Asymptotics of some function corresponding to refraction phenomena arising in inverse problems of wave equation of flat two layer medium

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§1. Introduction

Inverse problems concerning nondestructive testing play an important role in science or other fields of activities of human being. Some inverse problems are formulated as those of differential equations. One of famous and interesting problems is to find information of the inside of a conductive body from a given pair of the voltage potential and electric current on the boundary of the conductive body. In the author's best knowledge, this type of problems originate with Calderon [3]. This problem is formulated by boundary value problems for elliptic equations (representing static cases), which is called boundary inverse problems. The boundary inverse problems have the long history as in [9, 25, 26, 31] or the references in them. Even now, many researchers try to develop these fields.

In some context of boundary inverse problems, an interesting approach is introduced by Ikehata [6] and [7], which is called "the enclosure method" now. Recent works for inverse problems by using the enclosure method show that the shortest lengths (or longest one which depends on settings of problems) in some sense (depending on each setting of problems) can be picked up. In §2, we give a reason why the shortest lengths appear by using a model case closely related to the original work [7]. This is the main purpose of §2.

For time dependent problems such as heat equations and wave equations, it is natural to formulate inverse problems, which are investigated by many researchers. It is no doubt that the enclosure methods are also useful for time dependent problems, which are called "the time dependent enclosure method". For boundary inverse problems of the heat equations, as in [8, 15, 16, 18, 23], the enclosure method is applied. A survey for recent development of the time dependent enclosure method is given by Ikehata [14]. For inverse problems of differential equations including the enclosure method, see for example, [9, 26, 27, 30, 31] and earlier references given there.

In this article, we consider inverse problems governed by wave equations, which are related to find obstacles by using such waves as radar and sonar (see e.g. [1] for ground probing radar, and [4] for subsurface radar). You want to get information of obstacles D hidden by the ground or sea. In these cases, we need to take the layered boundary cases into account. But, as the first step, the simplest case is that there is no layer, i.e. obstacles are in whole (three dimensional) space $\mathbb{R}^3$ filled by homogeneous medium. We fix some region B emitting incident waves. Since B is apart from D, it is natural to assume that $\overline{B}$ is disjoint to $\overline{D}$. If the incident waves are emitted in B, the obstacle D reflects the waves and the reflected waves come back to B. A basic problem about this setting is to get information of D by observing the reflected waves in B. Ikehata gives serial papers concerning this problems (cf. [10, 11, 12] and for electromagnetic waves, [13]). In these investigations, it is assumed that medium $\mathbb{R}^3 \setminus \overline{D}$ is homogeneous, which means that the waves are governed by the usual wave equation

$$(\partial_t^2 - c^2\triangle)u(t,x) = 0 \quad \text{in} \quad [0,T) \times (\mathbb{R}^3 \setminus \overline{D})$$

with the propagation speed $c > 0$ of waves. In this case, it takes a time $|x-y|/c$ for waves emanating $y \in B$ and arriving at $x \in D$. Thus, we can guess that the shortest lengths in this case is given by the infimum

$$c^{-1}\mathrm{dist}(B, D) = c^{-1} \inf_{(x,y)\in D\times B} |x-y|.$$

As in [10, 11] (and section 2 of [24] for a case of less regularities), applying the enclosure method, we can show that this is true in §2.

In §3, we consider what happens if we have some layers between D and B. As the simplest case with layers, we handle the case that there exists an only one flat layer. This type of inverse problems are already studied by many authors. Here, we handle the problem by using the time dependent enclosure method. For other approaches, see e.g. [2, 5, 28, 29].

We put $\mathbb{R}^3_\pm = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid \pm x_3 > 0\}$. Assume that the propagation speed of the wave in $\mathbb{R}^3_\pm$ is given by $\sqrt{\gamma_\pm}$, where $\gamma_\pm > 0$ are constants with $\gamma_+ \neq \gamma_-$. Thus, the transmission boundary $\partial\mathbb{R}^3_\pm$ is corresponding to the flat layer (boundary).

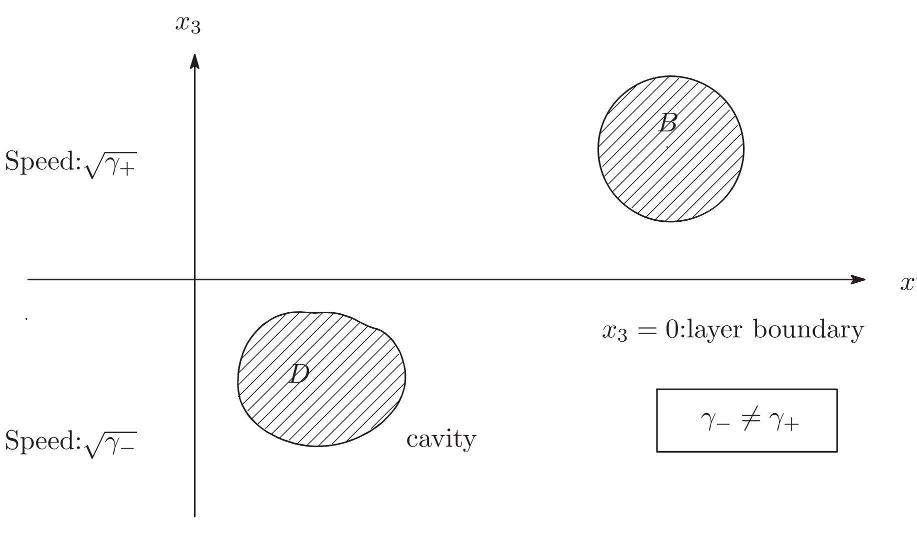

Fig. 1 Two-layered case

We call $\mathbb{R}^3_+$ (resp. $\mathbb{R}^3_-$) the upper (resp. lower) side of the flat layer $\partial\mathbb{R}^3_\pm$. If there is no obstacle D in this two-layered medium, we call the free media or the free case. The waves propagating the free media is governed by

$$\begin{cases} (\partial_t^2 - L_{\gamma_0})u_0 = 0 & \text{in } (0,\infty)\times\mathbb{R}^3, \\ u_0(0,x) = 0, \quad \partial_t u_0(0,x) = f(x) & \text{on } \mathbb{R}^3, \end{cases}$$

where $\gamma_0(x) = \gamma_\pm I_3$ for $\pm x_3 > 0$ and $L_{\gamma_0} u = \mathrm{div}(\gamma_0 \nabla u)$.

We consider the case that obstacles D is inside of the lower side. To get information of D, we produce waves in some region B of the upper side, and staying B, we measure the waves reflected by the obstacles D. Hence, we assume that $\overline{D} \subset \mathbb{R}^3_-$ and $\overline{B} \subset \mathbb{R}^3_+$, and $\mathbb{R}^3_- \setminus \overline{D}$ is connected. In the case that there are some inclusions (i.e. heterogeneous media in the background material occupying the lower side of the flat layer boundary), [20, 21] (and [24] for inclusions with less regularities) handle the time dependent enclosure method. In §3, we give a formulation of the time dependent enclosure method for the inverse problems detecting obstacles D in the lower side of two-layered media.

