THREE-TERM ARITHMETIC PROGRESSIONS OF PIATETSKI-SHAPIRO SEQUENCES

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Abstract. For every non-integral $\alpha > 1$, the sequence of the integer parts of n^α ($n = 1, 2, \ldots$) is called the Piatetski-Shapiro sequence with exponent α. Let $\mathrm{PS}(\alpha)$ be the set of all those terms. In a previous study, Matsusaka and the author studied the set of $\alpha \in I$ such that $\mathrm{PS}(\alpha)$ contains infinitely many arithmetic progressions of length 3, where I is a closed interval of $[2, \infty)$. As a corollary of their main result, they showed that the set is uncountable and dense in I. The aim of this article is to provide a direct proof of this result.

1. Introduction

We let $\lfloor x \rfloor$ denote the integer part of $x \in \mathbb{R}$. For every non-integral $\alpha > 1$, the sequence $(\lfloor n^\alpha \rfloor)_{n=1}^\infty$ is called the Piatetski-Shapiro sequence with exponent α, and we let $\mathrm{PS}(\alpha)$ be the set of all those terms. Let us fix $a, b, c \in \mathbb{N}$. In a previous study, Matsusaka and the author studied the set of all $\alpha \in [s, t]$ such that $ax + by = cz$ holds for infinitely many $(x, y, z) \in \mathrm{PS}(\alpha)^3$ with $\#\{x, y, z\} = 3$ [MS20]. They found explicit lower bounds of the Hausdorff dimension of the set [MS20, Theorem 1.1]. As a corollary of this result, they proved

Theorem 1.1 ([MS20, Corollary 1.3])**.** *For any closed set $I \subseteq [2, \infty)$, the set of $\alpha \in I$ such that $\mathrm{PS}(\alpha)$ contains infinitely many three-term arithmetic progressions is uncountable and dense in I.*

The aim of this article is to provide a direct proof of Theorem 1.1. For this purpose, specific knowledge of fractal geometry is not required. Instead, we apply the Baire category theorem which will be covered in Section 2.

Notation 1.2. Let $\mathbb{N}$ be the set of all positive integers, $\mathbb{Z}$ be the set of all integers, $\mathbb{Q}$ be the set of all rational numbers, and $\mathbb{R}$ be the set of all real numbers. For $x \in \mathbb{R}$, let $\{x\}$ denote the fractional part of x. Let $\sqrt{-1}$ denote the imaginary unit, and define $\mathrm{e}(x)$ by $e^{2\pi\sqrt{-1}x}$ for all $x \in \mathbb{R}$.

2. Proof of Theorem 1.1

Let X be a topological space. A set $\mathcal{U} \subseteq X$ is called a G_δ set if $\mathcal{U} = \bigcap_{j=1}^\infty U_j$ for some countable open sets $U_j \subseteq X$ ($j = 1, 2, \ldots$). In this section, we prove Theorem 1.1 assuming the following:

Theorem 2.1. *There exists a G_δ set $\mathcal{U} \subseteq (1, \infty)$ which is a subset of*

$$\{\alpha \in (1, \infty) \colon \mathrm{PS}(\alpha) \text{ contains infinitely many three-term arithmetic progressions}\},$$

and $\mathcal{U}$ is dense in $(1, \infty)$.

We prove Theorem 2.1 in Section 5.

Theorem 2.2 (the Baire category theorem)**.** *Let X be a complete metric space. If sets $U_j \subseteq X$ $(j = 1, 2, \dots)$ are open and dense in X, then $\bigcap_{j=1}^{\infty} U_j$ is dense in X.*

A proof of this theorem can be found in many textbooks on functional analysis. For example, see the book written by Rudin [Rud91]. We will apply the Baire category theorem to the proofs of Theorem 1.1 and Theorem 2.1.

Proof of Theorem 1.1 assuming Theorem 2.1. Let us fix any closed interval $I \subset [2, \infty)$. We define

$$\mathcal{E} = \{\alpha \in I \colon \mathrm{PS}(\alpha) \text{ contains infinitely many three-term arithmetic progressions}\}.$$

Let us introduce the Euclidean topology to I. We now consider that I is a complete metric space. Then from Theorem 2.1, there exists a G_δ set $\mathcal{U} \subseteq I$ such that $\mathcal{U} \subseteq \mathcal{E}$ and $\mathcal{U}$ is dense in I. Thus if $\mathcal{U}$ is uncountable, we reach the conclusion of Theorem 1.1.

