ON THE PRODUCT OF RIESZ SETS IN DUAL OBJECTS OF COMPACT GROUPS

AUTHOR(S):

YAMAGUCHI, HIROSHI

ON THE PRODUCT OF RIESZ SETS IN DUAL OBJECTS OF COMPACT GROUPS

HIROSHI YAMAGUCHI

ABSTRACT. Let E_i be a Riesz set in the dual object of a compact group $K_i (i = 1, 2)$. We show that the product set $E_1 \times E_2$ is a Riesz set in the dual object of $K_1 \times K_2$. We also give a result on compact groups related to a result of Glicksberg and Graham concerned with "small p set".

1. INTRODUCTION

Let $\mathbb{T}$ and $\mathbb{Z}$ be the circle group and the integer group respectively. $\mathbb{Z}^+$ denotes the semigroup of nonnegative integers. By a well-known theorem of Bochner, each measure on $\mathbb{T}^2$ whose Fourier-Stieltjes transform vanishes off $\mathbb{Z}^+ \times \mathbb{Z}^+$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{T}^2$. This shows that the product set $\mathbb{Z}^+ \times \mathbb{Z}^+$ of the Riesz set $\mathbb{Z}^+$ in $\mathbb{Z}$ is a Riesz set in $\hat{\mathbb{T}}^2 \cong \mathbb{Z} \times \mathbb{Z}$. This holds for locally compact abelian (LCA) groups. For a LCA group G, let $L^1(G)$ and $M(G)$ be the usual group algebra and the Banach algebra of bounded regular measures on G respectively. For $\mu \in M(G)$, $\hat{\mu}$ stands for the Fourier-Stieltjes transform of μ. Let m_G denote the Haar measure of G.

Definition 1.1. *Let G be a LCA group with the dual group $\hat{G}$, and let $p \in \mathbb{N}$ (the natural numbers). A closed subset E of $\hat{G}$ is called a small p set if*

$$\forall \mu \in M_E(G) \Longrightarrow \mu^p = \overbrace{\mu * \cdots * \mu}^{p} \in L^1(G), \tag{1.1}$$

where $M_E(G) = \{\mu \in M(G) : \hat{\mu} = 0 \, on \, E^c\}$. In particular, a small 1 set is called a Riesz set.

Theorem 1.1 (cf. [12, Corollary], [10, Theorem 6]). *Let G_1 and G_2 be LCA groups, and let $p \in \mathbb{N}$. Let E_1 and E_2 be small p sets in $\hat{G}_1$ and $\hat{G}_2$ respectively. Then $E_1 \times E_2$ is a small p set in $\widehat{G_1 \oplus G_2}$.*

A condition for a set in the dual group of a LCA group to be a small 2 set was obtained by Glicksberg([6]) and Graham([7]).

Theorem 1.2 (cf. [7, Therem 1(b)]). *Let G be a LCA group, and let E be a closed set in $\hat{G}$ satisfying the following:*

$$\{\gamma \in \hat{G} : m_{\hat{G}}(E \cap (\gamma - E)) < \infty\} \; is \, dense \, in \; \hat{G}. \tag{1.2}$$

Let $\mu, \nu \in M_E(G)$. Then $|\mu| * |\nu| \in L^1(G)$. In particular, E is a small 2 set.

On the other hand, the author proved that the product set of a Riesz set in the dual group of a compact abelian group and a Riesz set in the dual object of a compact group is a Riesz set ([16, Corollary 2.1]). In this paper, we shall show that results corresponding to Theorems 1.1 and 1.2 hold for (noncommutative) compact groups. In section 2, we state notation and our results. In section 3, we give the proofs of our results.

2. Notation and results

We often quote notation from the book of Hewitt and Ross ([9]). Let K be a compact group, and let Σ_K be the dual object of K, i.e., the set of equivalence classes of all continuous irreducible unitary representations of K. For a closed normal subgroup H of K, $A(\Sigma_K, H)$ denotes the annihilator of H in Σ_K (cf. [9, (28.7) Definition]). m_K stands for the Haar measure of K. Let $C(K)$ be the space of continuous functions on K and $M(K)$ the space of bounded regular measures on K. Let $L^1(K)$ be the group algebra. We identify $L^1(K)$ with the space of absolutely continuous measures in $M(K)$, by the Radon-Nikodym theorem. Set $M^+(K) = \{\mu \in M(K) : \mu \geq 0\}$. For $\mu \in M(K)$ and $f \in L^1(|\mu|)$, we often write $\mu(f)$ as $\int_K f(x)d\mu(x)$.

For $\sigma \in \Sigma_K$, $U^{(\sigma)}$ denotes a continuous irreducible unitary representation of K in σ with the representation space H_σ of dimension d_σ. For $\mu \in M(K)$, $\hat{\mu}$ denotes the Fourier transform of μ, i.e., for $\sigma \in \Sigma_K$ and $\xi, \eta \in H_\sigma$,

$$\langle \hat{\mu}(\sigma)\xi, \eta\rangle = \int_K \langle \overline{U}_x^{(\sigma)}\xi, \eta\rangle d\mu(x), \tag{2.1}$$

where $\overline{U}_x^{(\sigma)} = D_\sigma U_x^{(\sigma)} D_\sigma$ and D_σ is a conjugation on H_σ. Let $spec(\mu) = \{\sigma \in \Sigma_K : \hat{\mu}(\sigma) \neq 0\}$. Let $\overline{\sigma}$ denote the equivalence class in Σ_K that contains the representation $\overline{U}^{(\sigma)}$. For a subset E of Σ_K, set $M_E(K) = \{\mu \in M(K) : spec(\mu) \subset E\}$.

For $\sigma, \tau \in \Sigma_K, \sigma \times \tau$ is defined (cf. [9, (27.35) Definition]). $\sigma \times \tau$ is a finite subset of Σ_K. For a subset P of Σ_K, $[P]$ denotes the smallest subset of Σ_K that contains P and is closed under the operation '$\times$' and conjugation (cf. [9, (27.35) Definition]).

For $\sigma \in \Sigma_K, \mathfrak{T}_\sigma(K)$ is the linear span of all functions $x \to \langle U_x^{(\sigma)}\xi, \eta\rangle$, where $\xi, \eta \in H_\sigma$. Let $\mathfrak{T}(K)$ be the space of trigonometric polynomials on K, i.e., $\mathfrak{T}(K)$ is the set of finite linear combinations of functions $x \to \langle U_x^{(\sigma)}\xi, \eta\rangle$, where $\sigma \in \Sigma_K$ and $\xi, \eta \in H_\sigma$.

