# On Arithmetic Quantum Field Theory

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**Abstract**

We review fundamental aspects of arithmetic quantum field theory.

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# 1 Introduction

In recent developments of theoretical physics, it has been shown that number theory has connections with physics in various aspects (e.g., [23, 30]). Among others, "statistical mechanics" of numbers may be interesting, because it is related in a direct way to the Riemann zeta function and may give a key to solve the Riemann hypothesis ([17, 18, 20, 21, 22, 27, 28, 29] and references therein).

Spector [28] pointed out relationships between analytic number theory and a free supersymmetric quantum field theory, and further discussed these aspects with notions of partial supersymmetry and "duality"[29]. Motivated by these works of Spector, we started in [14] a research program developing analytic number theory as a field of infinite dimensional analysis or mathematically rigorous quantum field theory. We call this type of theory an *arithmetic quantum field theory.* In this paper we review some fundamental results in [14].

# 2 Arithmetical Functions in Boson Fock spaces

## 2.1 Partition functions and correlation functions

Let $\mathcal{H}$ be a separable infinite dimensional Hilbert space with inner product $(\cdot\,,\,\cdot)_{\mathcal{H}}$ (complex linear in the second variable) and $\otimes_s^n\mathcal{H}$ be the $n$-fold symmetric tensor product Hilbert space of $\mathcal{H}$ ($n = 0, 1, 2, \cdots$ ; $\otimes_s^0\mathcal{H} := \mathbf{C}$). Then the Boson Fock space over $\mathcal{H}$ is defined by $\mathcal{F}_B(\mathcal{H}) := \oplus_{n=0}^{\infty} \otimes_s^n \mathcal{H}$. Let $A$ be a nonnegative self-adjoint operator on $\mathcal{H}$ and

$$H_B(A) := d\Gamma_B(A) \tag{2.1}$$

be the second quantization of $A$ on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$ (e.g., [19, §5.2], [25, p. 302, Example 2]). We denote by $N_{\mathrm{B}}$ the number operator on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$: $N_{\mathrm{B}} := d\Gamma_{\mathrm{B}}(I)$, where $I$ denotes identity.

For $s > 0$, we define

$$Z_{\mathrm{B}}(s;A) := \mathrm{Tr}\, e^{-sH_{\mathrm{B}}(A)}, \quad \tilde{Z}_{\mathrm{B}}(s;A) := \mathrm{Tr}\left\{(-1)^{N_{\mathrm{B}}} e^{-sH_{\mathrm{B}}(A)}\right\}, \tag{2.2}$$

provided that $e^{-sH_{\mathrm{B}}(A)}$ is trace class on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$, where Tr denotes trace.

**Remark 2.1** In statistical mechanics of quantum fields, $Z_{\mathrm{B}}(s;A)$ is called the *partition function* of the Hamiltonian $H_{\mathrm{B}}(A)$ at temperature $1/s$ (physically $s$ denotes an *inverse temperature*). The function $\tilde{Z}_{\mathrm{B}}(s;A)$ is not so standard. We call it the *graded partition function* of the Hamiltonian $H_{\mathrm{B}}(A)$ at temperature $1/s$. This type of partition function was considered in a concrete case by Spector [29].

To treat the partition functions in a unified way, we introduce a more general partition function

$$Z_{\mathrm{B}}(s,z;A) := \mathrm{Tr}\left(\Gamma_{\mathrm{B}}(z)e^{-sH_{\mathrm{B}}(A)}\right) \tag{2.3}$$

with

$$z \in D := \{w \in \mathbf{C} \mid |w| \le 1\}, \tag{2.4}$$

provided that $e^{-sH_{\mathrm{B}}(A)}$ is trace class on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$, where $\Gamma_{\mathrm{B}}(z) := \oplus_{n=0}^{\infty} z^n$ acting on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$. We have

$$Z_{\mathrm{B}}(s,1;A) = Z_{\mathrm{B}}(s;A), \quad Z_{\mathrm{B}}(s,-1;A) = \tilde{Z}_{\mathrm{B}}(s;A). \tag{2.5}$$

In what follows, we assume the following.

**Hypothesis (A)** *The operator $A$ is strictly positive, self-adjoint and, for some $s > 0$, $e^{-sA}$ is trace class on $\mathcal{H}$.*

**Theorem 2.1** *Let $z \in D$. Then the operator $\Gamma_{\mathrm{B}}(z)e^{-sH_{\mathrm{B}}(A)}$ is trace class on $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$ and*

$$Z_{\mathrm{B}}(s,z;A) = \frac{1}{\det\left(I - ze^{-sA}\right)}, \tag{2.6}$$

*where $\det(I+S)$ is the determinant for $I+S$ with $S$ a trace class operator [26, §XIII.17].*

Using Theorem 2.1 and the product law of the determinant $\det(I + \cdot)$, we can derive relations of partition functions at different temperatures:

**Theorem 2.2** *For all $n \in \mathbf{N}$ and $z \in D$,*

$$Z_{\mathrm{B}}(s,z;A) = \det\left(\sum_{k=0}^{n-1} z^k e^{-ksA}\right) Z_{\mathrm{B}}(ns, z^n; A) \tag{2.7}$$

*and*

$$Z_{\mathrm{B}}(s,z;A)Z_{\mathrm{B}}(s,-z;A) = Z_{\mathrm{B}}(2s,z^2;A). \tag{2.8}$$

**Remark 2.2** In general, relationships among theories at different coupling constants are referred to as "duality" [29]. Eq.(2.8) is a duality relation, where the coupling constant is the inverse temperature.

