# ENDOSCOPIC CONGRUENCES AND ADJOINT $L$-VALUES FOR GSp(4)

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In this article, we present the result of [LO18] on the congruence between cuspidal automorphic representations of $\mathrm{GSp}_4$, which is a joint article with Francesco Lemma.

## 1. Congruence for $\mathrm{GL}_2$ over $\mathbb{Q}$ by Hida and related results

In this section, we recall a result by Hida on the relation between the existence of non-trivial congruence for a given eigen cuspform $f$ and the special value of adjoint $L$-function associated to $f$, which is a prototype of the work [LO18].

Let $f = \sum_{n=1}^{\infty} a_n(f)q^n \in S_k(\Gamma_1(M))$ be a primitive cuspform, with $k \geq 2$ and we set $\mathbb{Q}_f := \mathbb{Q}(\{a_n(f)\}_{n\geq 1})$ and $r := [\mathbb{Q}_f : \mathbb{Q}]$.

**Definition 1.1.** *We denote by $L(s, f, \mathrm{Ad})$ the adjoint $L$-function defined by the Euler product*

$$L(s, f, \mathrm{Ad}) = \prod_{\ell \nmid M} (1 - \frac{\alpha_\ell}{\beta_\ell}\ell^{-s})^{-1}(1 - \frac{\beta_\ell}{\alpha_\ell}\ell^{-s})^{-1}(1 - \ell^{-s})^{-1} \times (\textit{bad Euler factors})$$

*which is absolutely convergent for* $\mathrm{Re}(s) > 1$. *The $L$-function $L(s, f, \mathrm{Ad})$ is holomorphically continued to the whole $\mathbb{C}$-plane and does not vanish at $s = 1$.*

*We also set*

$$Z(s, f, \mathrm{Ad}) = \prod_{\sigma:\mathbb{Q}_f \to \overline{\mathbb{Q}}} L(s, {}^{\sigma}f, \mathrm{Ad}).$$

By the comparison theorem between the de Rham cohomology and the Betti cohomology, we have the Eichler-Shimura isomorphism as follows:

$$\mathrm{ES} : S_k(\Gamma_1(N); \mathbb{C}) \xrightarrow{\sim} H^1_{\mathrm{par}}(\Gamma_1(N); \mathrm{Sym}^{k-2}(\mathbb{R}^{\oplus 2}))$$

where the symbol "par" means imposing local triviality at all parabolic subgroups of $\Gamma_1(N)$.

On the de Rham side, we define a $\mathbb{C}$-subvector space $S_f \subset S_k(\Gamma_1(N); \mathbb{C})$ by

$$S_f := \bigoplus_{\sigma:\mathbb{Q}_f \to \overline{\mathbb{Q}}} \mathbb{C}f^{\sigma}.$$

On the Betti side, we define an $\mathbb{R}$-subvector space $W_f \subset H^1_{\mathrm{par}}(\Gamma_1(N); \mathrm{Sym}^{k-2}(\mathbb{R}^{\oplus 2}))$ by

$$W_f := \mathrm{ES}(S_f).$$

Note that $W_f$ is an $\mathbb{R}$-vector space of dimension $2r$.

We define a $\mathbb{Z}$-module $L_f$ by

$$L_f := H^1_{\mathrm{par}}(\Gamma_1(N); \mathrm{Sym}^{k-2}(\mathbb{Z}^{\oplus 2})) \cap W_f,$$

which is an intersection inside $H^1_{\mathrm{par}}(\Gamma_1(N); \mathrm{Sym}^{k-2}(\mathbb{R}^{\oplus 2}))$ and $L_f \cong \mathbb{Z}^{2r}$ is a lattice of $W_f$.