Let $\{\xi_1^{(\sigma)}, \cdots, \xi_{d_\sigma}^{(\sigma)}\}$ be a fixed orthonormal basis in H_σ, and let $u_{ij}^{(\sigma)}(1 \leq i, j \leq d_\sigma)$ be the coordinate function for $U^{(\sigma)} \in \sigma$ and $\{\xi_1^{(\sigma)}, \cdots, \xi_{d_\sigma}^{(\sigma)}\}$, i.e., $u_{ij}^{(\sigma)}(x) = \langle U_x^{(\sigma)}\xi_j^{(\sigma)}, \xi_i^{(\sigma)}\rangle$.

Definition 2.1. *Let p be a natural number and E a subset of Σ_K. E is called an s-small p set if*

$$\forall \mu_1, \cdots, \mu_p \in M_E(K) \Rightarrow \mu_1 * \cdots * \mu_p \in L^1(K). \tag{2.2}$$

In paticular, an s-small 1 set is called a Riesz set.

Remark 2.1. *When K is a compact abelian group, "s-small p set" and "small p set" are same notion (cf. [13, Lemma 1]).*

Theorem 2.1. *Let $p \in \mathbb{N}$, and let K_1 and K_2 be compact groups. Let E_1 and E_2 be s-small p sets in Σ_{K_1} and Σ_{K_2} respectively. Then $E_1 \times E_2$ is an s-small p set in $\Sigma_{K_1 \times K_2} \cong \Sigma_{K_1} \times \Sigma_{K_2}$.*

By the above theorem, we obtain the following corollary.

Corollary 2.1. *Let E_1 and E_2 be Riesz sets in Σ_{K_1} and Σ_{K_2} respectively. Then $E_1 \times E_2$ is a Riesz set in $\Sigma_{K_1 \times K_2} \cong \Sigma_{K_1} \times \Sigma_{K_2}$.*

Next we consider Theorem 1.2 for compact groups. When G is a compact abelian group, the condition (1.2) in Theorem 1.2 is equivalent to the following:

(1.2)' For any $\gamma_1, \gamma_2 \in \hat{G}$, $(\gamma_1 + S) \cap (\gamma_2 - S)$ is a finite set.

Theorem 2.2. *Let K be a compact group, and let Δ be a subset of Σ_K satisfying the following condition.*

$$\text{(2.3)} \qquad \text{For any } \sigma, \tau \in \Sigma_K, (\sigma \times \Delta) \cap (\tau \times \overline{\Delta}) \text{ is a finite set,}$$

where $\overline{\Delta} = \{\overline{\omega} : \omega \in \Delta\}$ and $\sigma \times \Delta = \{\sigma \times \eta : \eta \in \Delta\}$. Let $\mu, \nu \in M_\Delta(K)$. Then $|\mu| * |\nu| \in L^1(K)$. In particular, Δ is an s-small 2 set.

The following also holds (cf. [7, Theorem 2]).

Theorem 2.3. *Let K be a compact group, and let $p, q \in \mathbb{N}$. Let Δ be a subset of Σ_K satisfying the following condition.*

(2.3)' $(\sigma_1 \times \Delta) \cap \cdots \cap (\sigma_p \times \Delta) \cap (\tau_1 \times \overline{\Delta}) \cap \cdots \cap (\tau_q \times \overline{\Delta})$ *is a finite set for any* $\sigma_1, \cdots, \sigma_p, \tau_1, \cdots, \tau_q \in \Sigma_K$.

Let μ_i and ν_j be measures in $M_\Delta(K)$ $(i = 1, 2, \cdots, p; j = 1, 2, \cdots, q)$. Then $|\mu_1| * \cdots * |\mu_p| * |\nu_1| * \cdots * |\nu_q| \in L^1(K)$. In particular, Δ is an s-small $p + q$ set.

Example 2.1. *Let $K = \mathbb{T} \times SU(2)$, and let T^ℓ $(\ell = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots)$ be as in* [9, (29.13)]. *Then $\Sigma_K \cong \{\tau_{n,m} : n \in \mathbb{Z};\ m = 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$, where $\tau_{n,m}(e^{i\theta}, u) = e^{in\theta} T_u^{(m)}$. Let $\alpha > 0$, and set $\Delta = \{\tau_{n,m} \in \Sigma_K : n \geq 0,\ m \leq \alpha n\}$. Then, by* [9, (29.26)] *and the fact that $T^{(\ell)}$ are self-conjugate (cf.* [9, (29.25)]*), Δ satisfies the condition (2.3) in Theorem 2.2. (In fact, Δ is a Riesz set, by* [3, 3.4 *Example* (*a*)].)

We prove Theorem 2.2 in the next section. We can prove Theorem 2.3 by an argument similar to that in the proof of Theorem 2.2.

3. Proofs of Theorems

In this section, we prove Theorems 2.1 and 2.2. In order to prove Theorem 2.1, we use the theory of disintegration of measures.

Lemma 3.1. *Let K_1 and K_2 be compact groups, and let $p \in \mathbb{N}$. Let $\eta_n \in M^+(K_2)$, and let $\{\nu_h^{(n)}\}_{h \in K_2}$ be a family of measures in $M(K_1)$ with the following property $(n = 1, 2, \cdots, p)$:*

(1) $h \to (\nu_h^{(n)} \times \delta_h)(f)$ *is η_n-measurable for each $f \in C(K_1 \times K_2)$.*

Then

(2) $(h_1, \cdots, h_p) \to (\nu_{h_1}^{(1)} \times \delta_{h_1}) * \cdots * (\nu_{h_p}^{(p)} \times \delta_{h_p})(f)\ (= (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p}(f))$ *is $(\eta_1 \times \cdots \times \eta_p)$-measurable for each $f \in C(K_1 \times K_2)$.*

Proof. For $f_1, \cdots, f_p \in C(K_1 \times K_2)$, we define $f(z_1, \cdots, z_p) \in C((K_1 \times K_2)^p)$ by

$$f(z_1, \cdots, z_p) = f_1(z_1) \cdots f_p(z_p).$$

By (1),

(3) $(h_1, \cdots, h_p) \to (\nu_{h_1}^{(1)} \times \delta_{h_1}) \times \cdots \times (\nu_{h_p}^{(p)} \times \delta_{h_p})(f) = (\nu_{h_1}^{(1)} \times \delta_{h_1})(f_1) \cdots (\nu_{h_p}^{(p)} \times \delta_{h_p})(f_p)$ is $(\eta_1 \times \cdots \times \eta_p)$-measurable.