For two-layered case, we can guess that for $x \in D$ and $y \in B$, the shortest length $l(x, y)$ means the consuming time for waves starting at $y \in B$ and arriving at $x \in D$. Hence, the shortest length in this case is given by

$$l(D, B) = \inf_{x \in D, y \in B} l(x, y).$$

Note that $l(x, y)$ (resp. $l(D, B)$) are called the optical distance between $x \in D$ and $y \in B$ (resp. D and B). It is well known that the optical distance $l(x, y)$ is given by Fermat's principle,

$$l(x, y) = \inf_{z' \in \mathbb{R}^2} l_{x,y}(z') \qquad (x \in \mathbb{R}^3_-, y \in \mathbb{R}^3_+), \tag{1.1}$$

where

$$l_{x,y}(z') = \frac{1}{\sqrt{\gamma_-}} |\tilde{z}' - x| + \frac{1}{\sqrt{\gamma_+}} |\tilde{z}' - y| \quad (\tilde{z}' = (z_1, z_2, 0), z' = (z_1, z_2)). \tag{1.2}$$

As in [20, 21, 24], the shortest lengths are closely related to the asymptotic behavior of the fundamental solution $\Phi_\tau(x, y)$ of

$$(L_{\gamma_0} - \tau^2)\Phi_\tau(x, y) + \delta(x - y) = 0 \quad \text{in} \quad \mathbb{R}^3. \tag{1.3}$$

Hence, the problems are reduced to finding $l(x, y)$ from the fundamental solution. This is the most important part of this problem, and developed in [20, 21]. In §4, along with these previous works, we give a review how and why the optical distance is appeared in the fundamental solution of (1.3).

For incident waves starting from B, some refracted waves by the layered boundary hit the obstacles D. This produces reflected waves. Some of these waves come back to B through the layered boundary. Hence, we measure the refracted waves by the layered boundary of the reflected signals by the obstacles D. If $\gamma_- > \gamma_+$, all refracted waves are body waves, which means that the consuming time is given by (1.2). On the contrary, if $\gamma_- < \gamma_+$, we have totally refracted waves. These waves propagate on the layered boundary, and emanate into the upper side of the medium. Hence, the consuming time of them are different from usual time given by (1.2) (see e.g. (4.7) and figure 3 in §4). This fact requires additional arguments, however, eventually, we can show that these waves never affect the optical distance. We can say that this part is the most remarkable and interesting points among studies of the time dependent enclosure method for two-layered case.

§2. Shortest lengths and the enclosure methods

Let $\Omega \subset \mathbb{R}^n$ $(n = 2 \text{ or } 3)$ be bounded domain with Lipschitz boundary $\partial\Omega$. The set Ω stands for the whole shape of a homogeneous conductive body with some cavities inside of this body. All of these cavities are denoted by D. Hence, we assume that D is an open set satisfying $\overline{D} \subset \Omega$. Then the voltage potential $u(x)$ satisfies

$$\begin{cases} \triangle u(x) = 0 & \text{in } \Omega \setminus \overline{D}, \\ \partial_{\nu_x} u(x) = 0 & \text{on } \partial D, \\ u(x) = f(x) & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where ν_x is the unit outer normal of D (or Ω) at $x \in \partial D$ (or $x \in \partial\Omega$), and $\partial_{\nu_x} u = \nu_x \cdot \nabla_x u|_{\partial D}$ on ∂D(or $\partial_{\nu_x} u = \nu_x \cdot \nabla_x u|_{\partial\Omega}$ on $\partial\Omega$).

For the solution u of (2.1), $\partial_{\nu_x} u|_{\partial\Omega}$ is the electric current on the outside boundary. Thus, put a voltage potential f on the outside boundary $\partial\Omega$ of the conductive body Ω, and measure the current $\partial_{\nu_x} u|_{\partial\Omega}$ on $\partial\Omega$. By this step, one can have a pair $(f, \partial_{\nu_x} u|_{\partial\Omega})$ of f and $\partial_{\nu_x} u|_{\partial\Omega}$ as measurement. The problems is to find how to detect D or give information of D by measurements given in the above.

Take any ω and $\omega^\perp \in S^{n-1}$ with $\omega \cdot \omega^\perp = 0$ and put $f(x, \tau) = e^{\tau(x\cdot\omega + ix\cdot\omega^\perp)}$ $(x \in \mathbb{R}^n)$. For $f(x, \tau)$ on $\partial\Omega$ we find the solution $u(x, \tau)$ of (2.1). Next, we introduce the indicator function I_τ of the form:

$$I_\tau = \int_{\partial\Omega} \{\partial_{\nu_x} u(x, \tau) - \partial_{\nu_x} f(x, \tau)\} \overline{f(x, \tau)} dS_x.$$

In [7], Ikehata gives a procedure to obtain the support function of D defined by

$h_D(\omega) = \sup_{x\in D} x\cdot\omega$. We can see that D is contained the half space $\{x \in \mathbb{R}^n | x\cdot\omega < h_D(\omega)\}$. Since we can take all $\omega \in S^{n-1}$ theoretically, the convex hull D' of cavities D is given by $D' = \bigcap_{|\omega|=1}\{x | x\cdot\omega < h(\omega)\}$. This is the prototype of the idea. Note that in this case, to obtain $h_D(\omega)$, we need to take the limit $\tau \to \infty$ for I_τ. Thus, for any $\omega \in S^{n-1}$, we need infinitely many measurements corresponding to ω and large τ. For two dimensional case ($n = 2$), when we know that D is polygon a priori, even if we have only one pair of the measurement, we can also obtain the convex hull of D by introducing some other indicator functions (cf. Ikehata [6]).

Intuitively, we can see that I_τ gives information about "shortest lengths" in some sense. In this setting, $-h_D(\omega)$ can be regarded as the shortest lengths (hence, $h_D(\omega)$ itself is the longest one). Integration by parts implies that

$$0 = \int_{\Omega\setminus\overline{D}} \{\triangle u \cdot \overline{f} - \triangle f \cdot \overline{f}\}dx = I_\tau - \int_{\partial D} \{\partial_{\nu_x} u(x,\tau) - \partial_{\nu_x} f(x,\tau)\}\overline{f(x,\tau)}dS_x,$$

which yields

$$I_\tau = -\int_{\partial D} (\partial_{\nu_x} f)(x,\tau)\overline{f(x,\tau)}dS_x = -\tau\int_{\partial D} \nu_x\cdot(\omega + i\omega^\perp)e^{2\tau\omega\cdot x}dS_x.$$

From this, we can get $h_D(\omega)$ since we have

$$\lim_{\tau\to\infty} \frac{-1}{2\tau}\log|I_\tau| = h_D(\omega).$$

Usually, the enclosure methods give the shortest or longest lengths for the problems in hand. Even the time dependent problems seem to have the same structures as the stationary problems. For the boundary inverse problems for heat equations, see [15, 16, 18, 23]. These problems are finally reduced to giving asymptotic behavior for the solutions of some elliptic boundary value problems with large parameter. In these asymptotics, the shortest lengths appear. Perhaps, the prototype of these phenomena is given by Varadhan [32] to give short time asymptotics of the heat kernels. To accomplish this, Varadhan [32] shows that the solution $v(x;\tau)$ of

$$\begin{cases} (\tau^2 - \triangle)v(x;\tau) = 0 & \text{in } \Omega, \\ v(x;\tau) = 1 & \text{on } \partial\Omega \end{cases}$$

satisfies

$$\lim_{\tau\to\infty} \frac{\log|v(x;\tau)|}{\tau} = -\mathrm{dist}(x,\partial\Omega).$$

Some types of the time dependent enclosure method are closely related to such asymptotics. Take a bounded set D with $\overline{D} \subset \Omega$. Consider the following reduced problems:

$$\begin{cases} (\tau^2 - \triangle)v(x;\tau) = 0 & \text{in } \Omega \setminus \overline{D}, \\ \partial_\nu v(x;\tau) = 1 & \text{on } \partial\Omega, \\ \partial_\nu v(x;\tau) = 0 & \text{on } \partial D. \end{cases}$$

Since there is no signal on ∂D, only the reflected signals of original source given in $\partial\Omega$ come back to the inside. Hence, in this case, $\text{dist}(x, \partial\Omega)$ is the shortest length from the boundary to $x \in \Omega \setminus D$, which is justified in [19] (cf. Theorem 1.3 in [19]).

