TWO DUAL PAIR METHODS IN THE STUDY OF GENERALIZED WHITTAKER MODELS FOR IRREDUCIBLE HIGHEST WEIGHT MODULES

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INTRODUCTION

Let G be a connected simple linear Lie group of Hermitian type, and let K be a maximal compact subgroup of G. The Lie algebras of G and K are denoted by $\mathfrak{g}_0$ and $\mathfrak{k}_0$ respectively. The purpose of this note is to make an overview of our algebraic and geometric approach to the study of generalized Whittaker models for irreducible admissible representations of G with highest weights. We employ two kinds of dual pair methods in the course of our study.

To be more precise, we write $G_{\mathbf{C}}, K_{\mathbf{C}}$ (resp. $\mathfrak{g}, \mathfrak{k}$) for the complexifications of G, K (resp. $\mathfrak{g}_0, \mathfrak{k}_0$) respectively. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a complexified Cartan decomposition of $\mathfrak{g}$. The G-invariant complex structure on $K\backslash G$ gives a triangular decomposition $\mathfrak{g} = \mathfrak{p}_+ + \mathfrak{k} + \mathfrak{p}_-$ of $\mathfrak{g}$. It is well-known that $\mathfrak{p}_+$ admits precisely $r+1$ number of $K_{\mathbf{C}}$-orbits $\mathcal{O}_m$ $(m = 0, 1, \ldots, r)$ arranged as $\dim \mathcal{O}_0 = 0 < \dim \mathcal{O}_1 < \cdots < \dim \mathcal{O}_r = \dim \mathfrak{p}_+$, where r denotes the real rank of G.

These nilpotent $K_{\mathbf{C}}$-orbits $\mathcal{O}_m$ are essentially related to the highest weight representations. In reality, the Harish-Chandra module of an irreducible admissible G-representation with highest weight is isomorphic to the unique simple quotient $L(\tau)$ of generalized Verma module $M(\tau)$ attached to an irreducible representation (τ, V_τ) of K. Then, the associated variety (i.e., the support) $\mathcal{V}(L(\tau))$ of $L(\tau)$ coincides with the closure of a single $K_{\mathbf{C}}$-orbit $\mathcal{O}_{m(\tau)}$ in $\mathfrak{p}_+$, where $m(\tau)$ depends on τ. On the other hand, following the recipe by Kawanaka [12] (see also [23]), one can construct a generalized Gelfand-Graev representation $\Gamma_m = \operatorname{Ind}_{\mathfrak{n}(m)}^{G}(\eta_m)$ (GGGR for short; see Definition 4.1) attached to the nilpotent G-orbit $\mathcal{O}'_m$ in $\mathfrak{g}_0$ corresponding to each $K_{\mathbf{C}}$-orbit $\mathcal{O}_m$ through the Kostant-Sekiguchi bijection. The GGGR Γ_m is induced from certain one-dimensional representation η_m of a nilpotent Lie subalgebra $\mathfrak{n}(m)$ of $\mathfrak{g}$, and it is far from irreducible.

In this note, we are concerned with the following problem.

Problem. Describe the $(\mathfrak{g}, K)$-embeddings, i.e., the generalized Whittaker models, of $L(\tau)$ into these GGGRs Γ_m.

As for $L(\tau)$'s isomorphic to the irreducible generalized Verma modules $M(\tau)$, we already have a complete answer in [24, Part II]. Hence our main interest is in the case where the corresponding $M(\tau)$ is reducible.

In order to specify the embeddings, we use the invariant differential operator $\mathcal{D}_{\tau^*}$ on $K\backslash G$ of gradient type associated to the K-representation τ^* dual to τ (Definition 2.2). This operator $\mathcal{D}_{\tau^*}$ is due to Enright, Davidson and Stanke ([2], [3], [4]), and the K-finite kernel of $\mathcal{D}_{\tau^*}$ realizes the dual lowest weight module $L(\tau)^*$. Our first dual pair method, which comes essentially from a duality of Peter-Weyl type for irreducible $(\mathfrak{g}, K)$-modules,

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tells us that the space $\mathcal{Y}(\tau, m)$ of η_m-covariant solutions F of differential equation $\mathcal{D}_{\tau^*} F = 0$ is isomorphic to the space of $(\mathfrak{g}, K)$-homomorphisms in question. The space $\mathcal{Y}(\tau, m)$ can be intrinsically analyzed by an algebraic method, thanks to the Cayley transform on $G_{\mathbb{C}}$ which carries the bounded realization of $K\backslash G$ to the unbounded one.

As consequences, it is shown that $L(\tau)$ embeds into the GGGR Γ_m with nonzero and finite multiplicity if and only if the corresponding $\mathcal{O}_m$ is the unique open $K_{\mathbb{C}}$-orbit $\mathcal{O}_{m(\tau)}$ in the associated variety $\mathcal{V}(L(\tau))$. If $L(\tau)$ is unitarizable, we can specify the space $\mathcal{Y}(\tau) := \mathcal{Y}(\tau, m(\tau))$ in terms of the principal symbol at the origin Ke of the differential operator $\mathcal{D}_{\tau^*}$. This reveals a natural action on $\mathcal{Y}(\tau)$ of the isotropy subgroup $K_{\mathbb{C}}(X)$ of $K_{\mathbb{C}}$ at a certain point $X \in \mathcal{O}_{m(\tau)}$. Furthermore, we find that the dimension of $\mathcal{Y}(\tau)$ coincides with the multiplicity of $L(\tau)$ at the defining ideal of $\mathcal{V}(L(\tau))$. See Theorems 5.1 and 5.2.

If G is one of the classical groups $G = SU(p,q)$, $Sp(2n, \mathbb{R})$ and $SO^*(2n)$, the theory of reductive dual pair gives realizations of unitarizable highest weight modules $L(\tau)$ (cf. [11], [7], [3]). The generalized Whittaker models for such an $L(\tau)$ can be described more explicitly by using the oscillator representation of the pair (G, G') with a compact group G' dual to G. This is our second dual pair method. The case $SU(p,q)$ has been studied by Tagawa [20] motivated by author's observation in 1997 for the case $Sp(n, \mathbb{R})$. In this note we focus our attention on the remaining case $SO^*(2n)$.

The full detail of this overview will appear elsewhere (see [27]).

We organize this note as follows.

Section 1 concerns our first dual pair method. Namely, we provide with a kernel theorem (Theorem 1.2) which will be utilized for describing the generalized Whittaker models in later sections. We introduce in Section 2 the differential operator $\mathcal{D}_{\tau^*}$ on $K\backslash G$ of gradient type associated to τ^*, after [4]. Section 3 is devoted to characterizing the associated variety and multiplicity of irreducible highest weight module $L(\tau)$ by means of the principal symbol of $\mathcal{D}_{\tau^*}$ (Theorem 3.3). After introducing the GGGRs Γ_m in Section 4, we state our main results (Theorems 5.1 and 5.2) in Section 5. Also, we discuss the case of classical group $SO^*(2n)$ more explicitly in 5.2, through our second dual pair method.

1. The first dual pair method – Kernel theorem

In this section, let G be any connected semisimple Lie group with finite center. We employ the same notation as in Introduction. Conventionally, the complexification in $\mathfrak{g}$ of any real vector subspace $\mathfrak{s}_0$ of $\mathfrak{g}_0$ will be denoted by $\mathfrak{s}$ by dropping the subscript 0. We write $U(\mathfrak{m})$ (resp. $S(\mathfrak{v})$) for the universal enveloping algebra of a Lie algebra $\mathfrak{m}$ (resp. the symmetric algebra of a vector space $\mathfrak{v}$). A $U(\mathfrak{g})$-module $\boldsymbol{X}$ is called a $(\mathfrak{g}, K)$-module if the subalgebra $U(\mathfrak{k})$ acts on $\boldsymbol{X}$ locally finitely, and if the $\mathfrak{k}_0$-action gives rise to a representation of K on $\boldsymbol{X}$ through exponential map.

The group G acts on the space $C^\infty(G)$ of all smooth functions on G by left translation L and by right translation R as follows:

$$(1.1) \qquad g^L f(x) := f(g^{-1}x), \quad g^R f(x) := f(xg) \quad (g \in G, x \in G; f \in C^\infty(G)).$$

Through differentiation one gets two $U(\mathfrak{g})$-representations on $C^\infty(G)$ denoted again by L and R respectively. Let $C^\infty_K(G)$ be the space of all functions in $C^\infty(G)$ which are left K-finite and also right K-finite. Then $C^\infty_K(G)$ becomes a $(\mathfrak{g}, K)$-module through L or R.

The following well-known lemma says that a duality of Peter-Weyl type holds for irreducible $(\mathfrak{g}, K)$ modules.