Since $\{\sum_{i=1}^n f_{1i}(z_1) \cdots f_{pi}(z_p) : f_{ji} \in C(K_1 \times K_2)\,(1 \le j \le p;\, n = 1, 2, \cdots)\}$ is dense in $C((K_1 \times K_2)^p)$, (3) implies that

(4) $$(h_1, \cdots, h_p) \to (\nu_{h_1}^{(1)} \times \delta_{h_1}) \times \cdots \times (\nu_{h_p}^{(p)} \times \delta_{h_p})(f) \text{ is } (\eta_1 \times \cdots \times \eta_p)\text{-measurable for each } f \in C((K_1 \times K_2)^p).$$

We define $\pi_p : (K_1 \times K_2)^p \to K_1 \times K_2$ by $\pi_p(z_1, \cdots, z_p) = z_1 \cdots z_p$. Then

$$(\nu_{h_1}^{(1)} \times \delta_{h_1}) * \cdots * (\nu_{h_p}^{(p)} \times \delta_{h_p})(g) = (\nu_{h_1}^{(1)} \times \delta_{h_1}) \times \cdots \times (\nu_{h_p}^{(p)} \times \delta_{h_p})(g \circ \pi_p)$$

for each $g \in C(K_1 \times K_2)$. Thus (2) follows from (4). □

Lemma 3.2. *Let K_1 and K_2 be metrizable compact groups, and let $p \in \mathbb{N}$. Let $\mu_n \in M(K_1 \times K_2)$, $\eta_n \in M^+(K_2)$, and let $\{\nu_h^{(n)}\}_{h \in K_2}$ be a family of measures in $M(K_1)$ with the following properties $(n = 1, 2, \cdots, p)$:*

(1) *$h \to (\nu_h^{(n)} \times \delta_h)(f)$ is η_n-measurable for each $f \in C(K_1 \times K_2)$,*

(2) *$\|\nu_h^{(n)}\| \le 1$, and*

(3) *$\mu_n(f) = \int_{K_2} (\nu_h^{(n)} \times \delta_h)(f) d\eta_n(h)$ for all $f \in C(K_1 \times K_2)$.*

Let ρ be a measure in $M(K_1 \times K_2)$ defined by

(4) $\rho(f) = \int_{K_2} \cdots \int_{K_2} (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p}(f) d\eta_1(h_1) \cdots d\eta_p(h_p)$
for $f \in C(K_1 \times K_2)$. Then $\rho = \mu_1 * \cdots * \mu_p$.

Proof. Let (σ_1, σ_2) be any element in $\Sigma_{K_1} \times \Sigma_{K_2}$. For any $\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}, \xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)} \in H_{\sigma_1} \otimes H_{\sigma_2}$, we have

$$\begin{aligned}
&\langle \hat{\rho}(\sigma_1, \sigma_2)(\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}), \xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)} \rangle \\
&= \int_{K_1 \times K_2} \langle \overline{U}_x^{(\sigma_1)} \otimes \overline{U}_y^{(\sigma_2)} (\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}), \xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)} \rangle d\rho(x, y) \\
&= \int_{K_2} \cdots \int_{K_2} (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p} (\langle \overline{U}_x^{(\sigma_1)} (\xi_i^{(\sigma_1)}), \xi_j^{(\sigma_1)} \rangle \\
&\qquad \times \langle \overline{U}_y^{(\sigma_2)} (\xi_k^{(\sigma_2)}), \xi_\ell^{(\sigma_2)} \rangle) d\eta_1(h_1) \cdots d\eta_p(h_p) \qquad (5) \\
&= \int_{K_2} \cdots \int_{K_2} \langle (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)})\hat{}\,(\sigma_1)(\xi_i^{(\sigma_1)}), \xi_j^{(\sigma_1)} \rangle \\
&\qquad \times \langle \overline{U}_{h_1 \cdots h_p}^{(\sigma_2)} (\xi_k^{(\sigma_2)}), \xi_\ell^{(\sigma_2)} \rangle d\eta_1(h_1) \cdots d\eta_p(h_p).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
&\langle(\mu_1 * \cdots * \mu_p)\hat{}\,(\sigma_1,\sigma_2)(\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}), \xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)}\rangle \\
&= \int_{K_1\times K_2} \langle \overline{U}_x^{(\sigma_1)} \otimes \overline{U}_y^{(\sigma_2)}(\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}), \xi_j^{(\sigma_2)} \otimes \xi_\ell^{(\sigma_2)}\rangle d\mu_1 * \cdots * \mu_p(x,y) \\
&= \int_{K_1\times K_2} \cdots \int_{K_1\times K_2} \langle \overline{U}_{x_1\cdots x_p}^{(\sigma_1)} \otimes \overline{U}_{y_1\cdots y_p}^{(\sigma_2)}(\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}), \xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)}\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad d\mu_1(x_1,y_1)\cdots d\mu_p(x_p,y_p) \\
&= \int_{K_1\times K_2} \cdots \int_{K_1\times K_2} \langle \overline{U}_{x_1\cdots x_p}^{(\sigma_1)}(\xi_i^{(\sigma_1)}), \xi_j^{(\sigma_1)}\rangle \langle \overline{U}_{y_1\cdots y_p}^{(\sigma_2)}(\xi_k^{(\sigma_2)}), \xi_\ell^{(\sigma_2)}\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad d\mu_1(x_1,y_1)\cdots d\mu_p(x_p,y_p) \\
&= \int_{K_1\times K_2} \cdots \int_{K_1\times K_2} \int_{K_2} (\nu_{h_1}^{(1)} \times \delta_{h_1})(\langle \overline{U}_{x_1\cdots x_p}^{(\sigma_1)}(\xi_i^{(\sigma)}), \xi_j^{(\sigma_1)}\rangle \langle \overline{U}_{y_1\cdots y_p}^{(\sigma_2)}(\xi_k^{(\sigma_2)}, \xi_\ell^{(\sigma_2)}\rangle) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad d\eta_1(h_1)d\mu_2(x_2,y_2)\cdots\mu_p(x_p,y_p)
\end{aligned}
$$