To obtain this result, we essentially use the precise estimate given in [17], which gives the basis of developing some problems arising from the time dependent enclosure method (cf. [18, 23]). Thus, we can say that some inverse problems described above are closely connected to the problems finding the shortest lengths corresponding to the problems, which have a long history.

§3. Inverse problems for two layered case

In this section, we formulate the inverse problem for two layered case. Let D and B be the sets given in figure 1 of §1. We fix $T > 0$ and consider the following wave equation:

$$\begin{cases} (\partial_t^2 - L_{\gamma_0})u = 0 & \text{in } (0,T) \times (\mathbb{R}^3 \setminus \overline{D}), \\ \partial_{\nu_x} u = 0 & \text{on } (0,T) \times \partial D, \\ u(0,x) = 0, \quad \partial_t u(0,x) = f(x) & \text{on } \mathbb{R}^3 \setminus \overline{D}, \end{cases} \tag{3.1}$$

where ν_x is the unit outer normal of ∂D. In (3.1), the initial datum f is the source of the waves emitting from B, and $T > 0$ stands for the time measuring the reflected waves. As in [20, 21, 24], we introduce the indicator function

$$I_\tau = \int_{\mathbb{R}^3 \setminus D} f(x)(w(x;\tau) - v(x;\tau))dx, \tag{3.2}$$

where $w(x;\tau)$ and $v(x;\tau)$ is defined by

$$w(x;\tau) = \int_0^T e^{-\tau t} u(t,x)dt, \qquad v(x;\tau) = \int_0^\infty e^{-\tau t} u_0(t,x)dt. \tag{3.3}$$

Note that I_τ is obtained from the measurement $u(t,x)$ for $0 \le t \le T$ and $x \in B$.

To make sure that waves are emanated from B exactly, we assume that $f \in L^2(\mathbb{R}^3)$ satisfies the emission condition on B, which is described as

$$\begin{cases} f \in L^2(\mathbb{R}^3) \text{ with } \mathrm{supp} f \subset \overline{B} \text{ and there exists a constant } c_1 > 0 \\ \text{such that } f(x) \geq c_1 \ (x \in B) \text{ or } -f(x) \geq c_1 \ (x \in B). \end{cases}$$

Let us recall condition (C) introduced in [24], which describes regularities of boundaries in some sense. For $x \in \mathbb{R}^3$, $h > 0$, $\theta > 0$ and $n \in \mathbb{R}^3$ with $|n| = 1$, put $C(x, n, h, \theta) = \{y \in \mathbb{R}^3 | |y - x| \leq h, (y - x)\cdot n \geq |y - x| \cos\theta\}$. Note that $C(x, n, h, \theta)$ is a cone with vertex x, direction n, radius h and opening angle θ. We say that B satisfies condition (C) if B satisfies

$$\text{(C)} \quad \begin{cases} \text{for any } x \in \partial B, \text{ there exists a cone } C(x, n, h, \theta) \text{ with vertex } x \\ \text{satisfying } C(x, n, h, \theta) \setminus \{x\} \subset B. \end{cases}$$

Note that domains with C^1 or even $C^{0,1}$ boundary satisfy condition (C).

From asymptotic behavior of I_τ as $\tau \to \infty$, we can obtain the following result essentially given in [20, 21, 24].

Theorem 3.1 Suppose that D has C^1 boundary ∂D and B satisfies condition (C). Then, the indicator function defined by (3.2) satisfies
(i) for $T < 2l(D, B)$, $\lim_{\tau\to\infty} e^{\tau T} I_\tau = 0$,
(ii) for $T > 2l(D, B)$, $\lim_{\tau\to\infty} e^{\tau T} I_\tau = -\infty$.

Further, suppose $T > 2l(D, B)$, then

$$\lim_{\tau\to\infty} \frac{1}{\tau} \log |I_\tau| = -2l(D, B).$$

Remark 3.1 In the case of inclusions, we can handle the following two cases:
(a) D has C^1 boundary ∂D and B satisfies condition (C),
(b) D and B satisfy condition (C) and B is convex.
The case (a) is the same as Theorem 3.1. The case (b) means that we can relax the regularities of ∂D if B is convex (cf. [24]). It seems to be difficult to handle the case (b) for the case of cavities with the Neumann boundary conditions. Thus, we assume that D has C^1 boundary.

From Theorem 3.1, we can enclose D. In what follows, for $p \in \mathbb{R}^3$ and $r > 0$, we put $B_r(p) = \{x \in \mathbb{R}^3 | |x - p| < r\}$. Take $p \in \mathbb{R}^3_+$ and $r > 0$ so that $\overline{B_r(p)} \subset \mathbb{R}^3_+$, and put $B = B_r(p)$. Then, Theorem 3.1 implies D is in the set $\{x \in \mathbb{R}^3_- | l(x, p) >$

$l(D, B_r(p)) + r/\sqrt{\gamma_+}\}$. Since $l(x,p) \le |x-\tilde{z}'|/\sqrt{\gamma_-} + |p-\tilde{z}'|/\sqrt{\gamma_+}$ for any $z' \in \mathbb{R}^2$, it follows that

$$D \subset \bigcap_{p\in\mathbb{R}^3_+, r<p_3} \bigcap_{z'\in\mathbb{R}^2} \Big\{x \in \mathbb{R}^3_- \,\big|\, |x-\tilde{z}'| > \sqrt{\gamma_-}\Big(l(D, B_r(p)) + \frac{r - |p-\tilde{z}'|}{\sqrt{\gamma_+}}\Big)\Big\}.$$

Thus, in this case, D can be enclosed from the upper side. This result is reasonable with this setting, because we only can emanate signals to the cavities D and catch the reflected waves.

The steps getting Theorem 3.1 are the same as for the case of inclusions given in [20, 21, 24]. First we reduce the problem to estimate of $v(x;\tau)$ defined by (3.3).

Lemma 3.1 Suppose that D has $C^{0,1}$ boundary ∂D.
(i) There exist constants $C > 0$ and $C' > 0$ such that

$$|I_\tau| \le C\{\|\nabla_x v\|^2_{L^2(D)} + \tau^2\|v\|^2_{L^2(D)}\} + C'\tau^{-1}e^{-\tau T} \qquad (\tau \ge 1).$$

(ii) There exist constants $C_0 > 0$ and $C_0' > 0$ such that

$$I_\tau \ge C_0\{\|\nabla_x v\|^2_{L^2(D)} + \tau^2\|v\|^2_{L^2(D)}\} + C_0'\tau^{-1}e^{-\tau T} \qquad (\tau \ge 1).$$

Probably, the almost same lemma mentioned above would have been given somewhere, but we give a proof of Lemma 3.1 in the last of this section by similar method to section 2 of [15] and appendix in [20].

Now, we introduce the most important estimates of $v(x;\tau)$.

Proposition 3.1 Suppose that D has C^1 boundary and B satisfies condition (C). Then, there exists a constant $C > 0$ such that

$$C^{-1}\tau^{-7}e^{-2\tau l(D,B)} \le \|\nabla_x v(\cdot;\tau)\|^2_{L^2(D)} + \tau^2\|v(\cdot;\tau)\|^2_{L^2(D)} \le C\tau^2 e^{-2\tau l(D,B)} \quad (\tau \ge 1).$$

Proposition 3.1 is a key to Theorem 3.1. Lemma 3.1 and Proposition 3.1 imply Theorem 3.1.