Let us verify that $\mathcal{U}$ is uncountable. By the definition, there exist open sets $U_j \subseteq I$ $(j = 1, 2, \dots)$ which are dense in I such that $\mathcal{U} = \bigcap_{j=1}^{\infty} U_j$. Then we take any sequence $(b_j)_{j=1}^{\infty}$ composed of $b_j \in I$ for all $j \in \mathbb{N}$. Let $B = \{b_j \colon j = 1, 2, \cdots\}$. For all $j \in \mathbb{N}$, let $V_j = U_j \setminus \{b_j\}$. It is clear that V_j is open and dense in I. Since I is a complete metric space, by Theorem 2.2 the following set is dense in I:

$$\bigcap_{j=1}^{\infty} V_j = \bigcap_{j=1}^{\infty} U_j \setminus \bigcup_{j=1}^{\infty} \{b_j\} = \mathcal{U} \setminus B.$$

Therefore $\mathcal{U} \setminus B \neq \emptyset$ which means that $\mathcal{U} \neq B$. Hence $\mathcal{U}$ is uncountable since $\mathcal{U}$ is not coincident with an arbitrary countable subset of I. □

The rest of the article focuses on proving Theorem 2.1. In Section 3, we define the uniform distribution modulo 1 of the sequences, and describe some salient prior results. In Section 4, we obtain key lemmas. Finally, in Section 5, we provide a proof of Theorem 2.1.

3. Preparations

For all $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, define $\{\mathbf{x}\} = (\{x_1\}, \{x_2\}, \dots, \{x_d\})$. A sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ composed of $\mathbf{x}_n \in \mathbb{R}^d$ for all $n \in \mathbb{N}$ is called *uniformly distributed modulo 1* if for every $0 \le a_i < b_i \le 1$ $(i = 1, 2, \dots, d)$, we have

$$\lim_{N\to\infty} \frac{1}{N} \# \left\{ n \in \mathbb{N} \cap [1, N] \colon \{x_n\} \in \prod_{i=1}^{d} [a_i, b_i) \right\} = \prod_{i=1}^{d} (b_i - a_i). \tag{3.1}$$

The sequence $(\mathbf{x}_n)_{n\in\mathbb{N}}$ is uniformly distributed modulo 1 if and only if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathrm{e}(\langle \mathbf{h}, \mathbf{x}_n \rangle) = 0$$

for all $\mathbf{h} = (h_1, \dots, h_d) \in \mathbb{Z}^d \setminus \{(0, \dots, 0)\}$ where let $\langle \cdot, \cdot \rangle$ denote the standard inner product. This equivalence is called Weyl's criterion. A proof of that can be found in the book written by Kuipers and Niederreiter [KN74].

Lemma 3.1. *Let k be a positive integer, and $f(x)$ be a function defined for $x \ge 1$, which is k times differentiable for $x \ge x_0$. If $f^{(k)}(x)$ tends monotonically to 0 as $x \to \infty$ and if $\lim_{x\to\infty} x|f^{(k)}(x)| = \infty$, then the sequence $(f(n))_{n\in\mathbb{N}}$ is uniformly distributed modulo 1.*

Proof. See [KN74, Theorem 3.5]. □

By Lemma 3.1, we immediately obtain

Lemma 3.2. *For all $A \in \mathbb{R} \setminus \{0\}$ and non-integral $\alpha > 1$, the sequence $(An^\alpha)_{n=1}^\infty$ is uniformly distributed modulo 1.*

4. Lemmas

Lemma 4.1. *For all $1 < \beta < \gamma$, there exists $\alpha \in (\beta, \gamma)$ such that the equation $x^\alpha + 1 = 2z^\alpha$ has a solution of a pairwise distinct pair $(x, z) \in \mathbb{N}$.*

Proof. Let us fix any $1 < \beta < \gamma$, and let us define

$$J(x) = \left(\left(\frac{1 + x^{-\beta}}{2} \right)^{1/\beta} x,\ \left(\frac{1}{2} \right)^{1/\gamma} x \right) \cap \mathbb{N}$$

for all $x \in \mathbb{N}$. We can find a large enough $x \in \mathbb{N}$ so that $J(x)$ is non-empty. Let us fix such x and $z \in J(x)$. Let $f(\alpha) = x^\alpha + 1 - 2z^\alpha$ for all $\alpha \in \mathbb{R}$. Then f is continuous. In addition, $f(\beta) < 0$ and $f(\gamma) > 0$. Therefore by the intermediate value theorem, there exists $\alpha \in (\beta, \gamma)$ such that $f(\alpha) = 0$. □

Lemma 4.2. *Let $\alpha > 1$ be non-integral, and let x, z be positive integers. Suppose that $1 + x^\alpha = 2z^\alpha$. Then $\lfloor n^\alpha \rfloor + \lfloor (nx)^\alpha \rfloor = 2\lfloor (nz)^\alpha \rfloor$ for infinitely many $n \in \mathbb{N}$,*

Proof. Let us fix x, z, α given in the condition of Lemma 4.2. For all $n \in \mathbb{N}$,

$$\begin{aligned} \lfloor n^\alpha \rfloor + \lfloor (nx)^\alpha \rfloor - 2\lfloor (nz)^\alpha \rfloor &= n^\alpha(1 + x^\alpha - 2z^\alpha) - (\{n^\alpha\} + \{(nx)^\alpha\} - 2\{(nz)^\alpha\}) \\ &= -(\{n^\alpha\} + \{(nx)^\alpha\} - 2\{(nz)^\alpha\}). \end{aligned}$$