(6)

$$
\begin{aligned}
&= \int_{K_1\times K_2} \cdots \int_{K_1\times K_2} \int_{K_2} \int_{K_1} \langle \overline{U}_{x_1\cdots x_p}^{(\sigma_1)}(\xi_i^{(\sigma_1)}), \xi_j^{(\sigma_1)}\rangle d\nu_{h_1}^{(1)}(x_1) \\
&\qquad\qquad\qquad \times \langle \overline{U}_{h_1 y_2\cdots y_p}^{(\sigma_2)}(\xi_k^{(\sigma_2)}), \xi_\ell^{(\sigma_2)}\rangle d\eta_1(h_1)d\mu_2(x_2,y_2)\cdots d\mu_p(x_p,y_p) \\
&= \int_{K_1\times K_2} \cdots \int_{K_1\times K_2} \int_{K_2} \langle \hat{\nu}_{h_1}^{(1)}(\sigma_1)(\overline{U}_{x_2\cdots x_p}^{(\sigma_1)}(\xi_i^{(\sigma_1)})), \xi_j^{(\sigma_1)}\rangle \\
&\qquad\qquad\qquad \times \langle \overline{U}_{h_1 y_2\cdots y_p}^{(\sigma_2)}(\xi_k^{(\sigma_2)}), \xi_\ell^{(\sigma_2)}\rangle d\eta_1(h_1)d\mu_2(x_2,y_2)\cdots d\mu_p(x_p,y_p) \\
&= \int_{K_2} \int_{K_1\times K_2} \cdots \int_{K_1\times K_2} \langle \overline{U}_{x_2\cdots x_p}^{(\sigma_1)}(\xi_i^{(\sigma_1)}), \hat{\nu}_{h_1}^{(1)}(\sigma_1)^*(\xi_j^{(\sigma_1)})\rangle \\
&\qquad\qquad\qquad \times \langle \overline{U}_{y_2\cdots y_p}^{(\sigma_2)}(\xi_k^{(\sigma_2)}), \overline{U}_{h_1}^{(\sigma_2)*}(\xi_\ell^{(\sigma_2)})\rangle d\mu_2(x_2,y_2)\cdots\mu_p(x_p,y_p)d\eta_1(h_1)
\end{aligned}
$$

....

$$
\begin{aligned}
&= \overbrace{\int_{K_2} \cdots \int_{K_2}}^{r} \overbrace{\int_{K_1\times K_2} \cdots \int_{K_1\times K_2}}^{p-r} \langle \overline{U}_{x_{r+1}\cdots x_p}^{(\sigma_1)}(\xi_i^{(\sigma_1)}), (\nu_{h_1}^{(1)} * \cdots * \nu_{h_r}^{(r)})\hat{}\,(\sigma_1)^*(\xi_j^{(\sigma_1)})\rangle \\
&\times \langle \overline{U}_{y_{r+1}\cdots y_p}^{(\sigma_2)}(\xi_k^{(\sigma_2)}), \overline{U}_{h_1\cdots h_r}^{(\sigma_2)*}(\xi_\ell^{(\sigma_2)})\rangle d\mu_{r+1}(x_{r+1},y_{r+1})\cdots d\mu_p(x_p,y_p)d\eta_1(h_1)\cdots d\eta_r(h_r)
\end{aligned}
$$

....

$$= \int_{K_2} \cdots \int_{K_2} \langle \xi_i^{(\sigma_1)}, (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)})\hat{}(\sigma_1)^*(\xi_j^{(\sigma_1)})\rangle$$

$$\times \langle \xi_k^{(\sigma_2)}, \overline{U}_{h_1\cdots h_p}^{(\sigma_1)*}(\xi_\ell^{(\sigma_2)})\rangle d\eta_1(h_1)\cdots d\eta_p(h_p)$$

$$= \int_{K_2} \cdots \int_{K_2} \langle (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)})\hat{}(\sigma_1)(\xi_i^{(\sigma_1)}), \xi_j^{(\sigma_1)}\rangle$$

$$\times \langle \overline{U}_{h_1\cdots h_p}^{(\sigma_2)}(\xi_k^{(\sigma_2)}), \xi_\ell^{(\sigma_2)}\rangle d\eta_1(h_1)\cdots d\eta_p(h_p),$$

where $\hat{\nu}_{h_1}^{(1)}(\sigma_1)^*$ and $\overline{U}_{h_1\cdots h_r}^{(\sigma_2)*}$ are the adjoints of $\hat{\nu}_{h_1}^{(1)}(\sigma_1)$ and $\overline{U}_{h_1\cdots h_r}^{(\sigma_2)}$ respectively. By (5) and (6), we have

$$\langle \hat{\rho}(\sigma_1,\sigma_2)(\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}), \xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)}\rangle$$
$$= \langle (\mu_1 * \cdots * \mu_p)\hat{}(\sigma_1,\sigma_2)(\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}), \xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)}\rangle$$

for any $(\sigma_1,\sigma_2) \in \Sigma_{K_1} \times \Sigma_{K_2}$ and $\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}$, $\xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)} \in H_{\sigma_1} \otimes H_{\sigma_2}$. This yields $\rho = \mu_1 * \cdots * \mu_p$. □

Proposition 3.1. *Let K_1 and K_2 be metrizable compact groups, and let $p \in \mathbb{N}$. Let E_1 be an s-small p set in Σ_{K_1}, and let $\mu_1, \cdots, \mu_p \in M_{E_1\times\Sigma_{K_2}}(K_1 \times K_2)$. Then $\lim_{x\to e_1} \|\delta_{(x,e_2)} * \mu_1 * \cdots * \mu_p - \mu_1 * \cdots * \mu_p\| = 0$, where e_i is the unit element of K_i $(i = 1,2)$.*

Proof. Let $\pi : K_1 \times K_2 \to K_2$ be the projection, and let $\eta_n = \pi(|\mu_n|)$ $(n = 1,2,\cdots,p)$. Then, by the theory of disintegration of measures (cf. [1] or [14, Corollary 1.6]), there exists a family $\{\lambda_h^{(n)}\}_{h\in K_2}$ of measures in $M(K_1 \times K_2)$ with the following properties:

(1) $\qquad h \to \lambda_h^{(n)}(f)$ is η_n−measurable for each $f \in C(K_1 \times K_2)$,

(2) $\qquad \|\lambda_h^{(n)}\| \le 1$,

(3) $\qquad supp(\lambda_h^{(n)}) \subset \pi^{-1}(h)$, and

(4) $\qquad \mu_n(f) = \int_{K_2} \lambda_h^{(n)}(f)d\eta_n(h)$ for all $f \in C(K_1 \times K_2)$.