**Definition 1.2.** *Let $\delta_1, \dots, \delta_{2r}$ be an $\mathbb{R}$-basis of $S_f$ given by $\delta_i := f^{\sigma_i}$ $(i = 1, \dots, r)$ for $\sigma_1, \dots \sigma_r : \mathbb{Q}_f \to \overline{\mathbb{Q}}$ and $\delta_{r+i} := \sqrt{-1}\delta_i$ $(i = 1, \dots, r)$. Choose (non-canonically) a $\mathbb{Z}$-basis $\beta_1, \dots, \beta_{2r}$ of $L_f$. Note that $\beta_1, \dots, \beta_{2r}$ is also an $\mathbb{R}$-basis of $W_f$.*

*We define a complex period $u_f = u_f(\beta_1, \dots, \beta_{2r}; \delta_1, \dots \delta_{2r}) \in \mathbb{R}^\times/\mathbb{Z}^\times$ by*

$$u_f = u_f(\beta_1, \dots, \beta_{2r}; \delta_1, \dots \delta_{2r}) := \det(\mathrm{ES}|_{S_f}).$$

Put

$$C(f) := \left(2^{-(k+1)}(k-1)! N \mathrm{Cond}(\psi_f)\varphi(\tfrac{N}{\mathrm{Cond}(\psi_f)})\right)^r \times \frac{Z(1, f, \mathrm{Ad})}{\pi^{(k+1)r} u_f}$$

where $\psi_f$ is the Neben character of $f$. It is verified that $C(f)^2 \in \mathbb{Z}$ in [Hi81a].

**Theorem 1.3** (Hida [Hi81a])**.** *Let $f \in S_k(\Gamma_1(M))$ be a primitive cuspform, with $k \geq 2$ and with trivial Nebentypus. Let $p$ be a prime such that $p \geq k-1$ and $p \nmid 6M$. Then*

$$p | C(f)^2 \Rightarrow \exists \text{ a normalized eigen cuspform } g \in S_k(\Gamma_1(M))$$

*such that*

*(i) $\forall \sigma \in \mathrm{Aut}(\overline{\mathbb{Q}})$, $g \neq f^\sigma$,*

*(ii) $f \equiv g$ mod $\mathfrak{p}$ for some prime $\mathfrak{p}$ of $\overline{\mathbb{Q}}$ over $p$.*

**Remark 1.4.** *Ghate published a variant of the above theorem for each prime $\mathfrak{p}$ of $\overline{\mathbb{Q}}$ over $p$ which insists that*

$$\mathfrak{p} | c(f) \Rightarrow \exists \text{ a normalized eigen cuspform } g \in S_k(\Gamma_1(M)) \text{ such that } f \equiv g \text{ mod } \mathfrak{p}.$$

**Definition 1.5.** *(1) Let*

$$\langle\ ,\ \rangle : H^1_{\mathrm{par}}(\Gamma_1(N); \mathrm{Sym}^{k-2}(\mathbb{R}^{\oplus 2})) \times H^1_{\mathrm{par}}(\Gamma_1(N); \mathrm{Sym}^{k-2}(\mathbb{R}^{\oplus 2})) \longrightarrow H^2_{\mathrm{par}}(\Gamma_1(N); \mathbb{R}) \cong \mathbb{R}$$

*be the non-degenerate pairing induced by the $\Gamma_1(N)$-equivariant pairing*

$$\mathrm{Sym}^{k-2}(\mathbb{R}^{\oplus 2}) \times \mathrm{Sym}^{k-2}(\mathbb{R}^{\oplus 2}) \to \mathbb{R} : \ {}^t\begin{pmatrix} u \\ v \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \det \begin{pmatrix} u & x \\ v & y \end{pmatrix}.$$

*(2) We define the discriminant $d(f)$ by*

$$d(f) := \mathrm{disc}(\langle\ ,\ \rangle|_{W_f}) = \det(\langle \delta_i, \delta_j \rangle)_{1 \leq i,j \leq 2r}.$$

The strategy of the proof by Hida is summarized as follows:

Existence of mod $p$ congruence $f \equiv g$

$$\overset{\text{Step A}}{\Longleftarrow} p \mid \text{the cohomological discriminant } d(f) \overset{\text{Step B}}{\Longleftrightarrow} p \mid \text{Adjoint } L\text{-value}$$

We will explain two main steps of the proof (Step A and Step B).