In statistical mechanics, *correlation functions* are also important objects. We denote by $a_{\mathcal{H}}(f)$ $(f \in \mathcal{H})$ the annihilation operator on $\mathcal{F}_{\rm B}(\mathcal{H})$ (e.g., [19, §5.2], [25, §X.7]) ($a_{\mathcal{H}}(f)$ is antilinear in $f$). For all $t > s$ and $f, g \in D(A^{-1/2})$ ($D(A^{-1/2})$ denotes the domain of $A^{-1/2}$). We can define

$$R_{\rm B}(t, z; f, g; A) := \frac{\mathrm{Tr}\left(\Gamma_{\rm B}(z) a_{\mathcal{H}}(f)^* a_{\mathcal{H}}(g) e^{-tH_{\rm B}(A)}\right)}{Z_{\rm B}(t, z; A)}, \quad z \in D. \tag{2.9}$$

This is called a *two-point correlation function*. In the same manner as in [19, Proposition 5.2.28], we can show that

$$R_{\rm B}(t, z; f, g; A) = (g, ze^{-tA}(1 - ze^{-tA})^{-1} f)_{\mathcal{H}}. \tag{2.10}$$

## 2.2 Arithmetical aspects

By Hypothesis (A), the spectrum $\sigma(A)$ of $A$ is purely discrete with

$$\sigma(A) = \{E_n(A)\}_{n=1}^{\infty}, \tag{2.11}$$

$0 < E_1(A) \leq E_2(A) \leq \cdots$, $E_n(A) \to \infty \, (n \to \infty)$, counted with algebraic multiplicity. There exists a complete orthonormal system (CONS) $\{\phi_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ such that $\phi_n \in D(A)$, $A\phi_n = E_n(A)\phi_n$, $n \in \mathbf{N}$. We set

$$a_n := a_{\mathcal{H}}(\phi_n) \tag{2.12}$$

Then we have canonical commutation relations

$$[a_n, a_m^*] = \delta_{mn}, \quad [a_n, a_m] = 0, \quad [a_n^*, a_m^*] = 0, \quad n, m \geq 1, \tag{2.13}$$

on the finite particle subspace of $\mathcal{F}_{\rm B}(\mathcal{H})$.

We denote by

$$\mathcal{P} := \{p_n\}_{n=1}^{\infty} \tag{2.14}$$

the set of all prime numbers with $p_n < p_{n+1}$, $n \geq 1$ ($p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, p_5 = 11, \cdots$).

By definition, an arithmetical function is a complex-valued function on $\mathbf{N}$. An arithmetical function $f$ is called *completely multiplicative* if it satisfies

$$f(1) = 1, \quad f(mn) = f(m)f(n), \quad m, n \in \mathbf{N}.$$

Let $N \geq 2$ be a natural namber. Then, by the fundamental theorem of arithmetic, there exists a unique set $\{i_1, \cdots, i_n, \alpha_1, \cdots, \alpha_n\} \subset \mathbf{N}$ $(i_1 < \cdots < i_n)$ such that

$$N = (p_{i_1})^{\alpha_1} \cdots (p_{i_n})^{\alpha_n}. \tag{2.15}$$

Then we define an arithmetical function $\gamma(N)$ by $\gamma(1) := 0$ and

$$\gamma(N) := \sum_{k=1}^{n} \alpha_k, \quad N \geq 2. \tag{2.16}$$

The arithmetical function defined by $\lambda(1) := 1$ and

$$\lambda(N) := (-1)^{\gamma(N)}, \quad N \geq 2, \tag{2.17}$$

is called the *Liouville fucntion* [1, §2.12]. This function is completely multiplicative.

Using the representation (2.15) of $N$, we can define a vector $\Psi_N \in \mathcal{F}_B(\mathcal{H})$ by

$$\Psi_N := C_N(a_{i_1}^*)^{\alpha_1} \cdots (a_{i_n}^*)^{\alpha_n} \Omega_{\mathcal{H}}, \tag{2.18}$$

where $\Omega_{\mathcal{H}} := \{1, 0, 0, \cdots\}$ is the Fock vacuum in $\mathcal{F}_B(\mathcal{H})$ and $C_N := 1/\sqrt{\alpha_1! \cdots \alpha_n!}$ is a normalization constant so that $\|\Psi_N\| = 1$. We set $\Psi_1 := \Omega_{\mathcal{H}}$. A key fact is the following.

**Lemma 2.3** [28] *The set* $\{\Psi_N\}_{N=1}^{\infty}$ *is a CONS of* $\mathcal{F}_B(\mathcal{H})$.

**Lemma 2.4** *For all* $N \in \mathbf{N}$, $\Psi_N$ *is a unique eigenvector (up to constant multiples) of* $\Gamma_B(z)$ *with eigenvalue* $z^{\gamma(N)}$.

We introduce a function $F_A : \mathbf{N} \to (0, \infty)$ as follows: $F_A(1) := 1$ and if $N \geq 2$ is represented as (2.15), then

$$F_A(N) := \prod_{k=1}^{n} e^{\alpha_k E_{i_k}(A)}. \tag{2.19}$$

It is easy to see that $F_A$ is completely multiplicative.

**Lemma 2.5** *For all* $N \in \mathbf{N}$, $\Psi_N$ *is a unique eigenvector (up to constant multiples) of* $H_B(A)$ *with eigenvalue* $\log F_A(N)$.

By Lemmas 2.4 and 2.5, we have

$$Z_B(s, z; A) = \sum_{N=1}^{\infty} \frac{z^{\gamma(N)}}{F_A(N)^s}, \quad z \in D. \tag{2.20}$$

By this fact and Theorem 2.1, we obtain the following.