By (2) and (3), there exists a measure $\nu_h^{(n)} \in M(K_1)$, with $\|\nu_h^{(n)}\| \le 1$, such that

(5) $$\lambda_h^{(n)} = \nu_h^{(n)} \times \delta_h.$$

Let $\sigma_1 \notin E_1$. Let σ_2 be any element in Σ_{K_2}. For any $\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}$, $\xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)} \in H_{\sigma_1} \otimes H_{\sigma_2}$, we have

$$\begin{aligned}
0 = & \ \langle \hat{\mu}_n(\sigma_1, \sigma_2)(\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}), \xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)} \rangle \\
= & \int_{K_1 \times K_2} \langle \overline{U}_x^{(\sigma_1)} \otimes \overline{U}_y^{(\sigma_2)}(\xi_i^{(\sigma_1)} \otimes \xi_k^{(\sigma_2)}), \xi_j^{(\sigma_1)} \otimes \xi_\ell^{(\sigma_2)} \rangle d\mu_n(x,y) \\
= & \int_{K_1 \times K_2} \langle \overline{U}_x^{(\sigma_1)}(\xi_i^{(\sigma_1)}), \xi_j^{(\sigma_1)} \rangle \langle \overline{U}_y^{(\sigma_2)}(\xi_k^{(\sigma_2)}), \xi_\ell^{(\sigma_2)} \rangle d\mu_n(x,y) \\
= & \int_{K_2} \int_{K_1} \langle \overline{U}_x^{(\sigma_1)}(\xi_i^{(\sigma_1)}), \xi_j^{(\sigma_1)} \rangle d\nu_h^{(n)}(x) \\
& \qquad \times \langle \overline{U}_h^{(\sigma_2)}(\xi_k^{(\sigma_2)}), \xi_\ell^{(\sigma_2)} \rangle d\eta_n(h) \qquad \text{(by (4) and (5))} \\
= & \int_{K_2} \langle \hat{\nu}_h^{(n)}(\sigma_1)\xi_i^{(\sigma_1)}, \xi_j^{(\sigma_1)} \rangle \overline{u}_{\ell k}^{(\sigma_2)}(h) d\eta_n(h),
\end{aligned}$$

which yields

$$\int_{K_2} \langle \hat{\nu}_h^{(n)}(\sigma_1)\xi_i^{(\sigma_1)}, \xi_j^{(\sigma_1)} \rangle p(h) d\eta_n(h) = 0$$

for all $p \in \mathfrak{T}(K_2)$. Hence

$$\langle \hat{\nu}_h^{(n)}(\sigma_1)\xi_i^{(\sigma_1)}, \xi_j^{(\sigma_1)} \rangle \ = 0 \qquad \eta_n\text{–a.a. } h \in K_2 \quad (1 \leq \forall i, j \leq d_{\sigma_1}).$$

Thus

$$\hat{\nu}_h^{(n)}(\sigma_1) = 0 \qquad \eta_n\text{–a.a. } h \in K_2.$$

Since Σ_{K_1} is countable, we have

(6) $$\hat{\nu}_h^{(n)}(\sigma_1) = 0 \quad \text{for all } \sigma_1 \in \Sigma_{K_1} \setminus E_1 \quad \eta_n\text{–a.a. } h \in K_2.$$

Since E_1 is an s-small p set, we have

(7) $$\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)} \in L^1(K_1) \quad (\eta_1 \times \cdots \times \eta_p)\text{–a.a. } (h_1, \cdots, h_p) \in K_2^p.$$

It follows from Lemmas 3.1 and 3.2 that $(h_1, \cdots, h_p) \to (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p}(f)$ is $(\eta_1 \times \cdots \times \eta_p)$-measurable for each $f \in C(K_1 \times K_2)$ and

(8) $$\begin{aligned} & \mu_1 * \cdots * \mu_p(f) \\ & = \int_{K_2} \cdots \int_{K_2} (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p}(f) d\eta_1(h_1) \cdots d\eta_p(h_p) \end{aligned}$$

for all $f \in C(K_1 \times K_2)$. For $x \in K_1$, we note that $(h_1, \cdots, h_p) \to (\delta_x * \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p}(f)$ is $(\eta_1 \times \cdots \times \eta_p)$-measurable for each $f \in C(K_1 \times K_2)$. It follows from (8) that

(9) $$\begin{aligned} & \delta_{(x,e_2)} * \mu_1 * \cdots * \mu_p(f) \\ & = \int_{K_2} \cdots \int_{K_2} (\delta_x * \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p}(f) d\eta_1(h_1) \cdots d\eta_p(h_p) \end{aligned}$$

for all $f \in C(K_1 \times K_2)$. Let $\mathcal{A} = \{f_n\}$ be a countable dense set in $C(K_1 \times K_2)$. Since

$$\begin{aligned} &\|\delta_x * \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)} - \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}\| \\ &= \sup_{\substack{f_n \in \mathcal{A} \\ \|f_n\|_\infty \le 1}} |\{(\delta_x * \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p} - (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p}\}(f_n)|, \end{aligned}$$

we note that

$$(h_1, \cdots, h_p) \to \|\delta_x * \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)} - \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}\|$$

is $(\eta_1 \times \cdots \times \eta_p)$-measurable. Let $\{s_n\}$ be a sequence in K_1 such that $\lim_{n\to\infty} s_n = e_1$. Then, by (7),

$$\lim_{n\to\infty} \|\delta_{s_n} * \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)} - \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}\| = 0 \qquad (\eta_1 \times \cdots \times \eta_p)\text{–a.a. } (h_1, \cdots, h_p) \in K_2^p,$$

which, together with (8) and (9), yields

$$\begin{aligned} &\lim_{n\to\infty} \|\delta_{(s_n,e_2)} * \mu_1 * \cdots * \mu_p - \mu_1 * \cdots * \mu_p\| \\ &= \lim_{n\to\infty} \sup_{\substack{f \in \mathcal{A} \\ \|f\|_\infty \le 1}} |\ \delta_{(s_n,e_2)} * \mu_1 * \cdots * \mu_p(f) - \mu_1 * \cdots * \mu_p(f)\ | \\ &= \lim_{n\to\infty} \sup_{\substack{f \in \mathcal{A} \\ \|f\|_\infty \le 1}} |\int_{K_2} \cdots \int_{K_2} \{(\delta_{s_n} * \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p} - \\ &\qquad (\nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}) \times \delta_{h_1 \cdots h_p}\}(f) d\eta_1(h_1) \cdots d\eta_p(h_p)\ | \\ &\le \lim_{n\to\infty} \int_{K_2} \cdots \int_{K_2} \|\delta_{s_n} * \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)} - \nu_{h_1}^{(1)} * \cdots * \nu_{h_p}^{(p)}\| d\eta_1(h_1) \cdots d\eta_p(h_p) \\ &= 0. \qquad \text{(by the Lebesgue convergence theorem)} \end{aligned}$$

Since K_1 is metrizable, the proposition is obtained. □

Similarly we get the following proposition.