Lemma 1.1. *Let $\boldsymbol{X}$ be an irreducible $(\mathfrak{g}, K)$-module, and let f be in $C^\infty_K(G)$. Then the $(\mathfrak{g}, K)$-module $U(\mathfrak{g})^L f$ generated by f through L is isomorphic to $\boldsymbol{X}$ if and only if the* *corresponding $U(\mathfrak{g})^R f$ through R is isomorphic to the dual $(\mathfrak{g}, K)$-module $\boldsymbol{X}^*$ consisting of all K-finite linear forms on $\boldsymbol{X}$.*

For an irreducible $(\mathfrak{g}, K)$-module $\boldsymbol{X}$, we fix once and for all an irreducible finite-dimensional representation (τ, V_τ) of K which occurs in $\boldsymbol{X}$, and fix an embedding $i_\tau : V_\tau \hookrightarrow \boldsymbol{X}$ as K-modules. Then the adjoint operator i_τ^* of i_τ gives a surjective K-homomorphism from $\boldsymbol{X}^*$ to V_τ^*, where (τ^*, V_τ^*) denotes the representation of K contragredient to τ.

We now consider the C^∞-induced representation $\mathrm{Ind}_K^G(\tau^*)$ acting on the space

$$(1.2) \qquad C^\infty_{\tau^*}(G) := \{\Phi : G \xrightarrow{C^\infty} V_\tau^* \mid \Phi(kg) = \tau^*(k)\Phi(g)\ (g \in G,\ k \in K)\},$$

endowed with G- and $U(\mathfrak{g})$-module structures through right translation R. Equip $C^\infty_{\tau^*}(G)$ with a Fréchet space topology of compact uniform convergence of functions on G and each of their derivatives. Then the G-action on $C^\infty_{\tau^*}(G)$ is smooth. By the Frobenius reciprocity, there corresponds (to i_τ^*) a unique $(\mathfrak{g}, K)$-embedding A_{τ^*} from $\boldsymbol{X}^*$ into $C^\infty_{\tau^*}(G)$ through

$$(1.3) \qquad A_{\tau^*}(\varphi)(g) = \tilde{i}_\tau^*(\pi^*(g)\varphi) \quad (g \in G;\ \varphi \in \boldsymbol{X}^*).$$

Here $\tilde{i}_\tau^*$ denotes the unique continuous extension of $i_\tau^* : \boldsymbol{X}^* \to V_\tau^*$ to any irreducible admissible G-module $\boldsymbol{H}^*$ with K-finite part $\boldsymbol{X}^*$.

Let $\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G))$ be the space of $(\mathfrak{g}, K)$-homomorphisms from $\boldsymbol{X}$ into $C^\infty(G)$ (under the action L). The right action R on $C^\infty(G)$ naturally gives a G-module structure on this space of $(\mathfrak{g}, K)$-homomorphisms. For each element W in $\mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G))$, one can define $F \in C^\infty_{\tau^*}(G)$ by

$$(1.4) \qquad \langle F(g), v\rangle = ((W \circ i_\tau)(v))(g) \quad (g \in G,\ v \in V_\tau).$$

Here $\langle\ \cdot\ ,\ \cdot\ \rangle$ stands for the dual pairing on $V_\tau^* \times V_\tau$. Then it is easily seen that the assignment $W \mapsto F$ sets up a G-embedding

$$(1.5) \qquad \mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G)) \hookrightarrow C^\infty_{\tau^*}(G).$$

Lemma 1.1 together with our argument in [25, I, §2] allows us to prove the following kernel theorem.

Theorem 1.2. *Under the above notation, if $\mathcal{D}$ is any continuous G-homomorphism from $C^\infty_{\tau^*}(G)$ to a smooth Fréchet G-module $\boldsymbol{M}$ such that*

$$(1.6) \qquad A_{\tau^*}(\boldsymbol{X}^*) = \{F \in C^\infty_{\tau^*}(G) \mid F \text{ is right } K\text{-finite and } \mathcal{D}F = 0\},$$

then the full kernel space $\mathrm{Ker}\,\mathcal{D}$ *of $\mathcal{D}$ in $C^\infty_{\tau^*}(G)$ coincides with the image of the G-embedding* (1.5). *Hence one gets*

$$(1.7) \qquad \mathrm{Hom}_{\mathfrak{g},K}(\boldsymbol{X},\ C^\infty(G)) \simeq \mathrm{Ker}\,\mathcal{D} \quad \text{as } G\text{-modules.}$$

This claim can be deduced also from the work of Kashiwara and Schmid (cf. [10] and [19]) on the maximal globalization of Harish-Chandra modules, by noting that $\mathrm{Ker}\,\mathcal{D}$ gives the maximal globalization of the irreducible $(\mathfrak{g}, K)$-module $\boldsymbol{X}^*$.

Example 1.3. We mention that an operator $\mathcal{D}$ satisfying the requirement in Theorem 1.2 has been constructed when $\boldsymbol{X}^*$ is the $(\mathfrak{g}, K)$-module associated with: (a) discrete series ([18], [9]) and more generally Zuckerman cohomologically induced module ([22], [1]), with parameter "far from the walls", or (b) highest weight representation ([2], [4]; see also Theorem 2.5). In each of these cases, $\mathcal{D}$ is given as a G-invariant differential operator of gradient type acting on $C^\infty_{\tau^*}(G)$, where τ^* is the unique extreme K-type of $\boldsymbol{X}^*$.

We will apply the above kernel theorem later in order to describe the generalized Whittaker models for irreducible admissible highest weight representations.

2. Differential operators of gradient type

From now on, let us assume that G is of Hermitian type as in Introduction. We consider the irreducible highest weight $(\mathfrak{g}, K)$-modules $L(\tau)$ with extreme K-types τ. In this section we construct, following [4], the differential operators $\mathcal{D}_{\tau^*}$ of gradient type on $K\backslash G$ whose K-finite kernels realize the dual lowest weight $(\mathfrak{g}, K)$-modules $L(\tau)^*$ (Theorem 2.5).

2.1. **Generalized Verma modules.** First, we fix some notation concerning simple Lie algebras of Hermitian type (cf. [24, Part I, §5] and [8, 3.3]). Take the complexification $G_{\mathbf{C}}$ of G, and the analytic subgroup $K_{\mathbf{C}}$ of $G_{\mathbf{C}}$ with Lie algebra $\mathfrak{k} = \mathfrak{k}_0 \otimes_{\mathbf{R}} \mathbb{C}$. Then there exists a unique (up to sign) central element Z_0 of $\mathfrak{k}_0$ such that $\operatorname{ad} Z_0$ restricted to $\mathfrak{p}_0$ gives an $\operatorname{Ad}(K)$-invariant complex structure on $\mathfrak{p}_0$. One gets a triangular decomposition of $\mathfrak{g}$ as follows:

$$(2.1)\qquad \begin{aligned} &\mathfrak{g} = \mathfrak{p}_- \oplus \mathfrak{k} \oplus \mathfrak{p}_+ \quad \text{such that} \\ &[\mathfrak{k}, \mathfrak{p}_\pm] \subset \mathfrak{p}_\pm, \quad [\mathfrak{p}_+, \mathfrak{p}_-] \subset \mathfrak{k}, \quad [\mathfrak{p}_+, \mathfrak{p}_+] = [\mathfrak{p}_-, \mathfrak{p}_-] = \{0\}, \end{aligned}$$

where $\mathfrak{p}_\pm$ denotes the eigenspace of $\operatorname{ad} Z_0$ on $\mathfrak{g}$ with eigenvalue $\pm\sqrt{-1}$ respectively.

Let $\mathfrak{t}_0$ be a compact Cartan subalgebra of $\mathfrak{g}_0$ contained in $\mathfrak{k}_0$. We write Δ for the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$. For each $\gamma \in \Delta$, the corresponding root subspace of $\mathfrak{g}$ will be denoted by $\mathfrak{g}(\mathfrak{t}; \gamma)$. We choose root vectors $X_\gamma \in \mathfrak{g}(\mathfrak{t}; \gamma)$ $(\gamma \in \Delta)$ such that

$$(2.2)\qquad X_\gamma - X_{-\gamma},\ \sqrt{-1}(X_\gamma + X_{-\gamma}) \in \mathfrak{k}_0 + \sqrt{-1}\mathfrak{p}_0, \quad [X_\gamma, X_{-\gamma}] = H_\gamma,$$

where H_γ is the element of $\sqrt{-1}\mathfrak{t}_0$ corresponding the coroot $\gamma^\vee := 2\gamma/(\gamma, \gamma)$ through the identification $\mathfrak{t}^* = \mathfrak{t}$ by the Killing form B of $\mathfrak{g}$. Let Δ_c (resp. Δ_n) denote the subset of all compact (resp. noncompact) roots in Δ.