Since $v(x;\tau)$ is characterized as the L^2-solution of

$$(L_{\gamma_0} - \tau^2)v(x;\tau) + f(x) = 0 \qquad \text{in } \mathbb{R}^3, \tag{3.4}$$

$v(x;\tau)$ can be represented by

$$v(x;\tau) = \int_B \Phi_\tau(x,y)f(y)dy,$$

where $\Phi_\tau(x,y)$ is the fundamental solution of (1.3). Hence, the problem is eventually reduced to finding asymptotic behavior of $\nabla_x\Phi_\tau(x,y)$ ($x\in D$, $y\in B$) as $\tau\to\infty$. This is given as Proposition 1 in [20, 21]. Before introducing this, we need to express $l(x,y)$ defined by (1.1) in terms of Snell's law (as in Lemma 4.1 of [20]).

For $x=(x',x_3)\in\mathbb{R}^3_-$ and $y=(y',y_3)\in\mathbb{R}^3_+$, there exists a unique point $z'(x,y)\in\mathbb{R}^2$ satisfying $l(x,y)=l_{x,y}(z'(x,y))$, and the point $z'(x,y)$ is on the line segment $x'y'$ and C^∞ for $(x,y)\in\mathbb{R}^3_-\times\mathbb{R}^3_+$ (cf. Lemma 4.1 of [20]). Since $z'=z'(x,y)$ is a unique point attaining $l(x,y)=\inf_{z'\in\mathbb{R}^2}l_{x,y}(z')$, $\tilde{z}'(x,y)=(z'(x,y),0)$ satisfies $\partial_{z'}l_{x,y}(z')=0$, which yields

$$\frac{1}{\sqrt{\gamma_-}}\frac{z'(x,y)-x'}{|\tilde{z}'(x,y)-x|}+\frac{1}{\sqrt{\gamma_+}}\frac{z'(x,y)-y'}{|\tilde{z}'(x,y)-y|}=0. \tag{3.5}$$

We define $0\le\theta_\pm<\pi/2$ by

$$\sin\theta_-=\frac{|z'(x,y)-x'|}{|\tilde{z}'(x,y)-x|},\qquad \sin\theta_+=\frac{|z'(x,y)-y'|}{|\tilde{z}'(x,y)-y|}.$$

Then, the relation (3.5) implies

$$\frac{\sin\theta_-}{\sqrt{\gamma_-}}\frac{z'(x,y)-x'}{|z'(x,y)-x'|}+\frac{\sin\theta_+}{\sqrt{\gamma_+}}\frac{z'(x,y)-y'}{|z'(x,y)-y'|}=0.$$

This means that $z'(x,y)\in\mathbb{R}^2$ is on the line segment $x'y'$ on $\mathbb{R}^2$, and

$$\frac{\sin\theta_-}{\sqrt{\gamma_-}}=\frac{\sin\theta_+}{\sqrt{\gamma_+}},$$

which is Snell's law exactly.

Let us recall the asymptotics of $\Phi_\tau(x,y)$ given in Proposition 1 of [20, 21]. Take open sets B' and D' with $\overline{B'}\subset\mathbb{R}^3_+$, $\overline{D'}\subset\mathbb{R}^3_-$, $\overline{B}\subset B'$ and $\overline{D}\subset D'$.

Proposition 3.2 Assume that $\gamma_+\neq\gamma_-$. Then for $k=0,1$, we have

$$\nabla_x^k\Phi_\tau(x,y)=\frac{e^{-\tau l(x,y)}}{8\pi\gamma_+\gamma_-\sqrt{\det H(x,y)}}\Big(\frac{-\tau}{\sqrt{\gamma_-}}\Big)^k\Big(\sum_{j=0}^N\tau^{-j}\Phi_j^{(k)}(x,y)+Q_{N,\tau}^{(k)}(x,y)\Big),$$

where $H(x,y)=\mathrm{Hess}(l_{x,y})(z'(x,y))$ is the Hessian of $l_{x,y}$ given by (1.2) at $z'=z'(x,y)$, $\Phi_j^{(k)}(x,y)$ ($k=0,1$) are C^∞ in $\overline{D'}\times\overline{B'}$, for any $N\in\mathbb{N}\cup\{0\}$, $Q_{N,\tau}^{(k)}(x,y)$ ($k=0,1$) are continuous in $\overline{D'}\times\overline{B'}$ with a constant $C_N>0$ satisfying

$$|Q_{N,\tau}^{(0)}(x,y)|+|Q_{N,\tau}^{(1)}(x,y)|\le C_N\tau^{-(N+1)}\qquad (x\in\overline{D'},y\in\overline{B'},\tau\ge1).$$

Moreover, $\Phi_0^{(k)}(x,y)$ $(k=0,1)$ are given by

$$\Phi_0^{(0)}(x,y)=\frac{E_0(x-\tilde{z}'(x,y))}{|x-\tilde{z}'(x,y)||\tilde{z}'(x,y)-y|},$$

and

$$\Phi_0^{(1)}(x,y)=\Phi_0^{(0)}(x,y)\frac{x-\tilde{z}'(x,y)}{|x-\tilde{z}'(x,y)|},$$

where

$$E_0(x-\tilde{z}')=\frac{4\sqrt{\gamma_-}|x_3|\sqrt{a_0^2|x-\tilde{z}'|^2-|x'-z'|^2}}{|x-\tilde{z}'|\big(\sqrt{a_0^2|x-\tilde{z}'|^2-|x'-z'|^2}+a_0^2|x_3|\big)},\quad a_0=\sqrt{\frac{\gamma_-}{\gamma_+}}. \tag{3.6}$$

As stated in Proposition 3.2, the asymptotics of $\Phi_\tau(x,y)$ does not depend on whether γ_+ is larger than γ_- or not. But, their proof is different. For a proof see Proposition 1 of [20] for the case of $\gamma_+<\gamma_-$ (resp. Proposition 1 of [21] for the case of $\gamma_+>\gamma_-$). The differences between $\gamma_+<\gamma_-$ and $\gamma_+>\gamma_-$ concern the total reflection phenomena for incident waves coming to the layer boundary from the lower side. In line with section 2 of [21], we give some explanation in §4.

Proposition 3.1 is given by Proposition 3.2. For a proof, see the proof of Theorem 1.3 in [20] and section 4 of [24]. Algebraic orders in estimates of $\|\nabla_x v(\cdot;\tau)\|^2_{L^2(D)}+\tau^2\|v(\cdot;\tau)\|^2_{L^2(D)}$ are different from those in [20, 21]. These differences come from lower order estimates given as the following lemma:

Lemma 3.2 (1) (Proposition 3.2 of [15]) For any $a,p\in\mathbb{R}^3$ and $r>0$ with $|a-p|=r$, there exists a constant $C>0$ such that

$$\int_{B_r(p)}e^{-\tau|y-a|}dy\ge C\tau^{-2}\qquad(\tau\ge 1),$$

where $B_r(p)=\{x\in\mathbb{R}^3\,|\,|x-p|<r\}$.
(2) (Lemma 2.4 of [24]) For any cone $C(a,n,h,\theta)=\{x\in\mathbb{R}^3||x-a|\le h,(x-a)\cdot n\ge |x-a|\cos\theta\}$, there exists a constant $C>0$ such that

$$\int_{C(a,n,h,\theta)}e^{-\tau|x-a|}dx\ge C\tau^{-3}\qquad(\tau\ge 1).$$

In the rest of this section, we give a proof of Lemma 3.1.

Proof of Lemma 3.1: We put $f_T(x;\tau)=\partial_t u(T,x)+\tau u(T,x)$. From (3.1) and (3.3), it follows that $w(x;\tau)$ is the L^2-solution of

$$\begin{cases}(L_{\gamma_0}-\tau^2)w+f(x)=e^{-\tau T}f_T(x;\tau) & \text{in } \mathbb{R}^3\setminus\overline{D},\\ \partial_{\nu_x}w=0 & \text{on } \partial D,\end{cases} \tag{3.7}$$

and $v(x;\tau)$ is the L^2-solution of (3.4).

