , namely,

$$A^{\otimes k} := \underbrace{A \otimes A \otimes \cdots \otimes A}_{k \text{ times}}.$$

For a matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{n \times n}$, we define the tropical determinant of A by

$$\text{tropdet}(A) = \bigoplus_{\sigma \in S_n} a_{1\sigma(1)} \otimes a_{2\sigma(2)} \otimes \cdots \otimes a_{n\sigma(n)}$$

where S_n denotes the symmetric group of order n .

Although it has different operators to the conventional algebra, the eigenvalue problem for matrices is fundamental in both algebra.

Definition 2.1. For a max-plus matrix $A \in \mathbb{R}_{\max}^{n \times n}$, a scalar $\lambda \in \mathbb{R}_{\max}$ is called an eigenvalue of A if there is a vector $\mathbf{x} \neq (\varepsilon, \dots, \varepsilon)^\top \in \mathbb{R}_{\max}^n$ satisfying

$$A \otimes \mathbf{x} = \lambda \otimes \mathbf{x}.$$

Such vector \mathbf{x} is called an eigenvector of A with respect to λ .

The max-plus eigenvalues were shown, for example, in Baccelli et al. [1] to have a close relationship with the weighted directed graphs.

For a max-plus matrix $A = (a_{ij}) \in \mathbb{R}_{\max}^{n \times n}$, we define a weighted directed graph (briefly digraph) $G(A) = (V, E, w)$ as follows. The vertex set and the edge set are $V = \{1, 2, \dots, n\}$ and $E = \{(i, j) | a_{ij} \neq \varepsilon\} \subset V \times V$, respectively, and the weight function $w : E \rightarrow \mathbb{R}$ is defined by $w((i, j)) = a_{ij}$ for $(i, j) \in E$. A sequence $C = (i_1, i_2, \dots, i_s, i_1)$ of vertices is called a circuit if $(i_k, i_{k+1}) \in E$ for all $k = 1, 2, \dots, s$ where $i_{s+1} = i_1$ and $i_p \neq i_q$ for $1 \leq p < q \leq s$. The number $\ell(C) := s$ is called the length of C and $w(C) := w((i_1, i_2)) + w((i_2, i_3)) + \cdots + w((i_s, i_1))$ is called the weight of C . We define the average weight of C by $\text{ave}(C) := w(C)/\ell(C)$.

Proposition 2.2 (cf. F. Baccelli et al. [1]). For a max-plus matrix $A \in \mathbb{R}_{\max}^{n \times n}$, $\lambda \neq \varepsilon$ is an eigenvalue of A if there exists a circuit in the weighted digraph $G(A)$ whose average weight is equal to λ . In particular, if a max-plus matrix A is irreducible, A has precisely one eigenvalue and is equal to the maximum average weight of circuits in $G(A)$.

§ 3. Max-plus walk [9]

In this section, we give a brief review of the max-plus walk which is introduced in [9]. We consider the one dimensional lattice on \mathbb{Z} and a max-plus vector $\psi_k^n \in \mathbb{R}_{\max}^2$ given on a lattice point. The vector ψ_k^n is called the *state* at a position $k \in \mathbb{Z}$ and a discrete time n . We set the initial state ψ_k^0 as

$$\psi_k^0 = \begin{cases} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, & k = 0, \\ \begin{bmatrix} \varepsilon \\ \varepsilon \end{bmatrix}, & k \neq 0. \end{cases}$$

In the max-plus walk, the state ψ_k^n changes according to two max-plus matrices $P, Q \in \mathbb{R}_{\max}^{2 \times 2}$ given as

$$P = \begin{bmatrix} a & b \\ \varepsilon & \varepsilon \end{bmatrix}, \quad Q = \begin{bmatrix} \varepsilon & \varepsilon \\ c & d \end{bmatrix}.$$

Then, the state ψ_k^n is determined by the following evolution equation.