Take a positive system Δ^+ of Δ compatible with the decomposition (2.1):

$$(2.3)\qquad \mathfrak{p}_\pm = \bigoplus_{\gamma \in \Delta_n^+} \mathfrak{g}(\mathfrak{t}; \pm\gamma) \quad \text{with} \quad \Delta_n^+ := \Delta^+ \cap \Delta_n,$$

and fix a lexicographic order on $\sqrt{-1}\mathfrak{t}_0^*$ which yields Δ^+. Using this order we define a fundamental sequence $(\gamma_1, \gamma_2, \ldots, \gamma_r)$ of strongly orthogonal (i.e., $\gamma_i \pm \gamma_j \notin \Delta \cup \{0\}$ for $i \neq j$) noncompact positive roots in such a way that γ_k is the maximal element of Δ^+, which is strongly orthogonal to $\gamma_{k+1}, \ldots, \gamma_r$. Then r is equal to the real rank of G.

Let (τ, V_τ) be any irreducible finite-dimensional representation of K with Δ_c^+-highest weight $\lambda = \lambda(\tau)$. We consider the *generalized Verma* $U(\mathfrak{g})$*-module* induced from τ:

$$(2.4)\qquad M(\tau) := U(\mathfrak{g}) \otimes_{U(\mathfrak{k}+\mathfrak{p}_+)} V_\tau.$$

Here τ is extended to a representation of the maximal parabolic subalgebra $\mathfrak{k} + \mathfrak{p}_+$ by the null $\mathfrak{p}_+$-action on V_τ. $M(\tau)$ admits a natural $(\mathfrak{g}, K)$-module structure. Let $N(\tau)$ be the unique maximal proper $(\mathfrak{g}, K)$-submodule of $M(\tau)$. Then the quotient $L(\tau) := M(\tau)/N(\tau)$ gives an irreducible $(\mathfrak{g}, K)$-module with Δ^+-highest weight λ.

Note that $M(\tau) = U(\mathfrak{p}_-)V_\tau$ is canonically isomorphic to the tensor product $S(\mathfrak{p}_-) \otimes V_\tau = S(\mathfrak{p}_-) \otimes_{\mathbf{C}} V_\tau$ as a K-module, where $S(\mathfrak{p}_-)$ ($\simeq U(\mathfrak{p}_-)$, since $\mathfrak{p}_-$ is abelian) denotes the symmetric algebra of $\mathfrak{p}_-$ looked upon as a K-module by the adjoint action. This isomorphism yields a gradation of the K-module $M(\tau)$:

$$(2.5)\qquad M(\tau) = \bigoplus_{j=0}^{\infty} M_j(\tau) \quad \text{with} \quad M_j(\tau) := S^j(\mathfrak{p}_-)V_\tau \simeq S^j(\mathfrak{p}_-) \otimes V_\tau.$$

Here we write $S^j(\mathfrak{p}_-)$ for the K-submodule of $S(\mathfrak{p}_-)$ consisting of all homogeneous elements of $S(\mathfrak{p}_-)$ of degree j. Observe that the submodule $N(\tau)$ is graded:

$$N(\tau) = \bigoplus_{j=0}^{\infty} N_j(\tau) \quad \text{with} \quad N_j(\tau) := N(\tau) \cap M_j(\tau). \tag{2.6}$$

Since $M(\tau) = S(\mathfrak{p}_-)V_\tau$ is finitely generated over the Noetherian ring $S(\mathfrak{p}_-)$, so is the submodule $N(\tau)$, too. This implies that, if $N(\tau) \neq \{0\}$, there exist finitely many irreducible K-submodules $W_1, \ldots, W_q$ of $N(\tau)$ such that

$$N(\tau) = \sum_{u=1}^{q} S(\mathfrak{p}_-)W_u \quad \text{with} \quad W_u \subset S^{i_u}(\mathfrak{p}_-)V_\tau \simeq S^{i_u}(\mathfrak{p}_-) \otimes V_\tau \tag{2.7}$$

for some positive integers i_u ($u = 1, \ldots, q$) arranged as

$$i(\tau) := i_1 \leq i_2 \leq \cdots \leq i_q. \tag{2.8}$$

We call $i(\tau)$ *the level of reduction* of $M(\tau)$.

For unitarizable $L(\tau)$'s, Joseph [5] gives a simple description of the maximal submodule $N(\tau)$ as follows. Assume that $L(\tau)$ is unitarizable and that $N(\tau) \neq \{0\}$. Then the level $i(\tau)$ of reduction of $M(\tau)$ turns to be an integer such that $1 \leq i(\tau) \leq r$, where r is the real rank of G. Let $Q_{i(\tau)}$ be the irreducible K-submodule of $S^{i(\tau)}(\mathfrak{p}_-)$ with lowest weight $-\gamma_r - \ldots - \gamma_{r-i(\tau)+1}$. Then the tensor product $Q_{i(\tau)} \otimes V_\tau$ has a unique irreducible K-submodule W_1, called the PRV(Parthasarathy, Rao and Varadarajan)-component, with extreme weight $\lambda - \gamma_r - \ldots - \gamma_{r-i(\tau)+1}$. We regard W_1 as a K-submodule of $M_{i(\tau)}(\tau)$.

Theorem 2.1 ([5, 5.2, 6.5 and 8.3], see also [3, 3.1]). *Under the above assumption and notation, the maximal submodule $N(\tau)$ of $M(\tau)$ is a highest weight $(\mathfrak{g}, K)$-module generated over $S(\mathfrak{p}_-)$ by the PRV-component W_1.*

2.2. **A realization of the dual lowest weight module $L(\tau)^*$.** For each irreducible representation (τ, V_τ) of K, let $L(\tau)^*$ be the irreducible lowest weight $(\mathfrak{g}, K)$-module which is dual to $L(\tau)$. Since $L(\tau)^*$ contains the extreme K-type (τ^*, V_τ^*) with multiplicity one, there exists a unique (up to constant multiple) $(\mathfrak{g}, K)$-embedding A_{τ^*} form $L(\tau)^*$ into $C^\infty_{\tau^*}(G)$. We are going to introduce a differential operator of gradient type whose K-finite kernel coincides with the image $A_{\tau^*}(L(\tau)^*)$.

For this, we take a basis $X_1, \ldots, X_s$ of the $\mathbb{C}$-vector space $\mathfrak{p}_+$ such that $B(X_j, \overline{X}_k) = \delta_{jk}$ (Kronecker's δ), where $\overline{X}_i \in \mathfrak{p}_-$ denotes the complex conjugate of $X_i \in \mathfrak{p}_+$ with respect to the real form $\mathfrak{g}_0$. Set

$$X^\alpha := X_1^{\alpha_1} \cdots X_s^{\alpha_s} \in U(\mathfrak{p}_+) \quad \text{and} \quad \overline{X}^\alpha := \overline{X}_1^{\alpha_1} \cdots \overline{X}_s^{\alpha_s} \in U(\mathfrak{p}_-) \tag{2.9}$$

for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_s)$ of nonnegative integers $\alpha_1, \ldots, \alpha_s$. We call $|\alpha| := \alpha_1 + \cdots + \alpha_s$ the length of α. For each positive integer n we define the gradients ∇^n and $\overline{\nabla}^n$ of order n on $C^\infty_{\tau^*}(G)$ as follows.

$$\nabla^n F(x) := \sum_{|\alpha|=n} \overline{X}^\alpha \otimes (X^\alpha)^L F(x), \tag{2.10}$$

$$\overline{\nabla}^n F(x) := \sum_{|\alpha|=n} X^\alpha \otimes (\overline{X}^\alpha)^L F(x), \tag{2.11}$$

for $x \in G$ and $F \in C^\infty_{\tau^*}(G)$. It is then easy to see that $\nabla^n F$ and $\overline{\nabla}^n F$ are independent of the choice of a basis $X_1, \ldots, X_s$, and that the operators ∇^n and $\overline{\nabla}^n$ give continuous G-homomorphisms

$$\nabla^n : C^\infty_{\tau^*}(G) \to C^\infty_{\tau^*(-n)}(G), \quad \overline{\nabla}^n : C^\infty_{\tau^*}(G) \to C^\infty_{\tau^*(+n)}(G). \tag{2.12}$$

Here $\tau^*(\pm n)$ denotes the K-representation on the tensor product $S^n(\mathfrak{p}_\pm) \otimes V^*_\tau$ respectively.