**Step A**

We define a $\mathbb{Z}$-module $M_f$ by

$$M_f := \mathrm{Image}\Big[H^1_{\mathrm{par}}(\Gamma_1(N); \mathrm{Sym}^{k-2}(\mathbb{Z}^{\oplus 2})) \hookrightarrow H^1_{\mathrm{par}}(\Gamma_1(N); \mathrm{Sym}^{k-2}(\mathbb{R}^{\oplus 2})) \twoheadrightarrow W_f\Big],$$

which is a lattice of $W_f$ such that $L_f \subset M_f$. It is known that $d(f) = \#(M_f/L_f)$ and we see that

$$\begin{aligned} p|d(f) &\Leftrightarrow L_f \otimes_{\mathbb{Z}} \mathbb{Z}_p \text{ is not a direct summand as a module over a } p\text{-adic Hecke algebra,} \\ &\Rightarrow L_f \otimes_{\mathbb{Z}} \mathbb{Z}/(p) \cong L_g \otimes_{\mathbb{Z}} \mathbb{Z}/(p) \text{ as a module over a mod } p \text{ Hecke algebra.} \end{aligned}$$

**Step B**

Consider the Petersson inner product

$$(\ \ ,\ \ ) : \ S_k(\Gamma_1(N); \mathbb{C}) \times S_k(\Gamma_1(N); \mathbb{C}) \longrightarrow \mathbb{C}.$$

The Peterson inner product $(\ ,\ )$ on $S_k(\Gamma_1(N); \mathbb{C})$ is compatible with the pairing $\langle\ ,\ \rangle$ on $H^1_{\mathrm{par}}(\Gamma_1(N); \mathrm{Sym}^{k-2}(\mathbb{R}^{\oplus 2}))$ by Eichler-Sihmura isomorphism

$$\mathrm{ES} : S_k(\Gamma_1(N); \mathbb{C}) \xrightarrow{\sim} H^1_{\mathrm{par}}(\Gamma_1(N); \mathrm{Sym}^{k-2}(\mathbb{R}^{\oplus 2})).$$

Simura proves the following formula

$$(f, f) = 2^{-2k}(k-1)! N \mathrm{Cond}(\psi_f) \varphi\left(\tfrac{N}{\mathrm{Cond}(\psi_f)}\right) \frac{L(1, f, \mathrm{Ad})}{\pi^{k+1}},$$

which relates the divisibility of $C(f)$ by $p$ with the divisibility of $d(f)$ by $p$.

**Remark 1.6.** *We ask ourselves if the converse of Theorem 1.3 is true. "$\Leftarrow$" of Step A in the proof of Hida becomes "$\Leftrightarrow$" if the module $H^1_{\mathrm{par}}(\Gamma_1(N); \mathrm{Sym}^{k-2}(\mathbb{Z}_p^{\oplus 2}))$ is free over the p-adic Hecke algebra. This freeness (and thus "$\Leftarrow$" of Step A) holds true if $f$ is ordinary by Hida (Invent. 64/1981). Ribet (Invent. 71/1983) proves "$\Leftarrow$" of Step A if $p \geq k$.*

**Some historical remarks**

We list some of results after Hida on the congruence primes for the $GL(n)$ case.