From (3.4), integration by parts implies that

$$\int_{\mathbb{R}^3}\{\gamma_0\nabla_x v\cdot\nabla_x v+\tau^2|v|^2\}dx=\int_{\mathbb{R}^3}fvdx\le\frac{1}{2\tau^2}\|f\|^2_{L^2(\mathbb{R}^3)}+\frac{\tau^2}{2}\|v\|^2_{L^2(\mathbb{R}^3)},$$

which yields

$$\|\nabla_x v\|_{L^2(\mathbb{R}^3)}+\tau\|v\|_{L^2(\mathbb{R}^3)}\le C\tau^{-1}\|f\|_{L^2(B)}\qquad(\tau>0) \tag{3.8}$$

since $\mathrm{supp}f\subset\overline{B}$. From (3.7) and (3.4), $w-v$ satisfies

$$\begin{cases}(L_{\gamma_0}-\tau^2)(w-v)=e^{-\tau T}f_T(x;\tau) & \text{in } \mathbb{R}^3\setminus\overline{D},\\ \partial_{\nu_x}(w-v)=-\partial_{\nu_x}v & \text{on } \partial D.\end{cases} \tag{3.9}$$

Integration by parts and (3.9) imply that

$$\begin{aligned}&\int_{\mathbb{R}^3\setminus D}\{\gamma_0\nabla_x(w-v)\cdot\nabla_x(w-v)+\tau^2|w-v|^2\}dx\\ &\quad=-\int_{\partial D}\gamma_-(\partial_{\nu_x}(w-v))(w-v)dS_x-e^{-\tau T}\int_{\mathbb{R}^3\setminus D}f_T(x;\tau)(w-v)dx \qquad (3.10)\\ &\quad\le\int_{\partial D}\gamma_-(\partial_{\nu_x}v)(w-v)dS_x+\frac{e^{-2\tau T}}{2\tau^2}\|f_T\|^2_{L^2(\mathbb{R}^3\setminus D)}+\frac{\tau^2}{2}\|w-v\|^2_{L^2(\mathbb{R}^3\setminus D)},\end{aligned}$$

which yields

$$\begin{aligned}&\|\nabla_x(w-v)\|^2_{L^2(\mathbb{R}^3\setminus D)}+\tau^2\|w-v\|^2_{L^2(\mathbb{R}^3\setminus D)} \qquad (3.11)\\ &\qquad\le C\Big\{\int_{\partial D}\gamma_-(\partial_{\nu_x}v)(w-v)dS_x+e^{-2\tau T}\tau^{-2}\|f_T\|^2_{L^2(\mathbb{R}^3\setminus D)}\Big\}.\end{aligned}$$

Since ∂D is $C^{0,1}$, we note that the following estimates seems to be well known.

Lemma 3.3 (1) There exists a constant $C>0$ such that

$$\sqrt{\tau}\|\varphi\|_{L^2(\partial D)}\le C\{\|\nabla_x\varphi\|_{L^2(\mathbb{R}^3\setminus D)}+\tau\|\varphi\|_{L^2(\mathbb{R}^3\setminus D)}\}\qquad(\tau\ge1,\varphi\in H^1(\mathbb{R}^3\setminus D)).$$

(2) There exists a linear extension operator $E_\tau: H^{1/2}(\partial D)\to H^1(D)$ such that

$$\|\nabla_x(E_\tau g)\|_{L^2(D)}+\tau\|E_\tau g\|_{L^2(D)}\le C\{\|g\|_{H^{1/2}(\partial D)}+\sqrt{\tau}\|g\|_{L^2(\partial D)}\}$$
$$(\tau\ge1, g\in H^{1/2}(\partial D)).$$

Since $w, v \in H^1(\mathbb{R}^3 \setminus D)$, it follows that $w - v \in H^{1/2}(\partial D)$. We put $\eta = E_\tau g \in H^1(D)$, where $g = (w-v)|_{\partial D}$ and E_τ is given in (2) of Lemma 3.3. From

$$\mathrm{div}(\eta\gamma_0\nabla_x v) = \eta\mathrm{div}(\gamma_0\nabla_x v) + \gamma_0\nabla_x v\cdot\nabla_x\eta = \tau^2\eta v + \gamma_0\nabla_x v\cdot\nabla_x\eta \quad \text{in} \quad D,$$

it follows that

$$\begin{aligned}\Big|\int_{\partial D}\gamma_-(\partial_{\nu_x}v)(w-v)dS_x\Big| &\le \int_D |\gamma_0\nabla_x v\cdot\nabla_x\eta + \tau^2 v\eta|dx \\ &\le C\|\nabla_x v\|_{L^2(D)}\|\nabla_x\eta\|_{L^2(D)} + \tau^2\|v\|_{L^2(D)}\|\eta\|_{L^2(D)} \\ &\le C\{\|\nabla_x v\|_{L^2(D)} + \tau\|v\|_{L^2(D)}\}\{\|\nabla_x\eta\|_{L^2(D)} + \tau\|\eta\|_{L^2(D)}\}.\end{aligned}$$

Lemma 3.3 and usual trace theorem imply that

$$\begin{aligned}\|\nabla_x\eta\|_{L^2(D)} + \tau\|\eta\|_{L^2(D)} &\le C\{\|g\|_{H^{1/2}(\partial D)} + \sqrt{\tau}\|g\|_{L^2(\partial D)}\} \\ &\le C\{\|\nabla_x(w-v)\|_{L^2(\mathbb{R}^3\setminus D)} + \tau\|w-v\|_{L^2(\mathbb{R}^3\setminus D)}\} \qquad (\tau \ge 1).\end{aligned}$$

Combining these estimates, we obtain

$$\begin{aligned}\int_{\partial D}\gamma_-(\partial_{\nu_x}v)(w-v)dS_x &\le \varepsilon\big(\|\nabla_x(w-v)\|^2_{L^2(\mathbb{R}^3\setminus D)} + \tau^2\|w-v\|^2_{L^2(\mathbb{R}^3\setminus D)}\big) \\ &\quad + C\varepsilon^{-1}\big(\|\nabla_x v\|^2_{L^2(D)} + \tau^2\|v\|^2_{L^2(D)}\big)\end{aligned}$$

for all $\varepsilon > 0$ and $\tau \ge 1$. From this estimate, (3.8) and (3.11), we obtain

$$\begin{aligned}&\|\nabla_x(w-v)\|^2_{L^2(\mathbb{R}^3\setminus D)} + \tau^2\|w-v\|^2_{L^2(\mathbb{R}^3\setminus D)} \\ &\quad\le C\big\{\|\nabla_x v\|^2_{L^2(D)} + \tau^2\|v\|^2_{L^2(D)} + e^{-2\tau T}\tau^{-2}\|f_T\|^2_{L^2(\mathbb{R}^3\setminus D)}\big\} \\ &\quad\le C\tau^{-2}\big\{\|f\|^2_{L^2(B)} + e^{-2\tau T}\|f_T\|^2_{L^2(\mathbb{R}^3\setminus D)}\big\} \qquad (\tau \ge 1).\end{aligned} \tag{3.12}$$

Since

$$\begin{aligned}f(w-v) &= -\{((L_{\gamma_0}-\tau^2)v)w - ((L_{\gamma_0}-\tau^2)w - e^{-\tau T}f_T)v\} \\ &= -\mathrm{div}(\gamma_0(\nabla_x v)(w-v)) - \mathrm{div}(\gamma_0(\nabla_x v)v) + \mathrm{div}(\gamma_0(\nabla_x w)v) - e^{-\tau T}f_T v,\end{aligned}$$

(3.7), (3.10) and $(L_{\gamma_0}-\tau^2)v = 0$ in $\mathbb{R}^3_-$, it follows that

$$\begin{aligned}I_\tau &= \int_{\partial D}\gamma_-(\partial_{\nu_x}v)(w-v)dS_x + \int_{\partial D}\gamma_-(\partial_{\nu_x}v)vdS_x - \int_{\mathbb{R}^3\setminus D}e^{-\tau T}f_T v dx \\ &= -\int_{\partial D}\gamma_-(\partial_{\nu_x}(w-v))(w-v)dS_x + \int_{\partial D}\gamma_-(\partial_{\nu_x}v)vdS_x - \int_{\mathbb{R}^3\setminus D}e^{-\tau T}f_T v dx \\ &= \int_{\mathbb{R}^3\setminus D}\{\gamma_0\nabla_x(w-v)\cdot\nabla_x(w-v) + \tau^2|w-v|^2\}dx + e^{-\tau T}\int_{\mathbb{R}^3\setminus D}f_T(w-v)dx \\ &\qquad + \int_D\{\gamma_-|\nabla_x v|^2 + \tau^2|v|^2\}dx - e^{-\tau T}\int_{\mathbb{R}^3\setminus D}f_T v dx.\end{aligned}$$

From this equality, (3.8) and (3.12), we obtain Lemma 3.1.