Let W_u ($u = 1, \ldots, q$) be, as in (2.7), the irreducible K-submodules of $S^{i_u}(\mathfrak{p}_-)V_\tau \subset N(\tau)$ which generate $N(\tau)$ over $S(\mathfrak{p}_-)$ when $N(\tau) \neq \{0\}$. For each u, the adjoint operator P_u of the embedding

$$W_u \hookrightarrow S^{i_u}(\mathfrak{p}_-)V_\tau \simeq S^{i_u}(\mathfrak{p}_-) \otimes V_\tau \tag{2.13}$$

gives a surjective K-homomorphism:

$$P_u : S^{i_u}(\mathfrak{p}_+) \otimes V^*_\tau \simeq (S^{i_u}(\mathfrak{p}_-) \otimes V_\tau)^* \longrightarrow W^*_u, \tag{2.14}$$

where $\mathfrak{p}_+$ is identified with the dual space of $\mathfrak{p}_-$ through the Killing form B, which is nondegenerate on $\mathfrak{p}_+ \times \mathfrak{p}_-$.

Definition 2.2. Keep the above notation.

(1) Let $\mathcal{D}_{\tau^*}$ be a continuous G-homomorphism from $C^\infty_{\tau^*}(G)$ to $C^\infty_\rho(G)$ defined by

$$\mathcal{D}_{\tau^*}F(x) := \nabla^1 F(x) \oplus (\oplus_{u=1}^q P_u(\overline{\nabla}^{i_u} F(x))) \tag{2.15}$$

for $x \in G$ and $F \in C^\infty_{\tau^*}(G)$. Here we write $\rho = \rho(\tau^*)$ for the representation of K on

$$(\mathfrak{p}_- \otimes V^*_\tau) \oplus (\oplus_{u=1}^q W^*_u), \tag{2.16}$$

and $\mathcal{D}_{\tau^*}$ should be understood as $\mathcal{D}_{\tau^*} = \nabla^1$ if $N(\tau) = \{0\}$, or equivalently $M(\tau) = L(\tau)$. We call $\mathcal{D}_{\tau^*}$ the *differential operator of gradient type* associated to τ^*.

(2) Put for $X \in \mathfrak{p}_+$ and $v^* \in V^*_\tau$,

$$\sigma(X, v^*) := \sum_{u=1}^q P_u(X^{i_u} \otimes v^*) \in W^* := \oplus_{u=1}^q W^*_u. \tag{2.17}$$

We call σ the *principal symbol* of $\mathcal{D}_{\tau^*}$ at the origin. Here σ should be understood as $\sigma(X, v^*) = 0$ for every $X \in \mathfrak{p}_+$ and every $v^* \in V^*_\tau$, when $\mathcal{D}_{\tau^*} = \nabla^1$.

Remark 2.3. A function $F \in C^\infty_{\tau^*}(G)$ gives an anti-holomorphic section of the vector bundle on $K\backslash G$ associated to τ^* if and only if $\nabla^1 F = 0$. Hence the elements of $\operatorname{Ker}\mathcal{D}_{\tau^*}$ are necessarily anti-holomorphic. The converse is true when $N(\tau) = \{0\}$.

Remark 2.4. If $L(\tau)$ is unitarizable, one sees from Theorem 2.1 that

$$\mathcal{D}_{\tau^*} = \nabla^1 \oplus (P_1 \circ \overline{\nabla}^{i(\tau)}). \tag{2.18}$$

Here $i(\tau)$ is the level of reduction of $M(\tau)$, and the K-homomorphism P_1 is defined through the PRV-component $W_1 \subset S^{i(\tau)}(\mathfrak{p}_-) \otimes V_\tau$.

The following theorem, equivalent to [4, Prop.7.6] due to Davidson and Stanke, realizes the lowest weight module $L(\tau)^*$ by means of $\mathcal{D}_{\tau^*}$.

Theorem 2.5. *The image of the $(\mathfrak{g}, K)$-embedding A_{τ^*} from $L(\tau)^*$ into $C^\infty_{\tau^*}(G)$ coincides with the K-finite kernel of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type.*

3. Associated variety and principal symbol

This section concerns the relationship between the associated variety (with multiplicity) of $L(\tau)$ and the principal symbol σ of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type. The result is summarized as Theorem 3.3.

For every integer m such that $0 \le m \le r = \mathbb{R}\text{-rank}\, G$, we set

$$\mathcal{O}_m := \operatorname{Ad}(K_{\mathbb{C}})X(m) \quad \text{with} \quad X(m) := \sum_{k=r-m+1}^{r} X_{\gamma_k} \ \text{(see (2.2))}. \tag{3.1}$$

where $X(0)$ should be understood as 0. The following proposition is well-known.

Proposition 3.1. *The subspace $\mathfrak{p}_+$ splits into a disjoint union of $r+1$ number of $K_{\mathbb{C}}$-orbits $\mathcal{O}_m$ $(0 \le m \le r)$: $\mathfrak{p}_+ = \coprod_{0 \le m \le r} \mathcal{O}_m$, and the closure $\overline{\mathcal{O}_m}$ of each orbit $\mathcal{O}_m$ is equal to $\cup_{k \le m} \mathcal{O}_k$ for every m.*

Let $L(\tau)$ be the irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type (τ, V_τ). The annihilator $\operatorname{Ann}_{S(\mathfrak{p}_-)} L(\tau)$ of $L(\tau)$ in $S(\mathfrak{p}_-) = U(\mathfrak{p}_-)$ defines an affine algebraic variety

$$\mathcal{V}(L(\tau)) := \{X \in \mathfrak{p}_+ \mid D(X) = 0 \quad \text{for all } D \in \operatorname{Ann}_{S(\mathfrak{p}_-)} L(\tau)\} \subset \mathfrak{p}_+, \tag{3.2}$$

which is called the *associated variety* of the $(\mathfrak{g}, K)$-module $L(\tau)$. Here $S(\mathfrak{p}_-)$ is identified with the ring of polynomial functions on $\mathfrak{p}_+$ through the Killing form B of $\mathfrak{g}$. By noting that the ideal $\operatorname{Ann}_{S(\mathfrak{p}_-)} L(\tau)$ is stable under $\operatorname{Ad}(K_{\mathbb{C}})$, we see from Proposition 3.1 that there exists a unique integer $m(\tau)$ $(0 \le m(\tau) \le r)$ such that

$$\mathcal{V}(L(\tau)) = \overline{\mathcal{O}_{m(\tau)}}. \tag{3.3}$$

In particular, the variety $\mathcal{V}(L(\tau))$ is irreducible.

Now let I_m be the prime ideal of $S(\mathfrak{p}_-)$ that defines the irreducible variety $\overline{\mathcal{O}_m}$ $(0 \le m \le r)$. If M is a finitely generated $S(\mathfrak{p}_-)$-module, the *multiplicity* $\operatorname{mult}_{I_m}(M)$ of M at I_m is defined to be the length of the localization M_{I_m} as an $S(\mathfrak{p}_-)_{I_m}$-module. The associated variety $\mathcal{V}(L(\tau))$ with the multiplicity $\operatorname{mult}_{I_{m(\tau)}}(L(\tau))$ is called the *associated cycle* of $L(\tau)$.

For each $X \in \mathfrak{p}_+$, let $\mathfrak{m}(X)$ be the maximal ideal of $S(\mathfrak{p}_-)$ which defines the variety $\{X\}$ of a single element X. We set

$$\mathcal{W}(X, \tau) := L(\tau)/\mathfrak{m}(X)L(\tau). \tag{3.4}$$

Then we see that $\dim \mathcal{W}(X, \tau) < \infty$, and that the isotropy group $K_{\mathbb{C}}(X)$ of $K_{\mathbb{C}}$ at X acts on $\mathcal{W}(X, \tau)$ naturally. Let σ be the principal symbol of $\mathcal{D}_{\tau^*}$ as in Definition 2.2. The map $v^* \mapsto \sigma(X, v^*)$ gives a $K_{\mathbb{C}}(X)$-homomorphism $\sigma(X, \cdot\,)$ from V_τ^* to W^*. Hence $\operatorname{Ker} \sigma(X, \cdot\,)$ is a $K_{\mathbb{C}}(X)$-submodule of V_τ^*.

The following lemma relates the above kernel of σ with the $K_{\mathbb{C}}(X)$-module $\mathcal{W}(X, \tau)$.

Lemma 3.2. *For each $X \in \mathfrak{p}_+$, the natural map*

$$V_\tau \hookrightarrow M(\tau) \to L(\tau) = M(\tau)/N(\tau) \to \mathcal{W}(X, \tau) = L(\tau)/\mathfrak{m}(X)L(\tau) \tag{3.5}$$

from V_τ onto $\mathcal{W}(X, \tau)$ induces a $K_{\mathbb{C}}(X)$-isomorphism

$$\mathcal{W}(X, \tau)^* \simeq \operatorname{Ker} \sigma(X, \cdot\,) \subset V_\tau^* \tag{3.6}$$

through the contravariant functor $\operatorname{Hom}_{\mathbb{C}}(\,\cdot\,, \mathbb{C})$.