**Theorem 2.6** *For all* $z \in D$,

$$\sum_{N=1}^{\infty} \frac{z^{\gamma(N)}}{F_A(N)^s} = \frac{1}{\prod_{n=1}^{\infty}(1 - ze^{-sE_n(A)})}. \tag{2.21}$$

**Remark 2.3** Formula (2.21) may be regarded as a general form unifying arithmetical formulas known under the name of *Euler products* [1, Chapter 11]. See Section 2.3 below.

We introduce a function $\varrho(N, m) : \mathbf{N} \times \mathbf{N} \to \{0\} \cup \mathbf{N}$ by

$$\varrho(1, m) := 0, \quad \varrho(N, m) := \sum_{k=1}^{n} \alpha_k \delta_{i_k m} \tag{2.22}$$

if $N \geq 2$ is expressed as (2.15) ($N, m \in \mathbf{N}$).

**Theorem 2.7** *Let* $t > s$. *Then, for all* $m \in \mathbf{N}$ *and* $z \in D$,

$$\sum_{N=1}^{\infty} \frac{z^{\gamma(N)} \varrho(N, m)}{F_A(N)^t} = \frac{z}{e^{tE_m(A)} - z} Z_B(t, z; A). \tag{2.23}$$

Let $N \geq 2$ be given as (2.15). Then, each divisor $m$ of $N$ is of the form

$$m = p_{i_1}^{r_1} \cdots p_{i_n}^{r_n} \tag{2.24}$$

with $0 \leq r_j \leq \alpha_j$, $j = 1, \cdots, n$. We define a vector $\Psi_{N,m} \in \mathcal{F}_{\mathrm{B}}(\mathcal{H})$ by

$$\Psi_{N,m} := C_{N,m} a_{i_1}^{*\, r_1} \cdots a_{i_n}^{*\, r_n} \Omega_{\mathcal{H}}, \tag{2.25}$$

where $C_{N,m} > 0$ is a normalization constant. For an $m \in \mathbf{N}$ and $N \in \mathbf{N}$, we mean by $m|N$ that $m$ is a divisor of $N$. The set $\{\Psi_{N,m}\}_{m|N}$ of vectors is orthonormal. We introduce

$$\mathcal{F}_{\mathrm{B}}^{(N)}(\mathcal{H}) := \mathcal{L}\{\Psi_{N,m}\}_{m|N}, \tag{2.26}$$

where $\mathcal{L}\{\cdot\}$ means the subspace spanned algebraically by the vectors in the set $\{\cdot\}$. We set $\mathcal{F}_{\mathrm{B}}^{(1)} := \{\alpha\Omega_{\mathcal{H}} | \alpha \in \mathbf{C}\}$. We denote by $P_N$ the orthogonal projection from $\mathcal{F}_{\mathrm{B}}(\mathcal{H})$ onto $\mathcal{F}_{\mathrm{B}}^{(N)}(\mathcal{H})$.

**Proposition 2.8** *Let $z \in D$. Then, for all $N$,*

$$\mathrm{Tr}\left(P_N \Gamma_{\mathrm{B}}(z) e^{-sH_{\mathrm{B}}(A)} P_N\right) = \sum_{m|N} \frac{z^{\gamma(m)}}{F_A(m)^s}. \tag{2.27}$$

## 2.3 Connections with analytic number theory

A basic object in analytic number theory is the *Dirichlet series*

$$D(s,f) := \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \tag{2.28}$$

for an arithmetical function $f$ and $s \in \mathbf{C}$, provided that the infinite series converges. The *Riemann zeta function*

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad s > 1, \tag{2.29}$$

is a special case of $D(s,f)$. We first show that $\zeta(s)$ and $D(s,\lambda)$ can be represented as partition functions of $H_{\mathrm{B}}(A)$ with a suitable $A$. For this purpose, we consider the case where $\mathcal{H}$ is given by

$$\ell^2 := \oplus_{n=1}^{\infty} \mathbf{C} = \left\{ \psi = \{\psi_n\}_{n=1}^{\infty} \middle| \psi_n \in \mathbf{C}, n \geq 1, \sum_{n=1}^{\infty} |\psi_n|^2 < \infty \right\}. \tag{2.30}$$

On this Hilbert space we define an operator $\omega_{\mathcal{P}}$ as follows:

$$D(\omega_{\mathcal{P}}) = \left\{ \psi = \{\psi_n\}_{n=1}^{\infty} \in \ell^2 \middle| \sum_{n=1}^{\infty} |(\log p_n)\psi_n|^2 < \infty \right\}, \tag{2.31}$$

$$(\omega_{\mathcal{P}}\psi)_n = (\log p_n)\psi_n, \quad \psi \in D(\omega_{\mathcal{P}}),\ n \geq 1. \tag{2.32}$$

Then $\omega_{\mathcal{P}}$ is strictly positive and self-adjoint. Moreover, the spectrum of $\omega_{\mathcal{P}}$ is purely discrete with

$$\sigma(\omega_{\mathcal{P}}) = \{\log p_n\}_{n=1}^{\infty} \tag{2.33}$$

with the multiplicity of each eigenvalue $\log p_n$ being one. A normalized eigenvector of $\omega_{\mathcal{P}}$ with eigenvalue $\log p_n$ is given by

$$e_n := \{\delta_{nj}\}_{j=1}^{\infty} \in \ell^2. \tag{2.34}$$

**Theorem 2.9** *For all $s > 1$ and $z \in D$,*

$$Z_{\mathrm{B}}(s, z; \omega_{\mathcal{P}}) = \sum_{N=1}^{\infty} \frac{z^{\gamma(N)}}{N^s}. \tag{2.35}$$

Applying Theorem 2.6 with $A = \omega_{\mathcal{P}}$, we obtain the following.