§4. Asymptotics of some function derived by the fundamental solutions

In this section, in line with section 2 of [21], we give a sketch how the asymptotic behavior of $\Phi_\tau(x, y)$ can be obtained. For $z' \in \mathbb{R}^2$, $y \in \mathbb{R}^3_+$, we put

$$E_\tau^{\gamma_+,0}(z', y) = \frac{e^{-\tau|\tilde{z}'-y|/\sqrt{\gamma_+}}}{|\tilde{z}' - y|} \qquad (\tilde{z}' = (z', 0) \in \partial\mathbb{R}^3_+),$$

which is a usual fundamental solution of $(\gamma_+\triangle - \tau^2) + 4\pi\gamma_+\delta(x - y) = 0$. By partial Fourier transform, we can see that $\Phi_\tau(x, y)$ is given by

$$\Phi_\tau(x, y) = \frac{\tau}{4\pi\gamma_+}\int_{\mathbb{R}^2} E_\tau^{\gamma_-}(x, z')E_\tau^{\gamma_+,0}(z', y)dz'. \tag{4.1}$$

This integral representation describes that signals arriving at $x \in \mathbb{R}^3_-$ are made up of all signals starting from $y \in \mathbb{R}^3_+$, and refracted at $\tilde{z}' \in \partial\mathbb{R}^3_+$ (on the layered boundary). Thus, in (4.1), $E_\tau^{\gamma_-}(x, z')$ corresponds to the refracted part, which is important to find asymptotic behavior of $\Phi_\tau(x, y)$. Note that as in section 2 of [21], $E_\tau^{\gamma_-}(x, z')$ is given by

$$E_\tau^{\gamma_-}(x, z') = \frac{\tau}{2(2\pi)^2\gamma_-^{3/2}}\int_{\mathbb{R}} I_{\tilde{\tau},0}(x - \tilde{z}', \zeta_2)d\zeta_2,$$

where $\tilde{\tau} = \tau/\sqrt{\gamma_-}$, $a_0 = \sqrt{\gamma_-/\gamma_+}$,

$$I_{\tilde{\tau},0}(x - \tilde{z}', \zeta_2) = \int_{\mathbb{R}} e^{-\tilde{\tau}\sqrt{1+\zeta_2^2}(-i|x'-z'|\zeta_1+|x_3|\sqrt{1+\zeta_1^2})}Q_0(\zeta_1, \zeta_2)\frac{d\zeta_1}{\sqrt{1+\zeta_1^2}}, \tag{4.2}$$

$$Q_0(\zeta_1, \zeta_2) = \frac{4\sqrt{\gamma_-}\sqrt{1+\zeta_2^2}\sqrt{1+\zeta_1^2}P(\zeta_1, \zeta_2)}{P(\zeta_1, \zeta_2) + a_0^2\sqrt{1+\zeta_1^2}} \quad \text{and} \quad P(\zeta_1, \zeta_2) = \sqrt{\frac{a_0^2+\zeta_2^2}{1+\zeta_2^2} + \zeta_1^2}.$$

Thus, it suffices to obtain asymptotics for the refracted part $E_\tau^{\gamma_-}(x, z')$.

Since we need to take exponential decay term from integral (4.2), we change a contour of the integral to use the steepest descent method. For $x \in \mathbb{R}^3_-$ and $z' \in \mathbb{R}^2$, we define θ by the equations

$$\sin\theta = \frac{|x' - z'|}{|x - \tilde{z}'|}, \quad \cos\theta = \frac{|x_3|}{|x - \tilde{z}'|} \quad (0 \le \theta < \pi/2). \tag{4.3}$$

We put $r = |x - \tilde{z}'|\sqrt{1+\zeta_2^2}$ and

$$\lambda = \lambda(\zeta_1, x, z') = -i\sin\theta\zeta_1 + \cos\theta\sqrt{1+\zeta_1^2}, \tag{4.4}$$

where θ is defined by (4.3). Since (4.4) is equivalent to $\zeta_1 = i\lambda\sin\theta \pm \sqrt{\lambda^2-1}\cos\theta$, as (33) in [20], putting $\lambda = \sqrt{1+\rho^2}$ for $\lambda \geq 1$, we define a contour by

$$\zeta_1 = \zeta_1(\rho, x, z') = i\sqrt{1+\rho^2}\sin\theta + \rho\cos\theta \qquad (\rho \in \mathbb{R}, x \in \mathbb{R}^3_-, z' \in \mathbb{R}^2). \tag{4.5}$$

If $\gamma_- > \gamma_+$, $Q_0(\zeta_1, \zeta_2)$ given in (4.2) is holomorphic for $\zeta_1 \in \mathbb{C}\backslash((-i\infty, -i]\cup[i, i\infty))$. Hence, we can change the contour of integral (4.2) to the curve $\Gamma_{x,z'}$ defined by (4.5) (see the figure in the left side of figure 2).

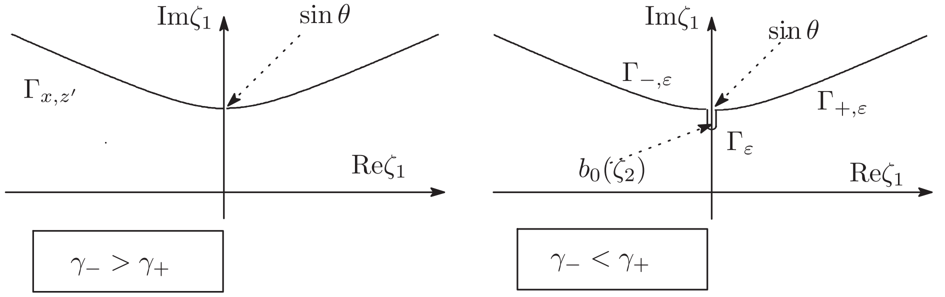

Fig. 2 Change of contour of integral (4.2)

Let us consider the case $\gamma_- < \gamma_+$. Since $a_0 < 1$, Q_0 are holomorphic for $\zeta_1 \in \mathbb{C}\setminus((-i\infty, -ib_0(\zeta_2)]\cup[ib_0(\zeta_2), i\infty))$, where

$$b_0(\zeta_2) = \sqrt{\frac{a_0^2+\zeta_2^2}{1+\zeta_2^2}} = \sqrt{1 - \frac{1-a_0^2}{1+\zeta_2^2}} < 1.$$

Thus, if $\sin\theta < a_0$, we can change the contour in (4.2) to $\Gamma_{x,z'}$, which is the same as for the case of $\gamma_- > \gamma_+$. Let us introduce the following set: for δ with $0 < \delta < a_0^{-1}$ and $x \in \mathbb{R}^3_-$,

$$\mathcal{U}_\delta(x) = \{\, z' \in \mathbb{R}^2 \,|\, |x' - z'| < a_0\delta|x - \tilde{z}'|\,\}(= \{\, z' \in \mathbb{R}^2 \mid \sin\theta < a_0\delta\,\}).$$

Since $\inf\{|ia_0 - \zeta_1| \,|\, \zeta_1 \in \Gamma_{x,z'}\} = a_0(1-\delta) > 0$ for any $0 < \delta < 1$, $x \in \mathbb{R}^3_-$ and $z' \in \mathcal{U}_\delta(x)$, the argument for getting Proposition 2 in [20] yields the following expansions of the refracted part for $z' \in \overline{\mathcal{U}_\delta(x)}$.