By applying the argument of Vogan in [21, Section 2] in view of Lemma 3.2, we can deduce the following theorem.

Theorem 3.3. *Let $L(\tau)$ be any irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type τ, and let $\sigma : \mathfrak{p}_+ \times V_\tau^* \to W^*$ be the principal symbol of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type associated to τ^*. Then it holds that*

$$\mathcal{V}(L(\tau)) = \{X \in \mathfrak{p}_+ \mid \operatorname{Ker}\sigma(X, \cdot) \neq \{0\}\}. \tag{3.7}$$

Moreover, if X is an element of the unique open $K_{\mathbf{C}}$-orbit $\mathcal{O}_{m(\tau)}$ of $\mathcal{V}(L(\tau))$, the dimension of $\operatorname{Ker}\sigma(X, \cdot)$ is equal to the multiplicity of $S(\mathfrak{p}_-)$-module $L(\tau)/I_{m(\tau)}L(\tau)$ at the prime ideal $I_{m(\tau)}$.

As for the unitarizable highest weight modules $L(\tau)$, some results of Joseph [15, Lem.2.4 and Th.5.6] (due to Davidson, Enright and Stanke [3] for $\mathfrak{g}$ classical) assure that the prime ideal $I_{m(\tau)}$ annihilates $L(\tau)$. Thus we obtain

Corollary 3.4. *One has* $\operatorname{mult}_{I_{m(\tau)}}(L(\tau)) = \dim \mathcal{W}(X, \tau)$ $(X \in \mathcal{O}_{m(\tau)})$ *for every irreducible unitarizable highest weight module $L(\tau)$.*

Remark 3.5. We can get the same kind of characterization of the associated cycle also for irreducible $(\mathfrak{g}, K)$-modules of discrete series, by using the results of [9] and [26]. We will discuss it elsewhere.

Remark 3.6. For classical groups $Sp(2n, \mathbb{R})$, $U(p, q)$ and $O^*(2p)$, Nishiyama, Ochiai and Taniguchi [17, Th.7.18 and Th.9.1] have described the associated cycle and the Bernstein degree of unitarizable highest weight module $L(\tau)$ by using the theory of reductive dual pairs (G, G') with compact G'. They deal with the case where the dual pair (G, G') is in the stable range with smaller G', through detailed study of K-types of $L(\tau)$. On the other hand, the above corollary gives another simple method for describing the multiplicity $\operatorname{mult}_{I_{m(\tau)}}(L(\tau))$ by means of the $K_{\mathbf{C}}(X)$-module $\mathcal{W}(X, \tau)$ (cf. 5.2).

4. Cayley transform and generalized Gelfand-Graev representations

In this section, we introduce the generalized Gelfand-Graev representations of G attached to the Cayley transforms of nilpotent $K_{\mathbf{C}}$-orbits $\mathcal{O}_m = \operatorname{Ad}(K_{\mathbf{C}})X(m)$ $(m = 0, \ldots, r)$ in $\mathfrak{p}_+$.

For this, we consider an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$:

$$X(m) = \sum_{k=r-m+1}^{r} X_{\gamma_k}, \quad H(m) := \sum_{k=r-m+1}^{r} H_{\gamma_k}, \quad Y(m) := \sum_{k=r-m+1}^{r} X_{-\gamma_k}, \tag{4.1}$$

and the *Cayley transform* $\mathbf{c} = \operatorname{Ad}(c)$ on $\mathfrak{g}$ defined by the element

$$c := \exp\left(\frac{\pi}{4} \cdot \sum_{k=1}^{r} (X_{\gamma_k} - X_{-\gamma_k})\right) \in G_{\mathbf{C}}. \tag{4.2}$$

We put

$$\begin{cases} X'(m) := -\sqrt{-1}\mathbf{c}^{-1}(X(m)) = \dfrac{\sqrt{-1}}{2}(H(m) - X(m) + Y(m)), \\ H'(m) := \mathbf{c}^{-1}(H(m)) = X(m) + Y(m), \\ Y'(m) := \sqrt{-1}\mathbf{c}^{-1}(Y(m)) = -\dfrac{\sqrt{-1}}{2}(H(m) + X(m) - Y(m)). \end{cases} \tag{4.3}$$

Then $(X'(m), H'(m), Y'(m))$ forms an $\mathfrak{sl}_2$-triple in the real form $\mathfrak{g}_0$ of $\mathfrak{g}$. Set $\mathcal{O}'_m := \operatorname{Ad}(G)X'(m)$. We note that the nilpotent G-orbit $\mathcal{O}'_m$ in $\mathfrak{g}_0$ corresponds to the $K_{\mathbf{C}}$-orbit $\mathcal{O}_m$ in $\mathfrak{p}_+ \subset \mathfrak{p}$ through the Kostant-Sekiguchi correspondence (cf. [8, Th.3.1]).

Now, let η_m be the one-dimensional representation (i.e., character) of abelian Lie subalgebra $\mathfrak{n}(m) := \mathbf{c}([\mathfrak{k}, Y(m)])$ defined by

$$\eta_m(U) := -\sqrt{-1}B(U, Y'(m)) = -B(\mathbf{c}^{-1}U, X(m)) \quad \text{for} \quad U \in \mathfrak{n}(m). \tag{4.4}$$

Then, we can form a C^∞-induced G- and $(\mathfrak{g}, K)$-representation Γ_m acting on the space

$$C^\infty(G; \eta_m) := \{f \in C^\infty(G) \mid U^R f = -\eta_m(U) f \quad (U \in \mathfrak{n}(m))\} \tag{4.5}$$

by left translation L. Note that

$$C^\infty(G; \eta_r) \subset C^\infty(G; \eta_{r-1}) \subset \cdots \subset C^\infty(G; \eta_0) = C^\infty(G), \tag{4.6}$$

since one sees $\mathfrak{n}(m) \subset \mathfrak{n}(m')$ and $\eta_{m'}|\mathfrak{n}(m) = \eta_m$ for $m \leq m'$.

Definition 4.1. We call $(\Gamma_m, C^\infty(G; \eta_m))$ the *generalized Gelfand-Graev representation* (GGGR for short) of G attached to the nilpotent G-orbit $\mathcal{O}'_m = \mathrm{Ad}(G)X'(m)$ in $\mathfrak{g}_0$.

Remark 4.2. The GGGRs attached to arbitrary nilpotent orbits have been constructed in full generality by Kawanaka [12] for reductive algebraic groups. See also [23] for the GGGRs of real semisimple Lie groups.

In order to describe the generalized Whittaker models for $L(\tau)$, we need the bounded and unbounded realizations of Hermitian symmetric space $K\backslash G$. To be more precise, let $P_\pm := \exp \mathfrak{p}_\pm$ be the connected Lie subgroups of $G_{\mathbf{C}}$ with Lie algebras $\mathfrak{p}_\pm$, respectively. Note that the exponential map gives holomorphic diffeomorphisms from $\mathfrak{p}_\pm$ onto $P_\pm$. Consider an open dense subset $P_+K_{\mathbf{C}}P_-$ of $G_{\mathbf{C}}$, which is holomorphically diffeomorphic to the direct product $P_+ \times K_{\mathbf{C}} \times P_-$ through multiplication. For each $x \in P_+K_{\mathbf{C}}P_-$, let $p_+(x)$, $k_{\mathbf{C}}(x)$, and $p_-(x)$ denote respectively the elements of P_+, $K_{\mathbf{C}}$, and P_- such that $x = p_+(x)k_{\mathbf{C}}(x)p_-(x)$. Set $\xi(x) := \log p_-(x) \in \mathfrak{p}_-$.

Proposition 4.3 (cf. [13, Chapter VII]). (1) *One has $Gc \cup G \subset P_+K_{\mathbf{C}}P_-$, where c is the Cayley element of $G_{\mathbf{C}}$ in* (4.2).