$$\psi_k^n = (P \otimes \psi_{k+1}^{n-1}) \oplus (Q \otimes \psi_{k-1}^{n-1}) = A_k^n \otimes \psi_0^0,$$

where $A_k^n \in \mathbb{R}_{\max}^{2 \times 2}$ is called the state decision matrix. It is easy to see that, at discrete time n , $A_k^n = \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix}$ for $k = -n+1, -n+3, \dots, n-3, n-1$. We define a set of positions whose state vector has at least one finite element as $\mathcal{K} = \{-n, -n+2, \dots, n-2, n\}$. Time evolution of the max-plus walk is illustrated in Figure 1. For example, $A_{-3}^3 = P^{\otimes 3}$ and $A_1^3 = Q \otimes (Q \otimes P \oplus P \otimes Q) \oplus P \otimes Q^{\otimes 2}$. Here we introduce two matrices $R, S \in \mathbb{R}_{\max}^{2 \times 2}$ as follows.

$$R = \begin{bmatrix} c & d \\ \varepsilon & \varepsilon \end{bmatrix}, \quad S = \begin{bmatrix} \varepsilon & \varepsilon \\ a & b \end{bmatrix}.$$

Let ℓ and m be the number of moving left (P) and right (Q), respectively, and $r = \lfloor \tilde{r}/2 \rfloor$, where \tilde{r} is the summation of the number of making a turn from left to right and from

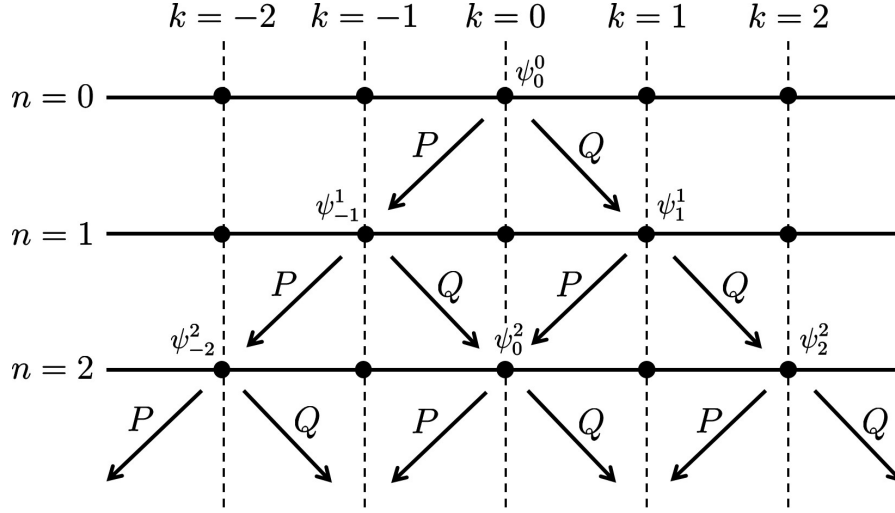


Figure 1. Max-plus walk.

right to left. Note here that $\ell = \frac{n-k}{2}$ and $m = \frac{n+k}{2}$. Then, the state decision matrices A_k^n at position $k \in \mathcal{K}$ and discrete time n can be given as

$$\begin{aligned}
 A_k^n = & \bigoplus_{r=1}^{(\ell-1) \wedge m} \{(\ell-r-1)a + rb + rc + (m-r)d\} \otimes P \\
 & \oplus \bigoplus_{r=1}^{\ell \wedge (m-1)} \{(\ell-r)a + rb + rc + (m-r-1)d\} \otimes Q \\
 & \oplus \bigoplus_{r=1}^{\ell \wedge m} \{(\ell-r)a + rb + (r-1)c + (m-r)d\} \otimes R \\
 & \oplus \bigoplus_{r=1}^{\ell \wedge m} \{(\ell-r)a + (r-1)b + rc + (m-r)d\} \otimes S,
 \end{aligned}$$

where $a \wedge b = \min\{a, b\}$. Let $U = P \oplus Q = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and impose the following condition to the entries $a, b, c, d \in \mathbb{R}_{\max}$ of U .