**Corollary 2.10** *For all $s > 1$ and $z \in D$,*

$$\sum_{N=1}^{\infty} \frac{z^{\gamma(N)}}{N^s} = \frac{1}{\prod_{p \in \mathcal{P}} (1 - zp^{-s})}. \tag{2.36}$$

An application of Theprem 2.7 gives the following.

**Corollary 2.11** *For all $s > 1$, $n \in \mathbf{N}$ and $z \in D$,*

$$\sum_{N=1}^{\infty} \frac{z^{\gamma(N)} \varrho(N, n)}{N^s} = \frac{z}{p_n^s - z} Z_{\mathrm{B}}(s, z; \omega_{\mathcal{P}}). \tag{2.37}$$

The operator $\omega_{\mathcal{P}}$ may be regarded as as a special case of a more general operator associated with a completely multiplicative function. Let $f$ be a completely multiplicative function such that $0 < f(n) < 1$ for all $n \geq 2$ and

$$\sum_{n=1}^{\infty} f(p_n) < \infty, \tag{2.38}$$

and define an operator $A_f$ on $\ell^2$ by

$$D(A_f) = \left\{ \psi = \{\psi_n\}_{n=1}^{\infty} \middle| \sum_{n=1}^{\infty} |\log f(p_n)|^2 |\psi_n|^2 < \infty \right\}, \tag{2.39}$$

$$(A_f \psi)_n = [-\log f(p_n)] \psi_n, \quad \psi \in D(A_f),\ n \geq 1. \tag{2.40}$$

Then $A_f$ is a strictly positive self-adjoint operator and $e^{-A_f}$ is trace class on $\ell^2$. It is easy to see that

$$F_{A_f}(N) = \frac{1}{f(N)}, \quad N \in \mathbf{N}. \tag{2.41}$$

Hence we have

$$Z_{\mathrm{B}}(1, z; A_f) = \sum_{n=1}^{\infty} z^{\gamma(n)} f(n), \quad z \in D. \tag{2.42}$$

Applying Theorem 2.6, we obtain the following fact.

**Corollary 2.12** *Let $f$ be as above. Then, for all $z \in D$,*

$$\sum_{n=1}^{\infty} z^{\gamma(n)} f(n) = \frac{1}{\prod_{p \in \mathcal{P}} (1 - zf(p))}. \tag{2.43}$$

Theorem 2.7 gives the following.

**Corollary 2.13** *Let $f$ be as above. Then, for all $n \in \mathbf{N}$ and $z \in D$,*

$$\sum_{N=1}^{\infty} z^{\gamma(N)} \varrho(N, n) f(N) = \frac{zf(p_n)}{1 - zf(p_n)} Z_{\mathrm{B}}(1, z; A_f). \tag{2.44}$$

Applying Proposition 2.8, we have for all $s > 1$

$$\mathrm{Tr}\left(P_N z^{N_{\mathrm{B}}} e^{-sH_{\mathrm{B}}(\omega_{\mathcal{P}})} P_N\right) = \sum_{m|N} \frac{z^{\gamma(m)}}{m^s}, \quad z \in D. \tag{2.45}$$

# 3 Arithmetical Functions in Fermion Fock Spaces

## 3.1 Partition functions and correlation functions

Let $\mathcal{K}$ be a separable infinite dimensional Hilbert space and $\otimes_{\rm as}^n \mathcal{K}$ be the $n$-fold antisymmetric tensor product Hilbert space of $\mathcal{K}$ ($n = 0, 1, 2, \cdots$; $\otimes_{\rm as}^0 \mathcal{K} := \mathbf{C}$). Then the Fermion Fock space over $\mathcal{K}$ is defined by $\mathcal{F}_{\rm F}(\mathcal{K}) := \oplus_{n=0}^\infty \otimes_{\rm as}^n \mathcal{K}$.

Let $T$ be a nonnegative self-adjoint operator on $\mathcal{K}$ and

$$H_{\rm F}(T) := d\Gamma_{\rm F}(T). \tag{3.1}$$

be the second quantization of $T$ in $\mathcal{F}_{\rm F}(\mathcal{K})$. The number operator on $\mathcal{F}_{\rm F}(\mathcal{K})$ is defined by $N_{\rm F} := d\Gamma_{\rm F}(I)$.

Let $s > 0$, $z \in D$ and

$$Z_{\rm F}(s, z; T) := \mathrm{Tr}\left(\Gamma_{\rm F}(z) e^{-sH_{\rm F}(T)}\right), \tag{3.2}$$

provided that $e^{-sH_{\rm F}(T)}$ is trace class on $\mathcal{F}_{\rm F}(\mathcal{H})$, where $\Gamma_{\rm F}(z) := \oplus_{n=0}^\infty z^n$ acting on $\mathcal{F}_{\rm F}(\mathcal{K})$.

In what follows, we assume the following.