(2) *The assignment $x \mapsto \xi(x)$ $(x \in G)$ sets up an anti-holomorphic diffeomorphism from $K\backslash G$ onto a bounded domain $\{\xi(x) \mid x \in G\}$ in $\mathfrak{p}_-$.*

(3) *Similarly, $x \mapsto \xi(xc)$ $(x \in G)$ induces an anti-holomorphic diffeomorphism from $K\backslash G$ onto an unbounded domain $\{\xi(xc) \mid x \in G\}$ in $\mathfrak{p}_-$.*

5. Generalized Whittaker models

For any irreducible finite-dimensional K-module (τ, V_τ), let $L(\tau) = M(\tau)/N(\tau)$ (see 2.1) be the irreducible highest weight $(\mathfrak{g}, K)$-module with extreme K-type τ. Consider the GGGRs $(\Gamma_m, C^\infty(G; \eta_m))$ $(m = 0, \ldots, r)$ induced from the characters $\eta_m : \mathfrak{n}(m) \to \mathbb{C}$. We say that $L(\tau)$ has a *generalized Whittaker model* of type η_m if $L(\tau)$ is isomorphic to a $(\mathfrak{g}, K)$-submodule of $C^\infty(G; \eta_m)$. In this section, we give an answer to the problem posed in Introduction.

5.1. **Main results.** We are going to describe the generalized Whittaker models for $L(\tau)$ by specifying the vector space of $(\mathfrak{g}, K)$-homomorphisms from $L(\tau)$ into $C^\infty(G;\, \eta_m)$. To do this, let $\mathcal{D}_{\tau^*} : C^\infty_{\tau^*}(G) \to C^\infty_\rho(G)$ be, as in Definition 2.2, the G-invariant differential operator of gradient type whose kernel realizes the maximal globarization of lowest weight module $L(\tau)^*$. We set

$$\mathcal{Y}(\tau, m) := \{F \in C^\infty_{\tau^*}(G) \mid \mathcal{D}_{\tau^*}F = 0, \;\; U^R F = -\eta_m(U) F \;(U \in \mathfrak{n}(m))\}. \tag{5.1}$$

Then the kernel theorem (Theorem 1.2) gives a linear isomorphism

$$\mathrm{Hom}_{\mathfrak{g},K}(L(\tau), C^\infty(G;\eta_m)) \simeq \mathcal{Y}(\tau, m) \tag{5.2}$$

through the correspondence (1.4). Thus our task amounts to specifying the space $\mathcal{Y}(\tau, m)$ for each τ and m.

Let $\mathcal{O}_{m(\tau)}$ be the unique open $K_{\mathbb{C}}$-orbit in the associated variety $\mathcal{V}(L(\tau))$ of $L(\tau)$. Among the generalized Whittaker models for $L(\tau)$, those of type $\eta_{m(\tau)}$ are most important. We obtain the following result on the corresponding linear space $\mathcal{Y}(\tau, m)$ with $m = m(\tau)$.

Theorem 5.1. *Let (τ, V_τ) be an irreducible finite-dimensional representation of K. Set $m = m(\tau)$ and $\mathcal{Y}(\tau) := \mathcal{Y}(\tau, m)$ for short. Then,*

(1) $\mathcal{Y}(\tau)$ is a nonzero, finite-dimensional vector space.

(2) For any $F \in \mathcal{Y}(\tau)$, there exists a unique polynomial function φ on $\mathfrak{p}_-$ with values in V_τ^* such that

$$F(x) = \exp B(X(m), \xi(xc))\tau^*(k_{\mathbb{C}}(xc))\varphi(\xi(xc)) \quad (x \in G). \tag{5.3}$$

(3) Let $\sigma : \mathfrak{p}_+ \times V_\tau^* \to W^*$ be the principal symbol of the differential operator $\mathcal{D}_{\tau^*}$ of gradient type, defined by (2.17). For $v^* \in V_\tau^*$, we write F_{v^*} for the function in (5.3) corresponding to the constant polynomial $\varphi : \mathfrak{p}_- \ni Z \mapsto v^* \in V_\tau^*$. Then the assignment $v^* \longmapsto \chi_\tau(v^*) := F_{v^*}$ $(v^* \in \mathrm{Ker}\,\sigma(X(m), \cdot\,))$ yields an injective linear map

$$\chi_\tau : \mathrm{Ker}\,\sigma(X(m), \cdot\,) \hookrightarrow \mathcal{Y}(\tau). \tag{5.4}$$

(4) Assume that $L(\tau)$ is unitarizable. Then the linear embedding χ_τ in (3) is surjective. Hence one gets

$$\mathrm{Hom}_{\mathfrak{g},K}(L(\tau), C^\infty(G;\eta_m)) \simeq \mathcal{Y}(\tau) \simeq \mathrm{Ker}\,\sigma(X(m), \cdot\,) \simeq \mathcal{W}(X(m), \tau) \tag{5.5}$$

as vector spaces, where $\mathcal{W}(X(m), \tau) = L(\tau)/\mathfrak{m}(X(m))L(\tau)$ is as in (3.4). Moreover, the dimension of the vector spaces in (5.5) equals the multiplicity $\mathrm{mult}_{I_m}(L(\tau))$ of the $S(\mathfrak{p}_-)$-module $L(\tau)$ at the unique associated prime I_m, by Corollary 3.4.

As for $\mathcal{Y}(\tau, m')$ with $m' \neq m(\tau)$, we can deduce the following

Theorem 5.2. *The linear space $\mathcal{Y}(\tau, m')$ vanishes (resp. is infinite-dimensional) if $m' > m(\tau)$ (resp. $m' < m(\tau)$).*

These two theorems are the main results of this note.

Remark 5.3. (1) Theorem 5.1 (4) recovers, to a great extent, our earlier work [24, Part II] on the generalized Whittaker models for the holomorphic discrete series.

(2) The vanishing of $\mathcal{Y}(\tau, m')$ $(m' > m(\tau))$ in Theorem 5.2 follows also from a general result of Matumoto [16, Th.1].

5.2. **The second dual pair method: case of $SO^*(2n)$.** Let G be the group $SO^*(2n)$ consisting of all matrices in $SL(2n, \mathbb{C})$ satisfying

$$g\begin{pmatrix} I_n & O \\ O & -I_n \end{pmatrix} {}^t\overline{g} = \begin{pmatrix} I_n & O \\ O & -I_n \end{pmatrix} \quad \text{and} \quad {}^tg\begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix} g = \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix},$$

where I_n denotes the identity matrix of size n. The totality of unitary matrices in G forms a maximal compact subgroup K. In this subsection, we describe the space $\mathcal{W}(X(m), \tau)$ in (5.5) by using the oscillator representation of the pair (G, G') with $G' = Sp(k)$.

5.2.1. First, we note that, under a natural identification, $K_{\mathbb{C}} = GL(n, \mathbb{C})$ acts on the space $\mathfrak{p}_+ = \mathrm{Alt}_n$ of all complex alternating matrices of size n by

$$g \cdot X = gX{}^tg, \quad g \in GL(n, \mathbb{C}), \ X \in \mathrm{Alt}_n. \tag{5.6}$$

For every positive integer k, we realize the compact group $G' = Sp(k)$ as

$$G' = \{g \in U(2k) \mid {}^tgJ_kg = J_k\} \quad \text{with } J_k = \begin{pmatrix} O & I_k \\ -I_k & O \end{pmatrix}. \tag{5.7}$$

The group $K_{\mathbb{C}} \times G'_{\mathbb{C}}$ acts on the vector space $M := M_{n,2k}$ by

$$(g, g') \cdot Z := gZg'^{-1}, \quad (g, g') \in K_{\mathbb{C}} \times G'_{\mathbb{C}}, \ Z \in M, \tag{5.8}$$

where $G'_{\mathbb{C}} = Sp(k, \mathbb{C})$ is the complexification of G', and $M_{p,q}$ denotes the space of all complex matrices of size $p \times q$.

We set $\psi(Z) := \frac{1}{2}ZJ_k{}^tZ$ for $Z \in M$. Note that $\psi : M \to \mathfrak{p}_+$ is a $K_{\mathbb{C}} \times G'_{\mathbb{C}}$-equivariant polynomial map of degree two, where the $G'_{\mathbb{C}}$-action on $\mathfrak{p}_+$ is trivial. For each $Y \in \mathfrak{p}_-$, let h_Y be a polynomial on M defined by

$$h_Y(Z) := B(\psi(Z), Y) \quad (B \text{ the Killing form of } \mathfrak{g}). \tag{5.9}$$

Let $\mathbb{C}[M]$ denote the ring of polynomial functions on the complex vector space M. One can define a $(\mathfrak{g}, K)$-representation ω on $\mathbb{C}[M]$ in the following fashion. First, the $\mathfrak{p}_-$ action on $\mathbb{C}[M]$ is given by multiplication:

$$\omega(Y)f(Z) := h_Y(Z)f(Z), \quad Y \in \mathfrak{p}_-, \tag{5.10}$$

for $f \in \mathbb{C}[M]$. Second, $\mathfrak{p}_+$ acts by differentiation:

$$\omega(X)f(Z) := \kappa(h_{\overline{X}}(\partial)f)(Z), \quad X \in \mathfrak{p}_+. \tag{5.11}$$

Here $h_{\overline{X}}(\partial)$ stands for the constant coefficient differential operator on M defined by the polynomial $h_{\overline{X}}$, and the constant κ depends only on the Lie algebra $\mathfrak{g}_0$ of G. Third, the complexification $K_{\mathbb{C}}$ acts on $\mathbb{C}[M]$ holomorphically as

$$\omega(g)f(Z) := (\det g)^{-k} f((g^{-1}, e) \cdot Z), \quad g \in K_{\mathbb{C}}. \tag{5.12}$$

On the other hand, $\mathbb{C}[M]$ has a natural $G'_{\mathbb{C}}$-module structure through

$$R(g')f(Z) := f((e, g'^{-1}) \cdot Z), \quad g' \in G'_{\mathbb{C}}. \tag{5.13}$$

Then it is easily seen that these two representations ω and R commute with each other. The resulting $(\mathfrak{g}, K) \times G'_{\mathbb{C}}$-representation (ω, R) on $\mathbb{C}[M]$ will be called the Fock model of the (infinitesimal) *oscillator representation* of the pair (G, G') (cf. [3, §7]).