$$(C1) \quad \begin{cases} a \otimes d = e, \\ b \otimes c = e. \end{cases}$$

Then the condition (C1) is an analogue of the property that the absolute value of the determinant of unitary matrices is 1 in conventional algebra, namely,

$$\text{tropdet}(U) = (a \otimes d) \oplus (b \otimes c) = e.$$

Under the condition (C1), A_k^n can be rewritten as

$$\begin{aligned}
A_k^n &= \{(\ell a + md - a) \otimes P\} \oplus \{(\ell a + md - d) \otimes Q\} \\
&\quad \oplus \{(\ell a + md - c) \otimes R\} \oplus \{(\ell a + md - b) \otimes S\} \\
&= \begin{bmatrix} \ell a + md & \max\{\ell a + md - a + b, \ell a + md - c + d\} \\ \max\{\ell a + md + a - b, \ell a + md + c - d\} & \ell a + md \end{bmatrix}, \\
&\quad n = 0, 1, \dots, \quad k = -n + 2, -n + 4, \dots, n - 4, n + 2.
\end{aligned}$$

Noting that $\ell a + md = (\ell - m)a = -ka$, then the state decision matrices A_k^n is given as follows.

$$A_k^n = \begin{cases} \begin{bmatrix} -ka & (-k-1)a+b \\ (-k+1)a-b & -ka \end{bmatrix} & \text{if } k = -n+2, -n+4, \dots, n-4, n-2, \\ \begin{bmatrix} na & (n-1)a+b \\ \varepsilon & \varepsilon \end{bmatrix} & \text{if } k = -n, \\ \begin{bmatrix} \varepsilon & \varepsilon \\ (-n+1)a-b & -na \end{bmatrix} & \text{if } k = n, \\ \begin{bmatrix} \varepsilon & \varepsilon \\ \varepsilon & \varepsilon \end{bmatrix} & \text{otherwise.} \end{cases}$$

Let $\lambda(A_k^n)$ be the eigenvalue of A_k^n . From Proposition 2.2, we have

$$\lambda(A_k^n) = -ak, \quad k \in \mathcal{K}.$$

It is remarkable here that the eigenvalues of A_k^n is not dependent of the discrete time n . Moreover, let Φ^n be the summation of eigenvalues $\lambda(A_k^n)$ of all the position $k \in \mathcal{K}$ at arbitrary discrete time n , namely

$$\Phi^n = \sum_{k \in \mathcal{K}} \lambda(A_k^n) = \sum_{k \in \mathcal{K}} -ak.$$

Then, we have the following theorem.

Theorem 3.1 (S. Watanabe, A. Fukuda, E. Segawa, I. Sato [9]). Under the condition (C1), the summation of all the eigenvalues Φ^n are invariant under the time evolution of the max-plus walk and is given as

$$\Phi^n = 0.$$

Namely, the summation of the eigenvalues are the conserved quantities of the max-plus walk.

In the quantum walk, the ℓ_2 -norm of the state vectors are probability and their summation is 1, which is conserved quantities of the quantum walk. Noting that, in the max-plus algebra, the identity element with respect to \otimes is 0, Theorem 3.1 is a max-plus analogue of the conserved quantities of the quantum walk. In the next section, using the analogue of the probability, we will discuss the limit measure of the max-plus walk. For the readers' reference, the following are also conserved quantities of the max-plus walk:

$$\bigoplus_{k \in \mathcal{K}} \left\{ \frac{1}{n} \lambda(A_k^n) \right\} = \bigoplus_{k \in \mathcal{K}} \left\{ \frac{1}{k} \lambda(A_k^n) \right\} = |a|,$$

which is written by using the max-plus analogue of the summation. This quantity corresponds to the pseudo velocity of traditional quantum walks [3].