**Hypothesis (T)** *For some $s > 0$, $e^{-sT}$ is trace class on $\mathcal{K}$.*

**Theorem 3.1** *For all $z \in D$, $\Gamma_{\rm F}(z) e^{-sH_{\rm F}(T)}$ is trace class on $\mathcal{F}_{\rm F}(\mathcal{K})$ and*

$$Z_{\rm F}(s, z; T) = \det(I + z e^{-sT}). \tag{3.3}$$

By Theorems 2.1 and 3.1, we have interesting relations between bosonic and fermionic partition functions:

**Corollary 3.2** *Consider the case $\mathcal{H} = \mathcal{K}$ and $A$ be an operator on $\mathcal{H}$ obeying Hypothesis (A) in Section 2. Then, for all $z \in D$,*

$$Z_{\rm B}(s, -z; A) = \frac{1}{Z_{\rm F}(s, z; A)}. \tag{3.4}$$

**Theorem 3.3** *For all $n \in \mathbf{N}$ and $z \in D$,*

$$\begin{aligned} Z_{\rm F}(ns, -z^n; T) &= \det\left(\sum_{k=1}^{n-1} z^k e^{-skT}\right) Z_{\rm F}(s, -z; T), & (3.5)\\ Z_{\rm F}(s, -z; T) Z_{\rm F}(s, z; T) &= Z_{\rm F}(2s, -z^2; T). & (3.6) \end{aligned}$$

**Remark 3.1** Relation (3.6) is a form of *duality* of fermionic partition functions. A special case is discussed in [29].

**Corollary 3.4** *Consider the case $\mathcal{H} = \mathcal{K}$ and $A$ be an operator on $\mathcal{H}$ obeying Hypothesis (A). Then*

$$Z_{\rm B}(2s, z^2; A) Z_{\rm F}(s, z; A) = Z_{\rm B}(s, z; A) \tag{3.7}$$

**Remark 3.2** Relation (3.7) is also a form of *duality* of fermionic and bosonic partition functions. For a special case, see [29].

Let $u, v \in \mathcal{K}$ and $z \in D$. Then a *fermionic two-point correlation function* is defined by

$$R_{\mathrm{F}}(s, z; u, v; T) := \frac{\mathrm{Tr}\left(\Gamma_{\mathrm{F}}(z) e^{-sH_{\mathrm{F}}(T)} b_{\mathcal{K}}(u)^* b_{\mathcal{K}}(v)\right)}{Z_{\mathrm{F}}(s, z; T)}. \tag{3.8}$$

where $b_{\mathcal{K}}(u)$ ($u \in \mathcal{K}$) the annihilation operator on $\mathcal{F}_{\mathrm{F}}(\mathcal{K})$ (e.g., [19, §5.2]). It is easy to see (e.g., cf. [19]) that

$$R_{\mathrm{F}}(s, z; u, v; T) = (v, ze^{-sT}(1 + ze^{-sT})^{-1}u)_{\mathcal{K}}. \tag{3.9}$$

## 3.2 Arithmetical aspects

By Hypothesis (T), the spectrum of $T$ is purely discrete with

$$\sigma(T) = \{E_n(T)\}_{n=1}^{\infty}, \tag{3.10}$$

$0 < E_1(T) \le E_2(T) \le \cdots$, $E_n(T) \to \infty\, (n \to \infty)$, counted with algebraic multiplicity. There exists a CONS $\{u_n\}_{n=1}^{\infty}$ of $\mathcal{K}$ such that $u_n \in D(T)$, $Tu_n = E_n(T)u_n$, $n \in \mathbf{N}$. We set

$$b_n := b_{\mathcal{K}}(u_n). \tag{3.11}$$

Then we have canonical anti-commutation relations

$$\{b_n, b_m^*\} = \delta_{mn}, \quad \{b_n, b_m\} = 0, \quad \{b_n^*, b_m^*\} = 0, \quad n, m \ge 1, \tag{3.12}$$

where $\{X, Y\} := XY + YX$. In particular, $b_n^2 = 0$, $b_n^{*2} = 0$, $n \in \mathbf{N}$.

For $N \in \mathbf{N}$ we define $\nu(N)$ by $\nu(1) := 1$ and

$$\nu(N) = n, \quad N \ge 2, \tag{3.13}$$

if $N$ is represented as (2.15) [1, p.247].

A natural number $m \ge 2$ is called *square free* if it is written as a product of mutually different prime numbers. As a convention, 1 is defined to be square free. We denote by $\mathcal{S}_0$ the set of square free elements in $\mathbf{N}$:

$$\mathcal{S}_0 := \{m \in \mathbf{N} | m \text{ is square free}\}. \tag{3.14}$$

For each $N \in \mathbf{N}$, we define a set $\mathcal{S}_0(N)$ as follows:

$$\mathcal{S}_0(1) := \{1\}, \tag{3.15}$$

$$\mathcal{S}_0(N) := \{m \in \mathcal{S}_0 | m \text{ is a divisor of } N\}, \quad N \ge 2. \tag{3.16}$$

Let $N \ge 2$ be given as (2.15). Then each element $m$ of $\mathcal{S}_0(N)$ is of the form

$$m = p_{i_1}^{q_1} \cdots p_{i_n}^{q_n}, \tag{3.17}$$

where $q_j = 0$ or $q_j = 1$ ($j = 1, \cdots, n$). Corresponding to this, we define a vector $\Phi_{N,m}$ by

$$\Phi_{N,m} := b_{i_1}^{*\,q_1} \cdots b_{i_n}^{*\,q_n} \Omega_{\mathcal{K}}, \tag{3.18}$$

where $\Omega_{\mathcal{K}} := \{1, 0, 0, \cdots\}$ is the Fock vacuum in $\mathcal{F}_{\mathrm{F}}(\mathcal{K})$.