5.2.2. Let (σ, V_σ) be an irreducible finite-dimensional representation of the compact group G'. Extend σ to a holomorphic representation of $G'_{\mathbb{C}}$ in the canonical way. We set

$$L[\sigma] := \mathrm{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{C}[M]), \tag{5.14}$$

which turns to be a $(\mathfrak{g}, K)$-module through the representation ω on $\mathbb{C}[M]$. Let $\Sigma(k)$ denote the totality of equivalence classes of irreducible finite-dimensional representations σ of G' such that $L[\sigma] \neq \{0\}$. Then one gets a natural isomorphism

$$\mathbb{C}[M] \simeq \bigoplus_{\sigma \in \Sigma(k)} L[\sigma] \otimes V_\sigma \quad \text{as } (\mathfrak{g}, K) \times G'_{\mathbb{C}}\text{-modules}. \tag{5.15}$$

The following theorem states the theta correspondence associated to (G, G').

Theorem 5.4 ([11], [6], [7]; cf. [3]). (1) *$L[\sigma]$ is an irreducible unitarizable highest weight $(\mathfrak{g}, K)$-module for every $\sigma \in \Sigma(k)$. In particular,* (5.15) *gives the irreducible decomposition of the $(\mathfrak{g}, K) \times G'_{\mathbb{C}}$-module $\mathbb{C}[M]$.*

(2) *Let $\sigma_1, \sigma_2 \in \Sigma(k)$. Then, $V_{\sigma_1} \simeq V_{\sigma_2}$ as $G'_{\mathbb{C}}$-modules if and only if $L[\sigma_1] \simeq L[\sigma_2]$ as $(\mathfrak{g}, K)$-modules.*

Let $\tau[\sigma]$ denote the extreme K-type of highest weight $(\mathfrak{g}, K)$-module $L[\sigma]$, i.e., $L[\sigma] = L(\tau[\sigma])$. We note that the correspondence $\sigma \leftrightarrow \tau[\sigma]$ can be explicitly described in terms of their highest weights. For this, see the articles cited in the above theorem.

For each $m = 0, \ldots, r = [n/2]$, the $K_{\mathbb{C}}$-orbit $\mathcal{O}_m$ in $\mathfrak{p}_+$ consists of all the matrices in $\mathfrak{p}_+ = \mathrm{Alt}_n$ of rank $2m$. Let $E_{s,t}(i,j)$ denote the (i,j)-matrix unit of size $s \times t$ whose (k,l)-matrix entry e_{kl} is equal to 1 if $(k,l) = (i,j)$; $e_{kl} = 0$ otherwise. We take an element $X(m) \in \mathcal{O}_m$ explicitly as

$$X(m) := \sum_{i=1}^{m} (E_{n,n}(i, m+i) - E_{n,n}(m+i, i))/2. \tag{5.16}$$

It is easily verified that the image $\psi(M)$ of the $K_{\mathbb{C}} \times G'_{\mathbb{C}}$-equivariant map $\psi : M \to \mathfrak{p}_+$ is a $K_{\mathbb{C}}$-stable, irreducible algebraic variety described as

$$\psi(M) = \overline{\mathcal{O}_{m_k}} \quad \text{with} \quad m_k := \min(k, r), \tag{5.17}$$

where M and ψ depend on k. By (5.10) and (5.15), we find that, for any $\sigma \in \Sigma(k)$, the associated variety of $L[\sigma]$ is equal to the closure of the $K_{\mathbb{C}}$-orbit $\mathcal{O}_{m_k} = \mathrm{Ad}(K_{\mathbb{C}})X(m_k)$.

5.2.3. We consider the maximal ideal:

$$\mathfrak{m} := \mathfrak{m}(X(m_k)) = \sum_{Y \in \mathfrak{p}_-} (Y - B(X(m_k), Y)) S(\mathfrak{p}_-) \subset S(\mathfrak{p}_-) \quad \text{(cf. (3.4))}, \tag{5.18}$$

for each positive integer k. For $m = 0, \ldots, r$, let $K_{\mathbb{C}}(m) := K_{\mathbb{C}}(X(m))$ be the isotropy subgroup of $K_{\mathbb{C}}$ at $X(m) \in \mathcal{O}_m$. We want to describe the $K_{\mathbb{C}}(m_k)$-modules

$$\mathcal{W}[\sigma] := \mathcal{W}(X(m_k), \tau[\sigma]) = L[\sigma]/\mathfrak{m}L[\sigma] \simeq \mathrm{Hom}_{G'_{\mathbb{C}}}(V_\sigma, \mathbb{C}[M]/\omega(\mathfrak{m})\mathbb{C}[M]). \tag{5.19}$$

Namely, our task is to decompose the quotient $K_{\mathbb{C}}(m_k) \times G'_{\mathbb{C}}$-module $\mathbb{C}[M]/\omega(\mathfrak{m})\mathbb{C}[M]$.

To do this, we note that $\omega(\mathfrak{m})\mathbb{C}[M]$ is equal to the ideal of $\mathbb{C}[M]$ generated by all matrix entries of the following polynomial function of degree two:

$$M \ni Z \longmapsto \psi(Z) - X(m_k) \in \mathfrak{p}_+. \tag{5.20}$$

We write $\mathcal{V}_k$ for the corresponding affine algebraic variety of M:

$$\mathcal{V}_k := \{Z \in M \mid \psi(Z) = X(m_k)\} = \psi^{-1}(X(m_k)). \tag{5.21}$$

Clearly, $\mathcal{V}_k$ is stable under the action of $K_{\mathbb{C}}(m_k) \times G'_{\mathbb{C}}$.

We define a subgroup $G'_{\mathbb{C}}(k-r)$ of $G'_{\mathbb{C}}$ by

$$G'_{\mathbb{C}}(k-r) := \begin{cases} \{I_{2k}\} \text{ (the unit group)} & \text{if } k \le r, \\ \left\{ \begin{pmatrix} I_k & O & O & O \\ O & h_{11} & O & h_{12} \\ O & O & I_k & O \\ O & h_{21} & O & h_{22} \end{pmatrix} \in G'_{\mathbb{C}} \;\middle|\; h_{ij} \in M_{k-r,k-r} \right\} & \text{if } k > r. \end{cases} \tag{5.22}$$

Note that if $k > r$, the group $G'_{\mathbb{C}}(k-r)$ is naturally isomorphic to $Sp(k-r, \mathbb{C})$.

Lemma 5.5. (1) *If $k \le r$, one has*

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot I_{n,2k}(2k) \simeq G'_{\mathbb{C}} \quad \text{as } G'_{\mathbb{C}}\text{-sets,} \tag{5.23}$$

where $I_{s,t}(l) := \sum_{i=1}^{l} E_{s,t}(i,i) \in M_{s,t} \quad (l = 0, \dots, \min(s,t))$.

(2) *If $k > r = n/2$ with even integer n, the variety $\mathcal{V}_k$ is described as*

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot \begin{pmatrix} I_{r,k}(r) & O \\ O & I_{r,k}(r) \end{pmatrix} \simeq G'_{\mathbb{C}}/G'_{\mathbb{C}}(k-r), \tag{5.24}$$

where $G'_{\mathbb{C}}(k-r) \simeq Sp(k-r, \mathbb{C})$ (cf. (5.22)) coincides with the isotropy subgroup of $G'_{\mathbb{C}}$ at the matrix $\begin{pmatrix} I_{r,k}(r) & O \\ O & I_{r,k}(r) \end{pmatrix}$ in $M = M_{2r,2k}$.