§ 4. Limit measure of max-plus walk

First let us give a short review on the limit theorem for the traditional quantum walk on \mathbb{Z} [4]. Let $\mu_n : \mathbb{Z} \rightarrow [0, 1]$ be the probability distribution of this quantum walk at time n with the mixed initial state at the origin. Then μ_n converges in law as follows:

For any $x \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \sum_{k \leq nx} \mu_n(k) = \int_{-\infty}^x f_K(u; |a|) du$$

where $f_K(u; |a|)$ is the Konno distribution with the parameter $|a|$

$$f_K(u; |a|) = \begin{cases} \frac{\sqrt{1 - |a|^2}}{\pi(1 - u^2)\sqrt{|a|^2 - u^2}}, & \text{if } -|a| \leq u \leq |a|, \\ 0, & \text{otherwise} \end{cases}$$

In the following, let us consider the corresponding limit theorem for the max-plus walk. Since the counterpart of the probability distribution $\mu_n(k)$ is the eigenvalues of the state decision matrix A_k^n , then we compute

$$\sum_{k \in \mathcal{K}, k \leq nx} \lambda(A_k^n) = \sum_{k \in \mathcal{K}, k \leq nx} (-ak).$$

Putting $p \in \mathbb{Z}$ by $nx - 2 < -n + 2p \leq nx$, we have

$$\begin{aligned} \sum_{k \in \mathcal{K}, k \leq nx} (-ak) &= -a\{-n + (-n + 2) + \cdots + (-n + 2p)\} \\ (4.1) \qquad \qquad \qquad &= a(p + 1)(n - p) \end{aligned}$$

Since $-n + 2p \leq nx < -n + 2p + 2$,

$$(4.2) \quad -\frac{1}{n} + \frac{x+1}{2} < \frac{p}{n} \leq \frac{x+1}{2},$$

taking the limit of $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} \frac{p}{n} = \frac{x+1}{2}.$$

Therefore

$$(4.3) \quad \begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k \in \mathcal{K}, k \leq nx} \lambda(A_k^n) &= \lim_{n \rightarrow \infty} \frac{1}{n^2} a(p+1)(n-p) \\ &= \lim_{n \rightarrow \infty} a \left(\frac{p}{n} + \frac{1}{n} \right) \left(1 - \frac{p}{n} \right) \\ &= \frac{a}{4} (1 - x^2). \end{aligned}$$

RHS of (4.3) is rewritten by

$$\frac{a}{4} (1 - x^2) = \int_{-\infty}^x f(u) du.$$

Here

$$f(u) = \begin{cases} -\frac{au}{2} & : u \in [-1, 1], \\ 0 & : \text{otherwise.} \end{cases}$$

We summarize the above statement in the following theorem.

Theorem 4.1. The limit measure of the max-plus walk is expressed by

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{k \in \mathcal{K}, k \leq nx} \lambda(A_k^n) = \int_{-\infty}^x f(u) du.$$

Here

$$f(u) = \begin{cases} -\frac{au}{2} & : u \in [-1, 1], \\ 0 & : \text{otherwise.} \end{cases}$$

§ 5. Concluding remarks

In this paper, we obtained the limit measure on the max-plus walk proposed by [9]. Theorem 3.1 implies that if the setting of the max-plus walk satisfies some conditions, then we obtain the conservation quantity with respect to the time step and we can define

the signed measure on \mathbb{Z} at each time step. This corresponds to the ℓ^2 -conservation property of the unitary time evolution of quantum walks. Let B_k^n be the state decision matrix of the usual quantum walk. (See for its explicit expression, for example, [4]). If we start the usual quantum walk from the mixed state at the origin, then the distribution at time n $\mu_n : \mathbb{Z} \rightarrow [0, 1]$ is described by $\|B_k^n\|_F^2 = 2 \sum_i (\sigma_i)^2$ which is quite similar to the measure of the max-plus walk. Here, $\|\cdot\|_F$ is the Frobenius norm and σ_i is a singular value of B_k^n .

By Theorem 4.1, the linearly spreading is also reflected in the max-plus walk since the scaling orders of both this measure of the max-plus walk and usual quantum walk are linearly proportional to the time steps. Moreover, it is also interesting that the limit measure of the max-plus walk depends only on the parameter a while the pseudo velocity of the usual quantum walk which is the parameter of the Konno distribution depends only on also “ a ”. How usual quantum walk’s behaviors are extracted in the max-plus walk’s measure more clearly is one of the interesting future’s problems.

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