- $GL(2)$ over imaginary quadratic fields by Urban [U95]
- $GL(2)$ over totally real fields by Ghate [Gh02] / Dimitrov [Di05]
- $GL(2)$ over number fields by Namikawa [N15]
- $GL(n)$ over number fields by Balasubramanyan-Raghram [BaR17]

**Remarks 1.7.** *Our result for* $\mathrm{GSp}_4$ *and its proof which will be presented in the next section has a similar flavor as* [BaR17]*. But, we have an advantage that all constants are explicit in our main theorem (In* [BaR17]*, there appears constants which is not made explicit). See Remark 2.5 and some comments at the end of the next section.*

## 2. Our results

Now we present the main result of [LO18]. First, we fix the setting of this article.

**Setting**

We set $G = \mathrm{GSp}_4$, $\mathbb{A} = \mathbb{R} \times \mathbb{A}_{\mathrm{fin}}$ the ring of adèles of $\mathbb{Q}$.
We consider $\pi = \bigotimes_{\ell}' \pi_\ell$: cuspidal automorphic representation of $G(\mathbb{A})$ and we assume the following conditions in the rest of the paper:

(a) The central character of $\pi$ is trivial.
(b) The representation $\pi$ is globally generic.
(c) The paramodular conductor $N$ of $\pi$ is square-free.
(d) $\exists k \geq \exists k' \geq 0$ such that $H^3(\mathfrak{g}, K_\infty, V_{k,k'}(\mathbb{C}) \otimes \pi_\infty) \neq 0$
where $V_{k,k'}$ is the algebraic representation of $G$ of highest weight $k, k'$.
(e) The representation $\pi$ is endoscopic.

**Remarks 2.1.** *In order to use it later in the proof, we remark that the endoscopic assumption* $(e)$ *above is translated into the situation where there exist elliptic cuspforms* $f_i$ *of weight* $k_i$ *and level* $N_i$ $(i = 1, 2)$ *such that*

- $k_1 = k + k' + 4,\ k_2 = k - k' + 2$
- $N_1 N_2 = N$
- $\rho_\pi \cong \rho_{f_1} \oplus \rho_{f_2}$
*where* $\rho_\pi$ *(resp.* $\rho_{f_1}, \rho_{f_2}$*) is the* $p$*-adic Galois representation of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *associated to* $\pi$ *(resp.* $f_1$, $f_2$*)*

We define $\mathbb{Q}_{\pi_{\mathrm{fin}}}$ to be the rationality Number field of $\pi_{\mathrm{fin}}$ in the sense of Clozel-Waldspurger (see [Wa85] for example) and we set $r := [\mathbb{Q}_{\pi_{\mathrm{fin}}} : \mathbb{Q}]$.

**Definition 2.2.** *We denote by* $L(s, \pi, \mathrm{Ad})$ *the adjoint* $L$*-function defined by the Euler product* $L(s, \pi, \mathrm{Ad}) = \prod_{\ell \nmid M} \det(\mathrm{Id} - \mathrm{Ad}(\widehat{t_\ell})\ell^{-s}; \mathfrak{sp}_4(\mathbb{C})) \times$ *(bad Euler factors)* *where* $\widehat{t_\ell} \in {}^L\mathrm{GSp}_4 = \mathrm{GSpin}_5(\mathbb{C})$ *is defined by using Satake-parameters of* $\pi$ *at* $\ell$*. Since* $\mathfrak{sp}_4(\mathbb{C})$ *is of dimension* 10*, the Euler product above is of degree* 10*. The function* $L(s, \pi, \mathrm{Ad})$ *is absolutely convergent for* $\mathrm{Re}(s) > 1$ *and is holomorphically continued to the whole* $\mathbb{C}$*, non-vanishing at* $s = 1$*.*

*We also set*

$$Z(s, \pi, \mathrm{Ad}) = \prod_{\sigma : \mathbb{Q}_{\pi_{\mathrm{fin}}} \to \overline{\mathbb{Q}}} L(s, {}^\sigma\pi, \mathrm{Ad}).$$

Let $K_N \subset G(\mathbb{A}_{\mathrm{fin}})$ be the subgroup of paramodular level $N$, $K(3) \subset G(\mathbb{A}_{\mathrm{fin}})$ the congruence subgroup of full level 3, $K_N(3) \subset G(\mathbb{A}_{\mathrm{fin}})$ the intersection $K_N \cap K(3)$. For an open subgroup $K \subset G(\mathbb{A}_{\mathrm{fin}})$, we denote by $S_K$ the Shimura variety of level $K$.