Let

$$\mathcal{F}_{\mathrm{F}}^{(1)}(\mathcal{K}) := \{c\Omega_{\mathcal{K}} | c \in \mathbf{C}\}, \quad \mathcal{F}_{\mathrm{F}}^{(N)}(\mathcal{K}) := \mathcal{L}\{\Phi_{N,m} | m \in \mathcal{S}_0(N)\}, \quad N \geq 2. \tag{3.19}$$

Then $\mathcal{F}_{\mathrm{F}}^{(N)}(\mathcal{K})$ is finite dimensional with $\dim \mathcal{F}_{\mathrm{F}}^{(N)}(\mathcal{K}) = 2^{\nu(N)}$. We denote by $R_N$ the orthogonal projection from $\mathcal{F}_{\mathrm{F}}(\mathcal{K})$ onto $\mathcal{F}_{\mathrm{F}}^{(N)}(\mathcal{K})$.

Let $N \geq 2$ be of the form (2.15),

$$\mathcal{K}_N := \mathcal{L}\{u_{i_k} | k = 1, \cdots, n\} \tag{3.20}$$

and $T_N$ be the restriction of $T$ to $\mathcal{K}_N$. Then we can show that

$$\mathrm{Tr}\left(R_N \Gamma_{\mathrm{F}}(z) e^{-sH_{\mathrm{F}}(T)} R_N\right) = \det(1 + ze^{-sT_N}). \tag{3.21}$$

Let $m \in \mathcal{S}_0, m \geq 2$ and

$$m = p_{i_1} \cdots p_{i_r} \tag{3.22}$$

be its factorization in prime numbers ($i_j \neq i_k$, $j \neq k$). Then we define a vector $\Phi_m$ in $\mathcal{F}_{\mathrm{F}}(\mathcal{K})$ by

$$\Phi_m := b_{i_1}^* \cdots b_{i_r}^* \Omega_{\mathcal{K}}. \tag{3.23}$$

For $m = 1$, we set $\Phi_1 := \Omega_{\mathcal{K}}$. For $m \notin \mathcal{S}_0$, we define $\Phi_m := 0$.

**Lemma 3.5** [28] *The set* $\{\Phi_m\}_{m \in \mathcal{S}_0}$ *is a CONS of* $\mathcal{F}_{\mathrm{F}}(\mathcal{K})$.

The *Möbius function* $\mu : \mathbf{N} \to \{0, \pm 1\}$ is defined as follows: $\mu(1) := 1$, $\mu(m) := 0$ if $m \notin \mathcal{S}_0$ and $\mu(m) := (-1)^r$ if $m$ is written as the product of mutually different $r$ prime numbers. We have

$$\mu(m) = (-1)^{\gamma(m)}, \quad m \in \mathcal{S}_0. \tag{3.24}$$

**Lemma 3.6** *For all* $m \in \mathcal{S}_0$, $\Phi_m$ *is an eigenvector of* $N_{\mathrm{F}}$ *with eigenvalue* $\gamma(m)$.

**Lemma 3.7** *For all* $m \in \mathcal{S}_0$, $\Phi_m$ *is an eigenvector of* $H_{\mathrm{F}}(T)$ *with eigenvalue* $\log F_T(m)$, *where* $F_T$ *is defined by (2.19) with* $A = T$.

It follows from Lemmas 3.6 and 3.7 that

$$Z_{\mathrm{F}}(s, z; T) = \sum_{m=1}^{\infty} \frac{z^{\gamma(m)} |\mu(m)|}{F_T(m)^s}, \quad z \in D, \tag{3.25}$$

where we have used that $\mu(m) = 0$ for all $m \notin \mathcal{S}_0$ and $|\mu(m)| = 1$ for all $m \in \mathcal{S}_0$. By (3.25) and Theorem 3.1, we obtain the following.

**Theorem 3.8** *Let* $z \in D$. *Then*

$$\sum_{m=1}^{\infty} \frac{z^{\gamma(m)} |\mu(m)|}{F_T(m)^s} = \prod_{n=1}^{\infty} \left(1 + ze^{-sE_n(T)}\right). \tag{3.26}$$

Theorems 3.8 and 2.6 imply the following.

**Corollary 3.9** *Let $z \in D$. Then,*

$$\sum_{m=1}^{\infty} \frac{z^{\gamma(m)}|\mu(m)|}{F_T(m)^s} = \frac{1}{\sum_{n=1}^{\infty} \frac{(-z)^{\gamma(n)}}{F_T(n)^s}}. \tag{3.27}$$

We introduce a function $\eta$ on $\mathbf{N} \times \mathbf{N}$ by

$$\eta(1,n) := 0, \tag{3.28}$$

$$\eta(m,n) := \sum_{k=1}^{r}(-1)^{k-1}\delta_{i_k n} \tag{3.29}$$

if $m \in \mathcal{S}_0$ is expressed as (3.22). If $m \notin \mathcal{S}_0$, then $\eta(m,n) := 0$ for all $n \in \mathbf{N}$.

**Theorem 3.10** *Let $z \in D$ and $n \in \mathbf{N}$. Then*

$$\sum_{m=1}^{\infty} \frac{z^{\gamma(m)}\eta(m,n)}{F_T(m)^s} = \frac{z}{e^{sE_n(T)}+z} Z_{\mathrm{F}}(s,z;T). \tag{3.30}$$

The left hand side of (3.21) is equal to $\sum_{m \in \mathcal{S}_0(N)} z^{\gamma(m)}/F_T(m)^s$. Hence we obtain

$$\sum_{m|N} \frac{z^{\gamma(m)}|\mu(m)|}{F_T(m)^s} = \det\left(1 + ze^{-sT_N}\right). \tag{3.31}$$

## 3.3 Connections with analytic number theory

Consider the case where $\mathcal{H} = \ell^2$ and $T = \omega_{\mathcal{P}}$. Let $z \in D$ and $s > 1$. Then we have

$$Z_{\mathrm{F}}(s,z;\omega_{\mathcal{P}}) = \sum_{m=1}^{\infty} \frac{z^{\gamma(m)}|\mu(m)|}{m^s}. \tag{3.32}$$

Let $f$ be a completely multiplicative function as in Section 2.3 and $z \in D$. Then, by (2.41), we have

$$Z_{\mathrm{F}}(1,z;A_f) = \sum_{m=1}^{\infty} z^{\gamma(m)}|\mu(m)|f(m). \tag{3.33}$$

By Theorem 3.8, we obtain the following.