Let $\delta(n)$ be the most right-hand side of the above. Let

$$B = \{n \in \mathbb{N} \colon \{n^\alpha/2\} < 1/8,\ \{(nx)^\alpha/2\} < 1/8\}.$$

Then for all $n \in A$, we obtain

$$\begin{aligned} |\delta(n)| &\le \{n^\alpha\} + \{(nx)^\alpha\} + 2\{(n^\alpha(1 + x^\alpha)/2)\} \\ &\le 2\{n^\alpha/2\} + 2\{(nx)^\alpha/2\} + 2\{n^\alpha/2\} + 2\{(nx)^\alpha/2\} < 1. \end{aligned}$$

Therefore if B is infinite, we arrive at the conclusion of Lemma 4.2. Let us show the infinitude of B.

Where $x^\alpha \in \mathbb{R} \setminus \mathbb{Q}$, the sequence $(n^\alpha/2,\ (nx)^\alpha/2)$ is uniformly distributed modulo 1 from Weyl's criterion and Lemma 3.2. Hence B is infinite by the definition of the uniform distribution modulo 1.

Where $x^\alpha \in \mathbb{Q}$, there exist $u, v \in \mathbb{N}$ such that $x^\alpha = u/v$. Then let

$$C = \{n \in \mathbb{N} \colon \{n^\alpha/(2v)\} < 1/(8uv)\}.$$

For all $n \in C$, we have

$$\begin{aligned} \{n^\alpha/2\} = \{vn^\alpha/(2v)\} \le v\{n^\alpha/(2v)\} < 1/8, \\ \{(nx)^\alpha/2\} = \{un^\alpha/(2v)\} \le u\{n^\alpha/(2v)\} < 1/8. \end{aligned}$$

Therefore $C \subseteq B$. By Lemma 3.2, the sequence $(n^\alpha/(2v))_{n\in\mathbb{N}}$ is uniformly distributed modulo 1. Hence C is infinite by the definition of the uniform distribution modulo 1. This yields that B is infinite. □

5. Proof of Theorem 2.1

Let $\mathcal{A} = \{\alpha \in (1,\infty)\colon$ there exists a distinct pair $(x,z) \in \mathbb{N}^2$ such that $x^\alpha + 1 = 2z^\alpha\}$. By Lemma 4.1, $\mathcal{A}$ is dense in $(1,\infty)$. Further, by Lemma 4.2, for all $\alpha \in \mathcal{A}$, there exist distinct $x_\alpha, z_\alpha \in \mathbb{N}$ and there exist positive integers $n_{1,\alpha} < n_{2,\alpha} < \cdots$ such that

$$\lfloor n_{j,\alpha}^\alpha \rfloor + \lfloor (n_{j,\alpha} x_\alpha)^\alpha \rfloor = 2 \lfloor (n_{j,\alpha} z_\alpha)^\alpha \rfloor$$

for all $j \in \mathbb{N}$. Let us take such x_α, y_α, $n_{j,\alpha}$. Then for all $j \in \mathbb{N}$ and $\alpha \in \mathcal{A}$, we define

$$\ell_{j,\alpha} = \min\left\{ \frac{\log\left(\lfloor (n_{j,\alpha} w)^\alpha \rfloor + 1\right)}{\log(n_{j,\alpha} w)} - \alpha \colon w = 1,\ x_\alpha, z_\alpha \right\}.$$

Then $\lfloor n_{j,\alpha}^t \rfloor + \lfloor (n_{j,\alpha} x_\alpha)^t \rfloor = 2 \lfloor (n_{j,\alpha} z_\alpha)^t \rfloor$ for all $t \in (\alpha, \alpha + \ell_{j,\alpha})$. For all $j \in \mathbb{N}$, let

$$U_j = \bigcup_{\alpha \in \mathcal{A}} (\alpha, \alpha + \ell_{j,\alpha}).$$

Then U_j is open and dense in $(1,\infty)$. By Theorem 2.2, $\mathcal{U} := \bigcap_{j=1}^\infty U_j$ is dense in $(1,\infty)$. In addition, let us take any $t \in \mathcal{U}$. Then for all $j \in \mathbb{N}$, there exists $\alpha_j \in \mathcal{A}$ such that $\lfloor n_{j,\alpha_j}^t \rfloor + \lfloor (n_{j,\alpha_j} x_{\alpha_j})^t \rfloor = 2 \lfloor (n_{j,\alpha_j} z_{\alpha_j})^t \rfloor$. Therefore $\mathrm{PS}(t)$ contains infinitely many three-term arithmetic progressions since $n_{j,\alpha_j} \geq j \to \infty$ as $j \to \infty$.

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