We give some sets of "bad primes"

- $S_{N,3} = \{p \text{ primes}, p | \# K_N / K_N(3)\}$
- $S_{\mathrm{weight}} = \{p \text{ primes}, p < k + k' + 3\}$
- $S_{\mathrm{tors}} = \{p \text{ primes}, p | \# H^3_!(S_{K_N(3)}, V_{k,k'}(\mathbb{Z}))_{\mathrm{tors}}\}$
- $S'_{\mathrm{tors}} = \{p \text{ primes}, p | \# H^4_c(S_{K_N(3)}, V_{k,k'}(\mathbb{Z}))_{\mathrm{tors}}\}$
- $S''_{\mathrm{tors}} = \{p \text{ primes}, p | \# (H^3(S_{K_N(3)}, V_{k,k'}(\mathbb{Z})) / H^3_!(S_{K_N(3)}, V_{k,k'}(\mathbb{Z})))_{\mathrm{tors}}\}$

By the comparison theorem between the de Rham cohomology and the Betti cohomology for $S_K$, we have the Eichler-Shimura isomorphism as follows:

$$\mathrm{ES}|_{S_\pi} : S_\pi \xrightarrow{\sim} W_\pi$$

where we set $S_\pi := \bigoplus_{\sigma : \mathbb{Q}_{\pi_{\mathrm{fin}}} \to \overline{\mathbb{Q}}} {}^\sigma\pi^{K_N}$ and $W_\pi$ is an $\mathbb{R}$-subspace of $H^3_!(S_{K_N(3)}, V_{k,k'}(\mathbb{R}))^{K(3)}$. The $\mathbb{R}$-vector spaces $S_\pi$ and $W_\pi$ are of dimension $2r$.

In order to define a period invariant as a determinant of $\mathrm{ES}|_{S_\pi} : S_\pi$, we will give a basis on each of $S_\pi$ and $W_\pi$.

**Definition 2.3.** *(1) Let* $\varphi = \underset{v\leq\infty}{\otimes}\varphi_v \in \pi^{K_N}$ *be the element whose Whittaker function is normalized as:*

$$\begin{aligned}
W_{\varphi_\ell}(1) =& 1 \text{ for any prime } \ell \nmid N,\\
W_{\varphi_\ell}(\mathrm{diag}(\ell^{-1},1,\ell^2,\ell)) =& 1 \text{ for any prime } \ell | N,\\
W_{\varphi_\infty}(1) =& e^{-2\pi}\int_{c_1-\sqrt{-1}\infty}^{c_1+\sqrt{-1}\infty}\frac{ds_1}{2\pi\sqrt{-1}}\int_{c_2-\sqrt{-1}\infty}^{c_2+\sqrt{-1}\infty}\frac{ds_2}{2\pi\sqrt{-1}}\\
&\times(4\pi^3)^{(-s_1+k+4)/2}(4\pi)^{(-s_2-k'-1)/2}\\
&\times\Gamma\left(\frac{s_1+s_2-2k'-1}{2}\right)\Gamma\left(\frac{s_1+s_2+1}{2}\right)\Gamma\left(\frac{s_1}{2}\right)\Gamma\left(\frac{s_2}{2}\right)\\
&(c_1, c_2 \in \mathbb{R} \text{ with } c_1+c_2+1>0 \text{ and } c_1>0>c_2).
\end{aligned}$$

*Put* $\delta_i := \varphi^{\sigma_i}$ $(1\leq i\leq r)$ *for* $\sigma_1,\dots\sigma_r : \mathbb{Q}_{\pi_{\mathrm{fin}}} \to \overline{\mathbb{Q}}$ *and we set* $\delta_{r+i} := \sqrt{-1}\delta_i$ $(1\leq i\leq r)$. *Then* $\delta_1,\dots,\delta_{2r}$ *is an* $\mathbb{R}$*-basis of* $S_\pi$.