(3) *If $k > r = (n-1)/2$ with odd integer n, $\mathcal{V}_k$ consists of two $G'_{\mathbb{C}}$-orbits. In fact, we set*

$$(z_1, z_2)^{\tilde{}} := \begin{pmatrix} I_r & O & O & O \\ O & O & I_r & O \\ o & z_1 & o & z_2 \end{pmatrix} \quad \text{for} \quad (z_1, z_2) \in M_{1,2(k-r)} = M_{1,k-r} \times M_{1,k-r}. \tag{5.25}$$

Then $\mathcal{V}_k$ decomposes as

$$\mathcal{V}_k = G'_{\mathbb{C}} \cdot \tilde{M}_{1,2(k-r)} = G'_{\mathbb{C}} \cdot (0 \dots 0, 0 \dots 0)^{\tilde{}} \coprod G'_{\mathbb{C}} \cdot (1\ 0 \dots 0, 0 \dots 0)^{\tilde{}}, \tag{5.26}$$

where $\tilde{M}_{1,2(k-r)} := \{(z_1, z_2)^{\tilde{}} \mid z_1, z_2 \in M_{1,k-r}\}$.

The above lemma implies in particular that the affine variety $\mathcal{V}_k$ is irreducible. This allows us to deduce the following proposition by applying [14, Lemma 4].

Proposition 5.6. *The ideal $\omega(\mathfrak{m})\mathbb{C}[M]$ of $\mathbb{C}[M]$ coincides with the defining ideal of $\mathcal{V}_k$ in $\mathbb{C}[M]$. Hence one gets a natural isomorphism*

$$\mathbb{C}[M]/\omega(\mathfrak{m})\mathbb{C}[M] \simeq \mathbb{C}[\mathcal{V}_k] \quad \text{as } K_{\mathbb{C}}(\mathfrak{m}_k) \times G'_{\mathbb{C}}\text{-modules,} \tag{5.27}$$

where $\mathbb{C}[\mathcal{V}_k]$ denotes the affine coordinate ring of $\mathcal{V}_k$.

5.2.4. We are now in a position to specify the $K_{\mathbb{C}}(\mathfrak{m}_k)$-modules $\mathcal{W}[\sigma]$ for every $\sigma \in \Sigma(k)$ $(k = 1, 2, \dots)$. Let us introduce a $G'_{\mathbb{C}}(k-r)$-stable subvariety $\mathcal{U}_k$ of $\mathcal{V}_k$ as

$$\mathcal{U}_k := \begin{cases} \{I_{n,2k}(2k)\} & (k \le r = [n/2]) \\ \left\{ \begin{pmatrix} I_{r,k}(r) & O \\ O & I_{r,k}(r) \end{pmatrix} \right\} & (k > r = n/2 \text{ with } n \text{ even}), \\ \tilde{M}_{1,2(k-r)} & (k > r = (n-1)/2 \text{ with } n \text{ odd}). \end{cases} \tag{5.28}$$

Then it follows from Lemma 5.5 that $\mathcal{V}_k = G'_{\mathbb{C}} \cdot \mathcal{U}_k$, and that the $G'_{\mathbb{C}}$-orbits $\mathcal{X}$ in $\mathcal{V}_k$ are in one-one correspondence with the $G'_{\mathbb{C}}(k-r)$-orbits $\mathcal{X} \cap \mathcal{U}_k$ in $\mathcal{U}_k$.

Now Proposition 5.6 together with (5.19) allows us to deduce the following

Proposition 5.7. *Under the above notation, let $\mathbb{C}[\mathcal{U}_k]$ be the coordinate ring of $G'_{\mathbb{C}}(k-r)$-stable variety $\mathcal{U}_k$ viewed as a $G'_{\mathbb{C}}(k-r)$-module in the canonical way. Then one has a linear isomorphism*

$$\mathcal{W}[\sigma] \simeq \mathrm{Hom}_{G'_{\mathbb{C}}(k-r)}(V_\sigma, \mathbb{C}[\mathcal{U}_k]) \simeq (V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k])^{G'_{\mathbb{C}}(k-r)} \quad (\sigma \in \Sigma(k)). \tag{5.29}$$

In particular, it holds that

$$\mathcal{W}[\sigma] \simeq \begin{cases} (V_\sigma^*)^{G'_{\mathbb{C}}(k-r)} & \text{if } n \text{ is even and } k > r, \\ V_\sigma^* & \text{if } k \le r. \end{cases} \tag{5.30}$$

Here $(V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k])^{G'_{\mathbb{C}}(k-r)}$ *denotes the subspace of* $V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k]$ *of* $G'_{\mathbb{C}}(k-r)$*-fixed vectors.*

Remark 5.8. For the case $k > r$ with odd n, $\mathbb{C}[\mathcal{U}_k]$ decomposes into a direct sum of the irreducible representations $V(l)$ $(l = 0, 1, \dots)$ of $G'_{\mathbb{C}}(k-r) = Sp(k-r, \mathbb{C})$ with highest weights $(l, 0, \dots, 0)$: $\mathbb{C}[\mathcal{U}_k] \simeq \oplus_{l \geq 0} V(l)$.

At the end, we are going to clarify how the isotropy subgroup $K_{\mathbb{C}}(m_k)$ acts on the space $\mathcal{W}[\sigma] \simeq \mathrm{Hom}_{G'_{\mathbb{C}}(k-r)}(V_\sigma, \mathbb{C}[\mathcal{U}_k])$. To do this, we note that the elements g of the subgroup $K_{\mathbb{C}}(m)$ $(0 \leq m \leq r)$ of $K_{\mathbb{C}}$ are written as follows.

$$g = \begin{pmatrix} g_{11} & g_{12} \\ O & g_{22} \end{pmatrix} \in K_{\mathbb{C}} = GL(n, \mathbb{C}) \quad \text{with} \quad g_{11} \in Sp(m, \mathbb{C}). \tag{5.31}$$

Define a group homomorphism

$$\alpha : K_{\mathbb{C}}(m_k) \to G'_{\mathbb{C}}, \quad g \mapsto \alpha(g), \tag{5.32}$$

by putting

$$\alpha(g) := \begin{pmatrix} p_{11} & O & p_{12} & O \\ O & I_{k-r} & O & O \\ p_{21} & O & p_{22} & O \\ O & O & O & I_{k-r} \end{pmatrix} \quad \text{with} \quad g_{11} = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}. \tag{5.33}$$

Here p_{ij} is a matrix of size k, and $\alpha(g)$ should be understood as g_{11} if $k \leq r$. Note that the elements of $\alpha(K_{\mathbb{C}}(m_k))$ commute with those of the subgroup $G'_{\mathbb{C}}(k-r)$.

Now we can deduce

Theorem 5.9. *If n is even or $k \leq r$, it holds that*

$$\mathcal{W}[\sigma] \simeq (\det(\,\cdot\,)^{-k} \otimes (\sigma^* \circ \alpha), \quad (V_\sigma^*)^{G'_{\mathbb{C}}(k-r)}) \quad \textit{as } K_{\mathbb{C}}(m_k)\textit{-modules.} \tag{5.34}$$

In particular, $\mathcal{W}[\sigma]$ is an irreducible $K_{\mathbb{C}}(m_k)$-module if $k \leq r$.

Next we consider the remaining case: $k > r$ with odd n. Then, $\beta(g) := g_{22}$ $(g \in K_{\mathbb{C}}(r))$ defines a group homomorphism β from $K_{\mathbb{C}}(r)$ to $GL(1, \mathbb{C}) = \mathbb{C}^\times$. The group $K_{\mathbb{C}}(r)$ acts on $\mathbb{C}[\mathcal{U}_k] \simeq \mathbb{C}[M_{1,2(k-r)}]$ naturally through the left multiplication composed with β. We denote by ν the resulting representation of $K_{\mathbb{C}}(r)$ on $\mathbb{C}[\mathcal{U}_k]$. Note that ν as well as $\sigma^* \circ \alpha$ commutes with the $G'_{\mathbb{C}}(k-r)$-action.

Theorem 5.10. *If $k > r$ with odd n, the reductive part of $K_{\mathbb{C}}(r)$ acts on $\mathcal{W}[\sigma] \simeq (V_\sigma^* \otimes \mathbb{C}[\mathcal{U}_k])^{G'_{\mathbb{C}}(k-r)}$ by the representation* $\det(\,\cdot\,)^{-k} \otimes (\sigma^* \circ \alpha) \otimes \nu$.

Similar descriptions of $\mathcal{W}[\sigma]$ can be obtained for the groups $G = SU(p,q)$ and $Sp(n, \mathbb{R})$ also. For this we refer to [20] and [27, Section 5].

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