Proposition 3.2. *Let K_1 and K_2 be metrizable compact groups, and let $p \in \mathbb{N}$. Let E_2 be an s-small p set in Σ_{K_2}, and let $\mu_1, \cdots, \mu_p \in M_{\Sigma_{K_1} \times E_2}(K_1 \times K_2)$. Then $\lim_{y\to e_2} \|\mu_1 * \cdots * \mu_p - \delta_{(e_1,y)} * \mu_1 * \cdots * \mu_p\| = 0$.*

Proposition 3.3. *Let K_1 and K_2 be metrizable compact groups, and let $p \in \mathbb{N}$. Let E_1 and E_2 be s-small p sets in Σ_{K_1} and Σ_{K_2} respectively. Then $E_1 \times E_2$ is an s-small p set in $\Sigma_{K_1 \times K_2} \cong \Sigma_{K_1} \times \Sigma_{K_2}$.*

Proof. Let $\mu_n \in M_{E_1 \times E_2}(K_1 \times K_2)$ $(n = 1, 2, \cdots, p)$. It follows from Propositions 3.1 and 3.2 that

(1) $$\lim_{x\to e_1} \|\mu_1 * \cdots * \mu_p - \delta_{(x,e_2)} * \mu_1 * \cdots * \mu_p\| = 0, \quad \text{and}$$

(2) $$\lim_{y\to e_2} \|\mu_1 * \cdots * \mu_p - \delta_{(e_1,y)} * \mu_1 * \cdots * \mu_p\| = 0.$$

Thus we have

$$\lim_{(x,y)\to(e_1,e_2)} \|\mu_1 * \cdots * \mu_p - \delta_{(x,y)} * \mu_1 * \cdots * \mu_p\|$$
$$\leq \lim_{(x,y)\to(e_1,e_2)} \{\|\mu_1 * \cdots * \mu_p - \delta_{(x,e_2)} * \mu_1 * \cdots * \mu_p\| + \|\delta_{(x,e_2)} * \mu_1 * \cdots * \mu_p - \delta_{(x,y)} * \mu_1 * \cdots * \mu_p\|\}$$
$$= 0,$$

which implies $\mu_1 * \cdots * \mu_p \in L^1(K_1 \times K_2)$. This completes the proof. □

Lemma 3.3. *Let K be a compact group, and let H be a closed normal subgroup of K. Let $\nu \in M(K/H)$, and let $\pi : K \to K/H$ be the canonical map. Then there exists a measure $\mu \in M(K)$ with the following :*

(1) $\pi(\mu) = \nu$,

(2) $\hat{\mu}(\sigma) = 0$ *for* $\sigma \in \Sigma_K \setminus A(\Sigma_K, H)$, *and*

(3) $\{\sigma \in A(\Sigma_K, H) : \hat{\mu}(\sigma) \neq 0\} = \{\sigma \in A(\Sigma_K, H) : \hat{\nu}(\sigma) \neq 0\}$.

Proof. Let $\nu \in M(K/H)$. For $f \in C(K)$, let $[f]$ be a continuous function in $C(K/H)$ defined by

$$[f](\dot{x}) = \int_H f(xy)dm_H(y),$$

and we define $\mu \in M(K)$ by

$$\mu(f) = \int_{K/H} [f](\dot{x})d\nu(\dot{x})$$

for $f \in C(K)$. It is easy to verify that

(4) $\pi(\mu) = \nu$.

Claim 1. $\hat{\mu}(\sigma) = 0$ for $\sigma \in \Sigma_K \setminus A(\Sigma_K, H)$.

Let $\sigma \in \Sigma_K \setminus A(\Sigma_K, H)$. For $\xi, \eta \in H_\sigma$, we have

$$\begin{aligned}\langle \hat{\mu}(\sigma)\xi, \eta\rangle &= \int_K \langle \overline{U}_x^{(\sigma)}\xi, \eta\rangle d\mu(x)\\ &= \int_{K/H}\int_H \langle \overline{U}_{xy}^{(\sigma)}\xi, \eta\rangle dm_H(y)d\nu(\dot{x})\\ &= \int_{K/H}\int_H \langle \overline{U}_y^{(\sigma)}\xi, \overline{U}_x^{(\sigma)*}\eta\rangle dm_H(y)d\nu(\dot{x})\\ &= \int_{K/H} \langle \hat{m}_H(\sigma)\xi, \overline{U}_x^{(\sigma)*}\eta\rangle d\nu(\dot{x})\\ &= 0. \qquad \text{(by [9, 28.72(g), p.112])}\end{aligned}$$

This shows that $\hat{\mu}(\sigma) = 0$.

Claim 2. Let $\sigma \in A(\Sigma_K, H)$. Then $\hat{\mu}(\sigma) \neq 0$ if and only if $\hat{\nu}(\sigma) \neq 0$.

For $\xi, \eta \in H_\sigma$, we have, by the fact that $\sigma \in A(\Sigma_K, H)$,

$$\begin{aligned}\langle \hat{\mu}(\sigma)\xi, \eta\rangle &= \int_{K/H}\int_H \langle \overline{U}^{(\sigma)}_{xy}\xi, \eta\rangle dm_H(y)d\nu(\dot{x}) \\ &= \int_{K/H} \langle \overline{U}^{(\sigma)}_{xH}\xi, \eta\rangle d\nu(\dot{x}) \\ &= \langle \hat{\nu}(\sigma)\xi, \eta\rangle.\end{aligned}$$

Thus Claim 2 follows. By (4) and Claims 1 and 2, the lemma is obtained. □

Lemma 3.4. *Let K be a compact group, and let H be a closed normal subgroup of K. Let $p \in \mathbb{N}$. If E is an s-small p set in Σ_K, then $E \cap A(\Sigma_K, H)$ is an s-small p set in $\Sigma_{K/H} \cong A(\Sigma_K, H)$.*

Proof. We note that $\Sigma_{K/H} \cong A(\Sigma_K, H)$ (cf. [9, (28.10) Corollary]).
Let $\nu_n \in M_{E\cap A(\Sigma_K,H)}(K/H)\,(n = 1, 2, \cdots, p)$, and let $\pi : K \to K/H$ be the canonical map. It follows from Lemma 3.3 that there exists $\mu_n \in M(K)$ such that

(1) $\qquad \pi(\mu_n) = \nu_n,$

(2) $\qquad \hat{\mu}_n(\sigma) = 0 \quad \text{for } \sigma \in \Sigma_K \setminus A(\Sigma_K, H), \quad \text{and}$

(3) $\qquad \{\sigma \in A(\Sigma_K, H) : \hat{\mu}_n(\sigma) \neq 0\} = \{\sigma \in A(\Sigma_K, H) : \hat{\nu}_n(\sigma) \neq 0\}.$

Then

$$\{\sigma \in \Sigma_K : \hat{\mu}_n(\sigma) \neq 0\} \subset E \cap A(\Sigma_K, H).$$

Since E is an s-small p set, $\mu_1 * \cdots * \mu_p$ belongs to $L^1(K)$, which yields that $\nu_1 * \cdots * \nu_p = \pi(\mu_1 * \cdots * \mu_p) \in L^1(K/H)$. This completes the proof. □

The following lemma is due to [16]. For a subset P of Σ_K, $A(K, P)$ denotes the annihilator of P in K.