*(2) Let us assume that* $p\notin S_{\mathrm{tors}}$. *We define*

$$L_\pi := H^3_!(S_{K_N(3)}, V_{k,k'}(\mathbb{Z}_{(p)}))^{K(3)} \cap W_\pi$$

*(an intersection inside* $H^3_!(S_{K_N(3)}, V_{k,k'}(\mathbb{R}))^{K(3)}$*).*
*Note that we have* $L_\pi \cong \mathbb{Z}_{(p)}^{2r}$ *and* $L_\pi$ *is a lattice of* $W_\pi$. *We choose (non-canonically) a* $\mathbb{Z}_{(p)}$*-basis* $\beta_1,\dots,\beta_{2r}$ *of* $L_\pi$. *Then* $\beta_1,\dots,\beta_{2r}$ *is also an* $\mathbb{R}$*-basis of* $W_\pi$.

*(3) We define a complex period* $u_\pi = u_\pi(\beta_1,\dots,\beta_{2r};\delta_1,\dots\delta_{2r}) \in \mathbb{R}^\times/\mathbb{Z}_{(p)}^\times$ *by*

$$u_\pi = u_\pi(\beta_1,\dots,\beta_{2r};\delta_1,\dots\delta_{2r}) := \det(\mathrm{ES}|_{S_\pi}).$$

Put

$$C(\pi) := \left(\frac{C_\infty C_{k,k'}C_N\pi^3}{2}\right)^r \times \frac{Z(1,\pi,\mathrm{Ad})}{u_\pi}$$

where

$$\begin{aligned}
C_\infty &= 2^{k+k'+5}(k+k'+5)^{-1}\pi^{3k+k'+12},\\
C_{k,k'} &= (-1)^{k+k'}3^{-3}5^{-k}(k+k'+4)!(k+k'+5)!,\\
C_N &= \prod_{\ell|N}(\ell+\ell^{-1})^{-1}(\ell^2+1)^{-1}.
\end{aligned}$$

In the proof of our result, we can check that $C(\pi)^2 \in \mathbb{Z}_{(p)}$.

**Theorem 2.4** (Lemma-O)**.** *Let* $\pi$ *be a cuspidal automorphic representation of* $\mathrm{GSp}_4(\mathbb{A})$ *satisfying the conditions (a) to (e). Assume that* $p\notin S_{N,3}\cup S_{\mathrm{weight}}\cup S_{\mathrm{tors}}\cup S'_{\mathrm{tors}}\cup S''_{\mathrm{tors}}$. *Assume further the following conditions*

- *Mod* $p$ *Galois representations* $\overline{\rho}_{f_1}$, $\overline{\rho}_{f_2}$ *of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *is irreducible.*
- *For any prime* $\mathfrak{p}$ *above* $p$ *in* $\overline{\mathbb{Q}}$, $\mathfrak{p}$ *does not divide* $c(f_1)$ *nor* $c(f_2)$.
- *The prime* $p$ *divides* $C(\pi)^2$.