Proposition 4.1 For any $0 < \delta < 1$, the refracted part $E_\tau^{\gamma_-}(x,z')$ for $x \in \mathbb{R}^3_-$ and $z' \in \overline{\mathcal{U}_\delta(x)}$ is expanded by

$$E_\tau^{\gamma_-}(x,z') = \frac{e^{-\tau|x-\tilde{z}'|/\sqrt{\gamma_-}}}{4\pi\gamma_-|x-\tilde{z}'|}\Big(\sum_{j=0}^{N-1} E_j(x-\tilde{z}')\Big(\frac{\sqrt{\gamma_-}}{\tau|x-\tilde{z}'|}\Big)^j + \tilde{E}_N(x,z';\tau)\Big),$$

and for $k = 1,2,3$,

$$\partial_{x_k} E_\tau^{\gamma_-}(x,z') = \frac{-\tau e^{-\tau|x-\tilde{z}'|/\sqrt{\gamma_-}}}{4\pi\gamma_-^{3/2}|x-\tilde{z}'|}\Big(\sum_{j=0}^{N-1} G_{k,j}(x-\tilde{z}')\Big(\frac{\sqrt{\gamma_-}}{\tau|x-\tilde{z}'|}\Big)^j + \tilde{G}_{k,N}(x,z';\tau)\Big),$$

where $E_j(x-\tilde{z}')$, $G_{k,j}(x-\tilde{z}')$ ($k = 1,2,3$ and $j = 0,1,2,\ldots$) are C^∞ functions for x and z' with $z' \in \overline{\mathcal{U}_\delta(x)}$. Here, the remainder terms $\tilde{E}_N(x,z';\tau)$ and $\tilde{G}_{k,N}(x,z';\tau)$ ($k = 1,2,3$) are estimated by

$$|\tilde{E}_N(x,z';\tau)| + \sum_{k=1}^{3} |\tilde{G}_{k,N}(x,z';\tau)| \le C_{N,\delta}\Big(\frac{\sqrt{\gamma_-}}{\tau|x-\tilde{z}'|}\Big)^N \quad (x \in \mathbb{R}^3_-,\, z' \in \overline{\mathcal{U}_\delta(x)})$$

for some constant $C_{N,\delta} > 0$ depending only on $N \in \mathbb{N}$ and δ. In particular, we have

$$\begin{cases} G_{k,0}(x-\tilde{z}') = E_0(x-\tilde{z}')\dfrac{x_k - z_k}{|x-\tilde{z}'|} & (k = 1,2), \\ G_{3,0}(x-\tilde{z}') = E_0(x-\tilde{z}')\dfrac{x_3}{|x-\tilde{z}'|}, \end{cases}$$

where $E_0(x-\tilde{z}')$ is given in (3.6).

Note that for the opposite case; $\gamma_- > \gamma_+$, in Proposition 4.1, we can replace $\mathcal{U}_\delta(x)$ to the whole space $\mathbb{R}^2$ since we can take the contour $\Gamma_{x,z'}$ as the left side of figure 2.

Next, we consider the remaining case, i.e. $\gamma_- < \gamma_+$ and $\sin\theta \ge a_0$, which has the case $\sin\theta > b_0(\zeta_2)$ for some $\zeta_2 \in \mathbb{R}$. Hence, we need to make a detour to connect $\Gamma_{x,z'}$ and the branch point $\zeta_1 = ib_0(\zeta_2)$ of $P(\zeta_1,\zeta_2)$. Existence of these detours corresponds to the total reflection phenomena and makes the problems more difficult than the previous case. One of the main part of section 2 in [21] is to handle these new problems.

Proposition 4.2 (Proposition 2 in [21]) Assume that $\gamma_+ > \gamma_-$. Then, for any fixed δ with $0 < \delta < 1$, the refracted part $E_\tau^{\gamma_-}(x,z')$ for $x \in \mathbb{R}^3_-$ and $z' \in \overline{\mathcal{U}_1(x) \setminus \mathcal{U}_\delta(x)}$ is expanded by

$$E_\tau^{\gamma_-}(x,z') = \frac{e^{-\tau|x-\tilde{z}'|/\sqrt{\gamma_-}}}{4\pi\gamma_-|x-\tilde{z}'|}\Big(E_0(x-\tilde{z}') + \tilde{E}_{0,0}^{\gamma_-}(x,z';\tau)\Big),$$

$$\partial_{x_k} E_\tau^{\gamma_-}(x,z') = \frac{-\tau e^{-\tau|x-\tilde{z}'|/\sqrt{\gamma_-}}}{4\pi\gamma_-^{3/2}|x-\tilde{z}'|}\Big(G_{k,0}(x-\tilde{z}') + \tilde{E}_{k,0}^{\gamma_-}(x,z';\tau)\Big) \quad (k=1,2,3).$$

In the above, E_0 and $G_{k,0}$ are the functions given in Proposition 4.1. For the remainder terms $\tilde{E}_{k,0}^{\gamma_-}(x,z';\tau)$, for any $0<\delta<1$, there exists a constant $C_\delta>0$ such that

$$|\tilde{E}_{k,0}^{\gamma_-}(x,z';\tau)| \le C_\delta\Big(\frac{\sqrt{\gamma_-}}{\tau|x-\tilde{z}'|}\Big)^{1/4} \quad (x\in\mathbb{R}^3_-, z\in\overline{\mathcal{U}_1(x)\setminus\mathcal{U}_\delta(x)}, k=0,1,2,3).$$

Proposition 4.3 (Proposition 3 in [21]) Assume that $\gamma_+>\gamma_-$. Then, there exists a constant $C>0$ such that the refracted part $E_\tau^{\gamma_-}(x,z')$ for $x\in\mathbb{R}^3_-$ and $z'\in\mathbb{R}^2\setminus\mathcal{U}_1(x)$ is estimated by

$$|\nabla_x^k E_\tau^{\gamma_-}(x,z')| \le C\tau^k e^{-\tau T_{x,z'}(\theta_0)} \qquad (x\in\overline{D}, z'\in\mathbb{R}^2\setminus\mathcal{U}_1(x), k=0,1),$$

where for $x\in\mathbb{R}^3_-$ and $z'\in\mathbb{R}^2$, $T_{x,z'}(\alpha)$ is defined by

$$T_{x,z'}(\alpha) = \frac{1}{\sqrt{\gamma_-}}\Big(|x_3|\cos\alpha + |z'-x'|\sin\alpha\Big).$$

It seems hard to obtain higher order expansion for θ near θ_0 and even to pick up the first term for $\theta>\theta_0$. For the stationary waves, reduced equation is given by

$$(L_{\gamma_0}+k^2)w = f(x) \qquad \text{in } \mathbb{R}^3. \tag{4.6}$$

In this case, recently, Isozaki, Kadowaki and Watanabe [22] give a precise asymptotics of the fundamental solution of (4.6) as $|x|\to\infty$ for $k>0$, and get the main term even in the case of $\theta>\theta_0$. Since our case corresponding to $k=i\tau$, $\tau>0$, the results given in [22] can not be applied to our case.