Lemma 3.5 (cf. [16, Lemma 3.3]). *Let K be a compact group. Let μ_0 be a nonzero measure in $M(K)$, and let μ and ν be mutually singular positive measures in $M(K)$. Let σ_0 be an element in Σ_K such that $\hat{\mu}_0(\sigma_0) \neq 0$. Then there exists a countable subset P of Σ_K, with $[P] = P$, such that*

(i) $\qquad \sigma_0 \in P,$

(ii) $\qquad \pi(\mu_0)\hat{}\,(\sigma_0) \neq 0, \quad$ *and*

(iii) $\qquad \pi(\mu) \perp \pi(\nu),$

where $H = A(K, P)$ and $\pi : K \to K/H$ is the canonical map. Moreover, for any $P' \supset P$ with $[P'] = P'$, we have

(iv) $\qquad \pi'(\mu) \perp \pi'(\nu),$

where $H' = A(K, P')$ and $\pi' : K \to K/H'$ is the canonical map.

Now we prove Theorem 2.1. Suppose there exist measures $\mu_n \in M_{E_1\times E_2}(K_1 \times K_2)\,(n = 1, 2, \cdots, p)$ such that $\mu_1 * \cdots * \mu_p$ does not belong to $L^1(K_1 \times K_2)$. Let

$$\mu_1 * \cdots * \mu_p = \mu_a + \mu_s$$

be the Lebesgue decomposition of $\mu_1 * \cdots * \mu_p$ with respect to $m_{K_1 \times K_2}$. Then $\mu_s \neq 0$. Thus there exists $\sigma_0 = (\sigma_1, \sigma_2) \in \Sigma_{K_1} \times \Sigma_{K_2}$ such that $\hat{\mu}_s(\sigma_0) \neq 0$. It follows from Lemma 3.5 that there exists a countable subset P of $\Sigma_{K_1 \times K_2}$, with $[P] = P$, such that

$$\sigma_0 = (\sigma_1, \sigma_2) \in P, \tag{3.1}$$

$$\pi(\mu_s)\hat{}(\sigma_0) \neq 0, \quad \text{and} \tag{3.2}$$

$$\pi(|\mu_s|) \perp \pi(m_{K_1 \times K_2}), \tag{3.3}$$

where $\pi : K_1 \times K_2 \to K_1 \times K_2/A(K_1 \times K_2, P)$ is the canonical map. Moreover, P can be chosen so that, for any $P' \supset P$ with $[P'] = P'$,

$$\pi'(|\mu_s|) \perp \pi'(m_{K_1 \oplus K_2}), \tag{3.4}$$

where $\pi' : K_1 \times K_2 \to K_1 \times K_2/A(K_1 \times K_2, P')$ is the canonical map. Let $\tau_i : \Sigma_{K_1} \times \Sigma_{K_2} (\cong \Sigma_{K_1 \times K_2}) \to \Sigma_{K_i}$ be the projection (i = 1, 2), and let P_i be a countable subset of Σ_{K_i} such that $\tau_i(P) \subset P_i$ and $[P_i] = P_i$ $(i = 1, 2)$. Set $H_i = A(K_i, P_i)$, and put $H = H_1 \times H_2$. Then H_i and H are closed normal subgroups of K_i and $K_1 \times K_2$ respectively. Let $\pi_H : K_1 \times K_2 \to K_1 \times K_2/H \cong K_1/H_1 \times K_2/H_2$ be the natural map. Since $P \subset P_1 \times P_2$, we have, by (3.4),

$$\pi_H(|\mu_s|) \perp \pi_H(m_{K_1 \times K_2}). \tag{3.5}$$

Since $\sigma_0 = (\sigma_1, \sigma_2) \in P_1 \times P_2$ and $\hat{\mu}_s(\sigma_0) \neq 0$, we note that

$$\pi_H(\mu_s)\hat{}(\sigma_0) \neq 0 \tag{3.6}$$

(cf. the proof of Lemma 3.3 in [16]). It follows from Lemma 3.4 that $E_i \cap A(\Sigma_{K_i}, H_i)$ is an s-small p set. Since P_i is countable, K_i/H_i is a metrizable compact group. Hence $(E_1 \cap A(\Sigma_{K_1}, H_1)) \times (E_2 \cap A(\Sigma_{K_2}, H_2))$ is an s-small p set in $\Sigma_{K_1 \times K_2/H} \cong A(\Sigma_{K_1}, H_1) \times A(\Sigma_{K_2}, H_2) (\cong P_1 \times P_2)$, by Proposition 3.3. Since $spec(\pi_H(\mu_n)) \subset (E_1 \cap A(\Sigma_{K_1}, H_1)) \times (E_2 \cap A(\Sigma_{K_2}, H_2))$, we have

$$\pi_H(\mu_1 * \cdots * \mu_p) = \pi_H(\mu_1) * \cdots * \pi_H(\mu_p) \in L^1(K_1 \times K_2/H). \tag{3.7}$$

On the other hand, (3.5) shows that $\pi_H(\mu_1 * \cdots * \mu_p) = \pi_H(\mu_a) + \pi_H(\mu_s)$ is the Lebesgue decomposition of $\pi_H(\mu_1 * \cdots * \mu_p)$ with respect to $\pi_H(m_{K_1 \times K_2})$. By (3.6), we have $\pi_H(\mu_s) \neq 0$, which contradicts (3.7). This shows that $E_1 \times E_2$ is an s-small p set in $\Sigma_{K_1 \times K_2}$, and the proof is complete.

Next we prove Theorem 2.2. We need several lemmas.
For $\mu \in M(K)$, define $\overline{\mu} \in M(K)$ by

$$\overline{\mu}(B) = \overline{\mu(B)} \tag{3.8}$$

for Borel sets B on K. Let $\sigma \in \Sigma_K$. We denote by $B(H_\sigma)$ the space of all bounded linear operators on H_σ. For $\mu \in M(K)$, we define $T_\mu \in B(H_\sigma)$ by

$$\langle T_\mu \xi, \eta \rangle = \int_K \langle D_\sigma \overline{U}_x^{(\sigma)} D_\sigma \xi, \eta \rangle d\mu(x) \tag{3.9}$$

for $\xi, \eta \in H_\sigma$. The following can be found in the proof of [9, (28.44) Theorem].

Lemma 3.6. *There exists an onto linear isometry* $C : H_{\overline{\sigma}} \to H_\sigma$ *such that* $\hat{\mu}(\overline{\sigma}) = C^{-1}T_\mu C$.

Lemma 3.7. *Let* $\mu \in M(K)$ *and* $\sigma \in \Sigma_K$. *Then* $\hat{\overline{\mu}}(\sigma) = D_\sigma T_\mu D_\sigma$.