*Then there exists a stable cuspidal automorphic representation $\pi'$ of $\mathrm{GSp}_4(\mathbb{A})$ with the same weight and the level as $\pi$ such that*

*(i) $\forall \sigma \in \mathrm{Aut}(\overline{\mathbb{Q}})$, $\pi' \not\cong {}^{\sigma}\pi$,*

*(ii) $\pi \equiv \pi'$ mod $\mathfrak{P}$ for some prime $\mathfrak{P}$ of $\overline{\mathbb{Q}}$ over $p$.*

The strategy of the proof is summarized as follows:

Existence of mod $p$ congruence $\pi \equiv \pi'$

$$\overset{\text{Step A}}{\Longleftarrow} p \mid \text{the cohomological discriminant } d(\pi) \overset{\text{Step B}}{\Longleftrightarrow} p \mid \text{Adjoint } L\text{-value}$$

Step A is similar to the argument of Hida but more complicated. In general, the automorphic contributions on the Betti cohomology of higher dimensional Simura varieties are much more complicated and not completely understood. However, the space $H^3(S_{K_N(3)}, V_{k,k'}(\mathbb{C}))$ is well-understood (Vogan-Zuckerman [VZ84] + Taylar (Invent. 1993)) fortunately. In any case, Wwe need some combinatorial arguments using our assumptions to prove that $\pi'$ is stable (see [LO18] for the detail).

Step B is due to a formula of Chen-Ichino

$$\langle \varphi, \varphi \rangle = 2^2 C_\infty \frac{L(1,\pi,\mathrm{Ad})}{\zeta(2)\zeta(4)}.$$

See their preprint [CI19] for the proof.

**Remarks 2.5.** *One of the most important advantages of our work is that $C_{k,k'}$ and $C_\infty$ is totally explicit. For example, in Balasubramanyan-Raghram (2017) for $\mathrm{GL}(n)$, the analogues of $C_{k,k'}$ and $C_\infty$ were non-zero numbers which depends only on the weight but we could not make them explicit.*

We end this article with some comments on the history of our research behind the determination of $C_{k,k'}$ and $C_\infty$.

(1) The constant $C_{k,k'}$ was much more complicated in an earlier version of our paper. (The actual simplified version was conjectured by Namikawa and proved with help of Yasuda)

(2) The constant $C_\infty$ was not explicit in the original paper by Ichino (2007) and it has become explicit in the updated article by Ichino-Chen [CI19].

As for the constant $C_{k,k'}$, we previously defined

$$(1) \quad C_{k,k'} = \frac{(-1)^{k+k'}(k+k')!(k+k'+4)!}{3!^2} \sum_{r=0}^{4} \sum_{i=0}^{k+k'} \sum_{\substack{0\le u,u'\le r \\ 0 \le i-u}} (-1)^{r+u+u'} a_r \binom{r}{u} \binom{r}{u'} r^{k,k'}_{i,u,r} r^{k,k'}_{i-u+u',u',r} s^{k,k'}_{i,u} t^{k,k'}_{i,u,r}$$

where

$$r^{k,k'}_{i,u,r} = \frac{(k+k'+u-i)!(k+k'+4-i)!(i+r-u)!}{(i-u)!(k+k'-i)!(k+k'+4-i-r+u)},$$

$$s^{k,k'}_{i,u} = \frac{(i-u)!}{(k+k'-i+u)!},\ t^{k,k'}_{i,u,r} = \frac{(k+k'+4+u-r-i)!}{(i+r-u)!},$$

$$a_0 = -1, a_1 = -\frac{1}{4}, a_2 = \frac{1}{72}, a_3 = -\frac{1}{72}, a_4 = -\frac{1}{576}.$$

We refer to the updated version [LO18] for the proof of the equality between the version (1) and the following simplified presentation of $C_{k,k'}$:

$$C_{k,k'} = (-1)^{k+k'} 3^{-3} 5^{-k} (k+k'+4)!(k+k'+5)! \tag{2}$$

but we remark that one of the most important the keys is the following formula (cf. §16 of the book "Calculus of finite differences" by C. Jordan (1965)):

$$\sum_{a=0}^{l} (-1)^a \binom{l}{a} (b-a)_m = \binom{m}{l} l!(b)_{m-l}$$

where $(x)_n$ is the Pochhammer symbol defined as follows:

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1) = \frac{(x+n-1)!}{(x-1)!}.$$

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