**Corollary 3.11** *For all $z \in D$,*

$$\sum_{m=1}^{\infty} z^{\gamma(m)}|\mu(m)|f(m) = \prod_{p \in \mathcal{P}}(1 + zf(p)). \tag{3.34}$$

Theorem 3.10 gives the following.

**Corollary 3.12** *For all $n \in \mathbf{N}$ and $z \in D$,*

$$\sum_{m=1}^{\infty} z^{\gamma(m)}\eta(m,n)f(m) = \frac{zf(p_n)}{1+zf(p_n)} Z_{\mathrm{F}}(1,z;A_f). \tag{3.35}$$

*Jordan's totient function* $J_s(N)$ ($s \geq 0, N \in \mathbf{N}$) is defined by $J_s(1) := 1$ and, for $N \geq 2$,

$$J_s(N) = N^s \prod_{p|N;p\in\mathcal{P}} \left(1 - \frac{1}{p^s}\right) \tag{3.36}$$

[1, p.48]. The special case

$$\varphi(N) = J_1(N) \tag{3.37}$$

is *Euler's totient function* [1, p.25, p.27]. We have

$$\det\left(1 - e^{-s(\omega_{\mathcal{P}})_N}\right) = \prod_{p|N;p\in\mathcal{P}} \left(1 - \frac{1}{p^s}\right), \quad s \geq 0,\ N \geq 2. \tag{3.38}$$

Hence we obtain

$$J_s(N) = N^s \det\left(1 - e^{-s(\omega_{\mathcal{P}})_N}\right), \quad s \geq 0,\ N \geq 2, \tag{3.39}$$

which, together with (3.21), implies that

$$J_s(N) = N^s \mathrm{Tr}\left(R_N(-1)^{N_{\mathrm{F}}} e^{-sH_{\mathrm{F}}(\omega_{\mathcal{P}})} R_N\right), \quad s \geq 0,\ N \in \mathbf{N}. \tag{3.40}$$

This gives an expression of Jordan's totient function in terms of Fock space objects. Formula (3.31) implies the well known identity [1, p.48]:

$$J_s(N) = \sum_{m|N} \mu(m) \left(\frac{N}{m}\right)^s, \quad s \geq 0,\ N \in \mathbf{N}. \tag{3.41}$$

# 4 Arithmetical Aspects of Boson-Fermion Fock Spaces

## 4.1 Some general aspects

Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces as before. Then the Boson-Fermion Fock space associated with the pair $\langle\mathcal{H},\mathcal{K}\rangle$ is defined by the tensor product Hilbert space

$$\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K}) := \mathcal{F}_{\mathrm{B}}(\mathcal{H}) \otimes \mathcal{F}_{\mathrm{F}}(\mathcal{K}). \tag{4.1}$$

Let $A$ and $T$ be nonnegative self-adjoint operators on $\mathcal{H}$ and $\mathcal{K}$ respectively. Then the operator

$$H(A,T) := H_{\mathrm{B}}(A) \otimes I + I \otimes H_{\mathrm{F}}(T) \tag{4.2}$$

on $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$ is nonnegative and self-adjoint.

We assume the following.

**Hypothesis (AT)** *The operators $A$ and $T$ satisfy Hypothesis (A) in Section 2 and Hypothesis (T) in Section 3 respectively.*

Under this assumption, $e^{-sH(A,T)}$ is trace class and we can define a partition function

$$Z(s,z,w;A,T) := \mathrm{Tr}\left(\Gamma_{\mathrm{B}}(z)\otimes\Gamma_{\mathrm{F}}(w)e^{-sH(A,T)}\right), \quad z,w\in D. \tag{4.3}$$

We have

$$Z(s,z,w;A,T) = Z_{\mathrm{B}}(s,z;A)Z_{\mathrm{F}}(s,w;T), \quad z,w\in D. \tag{4.4}$$

If one can represent the left hand side of (4.4) in various ways, (4.4) may produce nontrivial arithmetical relations for eigenvalues of $A$ and $T$. Moreover, different expressions of $\mathrm{Tr}\left(Xe^{-sH(A,T)}\right)$ with $X$ an operator on $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$ may yield interesting arithmetical relations. These are basic ideas to search for arithmetical relations by quantum field theoretical methods.

We carry over the notation in the preceding sections. Let $N\geq 2$ be of the form (2.15) and $m\in\mathcal{S}_0(N)$. Then we can write

$$m = (p_{i_1})^{q_1}(p_{i_2})^{q_2}\cdots(p_{i_n})^{q_n}, \tag{4.5}$$

where $q_j=0$ or $q_j=1$. Based on these factorizations, we define a vector

$$\Omega_{N,m} := C_{N,m}\left[(a_{i_1}^*)^{\alpha_1-q_1}\cdots(a_{i_n}^*)^{\alpha_n-q_n}\Omega_{\mathcal{H}}\right]\otimes\left[(b_{i_1}^*)^{q_1}\cdots(b_{i_n}^*)^{q_n}\Omega_{\mathcal{K}}\right], \tag{4.6}$$

where $C_{N,m}>0$ is a normalization constant. For $N=1$ and $m=1$, we set $\Omega_{1,1} := \Omega_{\mathcal{H}}\otimes\Omega_{\mathcal{K}}$.