Proof. For ξ, $\eta \in H_\sigma$, we have

$$\begin{aligned}\langle \hat{\overline{\mu}}(\sigma)\xi, \eta\rangle &= \int_K \langle \overline{U}_x^{(\sigma)}\xi, \eta\rangle d\overline{\mu}(x) = \overline{\int_K \overline{\langle \overline{U}_x^{(\sigma)}\xi, \eta\rangle} d\mu(x)} \\ &= \overline{\int_K \langle D_\sigma \overline{U}_x^{(\sigma)}\xi, D_\sigma\eta\rangle d\mu(x)} = \overline{\int_K \langle D_\sigma \overline{U}_x^{(\sigma)} D_\sigma D_\sigma \xi, D_\sigma\eta\rangle d\mu(x)} \\ &= \overline{\langle T_\mu D_\sigma\xi, D_\sigma\eta\rangle} = \langle D_\sigma T_\mu D_\sigma \xi, \eta\rangle.\end{aligned}$$

This completes the proof. □

Remark 3.1. *Let* $\mu \in M(K)$ *and* $\sigma \in \Sigma_K$. *It follows from Lemmas 3.6 and 3.7 that the following are equivalent.*

(i) $\hat{\mu}(\overline{\sigma}) \neq 0$.

(ii) $\hat{\overline{\mu}}(\sigma) \neq 0$.

Corollary 3.1. *Let* $\mu \in M(K)$. *Then* $spec(\overline{\mu}) = spec(\mu)^-$, *where* $spec(\mu)^- = \{\overline{\sigma} : \sigma \in spec(\mu)\}$.

Proof. For $\sigma \in \Sigma_K$, we note that $\overline{\overline{\sigma}} = \sigma$. Thus the corollary follows from Remark 3.1. □

The following lemma is due to [15].

Lemma 3.8 (cf. [15, Lemma 3.3]). *Let* $\sigma \in \Sigma_K$ *and* $\Delta \subset \Sigma_K$. *For* $f \in \mathfrak{T}_\sigma(K)$ *and* $\mu \in M(K)$ *with* $spec(\mu) \subset \Delta$, *we have* $spec(f\mu) \subset \sigma \times \Delta$.

Now we prove Theorem 2.2. Let $\mu, \nu \in M_\Delta(K)$. Then

$$(u_{ij}^{(\sigma)}\mu) * (u_{k\ell}^{(\tau)}\overline{\nu}) \in L^1(K) \tag{3.10}$$

for all $\sigma, \tau \in \Sigma_K$; $u_{ij}^{(\sigma)} \in \mathfrak{T}_\sigma(K), u_{k\ell}^{(\tau)} \in \mathfrak{T}_\tau(K)$. In fact, since $spec(\mu) \subset \Delta$, we have, by Lemma 3.8,

$$spec(u_{ij}^{(\sigma)}\mu) \subset \sigma \times \Delta.$$

Similary Corollary 3.1, together with the previous lemma, yields

$$spec(u_{k\ell}^{(\tau)}\overline{\nu}) \subset \tau \times \overline{\Delta}.$$

Hence we have

$$spec((u_{ij}^{(\sigma)}\mu) * (u_{k\ell}^{(\tau)}\overline{\nu})) \subset (\sigma \times \Delta) \cap (\tau \times \overline{\Delta}),$$

which implies (3.10), since $(\sigma \times \Delta) \cap (\tau \times \overline{\Delta})$ is finite by the hypothesis (2.3). It follows from (3.10) that

$$(f\mu) * (h\overline{\nu}) \in L^1(K) \quad \text{for any } f, h \in \mathfrak{T}(K). \tag{3.11}$$

On the other hand, there exist sequences $\{f_n\}$ and $\{h_n\}$ in $\mathfrak{T}(K)$ such that $\lim_{n\to\infty} \|f_n\mu - |\mu|\| = 0$ and $\lim_{n\to\infty} \|h_n\overline{\nu} - |\overline{\nu}|\| = 0$. Since $\lim_{n\to\infty} \|(f_n\mu) * (h_n\overline{\nu}) - |\mu| * |\overline{\nu}|\| = 0$, (3.11) yields $|\mu| * |\nu| = |\mu| * |\overline{\nu}| \in L^1(K)$. This completes the proof.

References

[1] N. Bourbaki, Intégration, Élements de Mathématique, Livre VI, ch 6, Paris, Herman, 1959.

[2] R.G.M. Brummelhuis, An F. and M. Riesz theorem for bounded symmetric domains, Ann. Inst. Fourier **37** (1987), 139-150.

[3] R.G.M. Brummelhuis, A note on Riesz sets and lacunary sets, J. Austral. Math. soc. **48** (1990), 57-65.

[4] G.A. Edgar, Disintegration of measures and vector-valued Radon-Nikodym theorem, Duke Math. J. **42** (1975), 447-450.

[5] G.B. Folland, Real Analysis: Modern Techniques and Their Applications, Wiley-Interscience, New York, 1984.

[6] I. Glicksberg, Fourier-Stieltjes transforms with small supports, Illinois J. Math. **9** (1965), 418-427.

[7] C.C. Graham, Fourier-Stieltjes transforms with small supports, Illinois J. Math. **18** (1974), 532-534.

[8] E. Hewitt and K.A. Ross, Abstract Harmonic Analysis, Vol. I, New York-Heidelberg-Berlin, Springer-Verlag, 1963.

[9] E. Hewitt and K.A. Ross, Abstract Harmonic Analysis, Vol. II, New York-Heidelberg-Berlin, Springer-Verlag, 1970.

[10] H.A. MacLean, Riesz sets and a theorem of Bochner, Pacific J. Math. **113** (1984), 115-135.

[11] W. Rudin, Fourier Analysis on Groups, Interscience, New York, 1962.

[12] Y. Takahashi and H. Yamaguchi, On measures which are continuous by certain translation, Hokkaido Math. J. **13** (1984), 109-117.

[13] H. Yamaguchi, On the product of a Riesz set and a small p set, Proc. Amer. Math. Soc. **81** (1981), 273-278.

[14] H. Yamaguchi, Idempotent multipliers on the space of analytic singular measures, Hokkaido Math. J. **14** (1985), 49-73.

[15] H. Yamaguchi, Absolute continuity of measures on compact transformation groups, Hokkaido Math. J. **22** (1993), 25-33.

[16] H. Yamaguchi, An F. and M. Riesz theorem on compact groups, Transactions of German-Jpanese Symposium on 'Infinite-Dimensional Harmonic Analysis' held in Tübingen, 249-273, Verlag D.+M. Gräbner, Bamberg, 1996.

Department of mathematics
Josai University
Sakado, Saitama
Japan