From (4.1) and Proposition 4.1, for $\gamma_->\gamma_+$, we can reduce the problem to applying the Laplace method of integrals of the form:

$$I_-(\tau;x;y) = \int_{\mathbb{R}^2} e^{-\tau l_{x,y}(z')} a_-(z')dz',$$

where $l_{x,y}$ is defined by (1.2) and $a_-(z')$ is a $\mathcal{B}^\infty$ function in $\mathbb{R}^2$ (i.e. $a_-\in C^\infty(\mathbb{R}^2)$ satisfying $\sup_{z'\in\mathbb{R}^2}|\partial_{z'}^\alpha a_-(z')|<\infty$ for any α). Noting (1.1), we can show Proposition 3.2 by using a usual Laplace's method (cf. the proof of Proposition 1 of [20]).

If $\gamma_- < \gamma_+$, from (4.1), Proposition 4.1, Proposition 4.2 and Proposition 4.3, we can reduce the problem to giving asymptotics for integrals of the forms:

$$I_+(\tau; x; y) = \int_{\mathcal{U}_{\delta_1}(x)} e^{-\tau l_{x,y}(z')} a_+(z') dz'$$

for some $1 > \delta_1 > 0$ sufficiently close to 1 (determined in Lemma 4.1 below), and evaluate integrals of the form:

$$I_{+,-\infty}(\tau; x; y) = \int_{\mathbb{R}^2 \setminus \mathcal{U}_{\delta_1}(x)} e^{-\tau \tilde{l}_{x,y}(z')} \tilde{a}_+(z') dz',$$

where

$$\tilde{l}_{x,y}(z') = \begin{cases} l_{x,y}(z') & (z' \in \mathcal{U}_1(x)), \\ T_{x,z'}(\theta_0) + \dfrac{|\tilde{z}' - y|}{\sqrt{\gamma_+}} & (z' \in \mathbb{R}^2 \setminus \mathcal{U}_1(x)), \end{cases}$$

$a_+ \in \mathcal{B}^\infty(\mathbb{R}^2)$ and $\tilde{a}_+ \in \mathcal{B}^0(\mathbb{R}^2)$. Hence, we find that the shortest lengths in this case is given by $\inf_{z' \in \mathbb{R}^2} \tilde{l}_{x,y}(z')$.

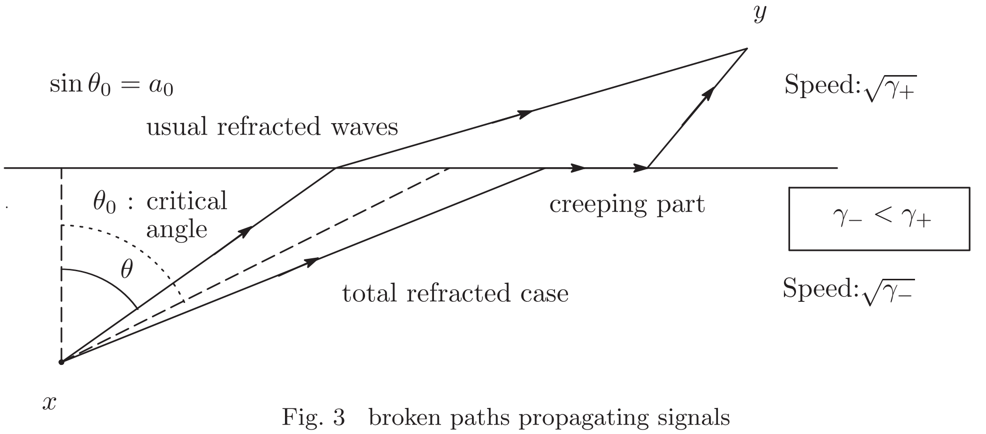

Fig. 3 broken paths propagating signals

As in section 3 of [21], for $z' \in \mathbb{R}^2 \setminus \overline{\mathcal{U}_1(x)}$, we can see that

$$\begin{aligned} \tilde{l}_{x,y}(z') &= \frac{|x - \tilde{z}'|}{\sqrt{\gamma_-}} \big\{ \cos\theta \cos\theta_0 + \sin\theta \sin\theta_0 \big\} + \frac{|\tilde{z}' - y|}{\sqrt{\gamma_+}} \\ &= \frac{|x - \tilde{z}'|}{\sqrt{\gamma_-}} \cos(\theta - \theta_0) + \frac{|\tilde{z}' - y|}{\sqrt{\gamma_+}} < \frac{|x - \tilde{z}'|}{\sqrt{\gamma_-}} + \frac{|\tilde{z}' - y|}{\sqrt{\gamma_+}} = l_{x,y}(z'), \end{aligned} \tag{4.7}$$

which means that for the case of the total refraction phenomena, the paths containing creeping part along the layered boundary are shorter than the usual paths.

Thus, we can say that the total refracted phenomena make some influence for the fundamental solution $\Phi_\tau(x,y)$. However, Proposition 3.2 shows that the total refracted phenomena do not affect the asymptotic behavior. From Remark 1 of [21], we obtain

$$T_{x,y}(\theta_0) = \frac{|\tilde{z}_0' - x|}{\sqrt{\gamma_-}} + \frac{|z_0' - z'|}{\sqrt{\gamma_+}},$$

where $z_0' \in \mathbb{R}^2$ is the point on the line segment $x'z'$ satisfying $|x' - z_0'|/|x - \tilde{z}_0{}'| = \sin\theta_0$ and $|x' - z'| = |x' - z_0'| + |z_0' - z'|$. Hence, we have

$$\tilde{l}_{x,y}(z') = T_{x,y}(\theta_0) + \frac{|\tilde{z}' - y|}{\sqrt{\gamma_+}} = \frac{|\tilde{z}_0' - x|}{\sqrt{\gamma_-}} + \frac{|z_0' - z'|}{\sqrt{\gamma_+}} + \frac{|\tilde{z}' - y|}{\sqrt{\gamma_+}} \geq l_{x,y}(z_0'),$$

which yields $\inf_{z' \in \mathbb{R}^2} \tilde{l}_{x,y}(z') = l_{x,y}(z'(x,y)) = l(x,y)$. More precisely, we can obtain the following properties:

Lemma 4.1 (Lemma 3.1 of [21]) Assume that $\gamma_- < \gamma_+$.
(1) For any $x \in \mathbb{R}^3_-, y \in \mathbb{R}^3_+$, $\inf_{z' \in \mathbb{R}^2} \tilde{l}_{x,y}(z') = l(x,y)$, and this infimum is attained at only $z' = z'(x,y)$.
(2) There exists a constant $0 < \delta_0 < 1$ such that $z'(x,y) \in \overline{\mathcal{U}_{\delta_0}(x)}$ $((x,y) \in \overline{D} \times \overline{B})$, and for any $\delta_1 > 0$ with $\delta_0 < \delta_1$, there exists a constant $c_0 > 0$ such that

$$\tilde{l}_{x,y}(z') \geq l(x,y) + c_0|z' - z'(x,y)| \qquad ((x,y) \in \overline{D} \times \overline{B}, z' \in \mathbb{R}^2 \setminus \mathcal{U}_{\delta_1}(x)).$$

From (2) of Lemma 4.1, it follows that there exist constants $c_0 > 0$ and $C > 0$ such that

$$|I_{+,-\infty}(\tau;x;y)| \leq Ce^{-\tau l(x,y)}e^{-c_0\tau} \qquad (\tau \geq 1).$$

This estimate means that the term $I_{+,0}(\tau;x;y)$ gives the principal part of $\Phi_\tau(x,y)$ as $\tau \to \infty$. Thus, in view of (1) of Lemma 4.1, the same argument as for the case $\gamma_- > \gamma_+$ is applicable even in the case containing the total refracted waves, and we obtain Proposition 3.2.

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