**Lemma 4.1** [28] *The set* $\{\Omega_{N,m}|N\geq 1,\ m\in\mathcal{S}_0(N)\}$ *is a CONS of* $\mathcal{F}_{\mathrm{BF}}(\mathcal{H},\mathcal{K})$.

The following fact is easily proven.

**Lemma 4.2** *Let* $N\in\mathbf{N}$, $m\in\mathcal{S}_0(N)$ *and* $z,w\in D$. *Then* $\Omega_{N,m}$ *is an eigenvector of* $\Gamma_{\mathrm{B}}(z)\otimes\Gamma_{\mathrm{F}}(w)$ *with eigenvalue* $z^{\gamma(N)-\gamma(m)}w^{\gamma(m)}$.

For each $N\in\mathbf{N}$, we define a function $Y_{A,T}(N,\cdot)$ on $\mathcal{S}_0(N)$ by

$$Y_{A,T}(N,m) := \prod_{k=1}^{n} e^{(\alpha_k-q_k)E_{i_k}(A)+q_kE_{i_k}(T)}, \quad m\in\mathcal{S}_0(N), \tag{4.7}$$

when $N$ and $m$ are represented as (2.15) and (4.5) respectively. Note that

$$Y_{A,T}(N,m) = F_A\left(\frac{N}{m}\right)F_T(m). \tag{4.8}$$

**Lemma 4.3** *Let* $N\in\mathbf{N}$ *and* $m\in\mathcal{S}_0(N)$. *Then* $\Omega_{N,m}$ *is an eigenvector of* $H(A,T)$ *with eigenvalue* $\log Y_{A,T}(N,m)$.

**Theorem 4.4** *Let* $z,w\in D$. *Then*

$$Z(s,z,w;A,T) = \sum_{N=1}^{\infty}\sum_{m|N}\frac{z^{\gamma(N)-\gamma(m)}w^{\gamma(m)}|\mu(m)|}{Y_{A,T}(N,m)^s}. \tag{4.9}$$

**Corollary 4.5** *Let* $z,w\in D$. *Then*

$$\sum_{N=1}^{\infty}\sum_{m|N}\frac{z^{\gamma(N)-\gamma(m)}w^{\gamma(m)}|\mu(m)|}{Y_{A,T}(N,m)^s} = Z_{\mathrm{B}}(s,z;A)Z_{\mathrm{F}}(s,w;T). \tag{4.10}$$

**Remark 4.1** If we put into the right hand side of (4.10) the formulas established in Sections 2 and 3, then we obtain explicit formulas, which are nontrivial.

**Remark 4.2** By rescaling as $T\to tT/s$ $(t>0)$ in (4.10), we can obtain relations at different temperatures $1/s$ and $1/t$. Hence (4.10) include "duality relations".

## 4.2 Connections with analytic number theory

We consider the case where $\mathcal{H} = \mathcal{K} = \ell^2$ and $A = T = \omega_{\mathcal{P}}$. Then we have $Y_{\omega_{\mathcal{P}},\omega_{\mathcal{P}}}(N,m) = N$. Hence Corollary 4.5 gives

$$\sum_{N=1}^{\infty} \sum_{m|N} \frac{z^{\gamma(N)-\gamma(m)} w^{\gamma(m)} |\mu(m)|}{N^s} = Z_{\mathrm{B}}(s,z;\omega_{\mathcal{P}}) Z_{\mathrm{F}}(s,w;\omega_{\mathcal{P}}), \quad s > 1. \tag{4.11}$$

This yields well known relations

$$\sum_{N=1}^{\infty} \frac{2^{\nu(N)}}{N^s} = \frac{\zeta(s)}{D(s,\lambda)}, \quad \sum_{N=1}^{\infty} \frac{\lambda(N) 2^{\nu(N)}}{N^s} = \frac{D(s,\lambda)}{\zeta(s)}, \quad s > 1.$$

Let $f$ be the completely multiplicative function considered in Section 2.3 and

$$H := H(A_f, A_f)$$

Then we have for all $s > 1$

$$\mathrm{Tr}\left(\Gamma_{\mathrm{F}} e^{-sH}\right) = 1, \quad \mathrm{Tr}\left(\Gamma_{\mathrm{B}} e^{-sH}\right) = 1, \tag{4.12}$$

which are supersymmetric identities [6, 28]. These relations imply the following:

$$\sum_{m=1}^{\infty} \mu(m) f(m) = \frac{1}{\sum_{n=1}^{\infty} f(n)}, \quad \sum_{m=1}^{\infty} |\mu(m)| f(m) = \frac{1}{\sum_{n=1}^{\infty} \lambda(n) f(n)}. \tag{4.13}$$

By Corollary 4.5 with rescaling $T \to tT/s$, we obtain

$$\sum_{N=1}^{\infty} \sum_{m|N} \frac{z^{\gamma(N)-\gamma(m)} w^{\gamma(m)} |\mu(m)|}{N^s m^{t-s}} = Z_{\mathrm{B}}(s,z;\omega_{\mathcal{P}}) Z_{\mathrm{F}}(t,w;\omega_{\mathcal{P}}), \quad t > s > 1. \tag{4.14}$$

**Remark 4.3** General theories on Boson-Fermion Fock spaces have been developed in [3, 5, 6, 7, 9, 11, 13, 15, 16]. See also [2, 4, 8, 10] for related aspects. Applications of these theories to arithmetic quantum field theories may yield interesting results in analytic number theory.

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